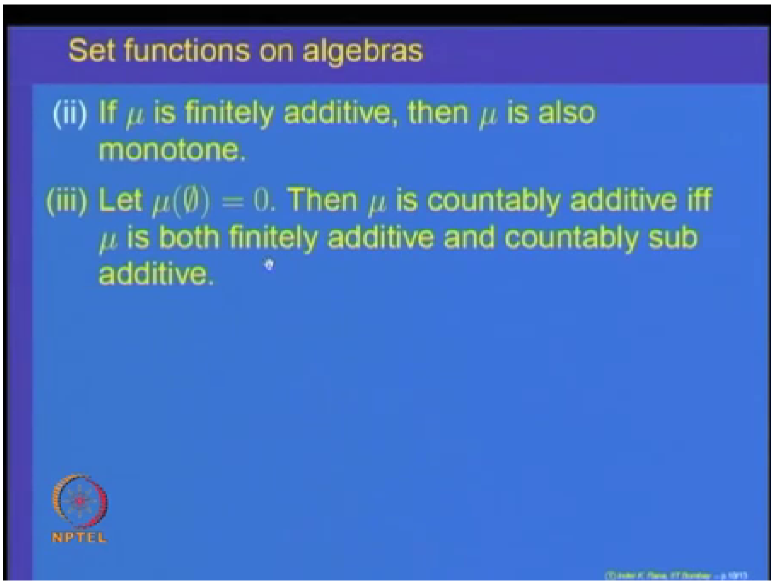
Measure & Integration
Prof. Inder K. Rana
Department of Mathematics
Indian Institute of Technology, Bombay

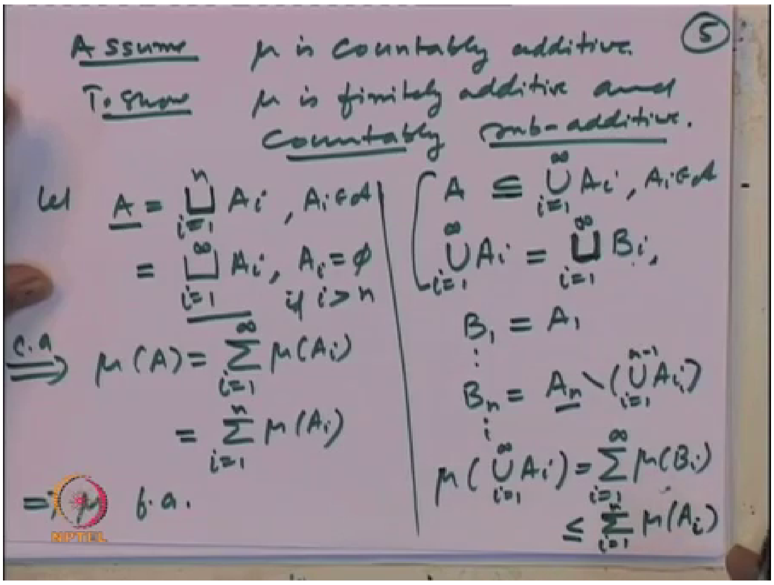
Lecture - 07 B
Countably Additive Set Functions on Intervals

(Refer Slide Time: 00:16)



So mu on monotone that we have already shown. So, let us look at the next property that that is a very important thing we will characterization of countable additiveness addictiveness of the set function. So, supp`ose mu of phi is equal to 0. Then we want to claim that mu is countably additive if and only if mu is both finitely additive and countably sub additive. So, we want to characterize countable additive property of the set function, define on a algebra in terms of it being finitely additive and countatbly sub additive. So, let us prove this properties. So, let us start well one way.

(Refer Slide Time: 01:07)



So, let us assume that mu is countably additive to show mu is finitely additive and countably sub additive.

