Measure & Integration
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Lecture - 08 A
Uniqueness Problem for Measure

Welcome to today's lecture on measure and integration this is the 8 th lecture on measure and integration. Today we will be looking at a problem called the uniqueness of measures on algebras and sigma algebras. So, for this we will need to define some term terminology. So, let us look at the uniqueness problem for topic for today's lecture. So, the problem is as follows.

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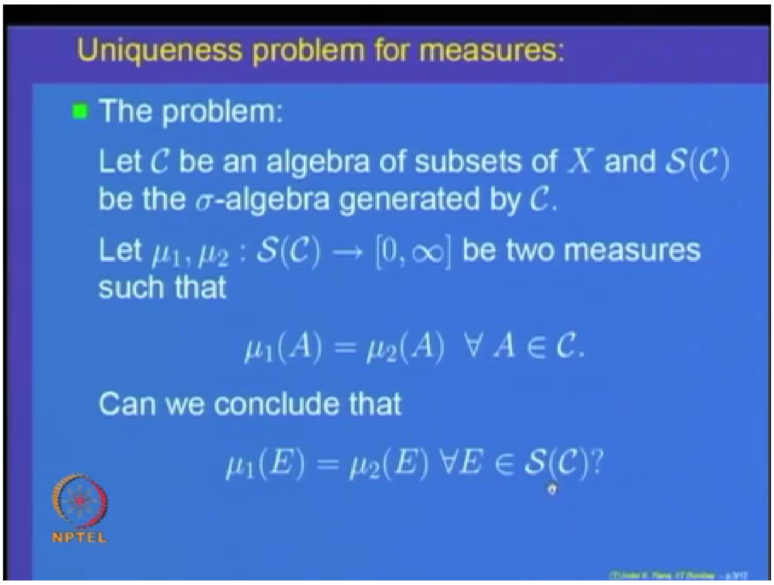

We are given C an algebra of subsets of a set X, and S of C is the sigma algebra generated by C. So, C is an algebra and S of C is the sigma algebra generated by C. We have got two measures mu 1 and m 2 defined on the sigma algebra generated by C such that mu 1 of a is equal to mu 2 of A for every A belonging to C. So, for all elements in C mu 1 and m 2 agree.

The question is can we conclude that mu 1 of E is equal to mu 2 of E for every element in the sigma algebra generated by C. So, this is the general uniqueness problem which place a role later on when we extend measures to general settings. So, to answer this question let us make some definitions. So, first of all this is not true in general for all measures, we are to put some conditions on the measures. So, let us look at what is called a totally finite measure.

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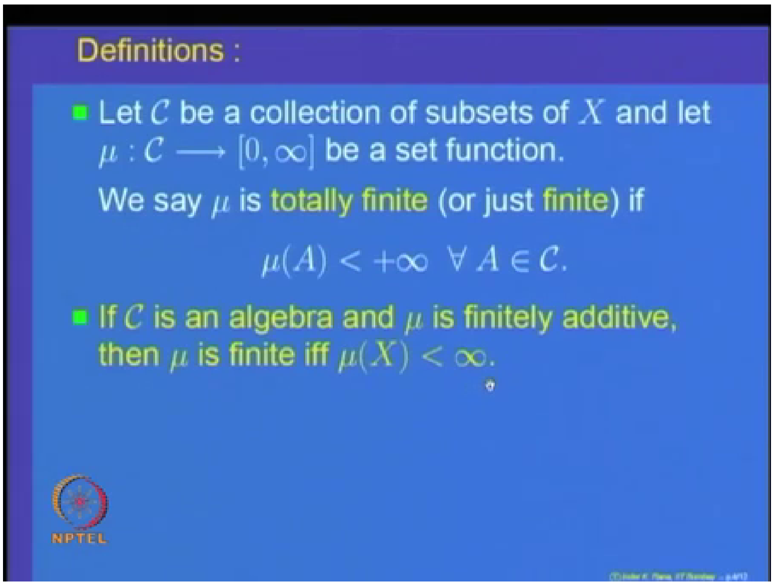

A measure C is called totally finite if mu of every subset A is finite in that domain of mu. So, C is a collection of subsets and mu is a set function. So, we said mu is totally finite or sometimes we also said is finite if mu of A is less than plus infinity for all A belonging to C and note that in case C is an algebra and mu is finitely additive then mu is finite if and only if mu of X is finite.

