Measure & Integration
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Lecture - 07 A
Countably Additive Set Functions on Intervals

Welcome to lecture 7 on measure and integration.

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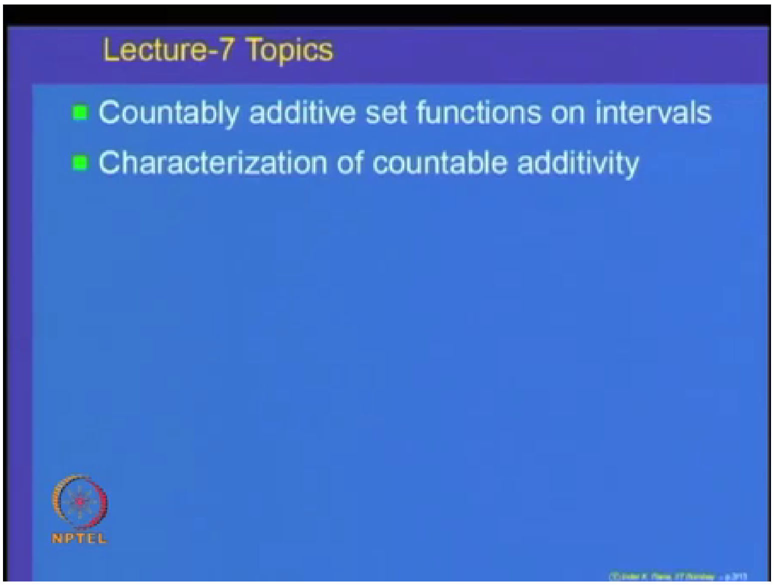

If you recall in the previous lecture we had started looking at the countable additive set functions on intervals and we proved some properties of such countably additive set functions. We will recall that the theorem that we were proving and then continue the proof and if time permits we look at a characterizations of a countably additive set functions defined on algebras in the later part of the lecture. So, let us just recall what we are proving in the last lecture.

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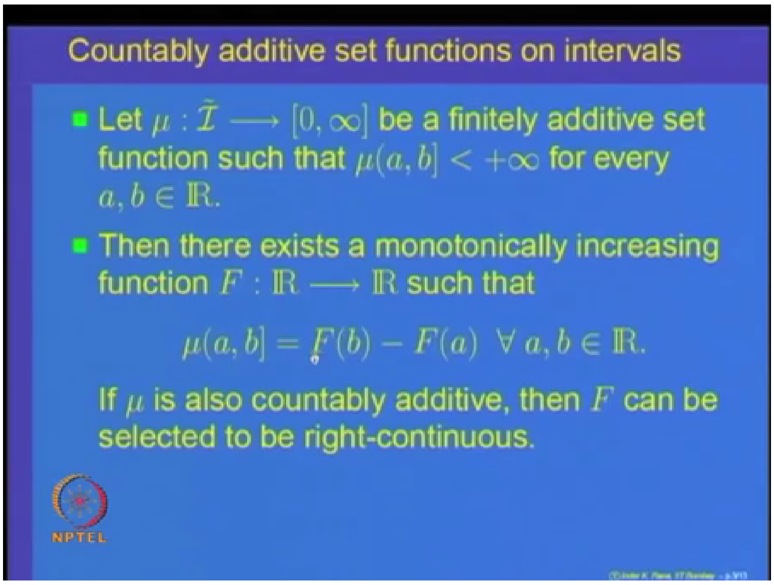

We were trying to show that if mu is a finitely additive set function defined on the collection of all left open right closed intervals which was denoted by i tilde, if such a finitely additive set function is given with a property that mu of any finite interval is finite. So, mu of left open right close interval a B is finite for every A and B then we wanted to characterize such countable additive properties of such functions and related to a class of functions on the real line.

So, the claim of the theorem is that then there exist a monotonically increasing function F from R to R such that the value mu of the open left open right closed interval a B is given by F B minus F of a for every A and B belonging to R. So, we want to show that given a we wanted to show that given a finitely additive set function on the class of all left open right close intervals it must arise from a monotonically increasing right continuous function f, with the relation that the value mu of a B is given by the difference F B minus F of a. And in addition if mu is here mu also only finitely additive if we assume mu is countably additive then F this function F can be selected to be right continuous.