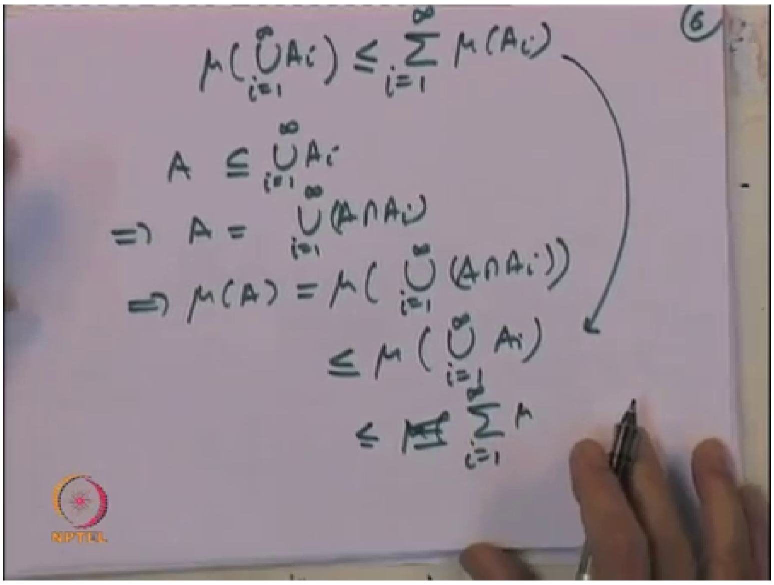
So, let us look at the first thing to show it is finitely additive. What we are to do it. So, let a be equal to a disjoint union A i i equal to 1 to n. So, whenever the union is disjoint sets where by disjoints we will write it as by a square union symbol for cup by the square instead of writing it as usual. So, where A i is belong to the algebra a now. So, I can also write it as union of A i i equal to 1 to infinity right where A i is equal to empty set if, i is bigger than n. From n onwards let us put them as empty sets then a is a countable union of pair wise disjoint sets. So, implies by countable additive property that mu of a is equal to summation mu of ais, i equal to 1 to infinity.

But that is same as sigma i equal to 1 to n mu of a i. Because for i bigger than or equal to n plus 1 this sets are empty and mu of the empty set is given to be 0. So, therefore, implies mu finitely additive. On the other side let us try to prove that mu is countably sub additive. So, let us take a set A in the algebra and let us say this is contained in union of a is i equal to 1 to infinity. Now let us observe the following namely, this union a i, i equal to 1 to infinity where a is are in the algebra, a if you recall we had shown that nay countable union of sets in the algebra can be written as a countable union of disjoint sets in the algebra.

Where again b I s are in the algebra, but this is a disjoint union. How did we do that let us just recall we defined b 1 to be equal to a 1 and in general b n to be equal to An minus union A i i equal to 1 to n minus 1 and so on. So, that is how we are defined those sets b I and note at every stage b 1 is a 1 a 1 in algebra. So, b 1 in the algebra similarly b n is An which is in the algebra finite union A i 1 to n minus 1 that is in the algebra and the difference of the 2 sets in the algebra is in again algebra. So, each b n is a element of the algebra these are disjoint and they are union because union of bn up to b n the same as union up to a 1 to An and that is true for every n. So, this is equal to true. So, once that is done.

So, using these 2 things now let us write that mu of a is a subset of this. So, this is mu of the union A i s i equal to 1 to infinity will be equal to summation mu of bi s i equal to 1 to infinity. Because this union A i is same as union bi s, and union of bi s this is a disjoint union. So, by countably additive property mu of the union is equal to this sum right. And now note b I is this each b n is a subset of a n. So, this is less than or equal to by finite additive property monotone property this is less than mu of An A i i equal to 1 to n. So, what we have shown is that sigma. So, what we have shown is the following namely that mu.

(Refer Slide Time: 06:22)

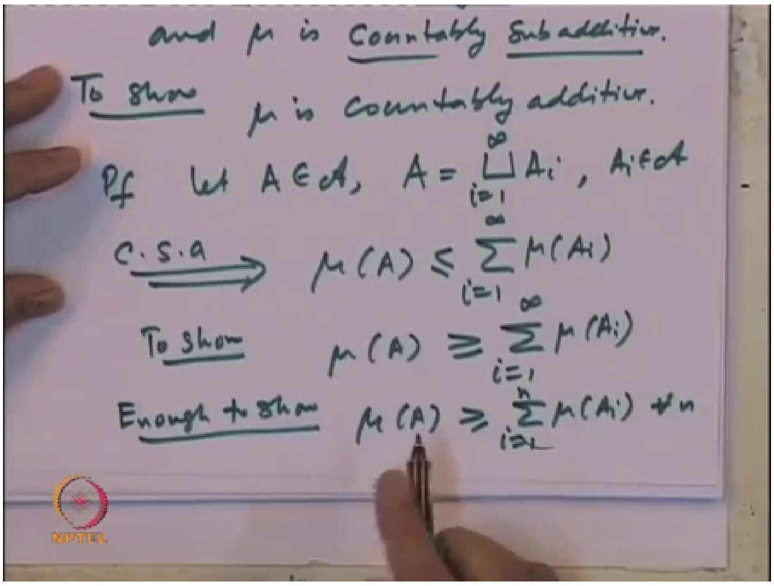


Of union A i i equal to 1 to infinity is less than or equal to sigma i equal to 1 to infinity mu of A i right. And now we just want to conclude that in fact, mu of a is less than or equal to this quantity.

