Measure & Integration
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Lecture – 09 A
Extension of Measure

Welcome to lecture 9 on measure and integration. As you recall we have been looking at classes of sub set, sub set X called semi algebra, algebra, sigma algebra and so on. And then we all also looked at set functions defined on these classes, which is a properties. So, in particular, we define the concept of measure. A measure is a set function defined on a collection of sub sets; such that the measure of mu of the empty set is equal to zero, and mu is countably additive. Today we are going to start the process which is called extension process. So, the topic for today's discussion is going to be extensions of measures, a basically the question arises from some properties on the real line.

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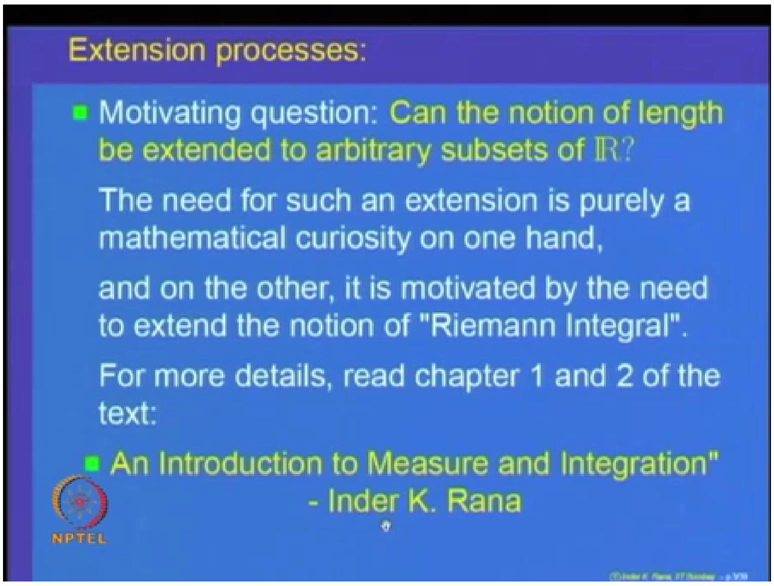

Let us look at mathematically the question. We know that notion of length is defined for all intervals. So, the question is, can the notion of length be extended to arbitrary subsets of the real line; that means, can we define the notion of the length for an arbitrary sub set of the real line. Of course, you should be compatible with the definition of the length for the interval. So, the need for such, and extension one; of course, is purely a mathematical curiosity that we have the notion of length for an interval. Can we define it for a arbitrary sub set. Another reason which is more important is that. It arises from some problems, in Riemann integration the concept of Riemann integral which is defined for a class of functions fails to satisfy some nice properties, like you if a function is the fundamental theorem of calculus, does not hold for Riemann integrable functions

So, in order to remove those difficulties on started for looking for an extension of Riemann integral, and that let though the problem of extending the notion of length from a class of sub sets; that is intervals to all sub sets possible. and if you are interested in looking at more details about that why Reimann integral should be extended to a wider class of functions, and how that leads to the concept of extending the notion of length to arbitrary sub sets, read chapter one and two of the text book, that we have mentioned earlier; namely an introduction to measure and integration by myself Inder K Rana.

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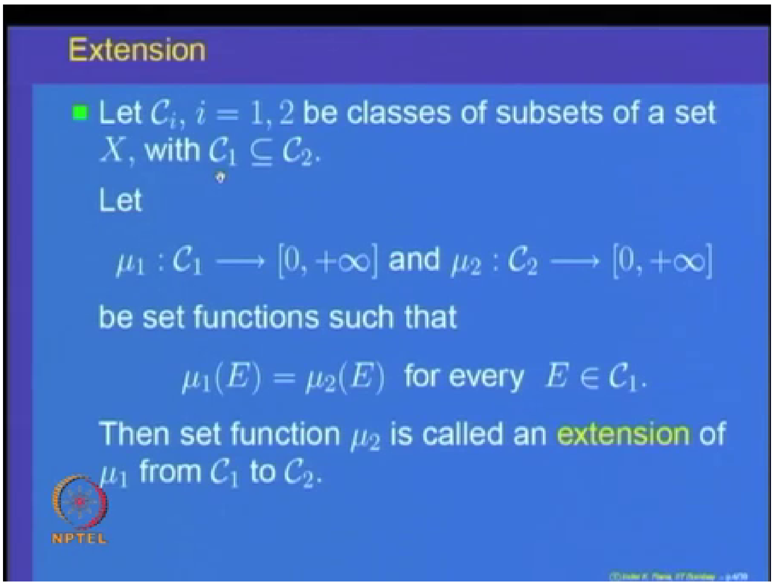