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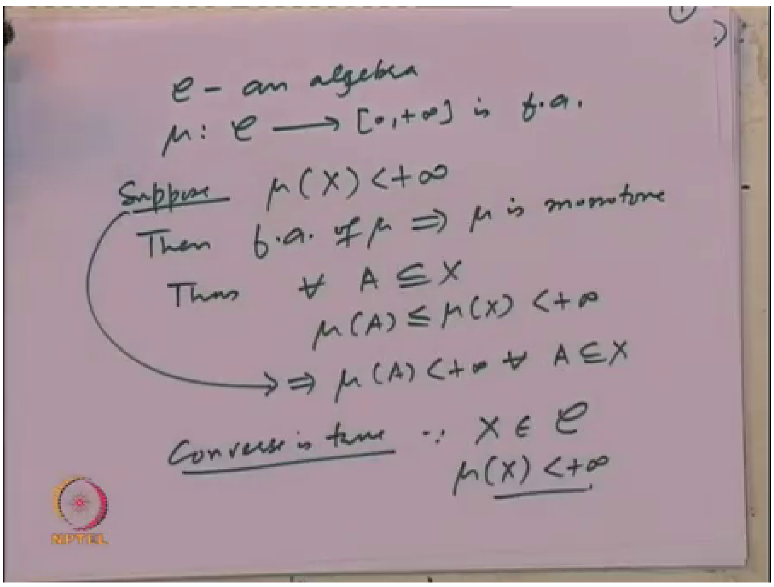

So, let us assume C is an algebra and mu from C to 0 to infinity is finitely additive. So, suppose mu of the whole space is fine note the whole space x belongs to C because C is an algebra. Then we had seen earlier finite additively of mu implies mu is monotone right we are seen this property earlier. So, we will not to going to the details of this again. So, because mu is monotone. So, thus for every a contained in x mu of A will be less than or equal to mu of x which is finite.

So, thus implies mu of A finite for every a subset of x and converse is obviously true. So, conversely is true converse is true because X belongs to C and so, mu of X is finite. So, whenever we are dealing with finitely additive set functions saying mu is totally finite it is enough to say that mu of X is finite and as a consequence mu of every subset will be finite. So, this is a easy consequence of saying for a finitely additive set function on an algebra, mu a of the whole space finite is same as saying mu of every subset a of x a in the algebra of course, is infinite.

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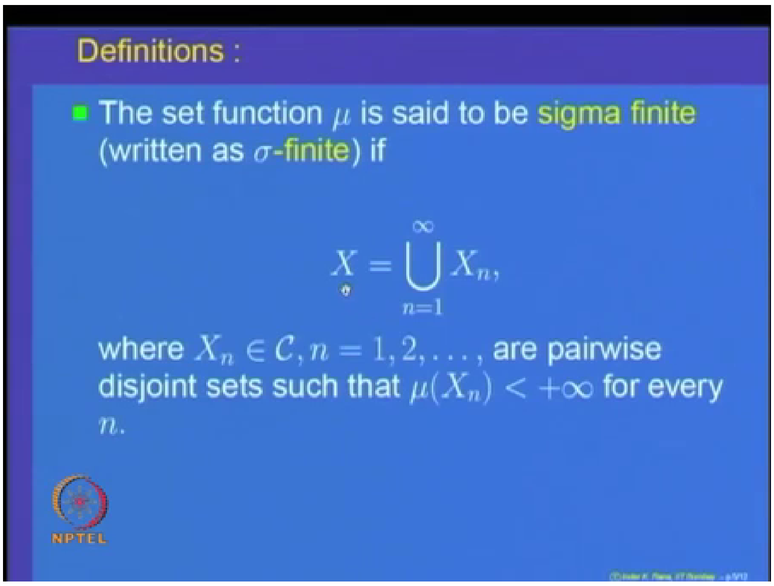

