Measure & Integration
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Lecture - 09 B
Extension of Measure

Finite, so let us assume.

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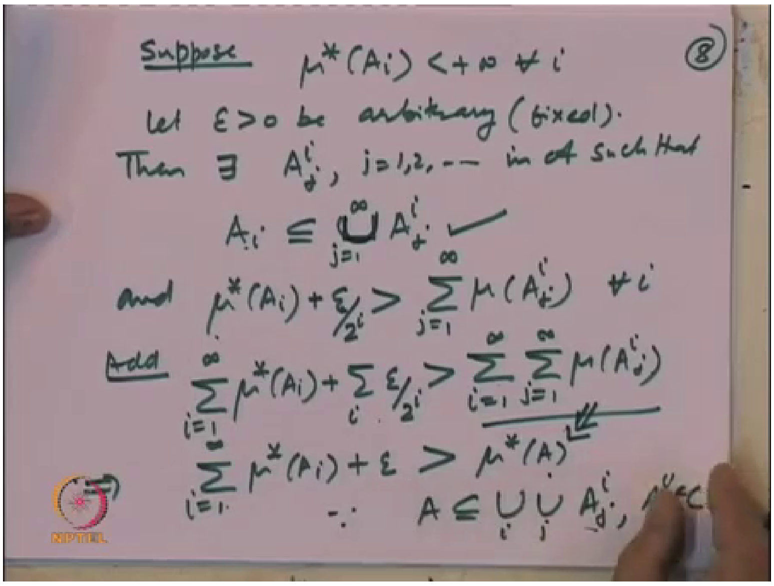

So, suppose mu star of each A i is finite for every I, now what is mu star of A i? mu star of A i is infimum of a certain collection. So, here we are going to use the property of something being infimum and that being finite. So, let epsilon greater than 0 be arbitrary of course, fixed we choose arbitrarily and fix it, then mu star of A i is infimum or all summations approximate sizes. So, then there exists at least one covering. So, there exist sets say A i j j equal to 1 2 so on in the algebra A, such that this A i is contained in this disjoint union of A i js. And mu star of A i which is infimum if to this I add the small number epsilon this become bigger than summation mu of A i js, j equal to 1 to infinity.

So, this is let me stress here this is the kind of definition or this is the kind of analysis will be coming across will be doing again and again. So, let us be very clear about this we have got some number which is the infimum over some collection, and if this infimum is finite then the infimum plus a small quantity epsilon cannot be the infimum because that is on the right side of it. So, that cannot be the infimum of that collection right otherwise alpha plus if alpha is infimum then alpha plus epsilon will be the infimum which contradicts the definition of the infimum. So, if alpha is the infimum alpha plus the small number epsilon any small number epsilon cannot be the infimum; that means, what? That means, there must be a member of the collection over which you are taking infimum which so that alpha plus epsilon becomes bigger than that number in the collection over which you are taking infimum.

So, that is what we are saying that it because mu star of A i is finite. So, given epsilon, the infimum plus epsilon must be bigger than a member of the collection over which you are taking infimum. So, then what is the collection that is obtained by taking a covering a disjoint covering of disjoint covering not really that is really disjoint actually any covering we are taking. So, any covering and say such that this is true.

So, given epsilon there exists a covering A i j, j equal to 1 to infinity of A i say that mu star of A i plus epsilon is bigger than this and this happens for every i. So, if you add up. So, add these equations over i. So, summation over i equal to 1 to infinity, mu star of A i plus sigma alpha over i is bigger than sigma over i equal to 1 to infinity sigma over j equal to 1 to infinity, mu of A i j right and that is what we wanted mu star of A i is bigger than something we have got that kind of inequality, now the problem is this we are going to add epsilon infinite number of times.

So, this will tend to become infinity and we do not want that. So, we go back and refine our estimates. So, given epsilon bigger than 0, this we can do it for any epsilon. So, in particular whenever we are looking at for A i, given epsilon there should exist a covering say that will refine it will make it 2 to the power i. So, we will change our epsilon that is true for every epsilon. So, in particular it should be true for this. So, what we are saying is given epsilon there is a covering such that A i is covered by that collection and mu star of A i plus epsilon divided by 2 to the power i is bigger than the approximate size that is mu the summation mu of A i j.