So, let us start with the question of what is an extension. So, let C 1 and C 2 be two classes of sub sets of a set X, and let us assume C 1 is a sub set of C 2. We have to measures two set functions mu 1 and mu 2. Mu 1 is defined on C 1, and mu 2 is defined on the collections C 2. So, mu 1 and mu 2 are set functions, as mu 1 defined on C 1 and mu 2 defined on C 2 with the property that mu 1 on C 1 is same as mu 2 for sub sets of C one. So, mu 1 and mu 2 agree on subsets of C 1. Recall C 1 as sub collection of C 2. So, in such a case we call mu 2 is an extension of mu 1. So, on C 1, which is a smaller class mu 1 and mu 2 are same. So, and mu 2 is defined on a bigger class; that is C 2. So, we say C 2 or mu 2 is an extension of the measure of the cresset function C 1

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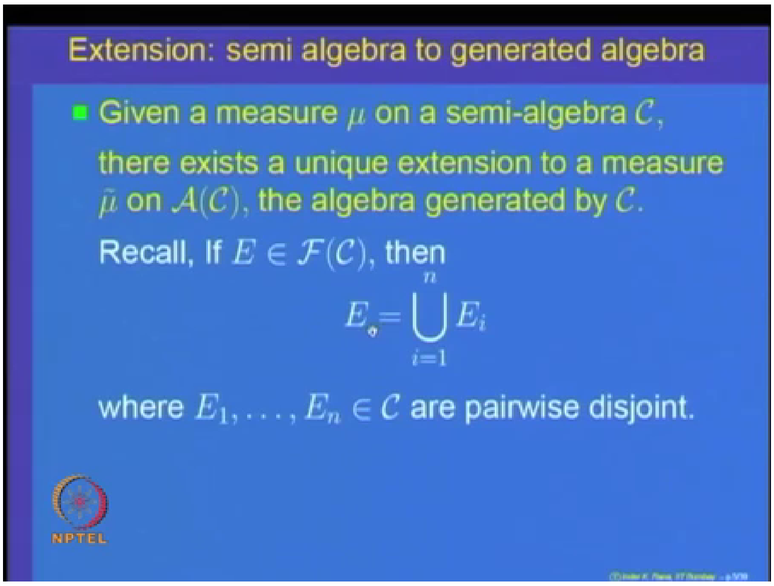

So, the problem is given a measure mu. We start with a measure mu on a semi algebra C of sub sets of a set x. We want to show you that there exists a unique extension to a measure mu tilde a of C, the algebra generated by it. So, this is going to be our first step of extension theory; namely given a measure on a semi algebra. We are going to extend it to a measure on the algebra generated by that semi algebra. So, let us see how this process is carried over. So, recall that a set E in the algebra generated by a semi algebra. We are characterized such sets can be given by a representation E is equal to union i 1 to m E i. So, every set in the algebra generated by a semi algebra is a finite union of sets, from the semi algebra, and in addition their pair wise disjoint. So, this was the result we had proved that the algebra generated by a semi algebra is nothing, but all finite disjoint union of sets in the semi algebra. So, let us take any set E in the semi algebra.

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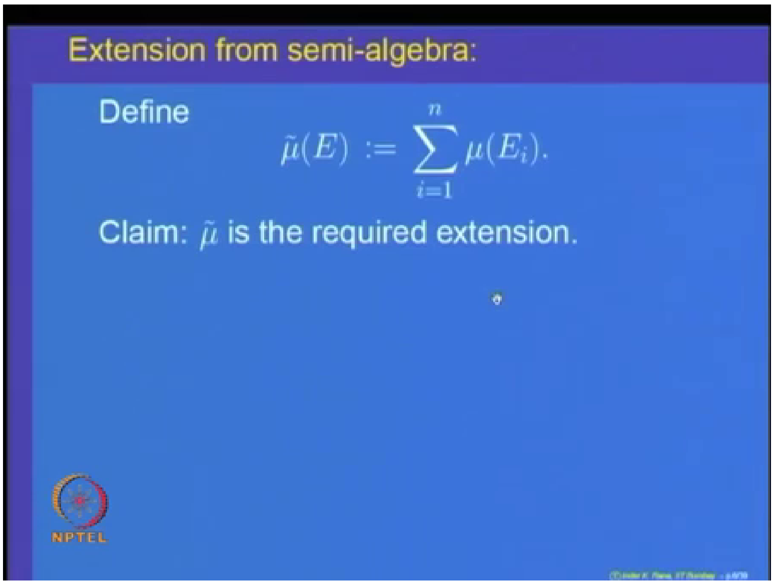

So, we define mu tilde of E to be a sigma i 1 to n mu of E i and the claim is that, this is the unique extension which we are looking for. So, let us see how do we do it.