So, now let is observe a is a subset of union a i. So, this implies I can write a is equal to union of a intersection a i, i equal to 1 to infinity. Right I can just intersect and then this is an equality so; that means, mu of a is equal to mu of union i equal to 1 to infinity a intersection A i and this is less than or equal to because this is this union is a subset of the union. So, this is less than mu of union i equal to 1 to infinity of ai s because each one is a subset of this. So, this union is subset of this, and now from here this is less than or equal to mu of summation i equal to 1 to infinity of mu of a i. So, we have shown that whenever a is an element in the algebra is a subset of union of ai s i equal to 1 to infinity then mu of a is less than or equal to summation mu of ai s.

So, that proves that mu is countably sub additive. So, we have shown if mu is countably additive then this implies mu is finitely additive and also mu is countably sub additive. So, that completes one part of the proof, let us prove the other way round implication namely. So, we want to show. So, assume.

(Refer Slide Time: 08:34).

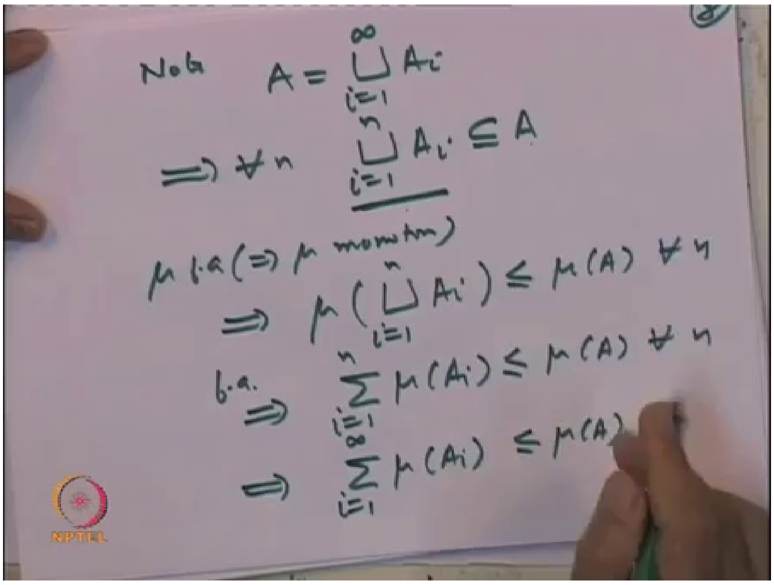


Mu is finitely additive and mu is countably sub additive. To show mu is countably additive. So, let us chose the proof. So, how to prove countable additivity what is show let a belong to algebra and a b equal to disjoint union ai s i equal to 1 to infinity ai is belonging to algebra. And we have to show mu of a is summation mu of ais.

Now, by countable additive countable sub additive property which is given to us this implies countable sub additive implies that mu of a is at least less than or equal to sigma i equal to 1 to infinity mu of Ai s. So, countable sub additivity implies this fact there is less than or less equal to this. So, we have to prove only the other way. So, to show that mu of a is also greater than or equal to sigma i equal to 1 to infinity mu of A i right. So, this is what we have to show and note. So, here is a small observation to show this enough to show it is enough to show that mu of a is bigger than or equal to sigma i equal to 1 to n mu of A i for every n. So, if you can show for every n mu of a is bigger than or equal to this, then it also will be true for i equal to 1 to infinity because this is nothing, but limit of this partial sums.

So, this is enough to show. So, we have to only show that mu of a is bigger than or equal to sigma mu of ai s i equal to 1 to n and to show that let us observe. So, note.