Let us look at the property that let us take a set function mu and we say that the sigma finite. So, we defined what is totally finite, and now we defined what is called sigma finite. So, a mu is set to be sigma finite if we can write the whole space as a union of sets X n and one to infinity, such that these sets are pair wise disjoint. So, we want this sets to be pair wise disjoint and mu of each X n so be finite. So, each X n should be element in the domain of mu in the class C and mu of X n should be finite. So, essentially saying that for totally finite we said mu of the whole space is finite and sigma finite means what that X can be cut up into pieces X 1, X 2, X n so on and mu of each X n is finite. So, this is what is called sigma finite set functions. So, what we are going to.

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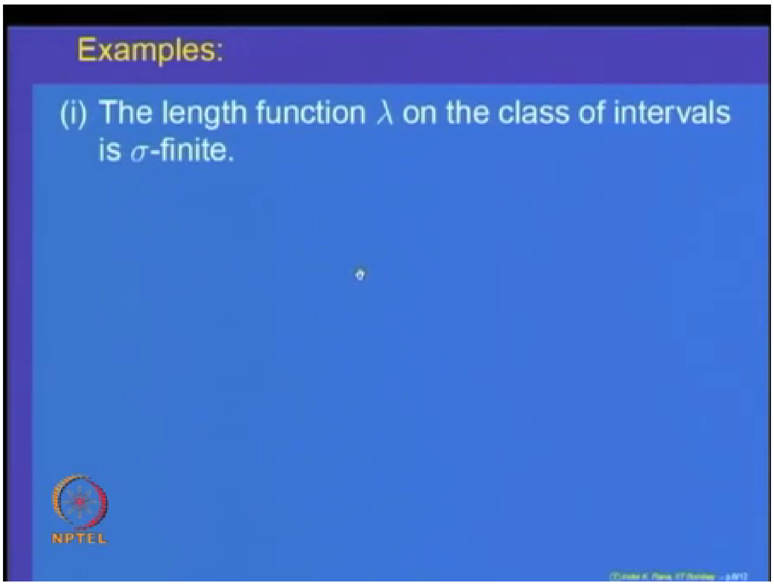

Let us look at some examples of set functions. So, the length function lambda on the class of all intervals is sigma finite.

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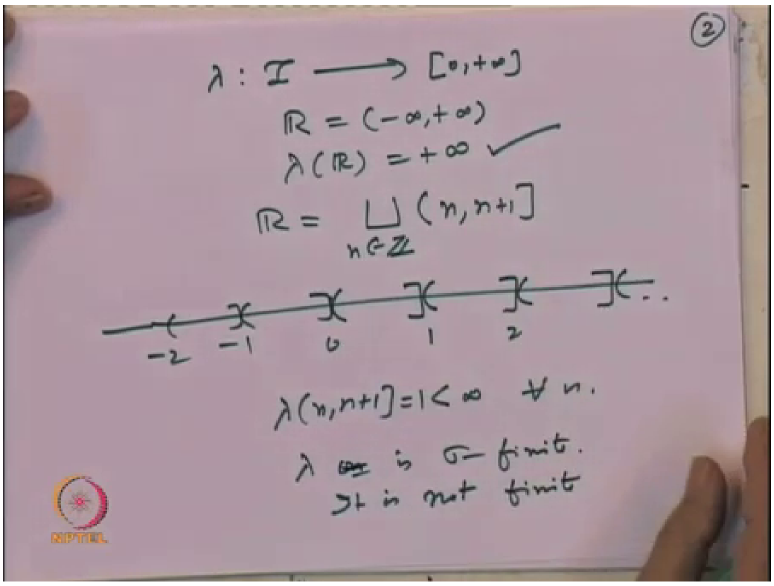