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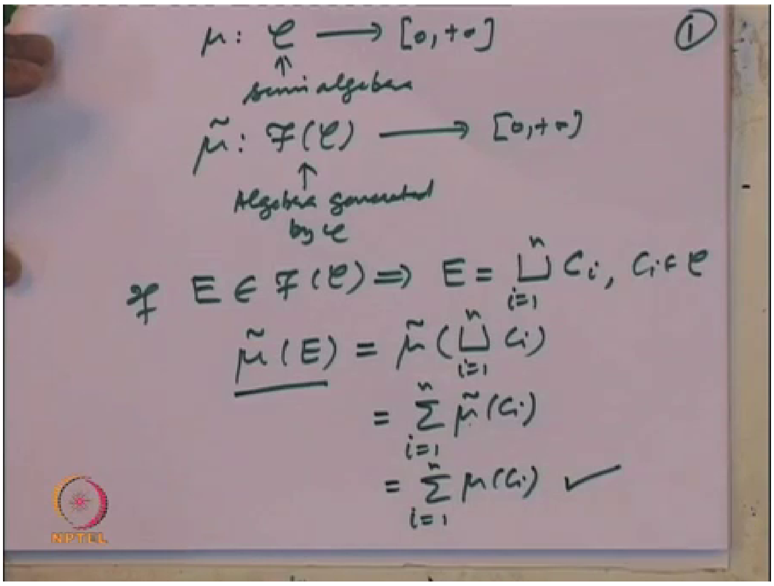

So, we have got mu on C, and this is a semi algebra. So, we define mu tilde on the algebra generated by C So, this is algebra generated by C So, we want to define a function here, a set function which should look like an extension. So, if E belongs to F of C, then we know this is set E looks like a disjoint union of element C i i equal to 1 to n for sum n C, i belonging to C. Now why we defined it the way we have defined mu tilde.

See if mu tilde of E is going to be defined, and it is going to be measure on the algebra F of C, then we know that every measure is also finitely additive.

So, by the finite additivity property of mu tilde, which we have not yet defined, but the finite additivity property will say that this should be equal to mu tilde of the union C i i equal to 1 to n, and this being finitely additive. We should have i equal to 1 to n mu tilde of C i, but mu tilde is going to be an extension. So; that means, mu tilde on C i is as same as mu on C i. So, this is same as 1 to n of mu of C i. So, that actually fixes, what is going to be the definition of mu tilde of e. So, if E is a finite disjoint union of elements which is C i's, then mu tilde of E must be given by this, and that also shows the uniqueness of the definition of mu tilde. So, mu mu tilde should be defined by this. It is necessary and we will show that actually this definition works also. So, let us prove this property that mu tilde.

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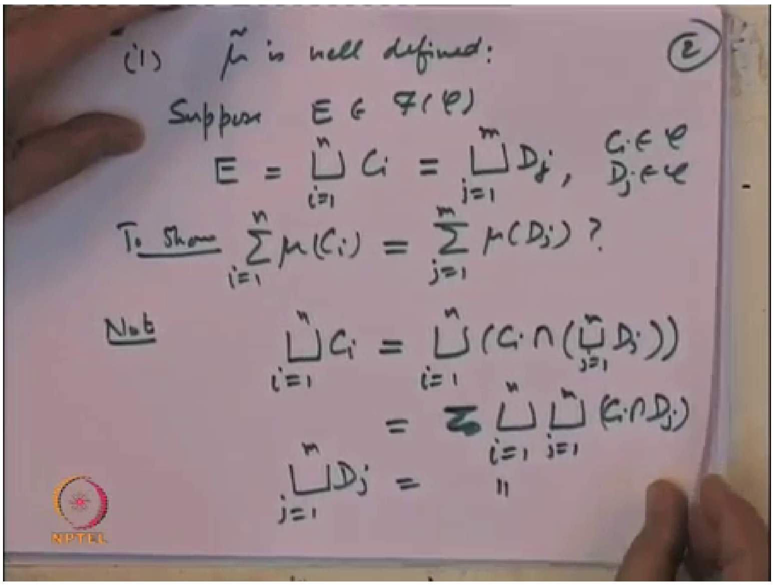

So, first we want to show that mu tilde is well defined. So, what does that mean? So, suppose E is a set which is in F of C, then we know that E can be written as a finite union of set C i's finite this disjoint union of set C i in C, but it is possible, it can have some other representation. So, it is possible the results were representable as j equal to 1 to m of some sets D i where C i is belong to C, and D j's also belong to, D j's also belong to C So, to show them, because our definition depended on the representation. So, we should show that mu of C i summation i equal to 1 to n is same as summation mu of D j, j equal to 1 to m. So, this is we should show, then only we can claim if you can able to show this, then only we can claim that our function mu tilde a is well defined

So, let us show this. So, now, note, because E is given by this two different representations. So, i can write union C i i equal to 1 to n, also as union C i intersection union D j's; j equal to 1 to m. So, that is equal to sigma. sorry that is equal to union i equal to 1 to n union j equal to 1 to m, C i intersection D j, and similarly union D j's j equal to 1 to m is also representable by the same way, because the two sets are same. So, it is same representation right.