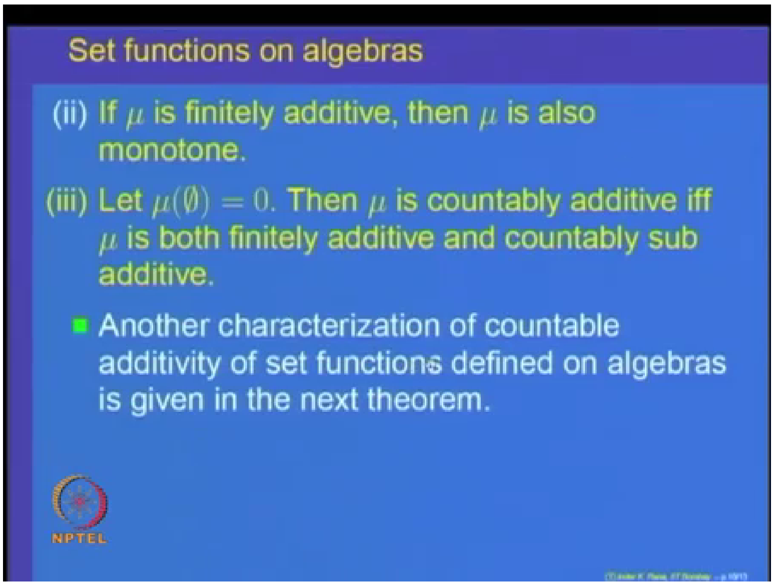
(Refer Slide Time: 11:06)



That a equal to union a i, i equal to 1 to infinity implies for every n the union A i, i equal to 1 to n is a subset of a for every n right and we are in algebra. So, this set is in the algebra this is in the algebra mu finitely additive implies mu monotone. And hence implies that mu of the union A i i equal to 1 to n will be less than or equal to mu of a for every n, but again by finite additivity this is nothing, but sigma i equal to 1 to n mu of A i is less than or equal to mu of a for every n and this is happening for every n. So, this is implies we can let and go to infinity.

So, i equal to 1 to infinity mu of A i is less than or equal to mu of a. So, that proves the other around inequality also of the required thing. So, this proves this. So, that proves that mu is countably additive. So, what we have proved is the following namely. So, we have given a characterization.

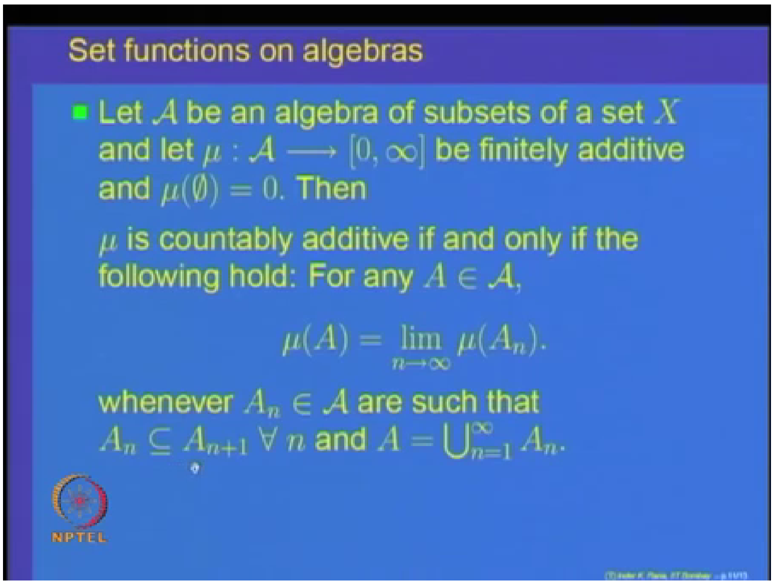
(Refer Slide Time: 12:34)



Of countable additive property of set functions which are finitely additive. So, if mu of empty set is equal to 0 then mu is countably additive if and only if. So, note here if and only if we have proved both ways. So, if and only if mu is both finitely additive and countably sub additive. So, this is a characterizations of countably additiveness additiveness of set functions, but of course, the domain of the set function should be an algebra that is important. So, this is a very useful criterion for accountable additivity.

We will prove another characterization of countable additivity of set functions in terms of a limits increasing and decreasing limits. So, that is given in the we will state next theorem, but again that theorem is again about set functions defined on algebras. So, the theorem says the following.

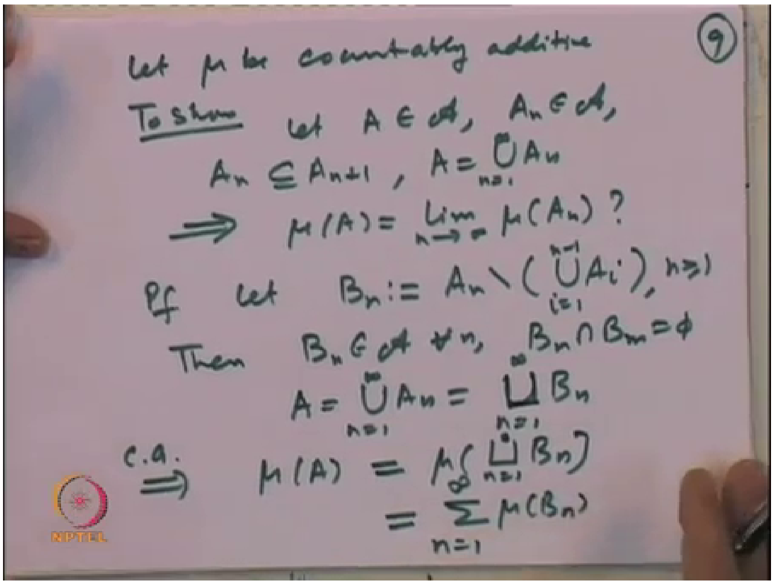
(Refer Slide Time: 13:42)