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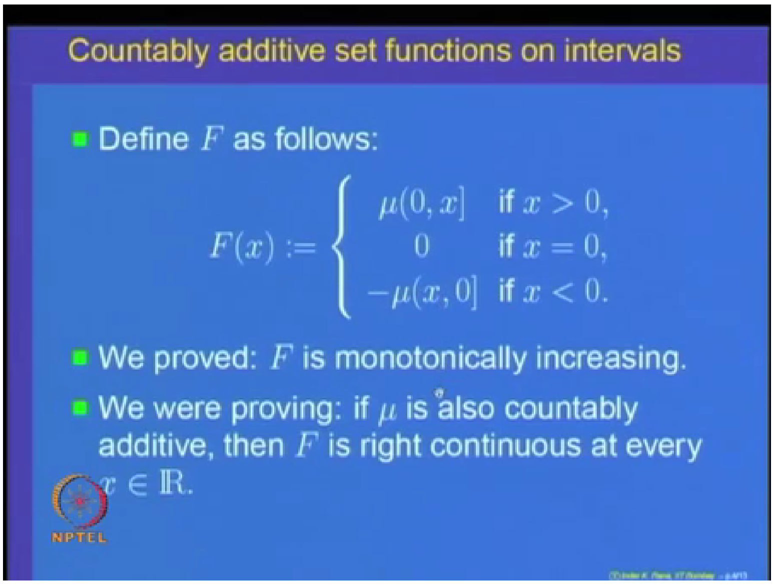

So, let us recall how if we define this function we looked at the function F defined by F at a point x is defined as the measure mu of the interval 0 to x if x is bigger than 0, and it is 0 if x is equal to 0 and is minus mu of x to 0 closed at 0 if x is less than 0.

So, this was the definition of the function F and we proved the property that this function F indeed is monotonically increasing. And for that if you recall we use the fact that F the measure mu this mu is a countable is a finitely additive set function. So, in next we if you assume that mu is countably additive we wanted to show that this function F is right continuous at every point x in R.

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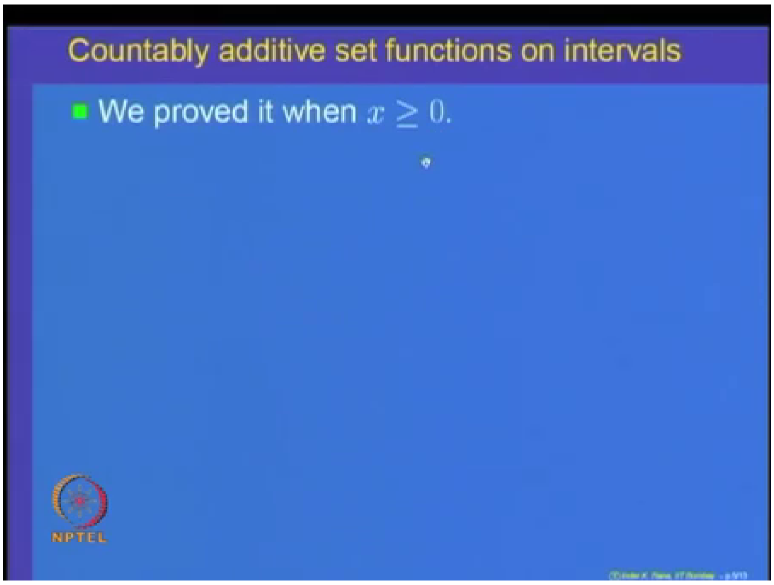

And we had started looking at the proof even x is bigger than or equal to 0. So, we had proved that for x bigger any point x bigger than or equal to 0 F is right continuous at the point x is equal to 0. So, let today we will start with proving the other part remaining part of the proof namely if x is less than 0 then also F is right continuous at x.

So, let us look at the proof.

