Measure & Integration
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Lecture - 06 B
The Length Function and it is Properties

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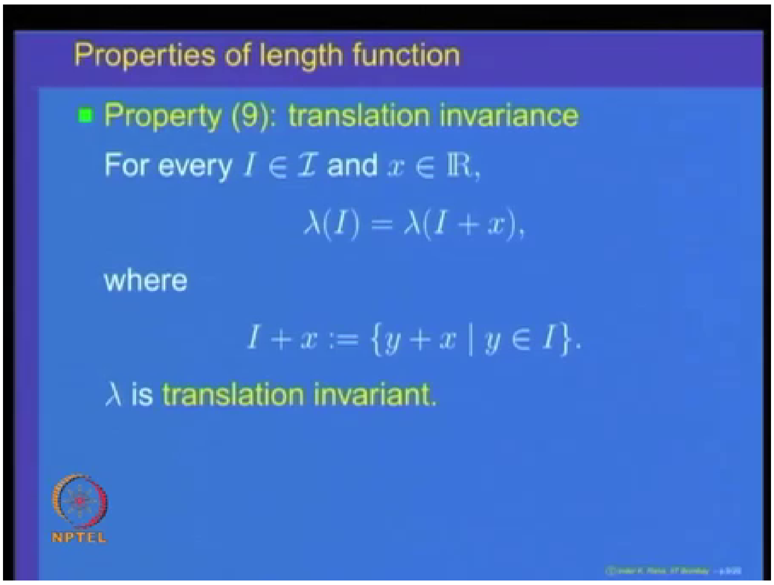

Here is a very important property of the length function which is called translation in variance. It says if I take a interval I and translate it by some number x then the length of it does not change. So, it says length of I is equal to length of I plus x. When I take a interval I and translate. So, this is a translated set just shift it push it by a distance x. So, I plus x is all y plus x y belonging to I and this property is obvious because if say for example, if I has got end points a and b.

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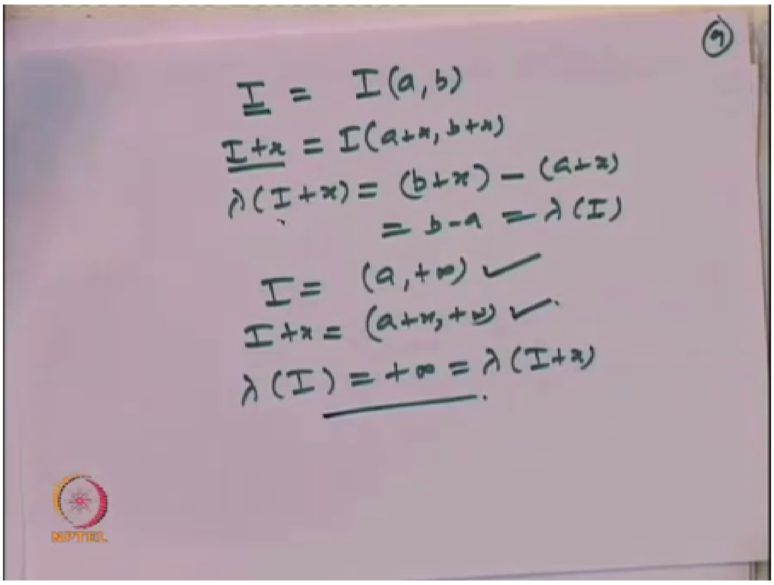

So, this property of translation in variance is quite obvious because of the fact that if I has got Let us say it is a interval with left end point a and right end point b. Then I plus x is the interval with the left end point a plus x and right end point b plus x.

So, length of I plus x is same as b plus x minus a plus x which is equal to b minus a which is equal to length of I. So, length of I is same as length of I plus x. So, that is for finite and the same proof where you continuous for infinite, because for example if I is equal to say a to infinity. Then what is a plus x? I plus x is a plus x to plus infinity and in either case length of I is equal to plus infinity which is same as length of I plus x. So, were would basically observing that if I is a infinite interval it is translation is remains an infinite interval. So, the values of both are equal to plus infinity.