Now, this is for every i now if i add here is epsilon 2 to the power 2 i so; that means, we have got this is now convergent. So, that implies that sigma i equal to 1 to infinity, mu star of A i plus epsilon is bigger than this sum and now note if i and j both vary. So, this is for every i now if I take the union over is that will be union over this. So, I will get a covering of union A j’s which will be covered by this right and a is inside this.

So, what we are claiming is this is bigger than mu star of A because a is contained in union over I, union over j, A i js and this A i js belong to C. So, a is covered by this countable union and this is one approximate size for mu of a. So, that is always bigger than the equal to mu star of A because that is a infimum. So, this quantity is implies that this is always bigger than this. So, I can claim that mu star of summation is bigger than this quantity now this epsilon is arbitrary that was fixed arbitrarily. So, I can let that go to infinity. So, one writes.

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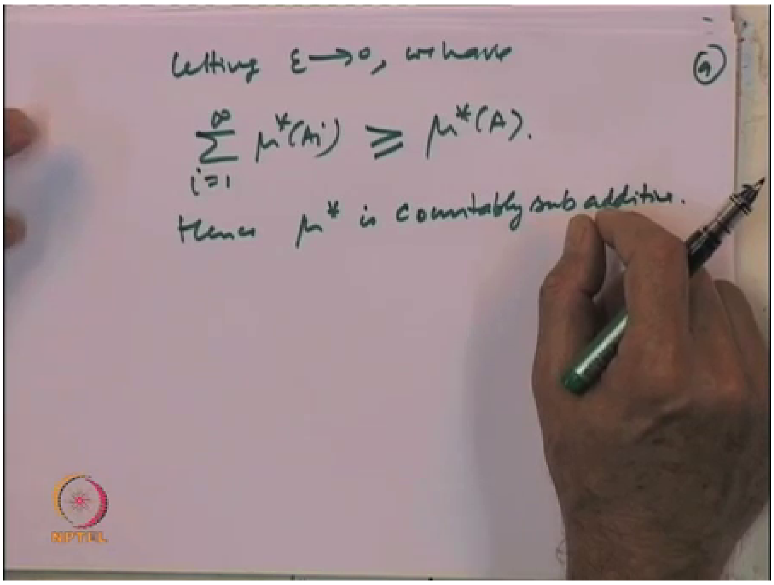

So, letting epsilon go to 0 we have sigma mu star of A i, i equal to 1 to infinity, this epsilon becomes 0 eventually now I will write bigger than or equal to because in the limit it can become bigger than or equal to mu star of A and that shows. So, hence mu star is countably sub additive. So, that we have proved is countably sub additive, I just want to go through the proof of this once again because this is an important kind of analysis may be doing again and again.

Let us just revise the proof once again that mu star is countably sub additive. So, to show that mu star is countably sub additive we have to show that if a is A sub set of X and A is contained in union A is, A is contained in X then I have to show that mu star of A is less than or equal to summation mu star of A is. Now to show this the first observation which would should keep in mind that whenever we are trying to show that one number is less than or equal to summation of a collection of numbers then and obvious case we arise namely one of the numbers may be equal to plus infinity.

So, if mu of A i is equal to plus infinity for some i then clearly this side is equal to plus infinity and mu star of A is always less than or equal to plus infinity. So, we get mu star of A less than or equal to plus infinity, and which is always less than this sum so; that means, that property is true. So, the obvious case is mu star then mu star of A i is finite for some i. So, what is the other possibility? Other is that mu star of A i is finite for every i.

Now here is the main construction part of the construction that we are going to use namely it is an infimum which is a real number. So, given epsilon bigger than 0 arbitrary, we can find a covering A i j of the set A i such that mu star of A i plus this small number and that small number will make it dependent on i, the stage at which we are doing epsilon divided by 2 to the power i bigger than the approximate sizes over which you are taking the infimum.

So, once again the property of infimum being a real number is used here nothing more than that. So, once that is done, you add both sides this is for every I take the summation on both sides. So, summation mu star of A i plus summation of this over i is less than is bigger than summation of mu of A i j. Now this is a convergence series it sum is equal to epsilon. So, this is mu star of A i summation plus epsilon and this the quantity on the right hand side is an approximate size of A that is this is bigger than or equal to mu star of A. mu star of A is infimum or all such numbers because a is covered by union over i union over j A i is covered by A i js.