So, that is easy to see y is the length function sigma finite. So, here is the length function lambda on the class of all intervals taking values in 0 to infinity, the whole space that is real line of course, is the interval minus infinity to plus infinity and length of R we know it is not finite it is equal to plus infinity right. But we can write R as a disjoint union of the intervals n to n plus 1 for example, n belonging to integers. So, the real line is cut up. So, here is the real line. So, here is 0. So, take 0 1 open 1 close 2, open 2 close 3 and so on and on this side minus 1 and this is minus 2 and so on.

So, we have cut up the real line we are divided the real line into countable many disjoint pieces, each one is a interval and length of n to n plus 1 for every n is equal to 1 which is less than of course, infinity for every n. So, the whole real line is written as a countable disjoint union of intervals, each one having finite length. So, lambda on the class of intervals is sigma finite. Of course, it is not finite right because the measure of the whole space length of the whole space is equal to plus infinity. So, this is an example of a set function which is sigma finite; we will give another example of a set function which is totally finite.

So, for that let us look at the length function restricted to any finite interval. So, let us example say for example, let us look at the interval a to b.

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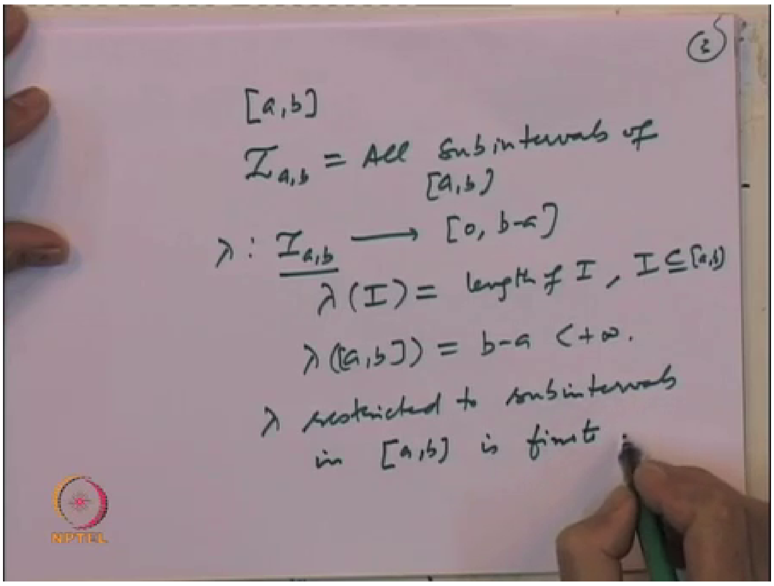

And let us look at all sub interval. So, let us look at I tilde a b, to be all sub intervals of a b and define of course, the length function as before. So, a b to. So, this will be a function from 0 to b minus a, length of I is equal to the usual definition of length I for I contained in a b, and we know length function is finitely additive its countably additive so on. So, so on this intervals also it is countable additive and lambda of the interval a to b we know is equal to b minus a which is finite. So, lambda. So, one says lambda restricted to sub intervals in a b is finite or totally finite for every a and b.

So, this is the example of a measure which is totally finite let us look at another example of a set function which is not. So, let us look at example.