Let A B and algebra of subsets of a set x and mu be finitely additive and with the property of course, mu of empty set is equal to 0. Then we want to prove that mu is countably additive if and only if once again it is a characterization if and only if the following property holds and the property says for any element a in the algebra a we should have mu of a is limit of mu of a ns, what are ns whenever An is a sequence of sets in the algebra which is decreasing. So, An is a subset of An plus 1 for every n the sequence An should be decreasing and a should be sorry An should be increasing An is a subset of An plus 1; that means, a ns are increasing sequence of sets in the algebra and a is the union of all this sets a n.

So, this is a characterization of countable additiveness of the set function mu provided, one can prove the following. For any set A and for any sequence An of sets in the algebra which is increasing and a is the union we should have mu of a equal to limit mu of a ns ns. So, let us prove this property once again. So, to prove this what we have to show.

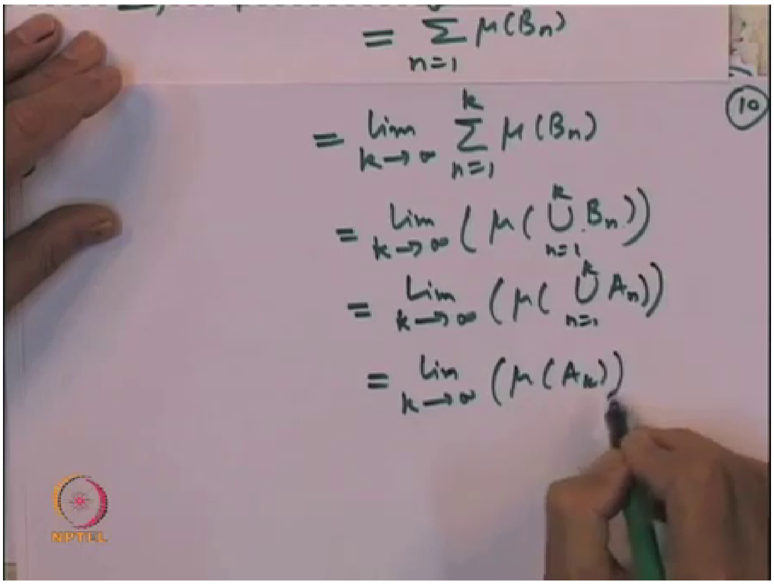
(Refer Slide Time: 15:15)



First let us prove. So, let mu be countably additive. So, let mu be countably additive. To show we have to show the following let take a set A belonging to the algebra take a sequence a ns belonging to the algebra such that An is sub set of An plus 1 and a is equal to union of a ns we should show that mu of a is equal to limit n going to infinity mu of a ns. So, that is what is to be shown. So, let us now let us observe a is union of a ns, and we are given something about countable additivity. So, the obvious thing is try to write this union has a countable disjoint union.

So, we do that. So, proof let b n we defined as An minus union a i, i equal to 1 to n minus 1 for every n bigger than or equal to 1. Then as as observed earlier each b n belongs to the algebra b ns are disjoint and a which is union of a ns is also equal to union of b ns of course, this is disjoint. So, let me write that equal to this. So, implies by countable additive property mu of a is equal to mu of this union b ns, and that is equal to by accountable additive property that is summation n equal to 1 to infinity mu of b ns. So, that is by accountable additive property and now, but we do not want b ns we want something in terms of a ns.

