Measure & Integration
Prof. Inder K. Rana
Department of Mathematics
Indian Institute of Technology, Bombay

Lecture – 08 B
Uniqueness Problem for Measure

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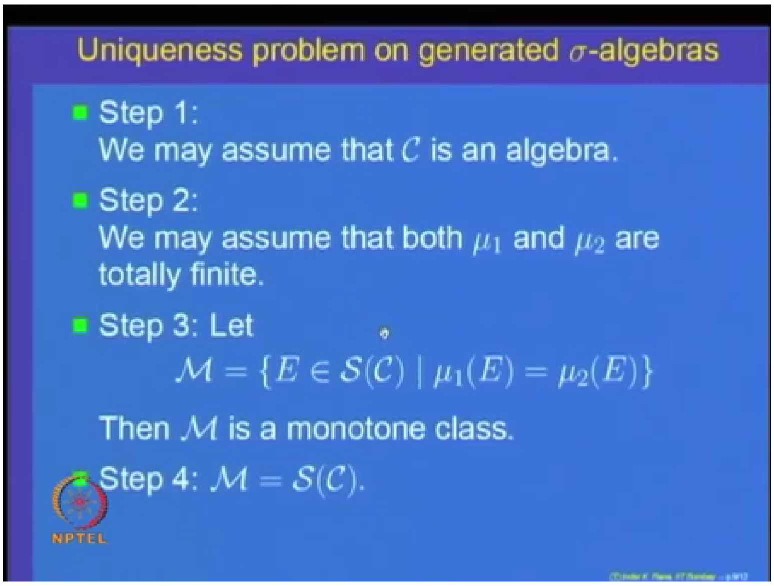

So, next step let us look at next step say that we may assume that both mu 1 and mu 2 are totally finite. We are given mu 1 and mu 2 are sigma finite.

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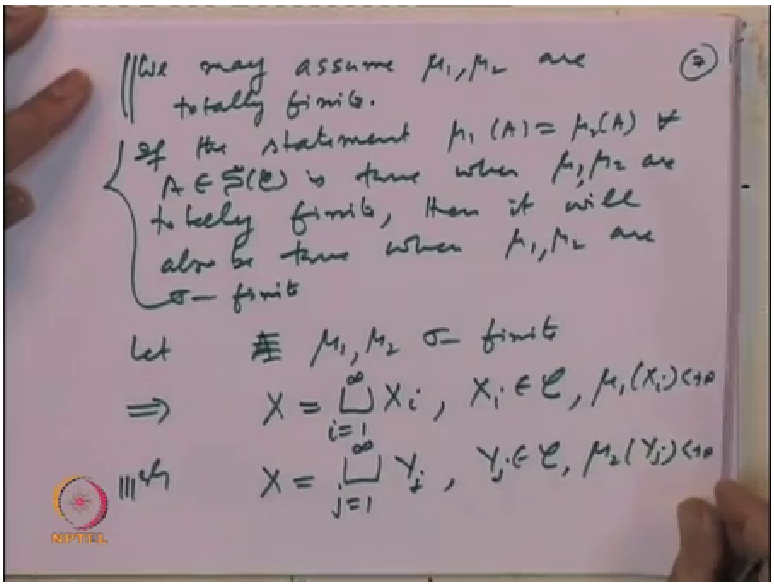

So, next step is that we may assume; mu 1, mu 2 are totally finite. So, what is the meaning of saying we may assume? That if the statement; so, this is same as saying; if the statement mu 1 of A equal to mu 2 of A; for every A belonging to S of C is true, when mu 1, mu 2 are totally finite; then it will also be true; when mu 1, mu 2 are sigma finite.

So, that is the meaning of saying that we may assume that mu 1 and mu 2 are totally finite. So, let us see why that is the case. So, let us take a set A contained in; so, what we are given is mu 1 and mu 2 are sigma finite. So, mu 1, mu 2 sigma finite; imply I can write X; mu 1 is sigma finite. So, I can write X as union of X i's; i equal to 1 to infinity where X i's belong to C and mu 1 of each X i is finite. Similarly mu 2 is sigma finite; so, I can write X as union some j equal to 1 to infinity; Y j where Y j's are subsets in C and mu 2 of each Y j is finite.