So, union over a is will be covered by this union and a is inside it. So, this is. So, this implies that summation mu star of mu of A i js summation over i and j is bigger than mu star of A; and once that is done that means, that we have got and let go epsilon go to 0. So, we get this quantity. So, that says that mu star is countably sub additive. So, we let us.

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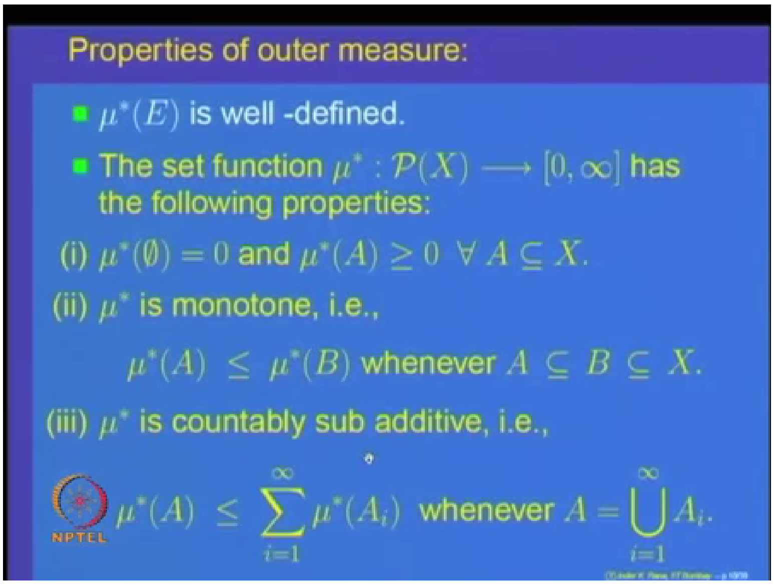

So, we have proved this property that mu star is countably sub additive. So, mu star now the only thing left to be shown is that that mu star actually is an extension otherwise all this process will be a waste. So, we want to claim that mu star is indeed and extension of mu.

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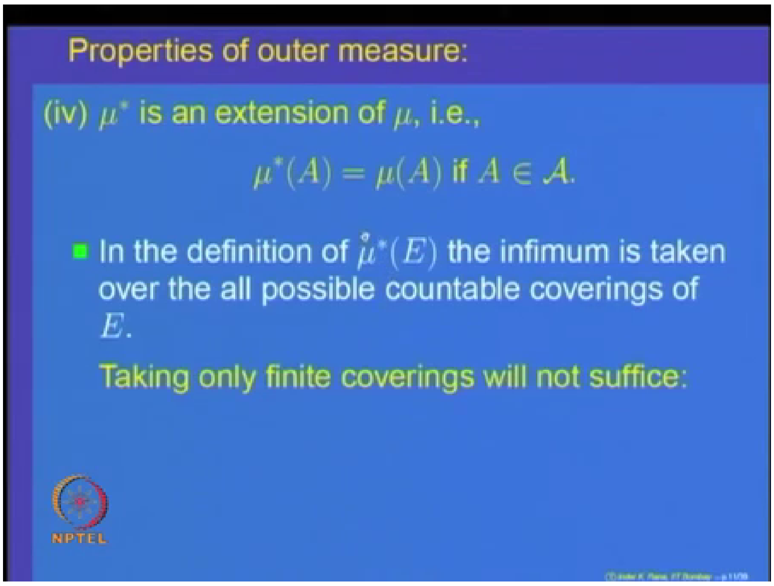

Mu star is not countably additive, but at least we should check it is an extension and it is countably sub additive that we already checked. So, we want to check that mu star of A is equal to mu of A, if A is in A. So, to check that let us look at the proper definition.