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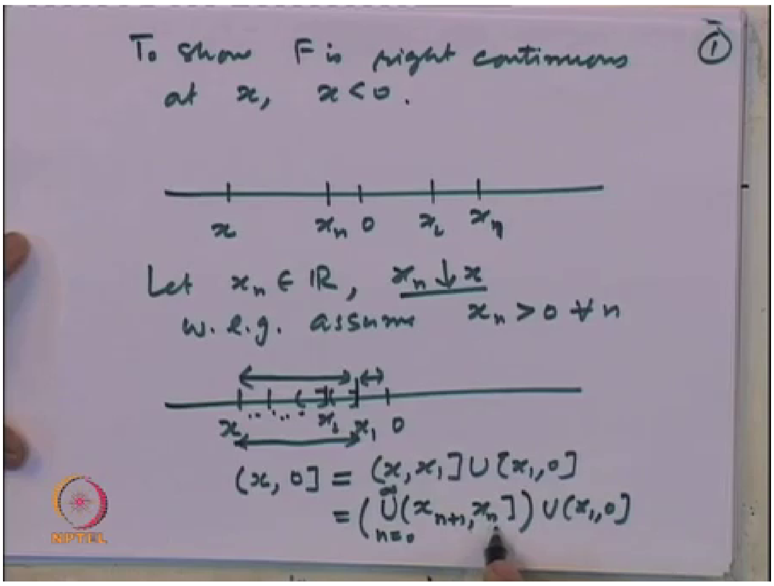

So, we want to show F is right continuous at a point x where x is less than 0. So, here is the point 0 and here is x. So, to show right continuity at the point x let us take. So, let x n belong to R. So, let us points x n a sequence in R say that x n decrease is to x; that means, x n is converging all the x ns are on the right side of the x and is converging to x. So, because it all the points x ns are on the right side of. So, here may be x n x 1 here may be x 2. So, on. So, after some stage x n has to cross over the point 0 the value 0. So, what we are saying is without lost of generality assume that all the x ns are bigger than 0 for every n because x n is going to converge to x and x is less than 0.

So, at some stage it has to cross over. So, analyzing we can start analyzing the sequence from that point onwards or one writes this has without loss of generality the proof is not change if you assume x n is less than 0 for n. So, here is the situation here is the point x here is the point 0 and here is the point x 1. So, now, let us look at. So, here is x 2 and so on.e. So, let us observe that the interval left open right close 0 can be written as x 2 x 1 union x 1 to 0. So, I can write this as from this point to x 1 and from this point onwards to this one. And now this interval x to x 1 and going to split further into a union of intervals. So, my claim is that this x to x 1 is same as x 1 to x 2, union x 2 to x 3 union x 3 to x 4 and so on. So, we are going to claim is that this is same as x n plus 1 comma x n left open right close union n n equal to 0 to infinity union x 1 0.

So, the interval x to x 1 this part we are splitting it into left open right close left open right close and so on and because this is true this this is equality because x n is decreasing to x. So, at any point here in between if i take from any point in between x and x 1 then that stage has to be crossed over by some x n. So, that point will belong here. So, this the interval x to x 1 is a union of the intervals left upon x n plus 1 to close x n n equal to 0 to infinity. So, and also observe that these intervals are all disjoint. So, these are all disjoint intervals.

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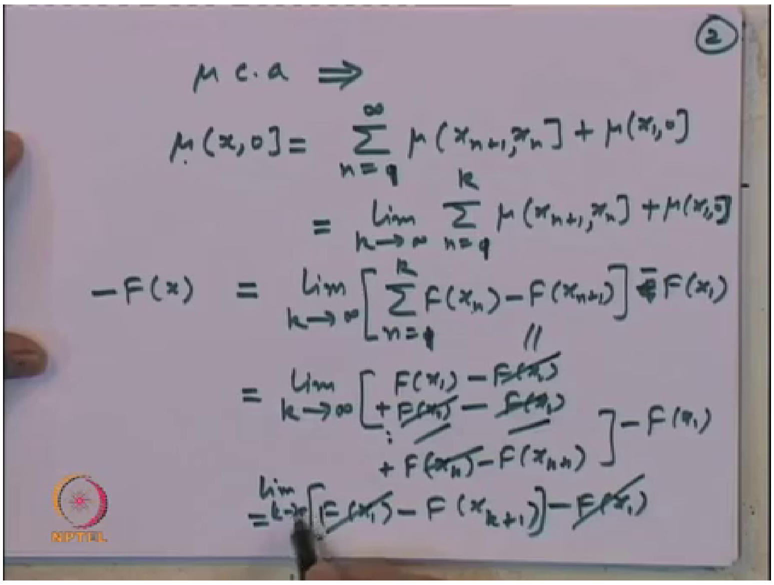