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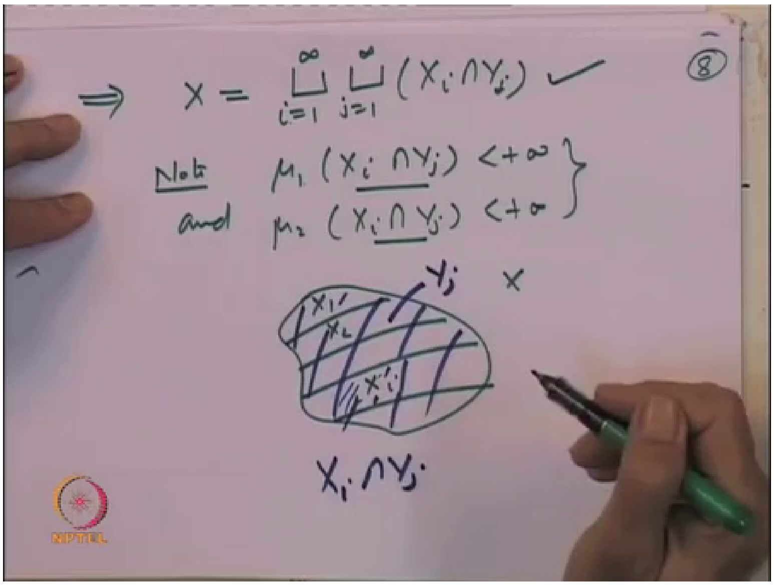

But, then from both of these statements; I can write X as; so, this implies, we can write X as union over i; 1 to infinity of X i's, but that I can decompose into union of Y j's. So, X i; intersection Y j; I can write that.

So, I can write this as a decomposition of X into subsets X i intersection Y j; now what we have achieved is the following, mu 1 of each X i was finite; mu 2 of each Y j was finite, But now this implies note; that mu 1 of X i; intersection Y j is finite and mu 2 of X i intersection Y j is also finite. So, now both mu 1 and mu 2 are finite on this p; so, in the picture you can think of this as X. So, you divide; these are sets X 1, X 2, X i and so on and then you also have sets of Y j’s. So, they are decompositions like this; so, this piece is nothing, but X. So, this piece is Y j; so, this piece here is X i; intersection Y j. So, the whole space is cut up into pieces; so, this is what this statement means, where each one of them is finite.

And now let us note; so, here is observation.

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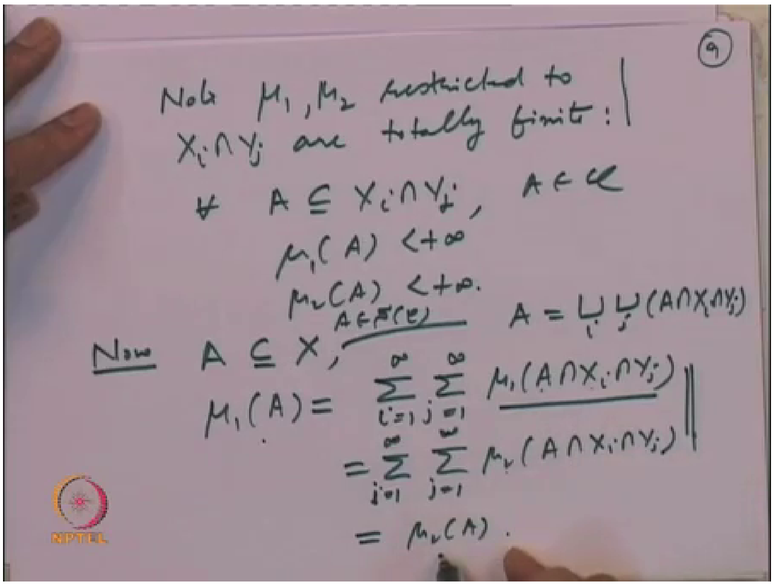

So, note mu 1, mu 2 restricted to X i; intersection Y j are totally finite. So, what is the meaning of this statement? They restricted; that means, if you look at the subsets on; that is for every A contained in X i intersection Y j; A belonging to C, mu of A is finite; mu 2 of A is finite. And for totally finitely measures, we have already proved that; we have already assumed that this statement is true.

And we are trying to show it for our sigma finite. So, now for any set A; contained in X mu 1 of A can be written as summation over i, summation over j; mu 1 of A; intersection X i, intersection Y j. So, that is because A is equal to union over i, union over j; A intersection X i; intersection Y j. So, this is a countable disjoint union; mu 1 is a measure. So, this must be true and now note that this A intersection; this is a set; in the set X i intersection Y j; where mu 1 is finite and then there we know that there the statement is true.