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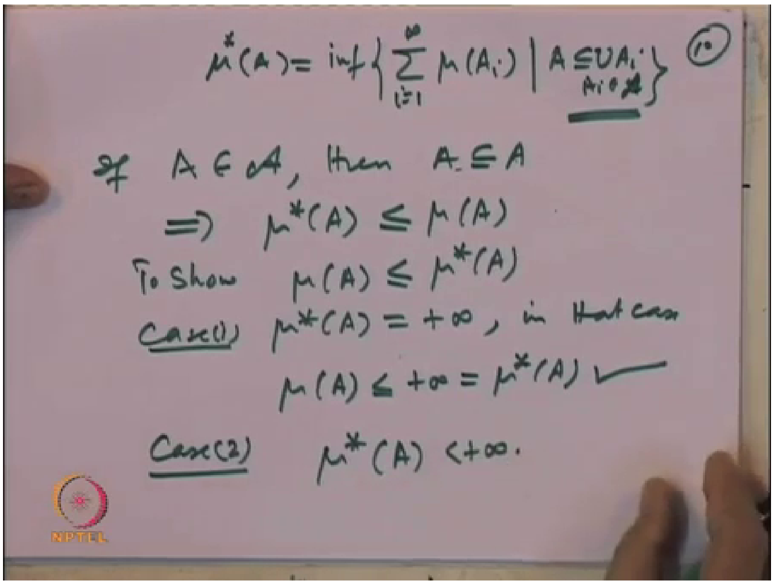

So, we had mu of mu star of A is equal to infimum over summation mu of A I, i equal to 1 to infinity where A is contained in union of A is of and A i is belong to c belong to the algebra A now. So, if A belongs to the algebra then A is actually equal to A. So, A is contained inside A. So, this is one of the elements are in the covering a itself covers it. So, it will appear in one of it will be one of the elements, over which you are going to take the infimum. So, that implies that mu star of A which is the infimum is less than or equal to mu of a right. So, that property is by the shear fact that A is covered by itself and A is in the algebra.

So, that is we want to prove other way round equality to show that mu of A is less than or equal to mu star of A. Now once again we want to show that one number is less than the other number. So, there is an obvious possibility case one that mu star of A is equal to plus infinity. So, in that case this is plus infinity and mu star mu of A is always less than or equal to plus infinity which is equal to mu star of A. So, that is obvious. So, the obvious case is when mu star of A is equal to plus infinity. So, let us look at case 2 mu star of A is finite. So, in that case again we are going to use the definition of infimum. So, mu star of A is the infimum of all possible approximate sizes summation so on.

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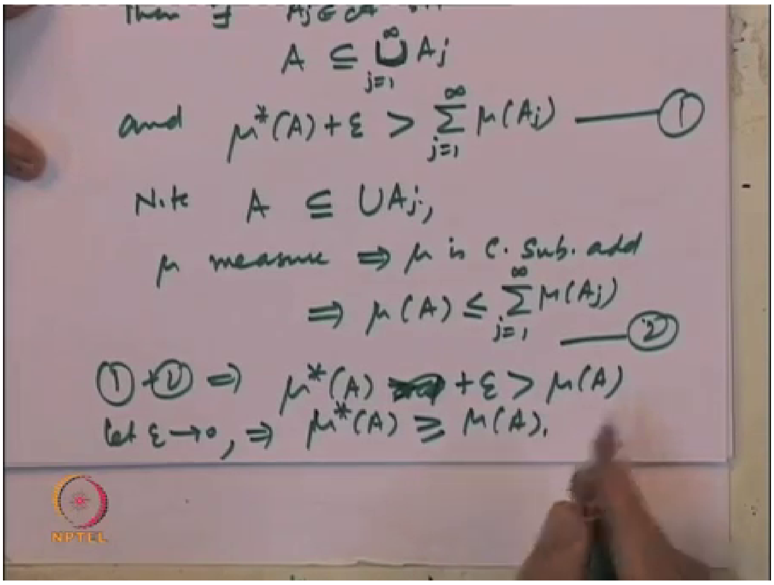

So, let Epsilon greater than 0 be arbitrary then there exist a covering. So, there exists sets A j's belonging to the algebra, such that a is contained in the disjoint union of A j's and the infimum says that mu star of A plus epsilon cannot be the infimum that has to be bigger than summation mu of A j's.

So, that is at least one such covering possible so that this is infinity this is not necessarily disjoint can make it we will show see it later on. So, this is finite, now note A is contained in union of A j's and all of them are elements in the algebra we assume A is in the algebra. So, everything is in the algebra. So, and mu is a measure and we showed every measure implies mu is countably sub additive and that implies that mu of A is less than or equal to summation mu of A j's, j equal to 1 to infinity. So, look at this equation one look at this equation 2.