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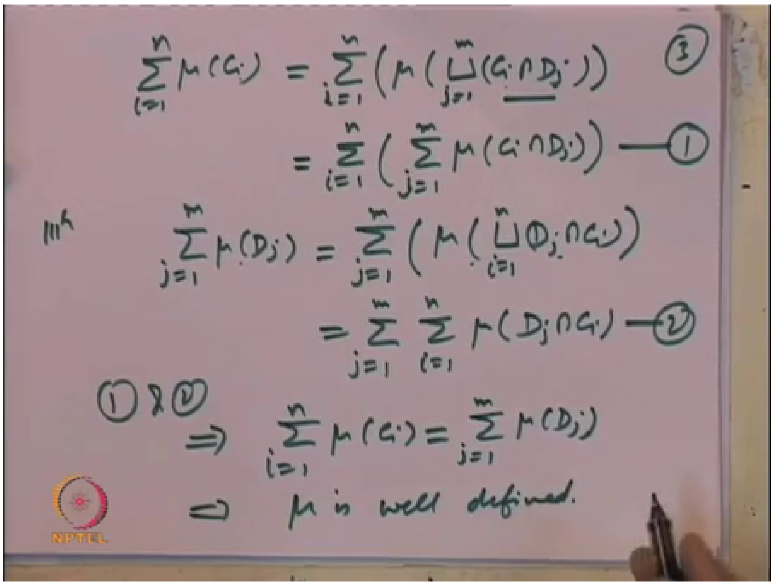

So, now let us compute. So, let us compute sigma mu of C i i equal to 1 to n, I can write it as sigma i equal to 1 to n. Now this mu of C i is this disjoint union of C i intersection D j; that is why, where we are using this representation that you just now wrote j equal to 1 to m, and this is a disjoint union C i is belong to the semi algebra, D j is belong to the semi algebra. So, this intersection belongs to the semi algebra, and their unions, is C i which is also in the semi algebra, and mu is a measure on the semi algebra. So, this is also finitely additive.

So, it is i equal to 1 to n. So, I can write this sigma equal to j equal to 1 to m mu of C i intersection D j. similarly we can also write j equal to 1 to m mu of D j to be equal to this is summation j equal to 1 to m, and mu of D j. So, that I can write as union of D j intersection C i i equal to 1 to n. And now again by finite additivity property this is j equal to 1 to m sigma i equal to 1 to n mu of D j intersections C i. So, look at this equation one, look at this equation two, one and two imply that sigma i equal to 2 to n mu of C i is equal to sigma j equal to 1 to m mu of D j

So, that says. So, implies mu is well defined. So, what we have shown is the following, that we take any set in the algebra generated by the semi algebra. So, that has got a representation in terms of the elements of the semi algebra. So, any element E in the algebra generated by the semi algebra can be represented as a finite disjoint union of elements in the semi algebra; say C i. So, pick up any such representation and define mu tilde of E to be equal to sum of and mu's of this P C C i sigma i equal to 1 to n. It does not matter which representation you choose, you will always get the same sum. So; that means, mu tilde of E is well defined. So, now, let us look at the next property, namely that mu tilde which is defined on the algebra, generated by the semi algebra is finitely additive. So, let us move that property.

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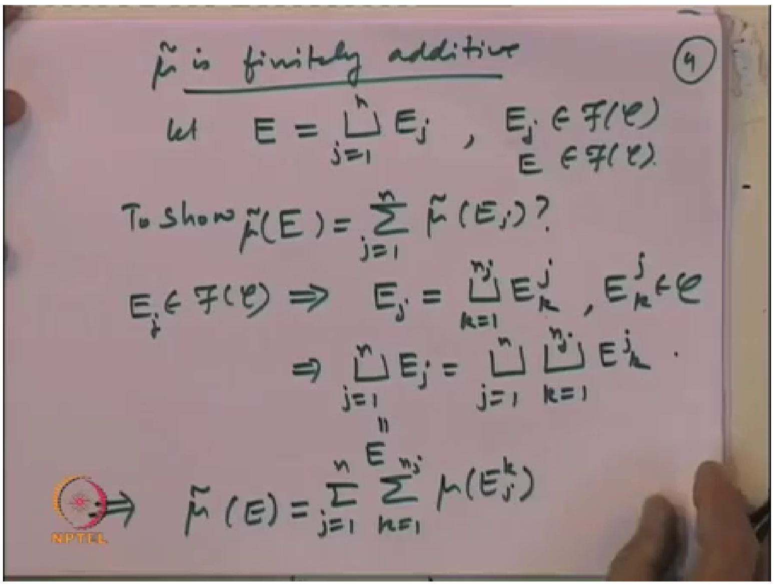

So, we want to prove that mu tilde. So, mu tilde is finitely additive. So, to prove that what we have to show. So, let E we written as a union of E j's, j equal to 1 to n, where each E j belongs to the algebra generated by C, and of course, E also belongs to the algebra generated by C. So, we want to show that E mu tilde of E is equal to summation j equal to 1 to n mu tilde of E j. So, this is what is to be shown. Now to show any such property we have to go back to the definition of mu tilde of any set. So, since E belongs to the algebra generated by C; that implies. Let us write each E j, E j belongs to the algebra. So, each E j can be written as a disjoint union of E j, and say k, k equal to 1 to n j, where E j k belong to C for every j n k.