So, here A is contained in X of course, A belonging to S of C; so, the statement is true. So; that means, what? So, by the assumption that statement is true for finite measures; we conclude that this is same as A intersection mu 2 of A; intersection X i; intersection Y j. And once again that is equal to mu 2 of A; so, here we use. So, a basic idea is for any set, we can bring it to the finite pieces; there we know it is true and go back to the original piece.

So, this is the proof of the second step that we may assume without loss of generality that our measures mu 1 and mu 2; both are finite. So, we have made two simplifications in our proof; the first one being, we may assume that C is an algebra and second one that mu 1 and mu 2 are totally finite.

So, what we want to prove now? So, we only are left with the case to prove that; if C is an algebra; mu 1 and mu 2 are totally finite; defined on the algebra C. And if they agree on C, then they will agree on the sigma algebra generated by C. So, that is the next step we want to show. So, for that let us write; so, to prove the final step; let us write M to be the class of all those elements of S of C, where mu 1 and mu 2 agree.

And what is the aim to proof? Our aim is to prove that this collection M is nothing, but S of C; we are picking up subsets of S of C. So, M is a subclass of S of C; we want to prove that this is equal to S of C. And that is proved as follows; first we will observe then M is a monotone class, we will prove that. Once we have proved M is a monotone class, we will also observe that we are given that mu 1 and mu 2 are equal on C.

So, C is a subclass of M and C is an algebra and C is contained in M; M is a monotone class, so that will mean what? That the monotone class generated by C must be inside M, but C is an algebra and the monotone class generated by an algebra is the sigma algebra generated by it. And that also we have proved, so that will prove as a step 4; that M is equal to S of C. So, let us prove step 3 and then conclude from its step 4. So, step 3; we want to prove, so we are given that mu 1, mu 2 totally finite.

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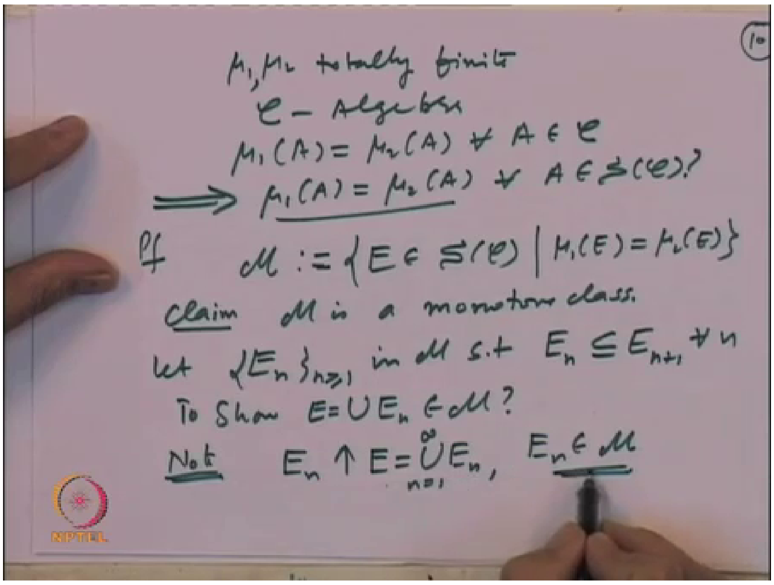

So, we are in this case; C algebra mu 1 of A equal to mu 2 of A for every A belonging to the algebra C; to show that mu 1 of A is equal to mu 2 of A; for every A belonging to the sigma algebra generated by C; so, that is the question.

So, we are saying the proof define M to be the class of all subsets belonging to S of C for which this property is true. So; that means, mu 1 of E is equal to mu 2 of E; so claim that this is M is a monotone class. So, what is a monotone class? Recall a monotone class is a collection of subsets of a set X, which is closed under increasing unions and decreasing intersections.

So, these two properties have to be checked; so, let us check that. So, let us take; so, let E n be a sequence in M such that; E n is increasing. So, E n is inside E n plus 1; for every n; to show union of E n's belong to M. Now, let us note that E n is an increasing sequence E n increases to E, which is union of E n's; E n's belong to M.

So, keep that in mind and we want to show; E belongs to, so that is the set E; we want to show that E belongs to M; that means, mu 1 of E is equal to mu 2 of E. So, let us note.