So, what does one and two imply mu star of A plus epsilon is bigger than this sum and that sum is bigger than mu of A. So, one and two imply that mu star of A is bigger than mu star of A plus epsilon. So, plus mu star of A plus epsilon is bigger than mu of A and epsilon is arbitrary. So, let epsilon go to 0 and that implies that mu star of A is bigger than or equal to mu of A. So, that proves the other way around equality also in the case when mu star of. So, once again mu star of A is less than or equal to mu of A because A is one of the members which is covering it.

So, mu of A is elements. So, mu of A is mu star of A is infimum. So, that is less than or equal to that is obvious property. And to show that the case when it is finite mu star of A is finite, we look at once again the definition given absolutely bigger than 0 there is a covering. So, that this holds the infimum plus epsilon is bigger than one of the elements over which you are taking the covering, and now using the fact that mu is countably sub additive this is bigger than or equal to mu of A and hence that proves the required property.

So, what we have shown is that there are mu star is indeed and extension of mu of A.

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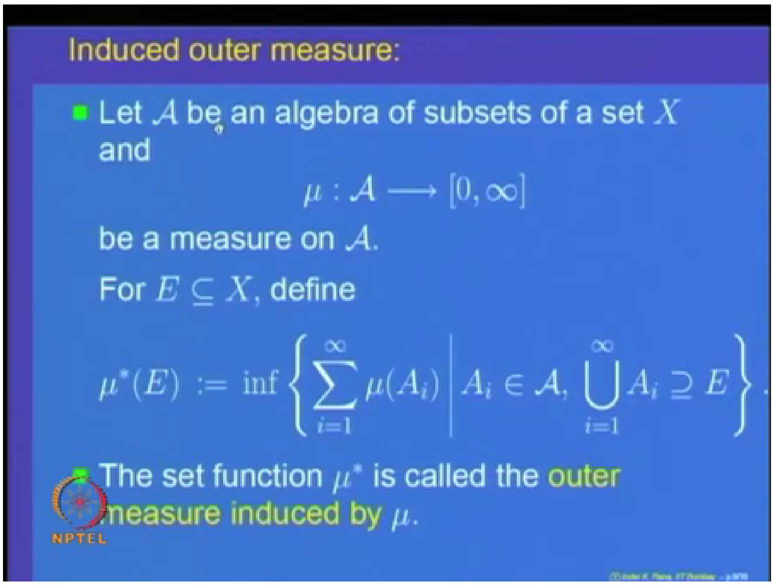

So, let us go back and have a look at what we have done is the following; we started with a measure mu on the algebra A measure means it is mu of empty set is equal to 0 and mu is countably additive. We are trying to extend it. So, we try to find out the size of any set by looking at sizes of sets in a. So, take any set E cover it by elements in the algebra A and look at the sizes of mu call it as mu A i, right. So, take the summation. So, this gives an approximate size of the set E look at the smallest possible of this numbers call it the infimum. So, mu star of E the induced outer measure is defined as the infimum over all this summations and the summations arise from coverings of E.

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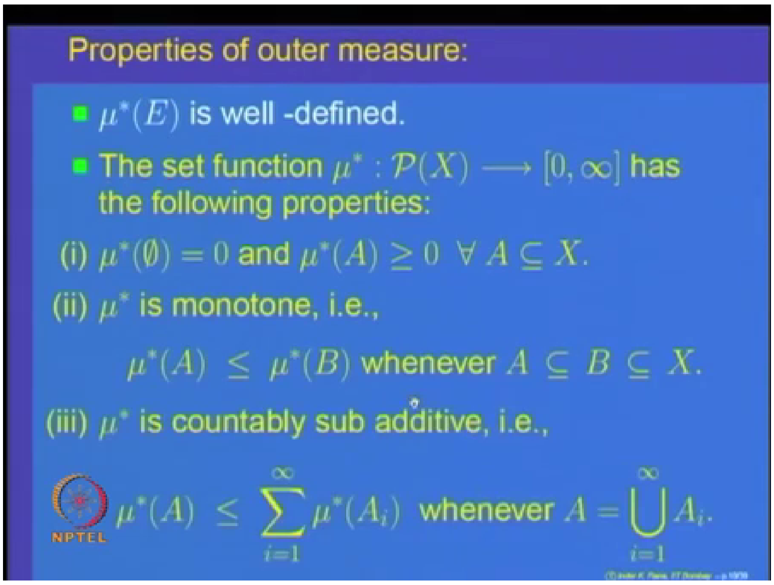