So, I can write mu of using countably additive property of the set function I can write this is equal to summation n equal to 0 to infinity mu of x n plus 1 x n plus mu of x 1 to 0. So, here we have used the fact that mu countably additive implies this property is true. And now this right inside is a sequence of a nonnegative real numbers possibly extended real numbers.

So, I can write this as limit k going to infinity sigma n equal to 0 2 k, mu of x n plus 1 x n plus mu of x 1 to 0 close here 0 . So, now, we will write everything in term of F. So, by definition mu of x to 0 is minus F of x is equal to limit k going to infinity summation n equal to 0 to k and this is nothing, but F of x n minus F of x n plus 1 plus F of F of x 1 to 0. So, that is in fact, minus F of x 1. So, now, let us x ah note what is this so this is limit k going to infinity and what is the sum this is starts with n equal to n equal to 0 will give x 0 that is not. So, let us. So, there was a well take care i should have written as union from n equal to one because it is one to 2 right one to 2 and so on. So, that was the mistake here.

So, this sum is from n equal to one to n equal to one to n equal to one to k. So, what is the sum. So, n equal to 1. So, that gives you F of x 1 minus F of x 2 plus F of x 2 minus F of x 3 and so on. Plus, F of x n equal to k. So, that is x k minus F of x k plus 1. And now so, that is this part the sum and minus F of x 1. So, now, we observe that in this x 1 x 2 x 2 this will cancel out and what is left with this is equal to F of x 1 minus F of x k plus 1 minus F of x 1. And now in this equation so this cancels with this so minus F of x oh sorry that is the limit outside. So, limit of this k going to infinity. So, what we get is F of x is equal to. So, this gives us that F of x.

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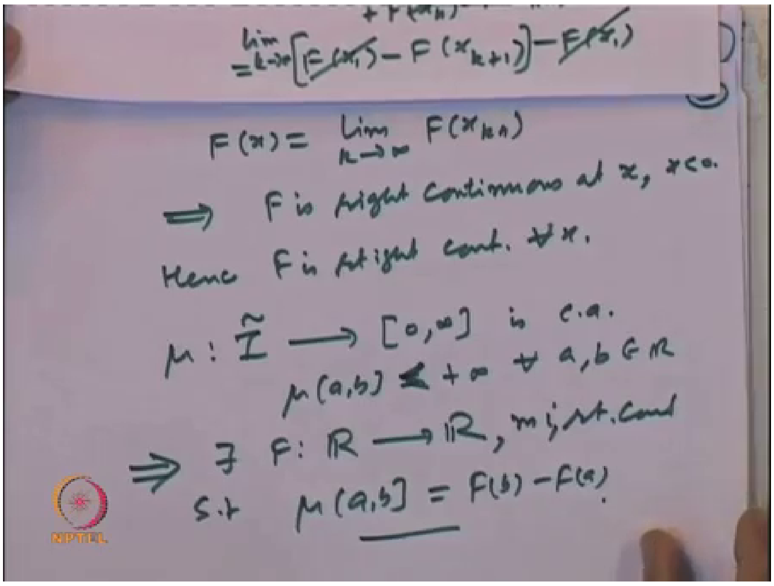

Is equal to limit k going to infinity of F of x k plus 1. And that proves the fact that F is right continuous at x in the case when x was less than 0.