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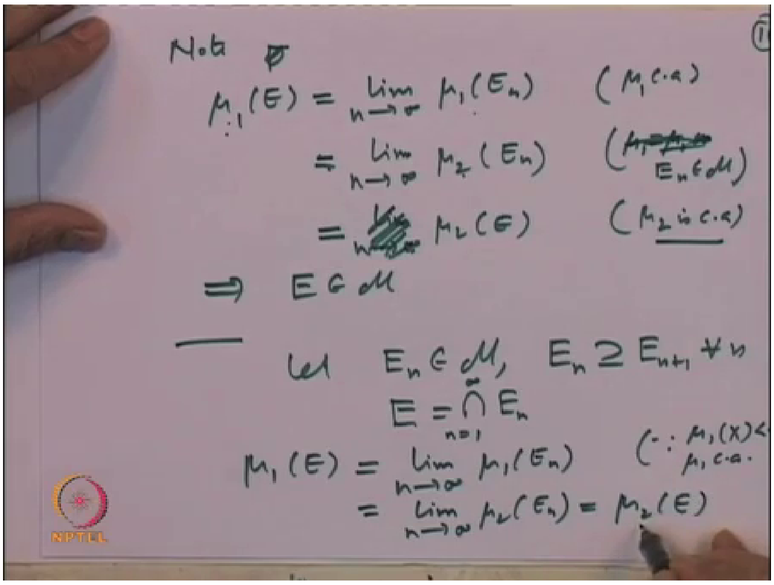

What is mu of E? How do we compute it? Let us observe that mu 1 of E is nothing, but limit of mu 1 of E n's and why is that? That is because mu 1 is a measure; it is countably additive. So, we had proved that countable additivity of the set function implies whenever a sequence E n increases to a set E, then mu of E must be limit of mu 1 of E n's; that was the characterization property for countable additivity. So, this go back and refer that was because of countable mu; countably additive. So, that is the property being used here.

The equivalent form of it; so, that and now each E n belongs to M. So, that implies that E 1; mu of 1 of E n is equal to mu 2 of E n. So, this is equal to limit; n going to infinity, mu 2 of E n. So, that is we are using mu 1 equal to mu 2; on mu 1 equal to mu 2 because sorry this is because E n belongs to M. And once again mu 2 is countably additive; so, there are mu 1 was known and that implies that this is so; this is mu 2 of E, using the fact that mu 2 is countably additive.

So, let us once again; we have used lot of things from which we have proved earlier; mu 1 is a measure, E n is increasing to E. So, by countable additivity; mu 1 of E must be equal to limit n going to infinity, mu 1 of E n; by countable additivity. Now each E n belongs to M; E n is a sequence in M so; that means, mu 1 of E n is equal to mu 2 of E n.

So, this is equal to this. So, this is the second step quality and now once again mu 2 is countably additive; E n increases to E. So, by countable additivity this limit must be equal to mu 2 of E. So, it says mu 1 of E is equal to mu 2 of E. So, that implies that E belongs to M whenever; E n is a sequence which is increasing to M. So, this is for increasing and the corresponding thing; we have to prove when it is decreasing and that is where we are going to use the fact; that mu 1 and mu 2 are totally finite.

So, for the second case; let E n's belong to M; E n include E n plus 1; for every n; decreasing and E b equal to intersection of E n's; n equal to 1 to infinity. So, we want to show that E also belongs to M; so, for that once again mu 1 of E is equal to limit n going to infinity mu 1 of E n; because mu 1 totally finite, mu 1 of X finite and mu 1 countably additive. And that as earlier is same as mu 2 of E n because each E n belongs to M; and that is equal to mu 2 of E; once again mu 2 is finite and E n is decreasing to E. So, that proves this also; so, this proves that M is a monotone class.

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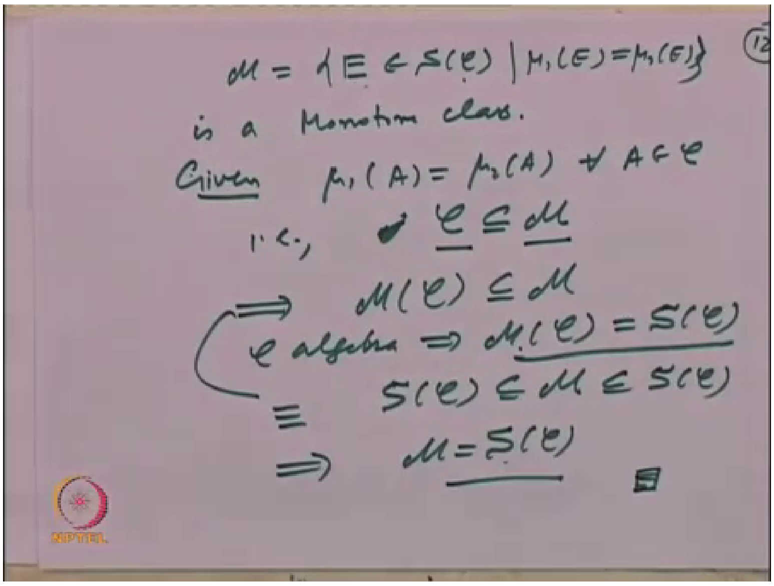