So, this is called the outer measure and we showed it is well defined, we showed it is it has the obvious property namely mu of empty set is equal to 0 mu star of A is bigger than or equal to 0, it is monotone and so; that means, mu star of A is less than or equal to mu of B and mu star is countably sub additive and finally, it is an extension. So, one let me point it out that we have mu star of A as the infimum over those summations and we have taken the coverings which are countable in number. One can ask the question cant we take only finite coverings now instead of countable coverings of E. So, let us give an example to show that that is not possible to do that the finite coverings will not suffice.

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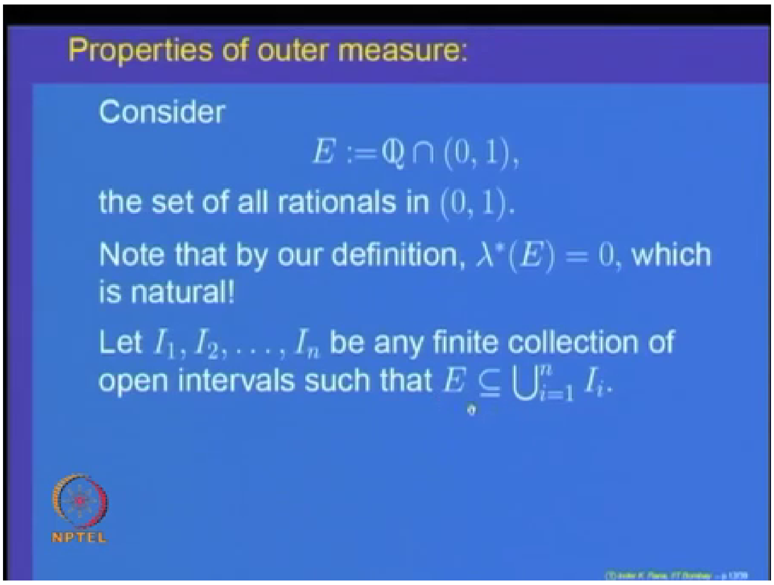

So, let us look at the set E the case is the real line we look at the set E which is rationals intersection with 0 to 1. So, we are looking at all the rationals in the set in the intervals 0 one

Clearly lambda star of E we expected to be equal to 0 why we expect size of this because is the countable set and the length of each singleton is equal to 0. So, we expect the length of each when add it together this also should remain small or lambda star of E is equal to 0 right its quite natural now suppose we define. So, this is when lambda star is defined by taking countable coverings. Now let us take a finite covering of E by interval. So, E is covered by finite number of intervals union E i.

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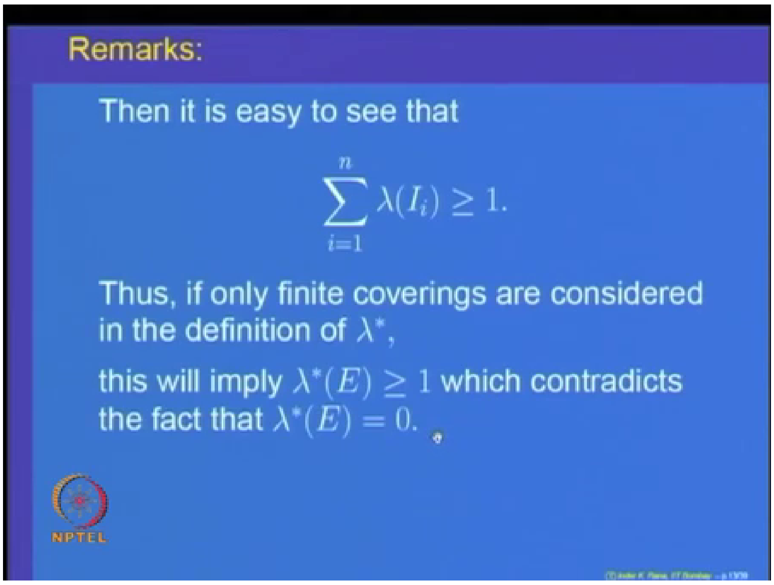

We claim that in that case this number the approximate size of E will always be bigger than or equal to 1, because of the following reason.