So, hence F is right continuous or every x. So, this proves the theorem that if mu on the class of all left open right closed intervals is a measure is countably additive with the property that mu of a B is equal mu of a B is finite for every a B in R then this is implies there is a function F which is monotonically increasing which is right continuous. So, monotonically increasing right continuous such that, mu of a B is equal to F B minus F of a. So, this is says. So, what we have shown is that to every countably additive set function mu on left open right closed intervals you can associate a monotonically increasing a right continuous function.

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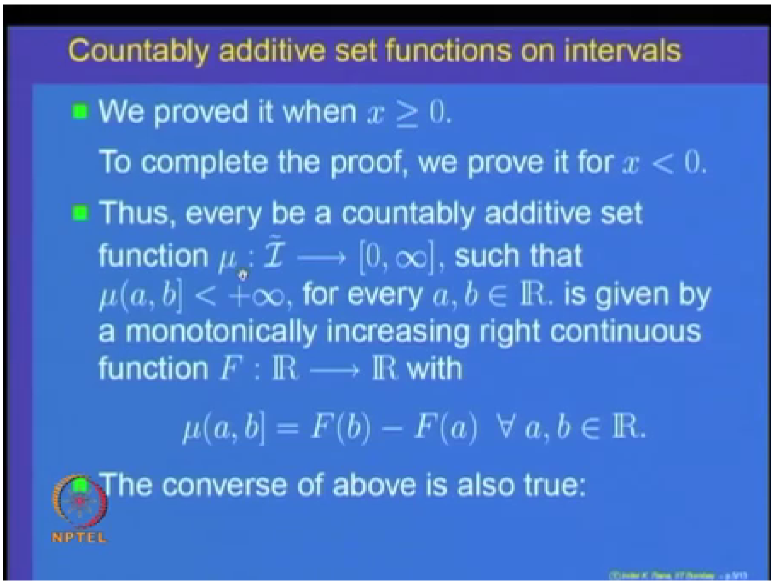

And we will next show so, this is proved at this completes the proof of the fact that to every monotonically increasing right continuous function, we can associate a to every a countably additive set function on the class of intervals, we can associate a monotonically increasing right continuous function with this property.

In fact, the converse of the statement also holds. So, what will be the converse of such a statement. The converse of such a statement would be that if you are given a monotonically increasing right continuous function F, then we can define a way a set function mu on left open right closed intervals in such a way that this equation is satisfied the is relation is satisfied. So, that will prove that the only way monotonically only way we can construct accountably additive set functions on the class of intervals is via monotonically increasing right continuous functions.

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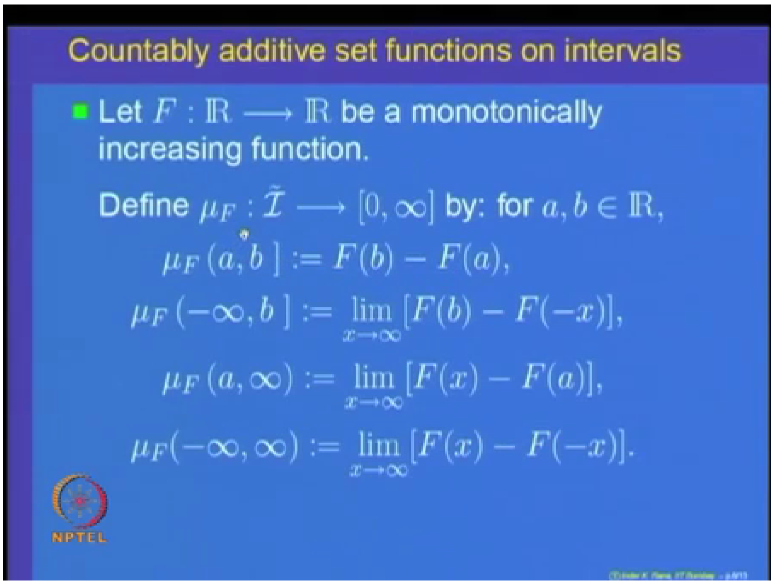

so the converse part of the theorem says the following let F B a monotonically increasing function from R to R. So, define mu F a set function from on the class of all left open right closed intervals has follows, for any 2 real numbers A and B we want to define what is mu F of the close left upon right closed intervals a b.