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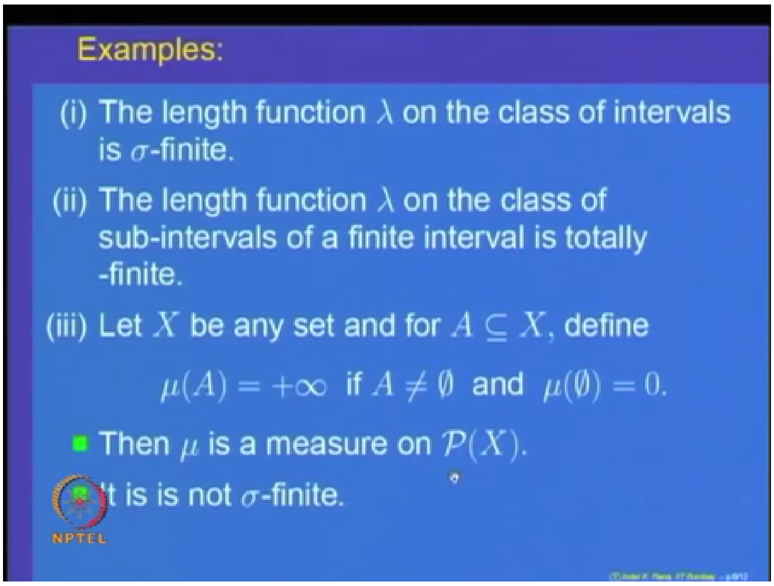

Of let x be any set and for any subset a of x let us define mu of A to be equal to plus infinity, if A is non empty and mu of 0 to be equal to 0 then; obviously, this is a measure on P of X this is a simple consequence the property that one can easily check, because mu of empty set is 0 is given and if A is any set which is a countable disjoint union or not. So, mu of the union will be equal to again plus infinity which is equal to sigma mu of is at least one of them has to be non empty. So, hope it is clear that this mu is countably additive this is a measure on P X.

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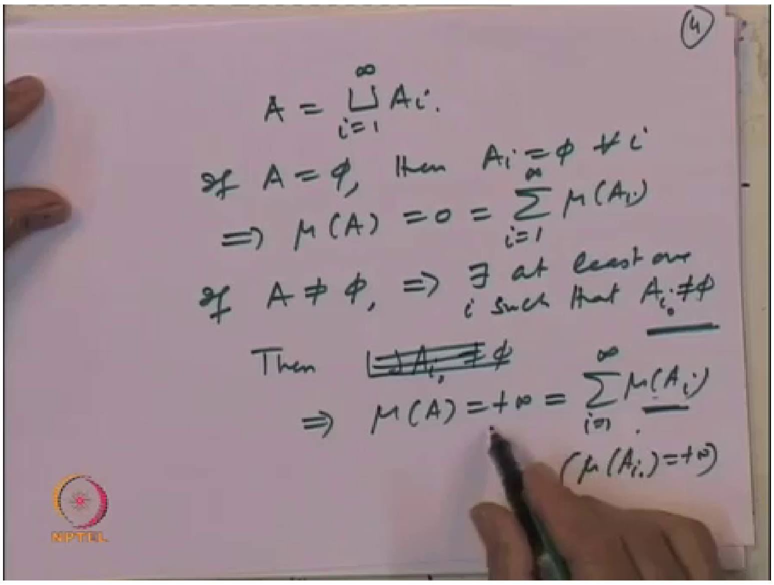

So, and if not let us look at supposing A is equal to union of A is i equal to 1 to infinity, if A is equal to empty set then A i is equal to empty set for every i.

So, implying mu of A which is 0 is same as sigma mu of A is i equal to one to infinity. The second possibility if A is not empty and A is equal to union. So, that implies there exist at least one i, i such that A i is not empty then. So, let us say that such that i says that. So, let us say that is A i naught there is a at least one i. So, let us that is i 0, then union A i naught is not empty and implying that ah not empty. So, that is A is not empty and even. So, this is not required then mu of A is equal to plus infinity is equal to summation mu of A is because at least one term here is equal to is not empty. So, that is equal to plus infinity.

So, mu of A i naught is equal to plus infinity. So, that say this is also plus infinity. So, they are same.