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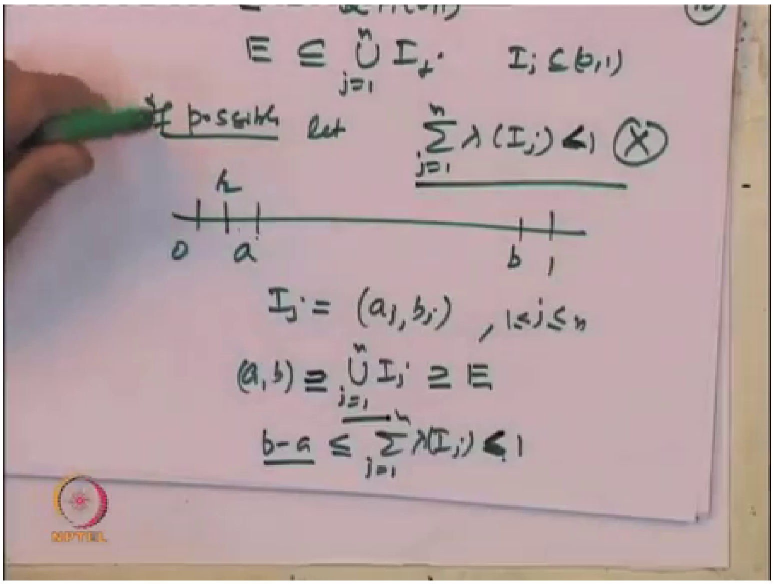

What is E? E is rationals inside 0 1, and suppose E is covered by union i js j equal to 1 to n right. So, suppose if possible let sigma lambda of i j, j equal to 1 to n be less than or equal to 1 now these are finite is a finite collection of. So, here is 0 here is 1 and i js are intervals of course, intervals in 0 1 which are covering. So, i js are in 0 one which are covering the set E which is rationals in 0 1. So, now,. So, let us say that i j say for the sake of just definition it is a j, B j it does not matter was it open or close you can just assume to be open it does not matter much actually then we have got this numbers between A j's and B j's. So, look at all the left end points and look at the smallest of them let us say the smallest is here that is a. So, what is a? A is the smallest of the numbers a 1 a 2 a n and B j look at the largest of B j's and call that as b.

Then this a b is equal to left may be closed does not matter is equal to union of i js or at least it will cover the union of i js j equal to 1 to n and they cover E now. So, and that covers E and now if this is less than or equal to this is the smallest and that is the largest ones which is covering. Now this number a so; that means, what? That means, b minus a is less than or equal to summation length of i js j equal to 1 to n and if that is less than or equal to 1 that means, b minus a is strictly less than 1 that means, it has to be like this, but then there is a rational here between 0 and 1 which belongs to E and E is inside a b. So, that is not possible.

So, that will be contradiction. So, this situation is not possible; that means, whenever this are covering we have to have lambda of i is bigger than or equal to 1, but; that means, all approximate sizes of E is bigger than one, that will imply that lambda star of E is bigger than or equal to 1 that means, but that is not possible because we just now said lambda star of E should be equal to 0. So, in the definition of the outer measure we cannot limit ourselves to only finite coverings, we have to allow all countable coverings also.

So, today we have tried to go beyond algebra. So, we started with the semi algebra and a measure on it, we extended it to a measure on the algebra generated by eight as a first step. As a next step we started with the measure on an algebra and we showed that by an example so on the real line ulam's theorem that you cannot extend it to all sub sets of real line. So, let us try to go as much as far as possible. So, we define given a measure mu on an algebra we defined the notion of an outer measure for any sub set a of that set x, and we showed this outer measure has some nice properties, it extends one the given measure it is monotone which is countably sub additive.

So, in the next lecture we will see how to get from it an actual extension which is a measure.

Thank you.