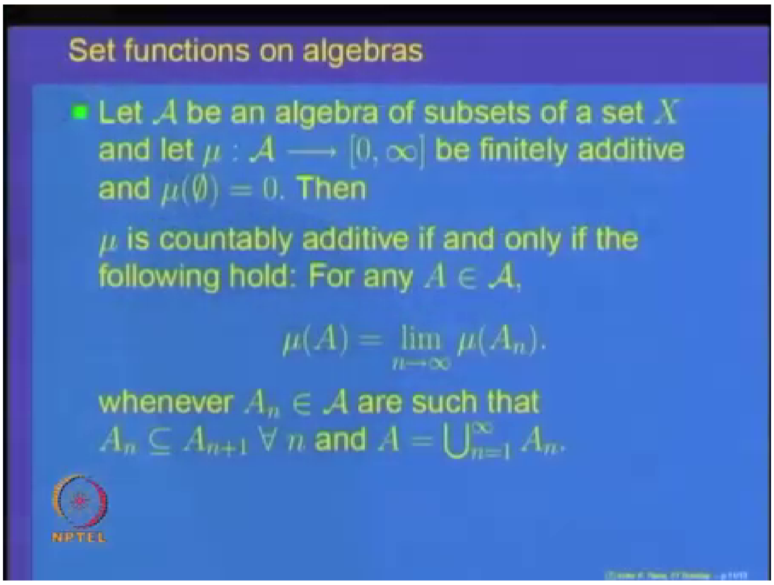
(Refer Slide Time: 17:45)



So, here is an observation this summation I can write as limit k going to infinity of the partial sums. So, n equal to 1 to k of b of b ns, but b ns are disjoint. So, this is same as limit k going to infinity of mu of union b b n n equal to 1 to k because b ns are disjoint by finite additive property this must be true and we are one. So, note once again because mu is given to be countable additive and hence it is finite additive and by finite additive property this is true. And this is and now the observation is that the union of b ns n equal to 1 to k is same as the union of as.

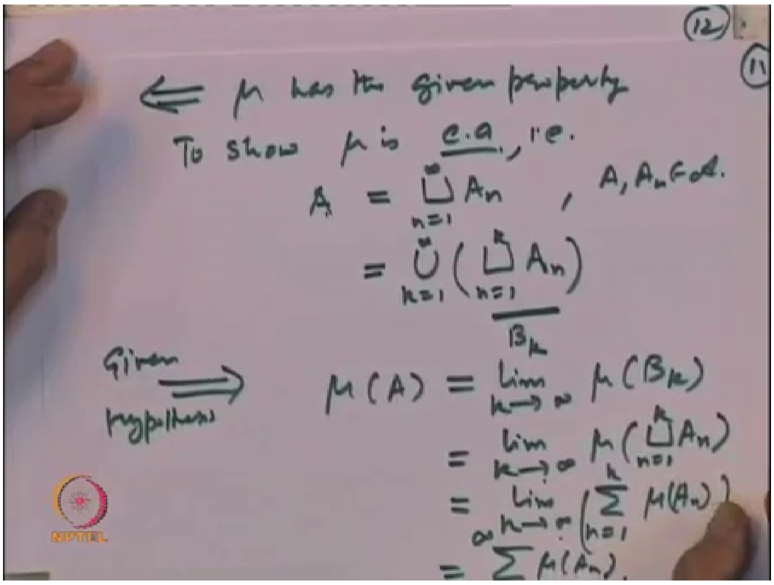
So, this is same as k going to infinity mu of union An union of a, n equal to 1 to k, but note we are not use anywhere the fact that a ns are increasing. So, in since a ns are increasing what is this union this union is precisely mu of the largest set that is a k. So, that is mu of a k. So, what we are shown is that mu of a. So, we are shown mu of a is limit of mu of a k is going to infinity, whenever An is a sequence which is increasing whenever a ns is increasing and a is equal to union. So, we have proved one way countable additivity implies the required property let us look at the converse. So, conversely.

(Refer Slide Time: 19:40)



So, let is assume mu has.

(Refer Slide Time: 19:43)



The given property and what is the given property.

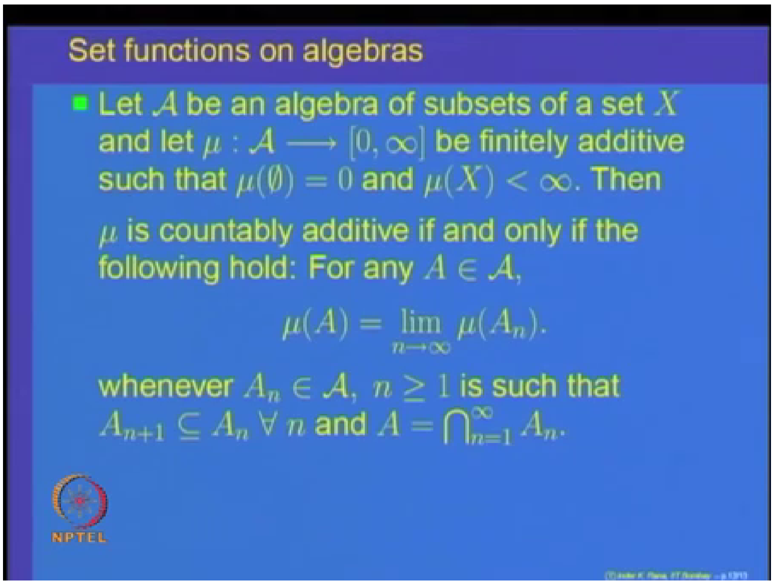
Given property says whenever a set A is written as union of a ns a ns are increasing then mu of a is equal to mu of the union. So, we want to prove to show. So, to show mu is countably additive that is let us take a set A which is disjoint union of sets An n equal to 1 to infinity where a and all a ns are in the algebra right. We have to show that mu of a is equal to summation mu of a ns, but this I can write it as union over k one to infinity union An n equal to 1 to k.