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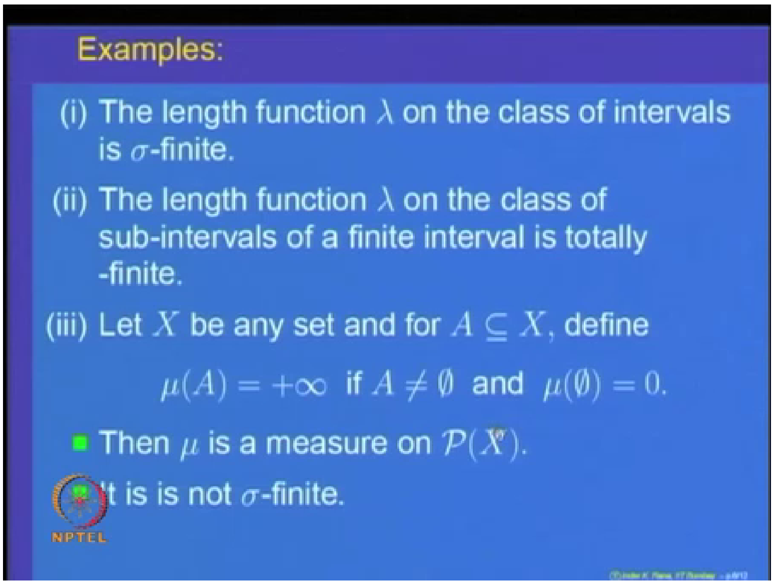

So, this is a measure on the class of all subsets. So, mu of A plus infinity if A is not empty and mu of empty set is equal to 0 is a measure and this; obviously, is not sigma finite because there are no subsets anyway whose mu is finite. So, this is example of ah non sigma finite measure. So, the theorem we want to prove today is the following namely.

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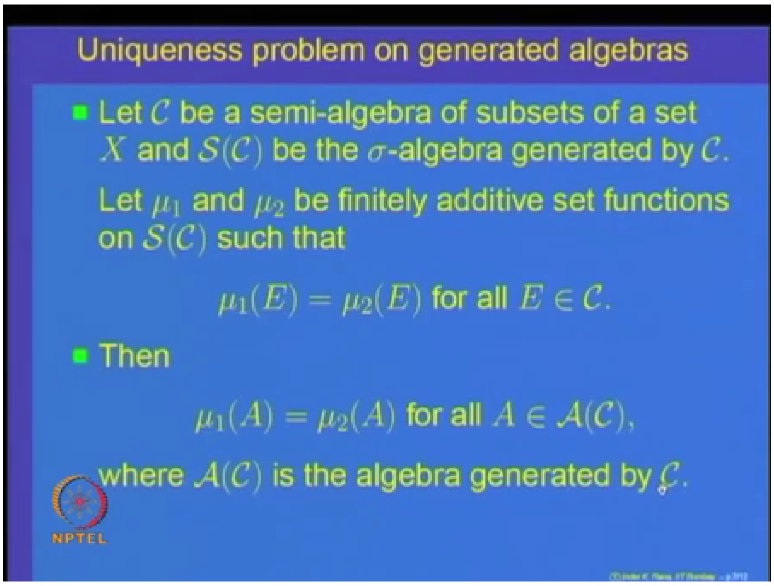

Let us take C a semi algebra of sub sets of a set x, and S of C be the sigma algebra generated by C. Let mu 1 and mu 2 between finitely additive set functions on S of C such that mu 1 of E is equal to mu 2 of E for all E belonging to C then we want to show that mu 1 of A is equal to mu 2 of A for all A belonging to first A of C, where A of C is the sigma algebra generated by C.

So, we are saying a first step we are going to prove that if two measures mu 1 and mu 2 defined on a semi algebra agree, then they also agree on the sigma on the algebra generated by that semi algebra. So, this is what we want to prove. So, let us see the proof of that.