So, the class M which was equal to all subsets; E belonging to S of C such that mu 1 of E is equal to mu 2 of E is a monotone class. And we are given that mu 1 of A equal to mu 2 of A; for every A belonging to C. So, what is that mean? An equivalent way of setting that is saying that the collection C is inside the collection M.

So, that is what it means by every definition. So, M is a monotone class; C is inside it, so, that implies that the monotone class generated by C must be inside M. So, because recall; what is monotone class generated by a collection of subsets of C? It is the smallest monotone class of subsets of X, which includes C; being the smallest, it must be inside it.

But note; C algebra implies M of C is equal to S of C. So, this is an important theorem which we had proved; that if you take an algebra and generate a monotone class out of it; that is same as generating the sigma algebra out of it. So, this is same as saying that S of C is contained in M, but M is a collection of subsets of S of E; so, that is inside S of C. So, that is same as saying that M is equal to S of C; the sigma algebra generated by C and; that means what? For all elements in S of C; mu 1 is equal to mu 2 of E.

So, that proves the theorem; the uniqueness theorem namely f two; so, we have finally, proved in this 4 steps, the theorem; that if mu 1 and mu 2 are two measures defined on a semi algebra of subsets of a set X and mu 1 and mu 2 are both sigma finite and they agree on the semi algebra, then they also agree on the sigma algebra generated by C. So, this is an important theorem, which we are going to use quite often. So, with this we come to an end a part of our course. So, this is probably the right stage to revise; what all we have done till now. So, let us revise what we have done till now.

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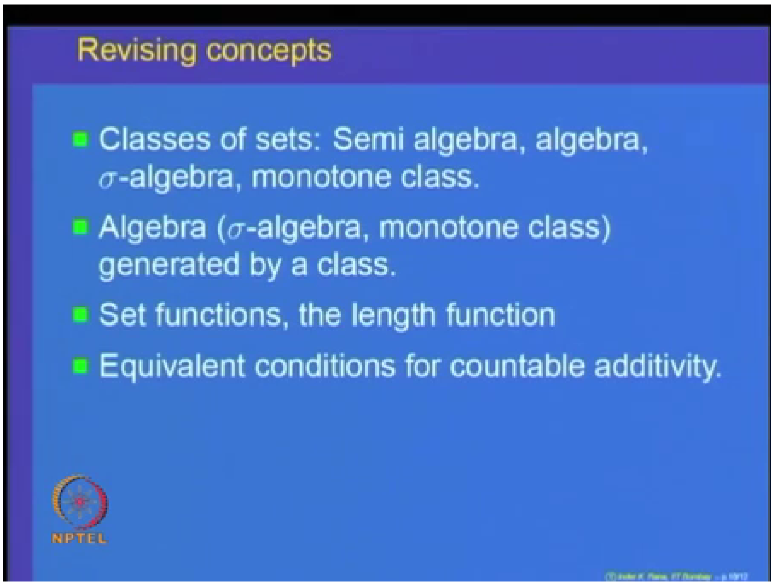

So, we started with looking at collections of subsets of a set X; we defined what is a semi algebra. So, what was a semi algebra? Semi algebra was a collection of subsets of a set X; with the properties, the whole space belongs to it; the empty set belongs to it. It is closed under intersections and the complement of a set is inside; not inside not necessarily inside it, but can be represented.