So, because this is the property has to be satisfied by F. So, that itself gives us the defining property of the set function mu. So, mu F of the left open right closed interval is defined as the difference F B minus F of a for all real numbers a and b. And now the question comes what happens if B is equal to plus infinity or a is equal to minus infinity or both of them. So, in that case we write this as for mu F of the infinite interval minus infinity to be So, it is open on the left side and close on the right side b. So, it is left open right closed interval on the real line. So, it what we do is we the definition as F B minus F of minus x x going to infinity. So, as x goes to infinity minus x will go to minus infinity.

So, we are using we are defining it via limits. So, look at the interval minus x to be left open. So, that is the value of the mu of F and then take the limit of that as x goes to infinity. So, this is the definition of mu F of minus infinity to b. And similarly if it is on the right side. So, if a to infinity. So, what we defined as take the interval a to closed x. So, then the value of that will be F of x minus F of a, and now take the limit of that as goes to infinity. So, for the infinite interval unbounded on the right side left open right close. So, a to infinity is defined as limit x going to infinity F of x minus F of a. And if it is the whole real line then we define mu of F of the whole real line to be limit x going to infinity of F of x minus F of minus x. So, look at the interval minus x to x and let both sides go to infinity.

So, this is the way we define a mu of F. And now note that this is a generalization of the length function if F is the identity function namely F of x is equal to x, that is a monotonically increasing function then this is nothing, but B minus a. So, mu of a B is nothing B minus a. So, this mu F is nothing, but the length function when F is a monotonically increasing function. And one can write down a proof of this on the lines of when we proved that the length function is countably additive. So, on the same lines one can write down the proof of the fact that this set function mu F is also countably additive. One can wonder where one will be using the fact that F is right continuous. So, the fact where will be using the right continuity of this F to prove that it is monotonically increasing to prove that mu F is countably additive.

So, this F for a monotonically increasing we can defined this mu F is finitely additive, but to prove countably additive we need F to be a right continuous function. So, if F is right continuous then, one can write down a proof similar to that of the case of the length function. One use this fact right continuity because one has to deal with the intervals which are left open end right closed. So, we says that if you are keen to know a proof of this you better write a proof yourself and trying to see that the steps given for the proof of the length function is countable additive can be suitably modified to do this so we leaved it as a exercise and if we feel it is too tough exercise let us assume this and go ahead.

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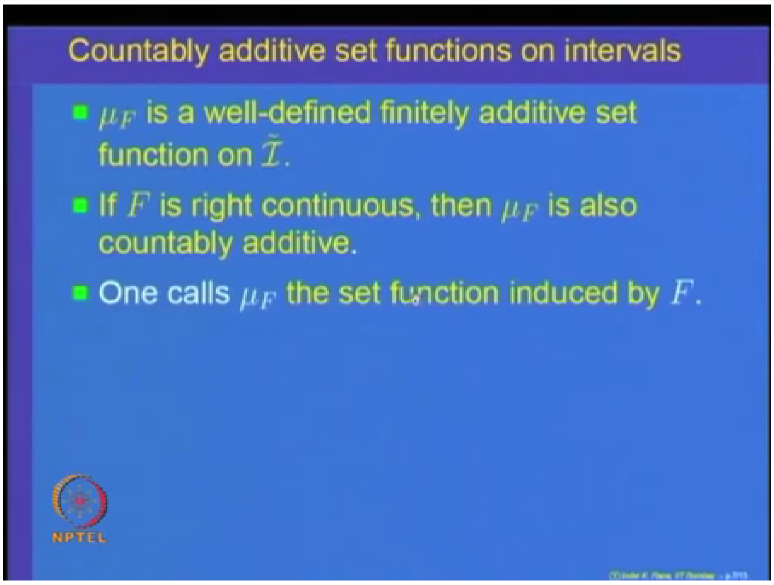

So, mu F is a finitely additive sets function and using if F is right continuous one proves that mu F is also countably additive.

So, this gives us this is this function mu F is called the set function induced by the increasing function F.