So, take instead of taking n equal to 1 to infinity take union of sets a 1 a 2 a k and then take the union over k both will be same right, but the advantage of this way is now if you call this as b k then b k is a set in the algebra, because it is a finite union of sets in the algebra b k is increasing because we are taking union of more and more sets and there union is equal to a. So, by the given property. So, by the given hypothesis mu of a is equal to limit k going to infinity mu of b k.

And now let us go back to represent b k as in terms of as. So, that is limit k going to infinity mu of union An n equal to 1 to k. And now we use the fact that mu is finitely additive. So, this is limit k going to infinity summation n equal to 1 to k of mu of a ns and which is same as sigma one to infinity of mu of a ns. So, that says whenever a is a disjoin union of countable disjoint union of sets in the algebra mu of a is mu of sigma mu of a ns and that is countable additive property of the set function. So, we have proved theorem completely namely if a is an algebra of subsets of a set x and mu is finitely additive with that property then mu is countably additive if and only if mu has the property that mu of a is the limit of mu of a ns whenever An is increasing and An is equal to union of a sets.

So, this is characterizing countable additivity in terms of limits of increasing sequence of sets and this property one says that mu is continuous from below at the point a. So, countable additivity for finitely additive set functions is same as saying they are continuous from below at the point a from below because a is union of these sets. So, from below there is a corresponding result for sequences which are decreasing. So, let us state that result and prove it also.

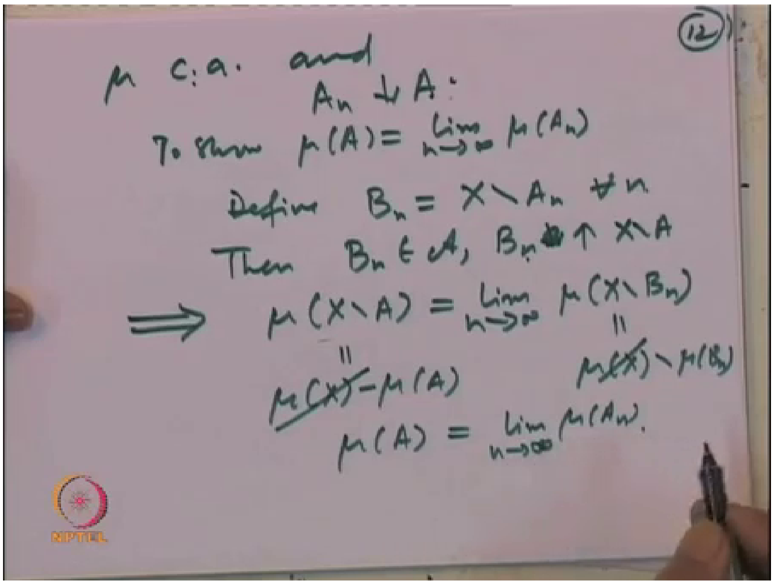
(Refer Slide Time: 23:34)



that if a is an algebra of subsets of a set x and mu is finitely additive. So, that conditions are same as it, plus we want additional condition that mu of the whole space is finite. So, this is a additional condition put to prove to state the result namely mu of the whole space is finite. So, it says mu is countably additive if and only if the following holds namely for any set A in a whenever mu of a is equal to limit n going to infinity mu of An and whenever a ns are decreasing.

So, An plus 1 is subset of An and a is the intersection we says. So, countable additivity is equal to sayng for every set A in the algebra if a is intersection of a decreasing sequence of sets a ns then mu of a must be equal to limit of a ns and the proof of this uses the earlier theorem. So, let us assume mu is countably additive and a ns decrease to a.