So, every element E j is in the algebra generated by C. So, it must be a finite decision to enough elements of C. So, that implies that the union E j, j equal to 1 to n is equal to union j equal to 1 to n union k equal to 1 to n j of E j k and this is my. So, this is our set E, E is equal to union. So, we have represented E as a finite this disjoint union of elements of C. So, that implies that mu of E mu tilde of E, I can choose any representations. So, n particulars this. So, it is equal to summation j equal to 1 to n summation k equal to 1 to m j of mu of E k j.

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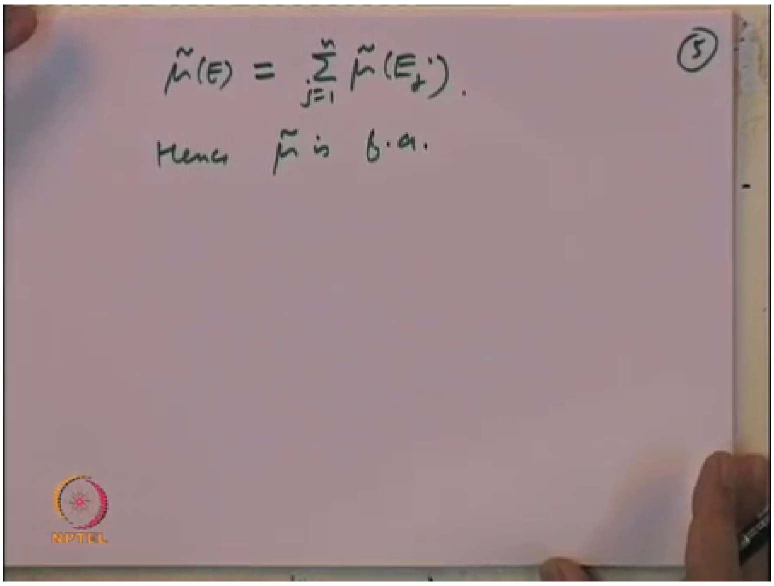

And now using the finite additive property of mu, we will write this. So, this is equal to look at this sum. So, look at this sum that is nothing, but j equal to 1 to n mu tilde of E j right; that is my definition, because E j is union of l k j over k. So, by definition I can take that representration right. This is equal to this. So, that says mu tilde of E is equal to this. So, hence mu tilde is finitely additive. So, we have proved that mu tilde is finitely additive. Now uniqueness we have already shown.

So, thus we have shown that a measure which is defined on a semi algebra, can be in a unique way extended to the algebra, generated by a. And basically the idea is, because every element intuitively keep in to your mind that mu of a set is the size. So, any element in the algebra generated by the semi algebra is a union of disjoint basis in the semi algebra, and size of each of them is known. So, the size of the union must be equal to some of the sizes of the individual pieces, because they are disjoint. So, that was the idea and that helped us through extend a measure from a semi algebra to the algebra generated by it. So, that is the first step of the extension theory. So, as a consequence the length function can be extended that we have already shown length function, can be extended from the collection of all intervals to the collection of finite disjoint union of intervals; that is the algebra generated by it right.

So, now we will go to the next step of the extension. So, we will start with a measure, which is defined on a algebra, and we want to try to extend it to the sigma algebra generated by it.

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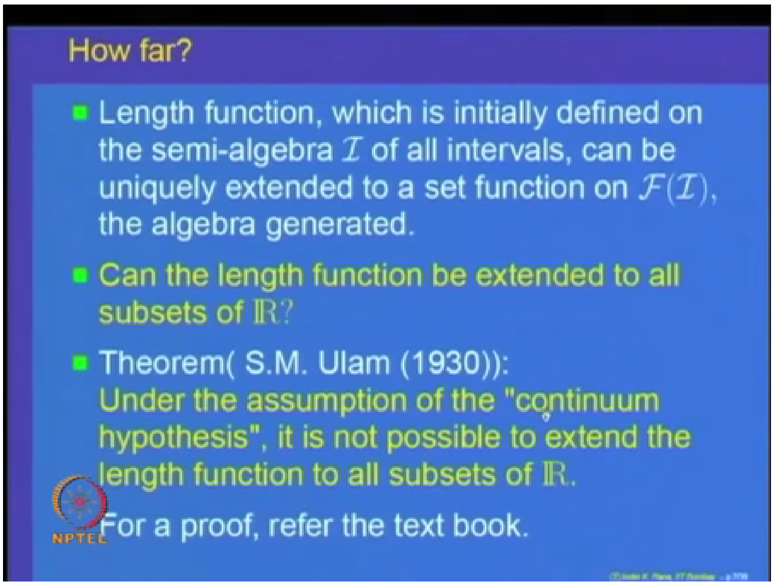