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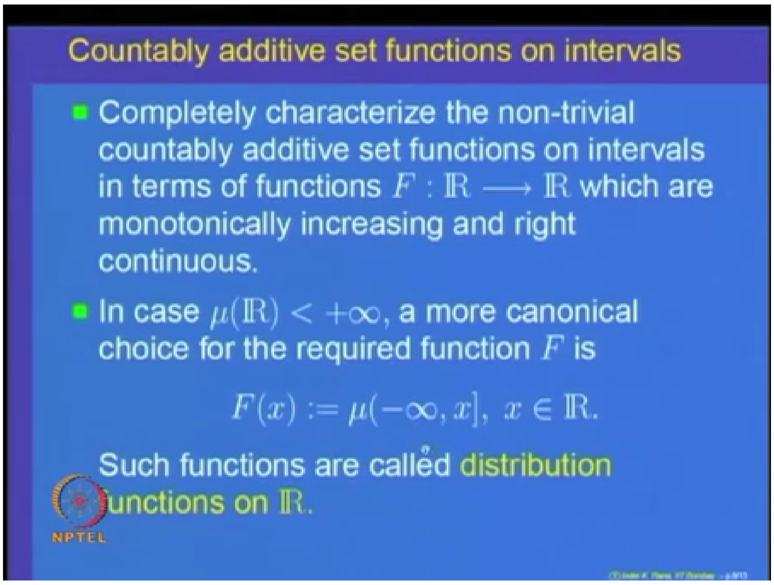

So, this gives us a complete characterizations of non trivial countably additive set functions why nontrivial because we are looking at mu of the left open right closed interval a B to be finite. In terms of functions which are monotonically increasing and right continuous. So, in some sense there is a correspondence between measures on the class of all intervals and monotonically increasing right continuous functions. In case that a countably additive set function mu has the property that mu of the whole real line is finite then one can select this monotonically increasing function to be mu of minus infinity to x because then this is defined.

We do not have to restrict the fact that mu of a B is finite that will be true anyway because this is finite. So, a more canonical choice for the monotonically increasing right function monotonically increasing right continuous function is mu of minus infinity to x when mu of the whole space R is finite. So, in that case this function F is called the distribution function on R, and this plays a role in the theory of probability where monotonically increasing right continuous functions are studied via what are called probability distributions. We will not go into that. So, we will just make a note of it in case we have finite condition that mu of R is finite we will take mu F of x to be that this function.

So, this proves the this is we have characterized all countably additive set functions on the class of all intervals. The aim what we will shall do now next is the following we will study what are called a set functions on general class of sets called algebras.

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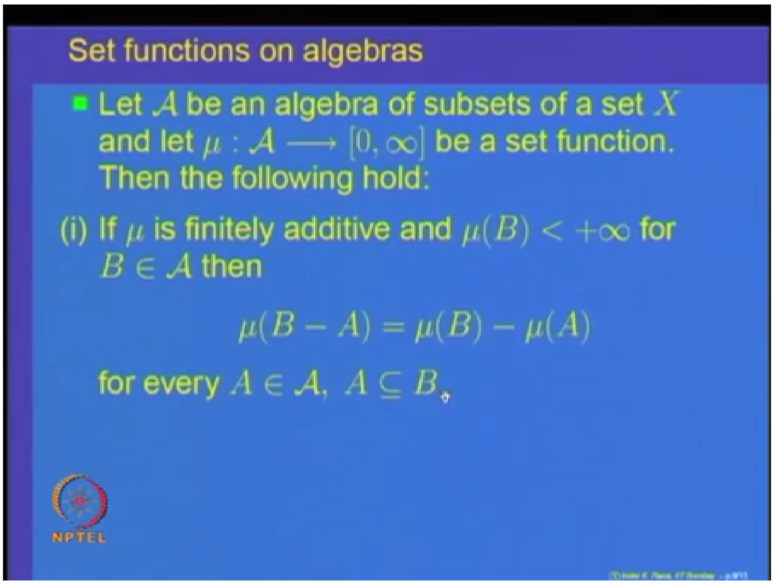

Where a so, let us start with looking at a an algebra of subsets of a set x and mu a set function defined on this algebra taking a negative real valued. So, taking values in 0 to infinity then we want to show that the following holds that if mu is finitely additive, and mu of the set B is finite for a set B in the algebra a then mu of the difference B minus a is equal to mu B minus mu of a whenever a is in the algebra and a is a subset of b. So, what we are saying is the following.