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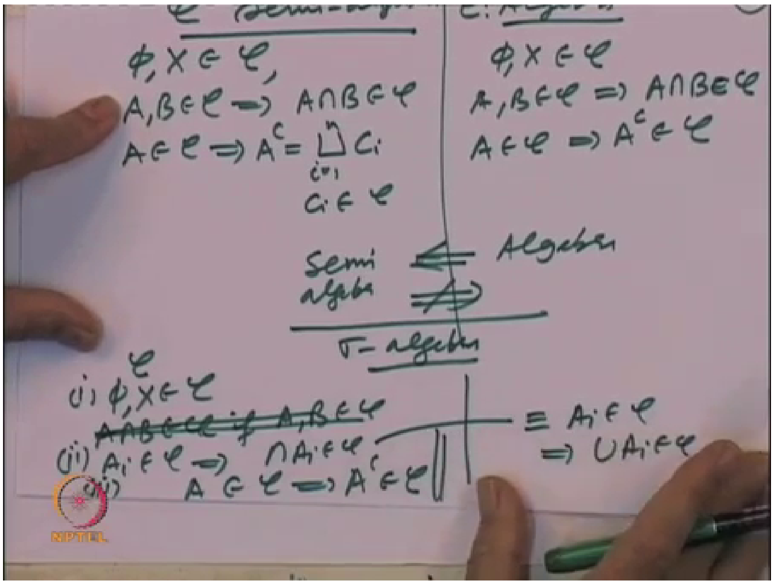

So, semi algebra C; semi algebra meant that empty set, the whole space belong to it; one property. The second one; A and B belonging to C should imply A intersection B belong to C; and the third property that A belonging to C implies, A compliment can be written as a finite disjoint union of elements of C. So n for some C i's; belonging to c. So, that is a semi algebra; then we defined what is called an algebra. So, a collection C is called an algebra; the first property as it is; empty set the whole space belong it. A and B belonging to C should imply, it is closed under intersections that also belongs to C.

And now we are something stronger instead of just saying that A belongs to C; its complement is representable. Actually we want that this compliment also belongs to C. So, this is something stronger; so, we said this is stronger property. So, semi algebra; so, algebra implies semi algebra and the converse need not be true; that we had checked. And then we defined what is called a sigma algebra; sigma algebra. So, a collection C is a sigma algebra if of course, it is an algebra first of all. So, it is phi X belong to C; it is closed under intersections A and B; belong to C; if A and B belong to C and then, but this is not enough. So, actually not a countable; this should be true for any countable collection. So, let us write whenever A i's belong to C; that should imply that intersection A i's belong to C and because it is going to be closed under compliments.

So, this is the property one; this is second and the third property is that whenever A belongs to C, should imply a compliment belong to C and that automatically implies that

C is also closed under. So, this property of countable intersections can be equivalently stated as; because of the compliments that A i's belong to C; imply union A i's also to C.

So, a sigma algebra is a collection which is closed under a countable unions and compliments and of course, empty set in the whole space belong to it; then we defined what is called a monotone class.

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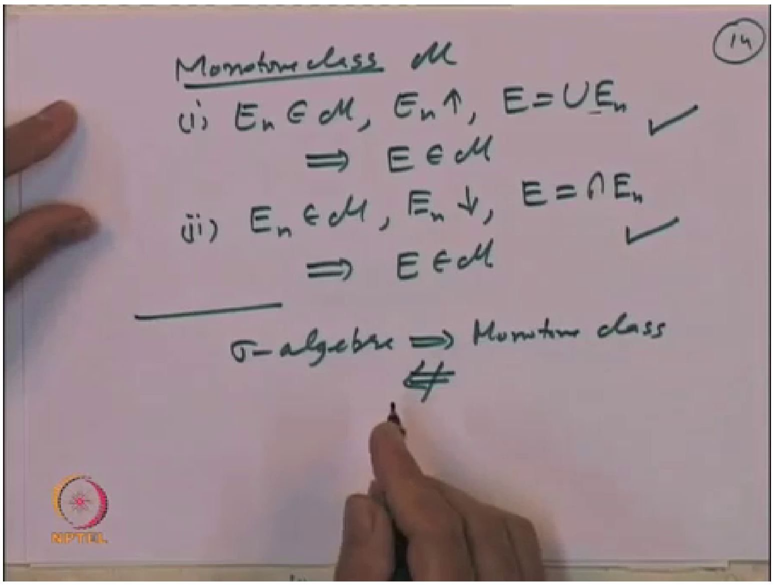

So, what was a monotone class? M was called a monotone class, if whenever a sequence E n belong to M; E n's increasing, E is equal to union E n's should imply that should imply that E also belongs to M and so, this is one property. And second property we want that whenever E n's belong to M; E n's are decreasing and E is equal to intersection of E n's should imply that E also belongs to M.