So, the next step in the extension theory is, can the length function, for example, we would like to say, can the length function be extended to all sub sets of the real line. We have done it from intervals to the algebra generated by intervals right. There is a theorem by mathematician called S M Ulam, and that theorem was proved in 1930. It says that under the assumption of continuum hypothesis, it is not possible to extend the notion of length to all sub sets of real line, and this is a very important theorem.

So, it uses two things; namely one is what is called continuum hypothesis. I will not go in to the discussion of what is called continuum hypothesis at this stage. I would say that one should read about this theorem from the text that we have just now mentioned, and introduction to measure an integration. So, this is very nice and important theorem, which says as a consequence that it is not possible to extend the length function to all sub sets of real line.

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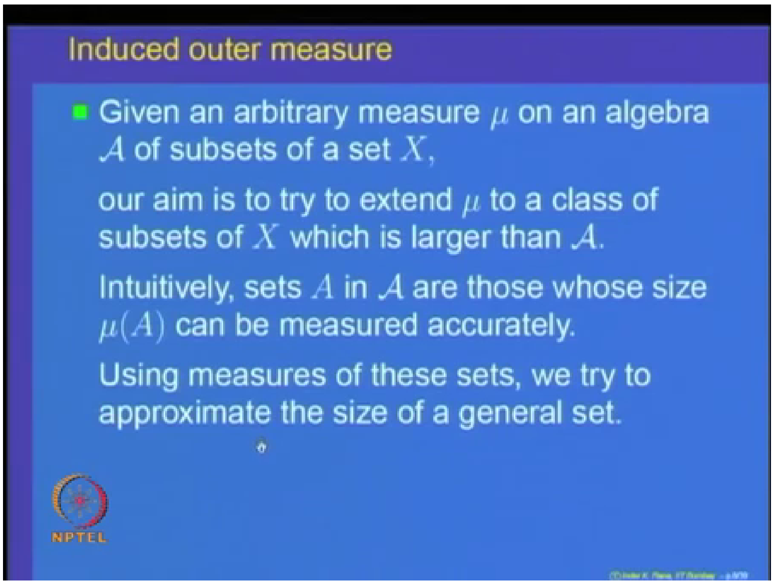

So, the question comes, if we cannot extend. So, in general A given a measure mu on an algebra of sub sets of X, we would like to extend it to a bigger class than A. It cannot be done it for all sub sets any way, but let us try to intuitively follow our idea of measuring the size of an object. So, intuitively given a measure mu on an algebra A, a collection of sub sets of a set a. Now if is at X mu of a is the size of the set a, which you can measure and given in a arbitrary, set E 1 may not be able to measure it size exactly using the mu, but we can at least try to approximate right

So, let us define what is called the outer measure induced by a measure.

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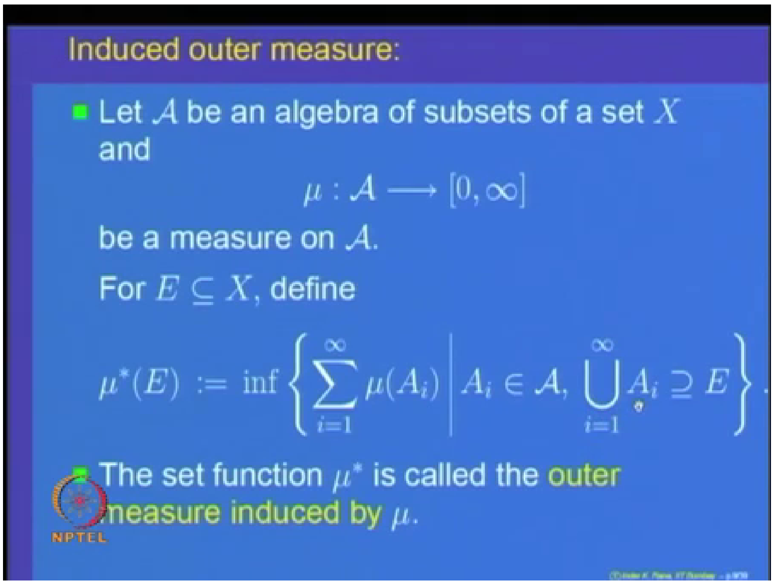

So, let us take mu and algebra of sub sets of the set X a, and algebra of sub sets of the set X and mu, a measure defined on it for any sub set E in X let us define what is called mu star of e. So, what is mu star of E, what we do is, given the set e. So, here is the set E you cover it by sets a i's. In the algebra you cover it by the sets, in the algebra take a covering of E by the sets a i'sin the algebra, and you know what is the size of the set a i. So, let us take the size of the set A i and add up all the sizes.