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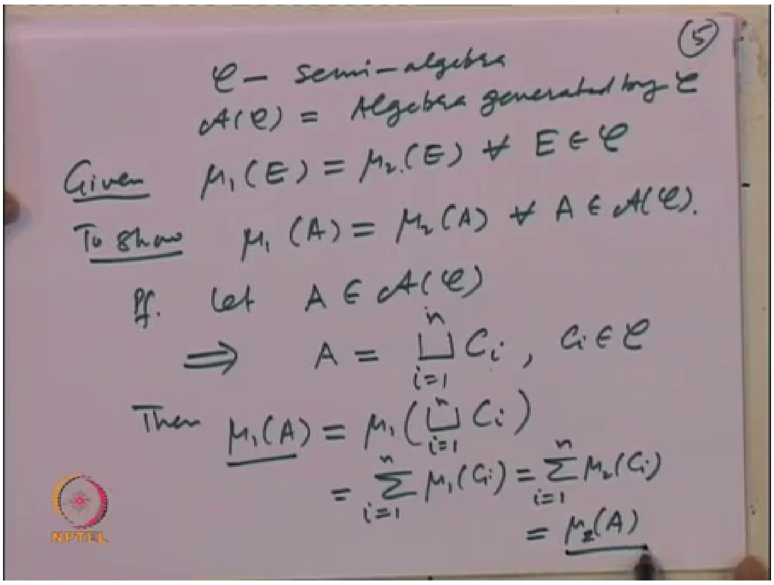

So, C semi algebra A of C that is the algebra generated by C and we are given mu 1 of E is equal to mu 2 of E for every E belonging to C to show mu 1 of A is equal to mu 2 of A where every A belonging to algebra generated by C. So, how do we prove it? So, let us start. So, let us take a set A,which belong to A of C then that implies. So, recall we had shown a characterizations of elements of the algebra generated by a semi algebra.

So, we showed that if A is a element of the algebra generated by a semi algebra, then this A must look like a finite disjoint union of element C i, i belonging to n there C is belong to the semi algebra c. So, every element A in the algebra generated by a semi algebra we had shown must have a representation like this, but then mu 1 of A is equal to mu 1 of this finite union and we know mu 1 is finitely additive. So, that implies this must be equal to sigma i equal to 1 to n mu 1 of C i, but each mu 1 is equal to mu 2 on each element of C and C is are elements of C. So, that implies that this must be equal to one to n mu 2 of C is, but again by using mu 2 is finitely additive I can write this as mu of A because A is a finite disjoint union of elements of this.

So, mu 1 of A is equal to mu 2 of A whenever A belongs to A of C. So, this proves theorem that whenever two measures finite whenever two finitely additive set functions mu 1 and mu 2 agree on a semi algebra then they also agree on the algebra generated by it.

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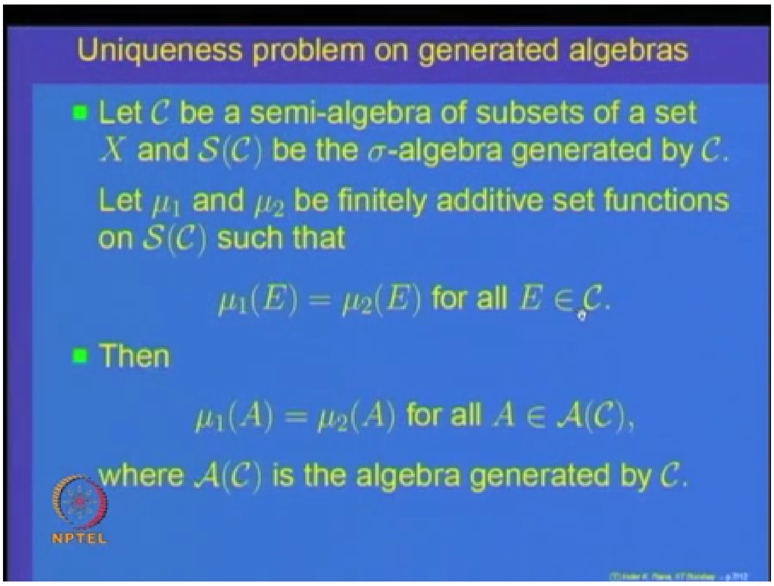