So, a monotone class is the collection of subsets of a set X with the property; it is closed under increasing unions and decreasing intersections. Of course, sigma algebra implies monotone class; the converse is not always true. Then we looked at; so, this was the first basic concepts or properties of collection of subsets of a set X; we looked at. And then we looked at; what are called the algebra generated by a collection of subsets; or the sigma algebra generated by a collection of subsets or the monotone class generated by a collection of subsets of a set X.

So, in all these cases basically given a collection C; so, let us just recall what was the meaning of saying generation?

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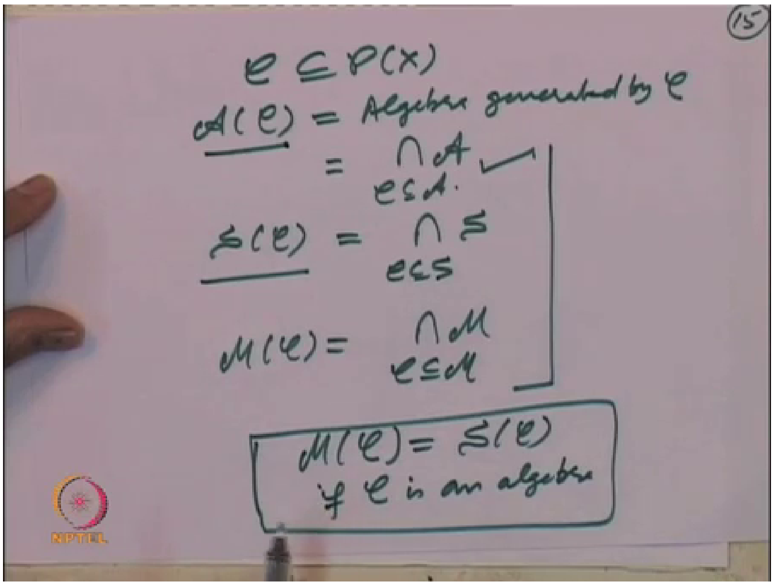

So, C any collection of subsets of a set X, so algebra generated by C generated is a smallest one. So, it was the intersection of all the algebras; which include C and we showed such a thing exists. Similarly, the sigma algebra generated by C; we said it is nothing, but look at all sigma algebras of sub sets of X which include C and take the intersection so, that is called the sigma algebra. So, another way of saying is the algebra generated by C is the smallest algebra of subsets of a set X, which include C.

Similarly S of C is the smallest sigma algebra of subsets of C; which includes C. And similarly we have monotone class generated by C, it is the smallest monotone class of subsets of X; which include C. And we showed by this properties that such a object always exist and then we proved a very important theorem; namely the monotone class generated by C is equal to the sigma algebra generated by C; if C is an algebra.

So, this was an important theorem that we had proved. So, these concepts were basically about collection of subsets of a set X. Then we looked at functions defined on such collection of subsets of this set X and we called them as set functions. So, set functions are functions defined on a collection of subsets of a set X. And the important class of set functions was; the length function and we showed that the length function had important properties, namely the length function which is defined on the class of all intervals in the real line was shown to be a countably additive set function; which is also in variant under translations; so, that was an important property.

And then finally, we had proved some equivalent conditions for countable additivity and these conditions are very useful. So, let us just recall this equivalent conditions, we have used one of them today also.

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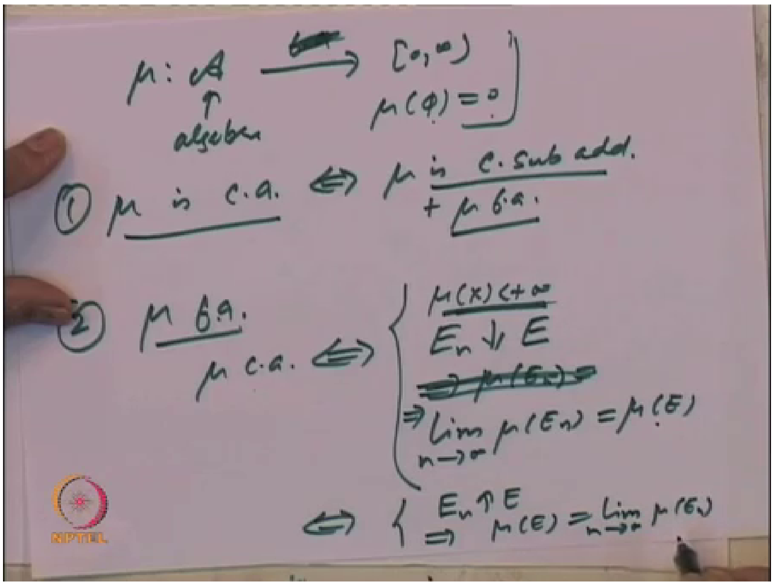