So, what do you think this sum will represent. This sum will represent in sums and the approximate size of the set E. Of course, it depends on the covering A i and now what we do is, we take the infimum of all this approximate sizes; that means, we take the infimum of this numbers, over all possible coverings of the set E, and define that number as mu star of E, and we will try to analyze what are the properties of this mu star of e. So, first of all let us give it a name this, mu star of E is called the outer measure induced by mu. Why the outer, because we are covering E right by sets. So, this things cover E we are going, maybe we are going outside e. So, this is outer and measure, because we are trying to measure the size of this in terms of induced by mu, because in terms of the non sizes mu

So, once again let us recall and look at carefully what this mu star is given a set E arbitrary, sub set X in X cover it by elements A i whose size is you know. So, take a covering of E by elements in the algebra. Look at the sizes of A i's, add up all this; that is the sum mu A i that is approximate size, and take the infimum of all this approximates sizes. So, that we are going to call as a outer measure induced by so.

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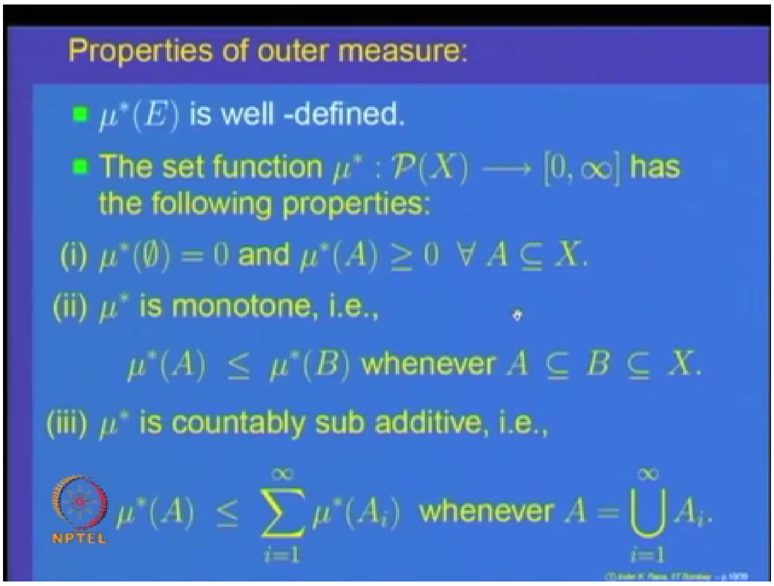

The first property we want to say is mu star, is well defined. Well what is the meaning of mu star is well defined. So, let us go back to the definition. This mu star is infimum of some numbers right, and infimum of a sub set of numbers exist in the real line, if it is non empty, and it should be bounded below. Of course, all this numbers are going to be bounded below, because all are nonnegative number. So, it is bounded below by zero. Why is this nonempty? Why is this collection non empty because a is an algebra. So, the whole space belong to it.

So, keep in mind a is an algebra and in the definition of an algebra, the whole space X is an element. So, E is covered by X itself. So, and X belongs to the algebra. So, at least there is one number in this collection over which you are taking infimums namely mu of x. So, it is X non empty collection of extended real numbers. So, its infimum always exists, and hence mu is a well-defined number of course, it could be equal to plus infinity. Keep in mind the numbers here; they are all extended real numbers. So, this is the set is a collection of nonnegative extended real numbers, and their infimum always exist, and infimum could be equal to plus infinity

So, we have shown that mu tilde is a mu star, the induced outer measure is well defined the next property. So, mu star is a well defined set function on the class of all sub sets of the set X, and we want to show some properties of it. So, the first property is mu star of empty set is equal to 0. So, that is true, because empty set belongs to the collection a in the algebra, and mu star of, there is equal to mu of a, and that is equal to 0 and for any set, that is a non infimum of non negative numbers. So, this infimum has to be bigger than or equal to 0. So, that first property is obvious second property, we want to check that mu star is monotone. So, let us check that mu star is a monotone function.