So, let us go to the next step of this uniqueness problem. So, that is saying that let C be a semi algebra of subsets of a set x once again and S of C be the sigma algebra generated by C. So, this is we have already prove sorry yes. So, let mu 1 and mu 2 be sigma finite measures on S of C such that mu 1 of E is equal to mu 2 of. So, this is a miss spread mu 1 of E should be equal to mu 2 of E for all E in C then mu 1 of E is equal to mu 2 of a for all A belonging to S of C whereas, of C is the sigma algebra generated by it. So, let me state and we will divide the proof into steps of course.

So, let us look at the statement of the theorem, once again we were saying that let mu one.

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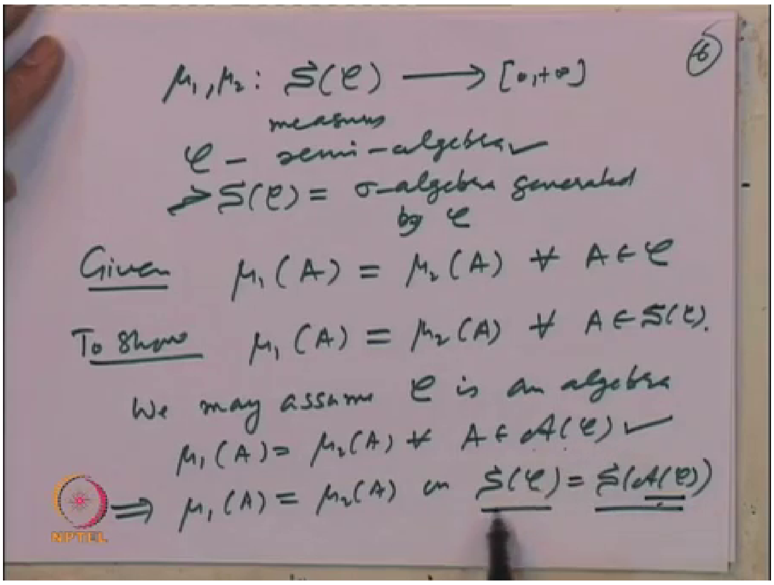

And mu 2 b two measures which are sigma finite defined on the sigma algebra generated by a semi algebra C. Measures C semi algebra S of C this S of C is equal to sigma algebra generated by C. Given mu 1 of A is equal to mu 2 of A for every A belonging to the semi algebra to show mu 1 of A is equal to mu 2 of A for every A in the sigma algebra generated by C. So, this is what we want to show. So, to show this the first step let us look at the first step. So, we may assume that C is an algebra. So, first. So, here we are given that C is a semi algebra.

So, step one says we may assume that C is an algebra and that is because of the fact that we have just now shown, that if mu 1 and mu 2 agree on the on a semi algebra then they also agree on the algebra generated by it. So, by the given hypothesis mu 1 of A is equal to mu 2 of A for every A belonging to the algebra generated by C.

So, we already mu 1 and mu 2 agree on the algebra generated by A of C and we want to show that this implies mu 1 of A is equal to mu 2 of A on S of C the sigma algebra generated by C, but note this is same as the sigma algebra generated by A of C that also we have shown that given a semi algebra you can directly generate the sigma algebra or you can generate the algebra first and then generate the sigma algebra both are same, and just now we showed whenever two measures agree on a semi algebra they agree on the algebra generated by it. So, mu 1 and mu 2 agree on the semi algebra they show that agree on the algebra generated by it, and we want to show that they agree on the sigma algebra generated by it and the which is nothing, but S of C. So, that proves the first step. So, as a first step in our proof we are saying that we can assume that the given class C on which mu 1 and mu 2 are defined is actually an algebra. So, that is the first simple equation in the proof, that without class of generality we may assume that C is an algebra.