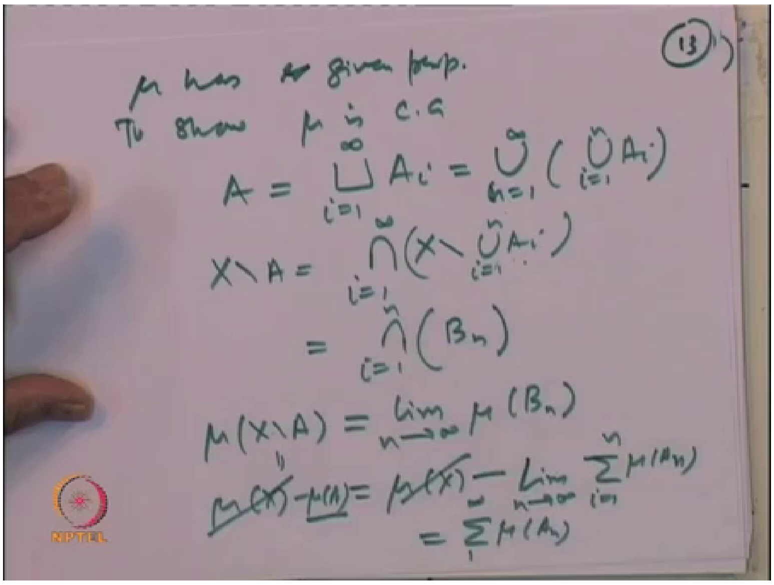
(Refer Slide Time: 24:45)



All in the algebra a, to show we want to show that mu of a is equal to mu of a ns to show mu of a is limit n going to infinity mu of An. Now we know something about to increasing sequences. So, from decreasing we want to manufacture a increasing sequence and that is done via compliments. So, define b n to be equal to x minus An for every n then b n each b n belongs to the algebra a b n is decreasing because a ns are b ns are sorry increasing as a ns are decreasing and where do the decrease. So, they b b ns increase to x minus a because a ns are decreasing to a.

So, by the earlier theorem we have. So, countable additivity implies whenever a sequence is increasing mu of x minus a must be equal to limit n going to infinity mu of x minus b n, but now we use the fact that mu of x is finite. So, this is same as mu of x minus mu of a and this thing is equal to mu of x minus mu of b n. And this is possible only because we have the fact that mu of the whole space is finite. So, everything is a finite quantity and we have already shown mu of the difference is equal to difference of mu's provided the things are finite. So, this is equal to limit of this. So, now, x cancels negative sign. So, limit. So, mu of a is equal to limit mu of a ns n going to infinity. So, this is countable additivity implies this the other way round property. So, let us assume this as the property that whenever a ns increase.

(Refer Slide Time: 27:06)



So, mu has required a given property to show mu is countably additive. So, let us take a equal to a disjoint union A i s. Then then this is what is x minus a that is intersection of i equal to 1 to infinity x minus. So, let us right this as union of i n equal to 1 to infinity union of ais i equal to 1 to n. So, these are the sets. So, then it is x minus union a i, i equal to 1 to n. So, that is equal to intersection i equal to 1 to n and this things nothing, but x minus. So, let us call this has a a set has b n. So, let us call this has b n right. So, now, note b ns are decreasing and they are in the algebra because the a ns are union n this will be increasing. So, this will be decreasing. So, mu of x minus a by the given hypothesis is limit n going to infinity mu of b ns. And what is mu of b n mu of b n is x minus this.

So, that is equal to mu of x minus. So, limit n going to infinity mu of the union that is this disjoint. So, summation i equal to 1 to n mu of a ns and this thing is equal to mu of x minus mu of a because everything is finite. So, this cancels with this. So, mu of a is limit of this, which is equal to sigma one to infinity mu of a ns. So, that proves countable additivity. So, we have proved that when mu is countably mu is countably additive if and only is for a decreasing sequence of steps An a equal to this intersection mu of a is the limit under the condition mu of x is finite. So, this is important this condition cannot be removed that is the. Next so, this kind of thing is called continuity from above and here is a remark that the condition mu of x is finite is necessary in the second part and cannot be removed. So, that we will request you to construct an example you can construct a very easily an example on the real line with length function has the set function. And here is an exercise for you to do that is finitely such that mu of is finite. So, last part we said An decreasing to a that you can actually reduce a bit says whenever a ns are decreasing to empty set that is also equivalent to saying that mu is countable additive. So, these 2 parts we will like you to explore and understand and answer this questions. So, thank you let us stop to it.

Thanks.