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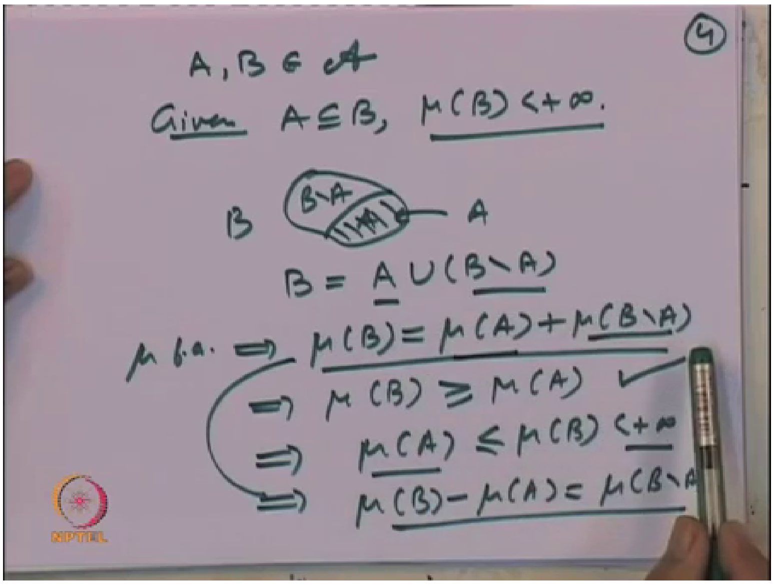

So, let us take twos let us take 2 sets A and B belonging to the algebra A and we are given of course, A is subset of B and mu of B is finite. So, here is the set B and A is a part of it.

So, this B. So, that is A. So, this part is A. So, we can write B as a union B minus a. So, this is the part B minus a right and note that A and B minus a both are disjoint sets. So, B is written as a finite union. In fact, union of the 2 sets A and B minus a and their pair wise disjoint a mu finitely additive implies that mu of B is equal to mu of a plus mu of B minus a and now let us note that this all are real numbers mu of B is a real number because a is finite mu of a is a real number because A is a subset of B. So, that is and mu of a will be less than or equal to mu or B that is finite. So, that is so these are all this is a equation in real numbers anyway anywhere that is not really important here.

But note that all are nonnegative quantities. So, that implies that mu of B is bigger than or equal to a mu of A that is one thing that we observe that because this is non negative. So, this is so this is also implies that mu of A is less than or equal to mu of B which is finite right. So, that implies mu of A is finite. So, in this equation now I can say all are real numbers. So, I can manipulate this as equation in real numbers. So, this equation implies that if i take it on the other side. So, mu of B minus mu of a is equal to mu of B minus A. So, that is what we wanted to prove, and note there we have used the fact that mu of B is finite and hence mu of every subset of it is finite whenever that set is in the algebra right. So, we can manipulate this as a equation only when they are real numbers if they are equal to plus infinity or minus plus infinity at any one of them, then I cannot transpose them on the other side and write this equation.

So, we have used the fact that mu is finitely additive and mu of B is finite that implies for every subset a of B which is in the algebra mu of a is also finite and mu of B minus a is equal to mu of B minus mu of a. And in particular suppose I take B equal to a. So, this gives mu of empty set is equal to 0. So, in particular mu of empty set is 0 if mu is finitely additive and mu for at least one set B is finite. So, these are consequences of a set function being finitely additive. So, what we are trying to show is if a set function is finitely additive what are the possible consequences we showed finite additivity implies monotone right, and if B is finite then I can interchange and right mu of B minus a to B equal to this finitely additive plus mu of at least one set infinite implies mu of phi is equal to 0.