So, one of the important conditions for example, if mu is defined on an algebra. So, this is a algebra and it is finitely additive, then we said that mu is countably additive if and only if mu is countably sub additive. Of course, plus; so, let us write plus mu finitely additive. So, let us remove this condition; let us right mu 2 be an algebra and mu of empty set equal to 0. So, let us put; so, mu is a set function defined on an algebra and mu of empty set is 0; then we proved that saying that mu is countably additive is equivalent to saying that mu is finitely additive and countably sub additive. And this is quite useful in proving the countable additivity of set functions.

So, this was one and second we proved; that mu finitely additive, we assume that. Then mu countably additive if and only if; whenever E ns it decrease of course, under the condition A is algebra to E should imply mu of E n's is equal; the decrease. So, let us write decrease to E; then limit n going to infinity mu of E n is equal to mu of E. This provided we had put an extra condition mu of X is finite. So, mu of X is finite E n's decrease to E and implies that limit of E n's is equal to mu of e.

So, this condition is equivalent to saying mu is countably additive; when we have this. If you dont put this condition then this may not be true, but then one can equivalently prove; another thing that if E n's increase to E; then that should imply mu of E is equal to limit n going to infinity mu of E n's.

So, this was the property of saying that when is something countably additive and finally, today we proved the uniqueness theorem.

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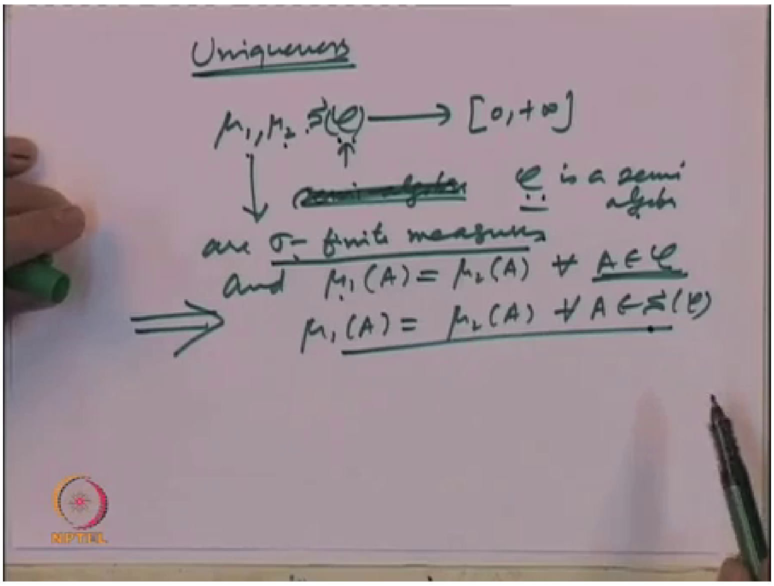

Saying that if mu 1 and mu 2 are two finite countably additive set functions defined on a semi algebra. So, this is a semi algebra; mu 1 and mu 2 are sigma finite measures; then and mu 1 of A is equal to mu 2 of A; for every A in the semi algebra, then this implies mu 1 of A is equal to mu 2 of A; for every A belonging to the sigma algebra generated by C. This is mu 1 and mu 2 should already be defined; I am sorry. So, we should say they are already defined in S of C; C is a semi algebra.

So, C is a semi algebra; so, let me state it once again. C is a semi algebra; mu 1 and mu 2 are defined on the sigma algebra generated by C, both mu 1 and mu 2 are sigma finite and mu 1 and mu 2 agree on the semi algebra, then they agree on the sigma algebra also. So, they agree on the whole domain. So, if they agree on the part of the domain which is the semi algebra; then they agree on the whole of the sigma algebra also. That is the uniqueness result and that we proved under the condition that mu 1 and mu 2 are sigma finite measures.

We will see this how that is used in extension theory in the next few lectures when we come to them. So, we stop today here; so, in the next lecture we will start a new topic called extension theory. So, we would like to extend a set function defined on a class to a bigger class. For example, on the real line; we have the notion of length defined on the collection of all intervals; we would like to define the notion of length for any set. So, that is the aim of; that is the motivating thing for extension theory. So, we will use that and prove the theorems in the next lectures.

Thank you.