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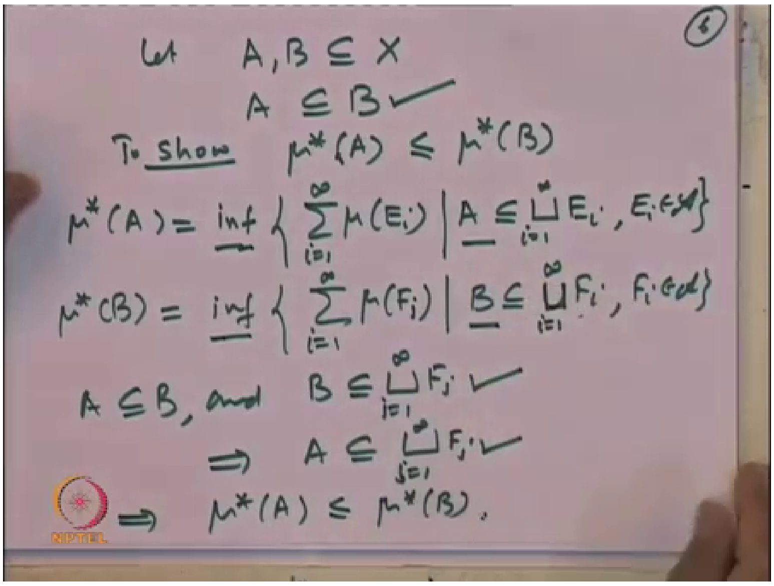

So, let us take let a and b be sub sets of X, and a sub set of b to show mu star of a is less than or equal to mu star of b. Now what is mu star of a, this is in all this properties we are going to use the definition of infimum critically. So, what is mu star of a. mu star of a is defined as by our definition, it is a infimum over sigma mu of E i's; say one to infinity where this set a is contained in union of E i's disjoint union, and of course, E i 's belong to the algebra right. And what is mu star of b; that is the infimum i equal to 1 to infinity of mu of say F i's, where b is contained a union of F i's i equal to 1 to infinity disjoint union, where F i's also belong to the algebra a

Now, note if a is given to be a sub set of b. If a is sub set of b and b is covered by F union j equal to, say going to infinity, then that implies a is also inside. So, this is also inside F of j. So, what we are saying is, every covering of b is also a covering of a. So, and this is the infimum over all possible coverings of b, and this is the infimum over all possible coverings of a, and every covering of b is also a covering of a. So, here we are taking infimum over a larger set, and here we are looking at the infimum over a smaller collection of numbers. And whenever you take infimum over a smaller collection of numbers; that is always bigger than or equal to infimum over a larger collection of numbers.

So, that is a simple property about infimum, if you are taking infimum of a larger collection, then that tends to be smaller than the infimum over a smaller collection. So, that property implies that mu star of a has to be less than or equal to mu star of b. So, that is purely a property of the infimum, over what collection you are taking every covering of b is also a covering of a. So, coverings of b form a sub set of coverings of a, and hence this property is true. So, that is the monotone property, namely mu star is monotone. Let us look at the next property namely mu star is countably sub additive.

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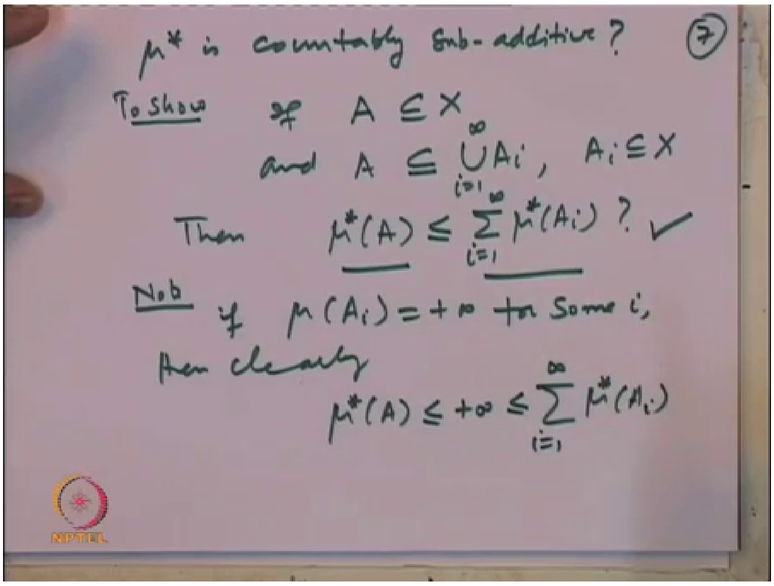

So, we want to prove mu star is countably sub additive. So; that means, what to show. So, what I have to show that, if a is a sub set of X and a is contained in union of A i's, a i's is also a substitute of X, then we want to show that mu of a is less than or equal to summation mu of A i's. So, this is what is to be shown right. Now let us observe. So, note we want to show one number mu of a is less than or equal to sum of these numbers. If one of these numbers is equal to plus infinity, then; obviously, this property is true.

So, note if mu of A i is plus infinity for some i, then clearly mu star of a is a number which is less than or equal to plus infinity, which is at least one of the mu over i's. So, that is less than or equal to mu of mu, so everything about star. So, its mu star we are looking. So, let us just write mu star, we are trying to prove that mu star countably additive. So, mu star of A i i equal to 1 to infinity. So, what we are saying is, this inequality is obvious if one of the terms in this sum, is equal to plus infinity. So, let us take the case when all of them are finite.