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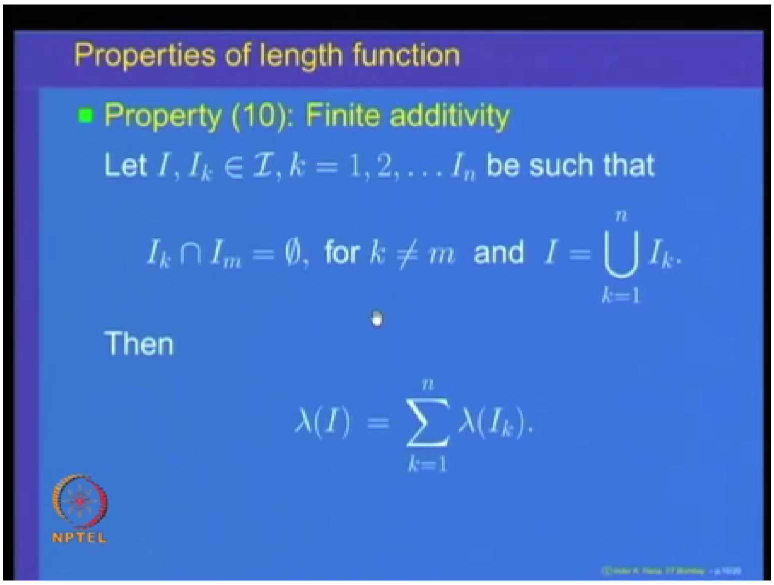

So, this is what is called the translation invariant property of a length function. And finally, let us prove what is called the finite additivity property of the length function. We have used finite additive property of the length function for finite intervals and we proved countable additivity property for the length function and I just want to exhibit that the countable additivity implies finite additivity when we have the fact that the length of the empty set is equal to 0.

So, basically what we are going to say is if I is a interval.

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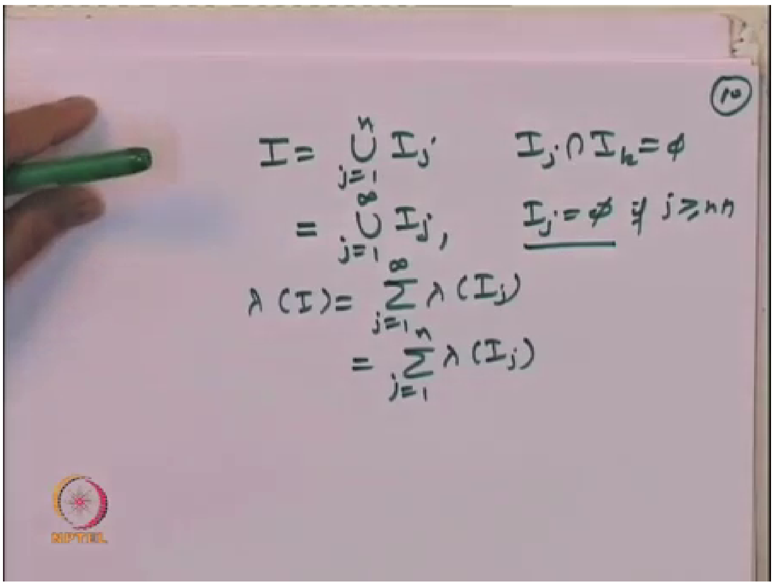

Which is union of I j, j equal to 1 to m, where I j intersection I k is equal to empty. Then I can also write it as union of j equal to 1 to infinity I j, where I can define I j to be equal to empty set if j is bigger than n plus 1 from n plus 1 onward put them everything equal to 0. So, then I is a countable disjoint union of intervals. So, length of I must be equal to summation length of I j by countable additivity property and that is same as summation j equal to 1 to n length of I j because from n plus onwards they are empty and so, the length is equal to 0. So, countable additive implies finite additivity whenever length of the empty set I can put it equal to 0.

So, let us just recapitulate the various properties of the length function that we have proved namely the length function is a function defined a set function defined on the class of all intervals in the real line. With the properties that it is countably additive, countably sub additive, finitely additive, finitely sub additive, and translation invariant the important properties that it is countably additive. So, in view of this the next question that arises is the following.

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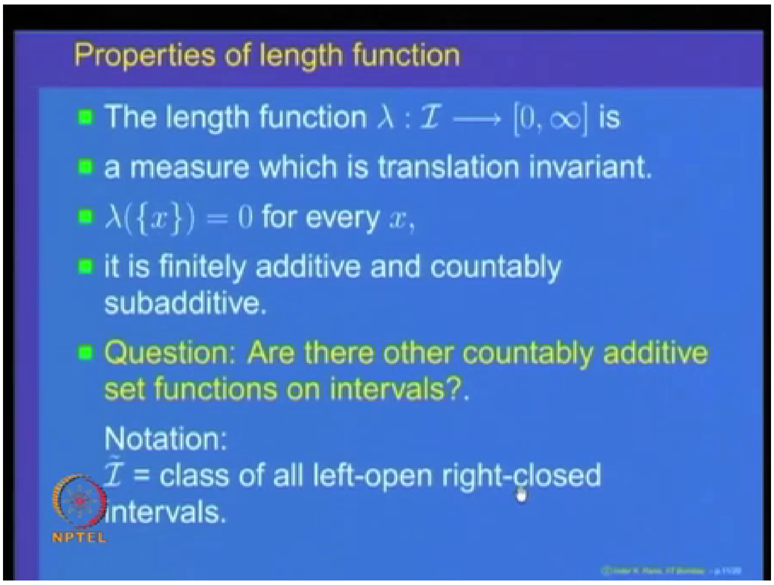

So, is a measure length function is a measure which is translation in variant because it is countably additive and length of the empty set is equal to 0. In view of this also observe that the length of the singleton is equal to 0, this is also an a property of the length function because the singleton set can be written as an open interval with the or a closed interval with the same end points. And so, length will be equal to x minus x which is equal to 0.

It is finitely additive and countably is sub additive. So, that we observed. So, here is the question that are there other countably additive set functions on the class of intervals. So, we would like to know length function which was yes now proved is one such function which is countably additive set function on the class of all intervals. So, are there other countable additive set functions on intervals. So, to answer this question let us make a notation. So, we will denote by I upper tilde. So, there is a wave kind of a sign. So, this symbol is called calligraphy I. So, the collection of all left open right closed intervals will be denoted by this symbol cal I this is calligraphy I right with the upper tilde. So, this is the collection of all left open right closed intervals. So, intervals was left end point is not included, but right end point is included. And keep in mind if it is infinite there in there is no right end point on the real line ok.

So, this is the collection of all left open right closed intervals.

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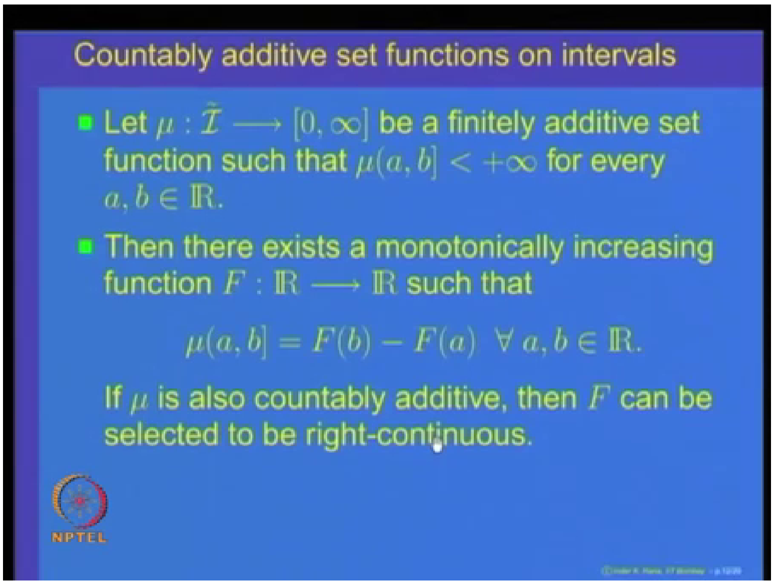

So, what we are going to prove is the following. Supposed we have got a set function mu on the class of all left open right closed intervals, such that let us say it is finitely additive. This mu is given to me finitely additive and also given the fact that mu for a finite interval is finite for every a and b.

Then we want to prove that this can be characterized by the existence of a monotonically increasing function F from R to R, such that mu of the interval left open right closed interval a b is given by F of b minus F of a for every a belonging to R. So, what we want to show is that if mu is given to be a finitely additive set function on the class of all left open right closed intervals and mu with the property that it is mu of a finite interval is finite, then we want to show that this must be given by a monotonically increasing function F. With the relation that mu of a b is nothing but F of b minus F of a. And keep in mind this looks like if lambda is the length function if mu is the length function, then the obvious choice for F is as a identity function y equal x.

So, then to be equal to b minus a. So, it is in some sense we are generalizing the length function; that means, if mu is any finitely additive set function that it must be given by this. So, to prove this, let us observe that let us observe that mu of a b is given by F of b minus F of a. So, that that itself tells us what should be the definition of the function F. For example, if I fix here a point a, if a is fix; that means, F of a is fixed then I can calculate F of b as equal to mu of a b minus aha plus F of a.

So, this relation itself gives me a hint how should I define mu of the function F. So, let us fix an a and the most convenient point is to fix a to be the origin. So, and we will also show later on that if mu is countable additive then this function can be chosen to be also not only monotonically increasing, but the right continuous function.

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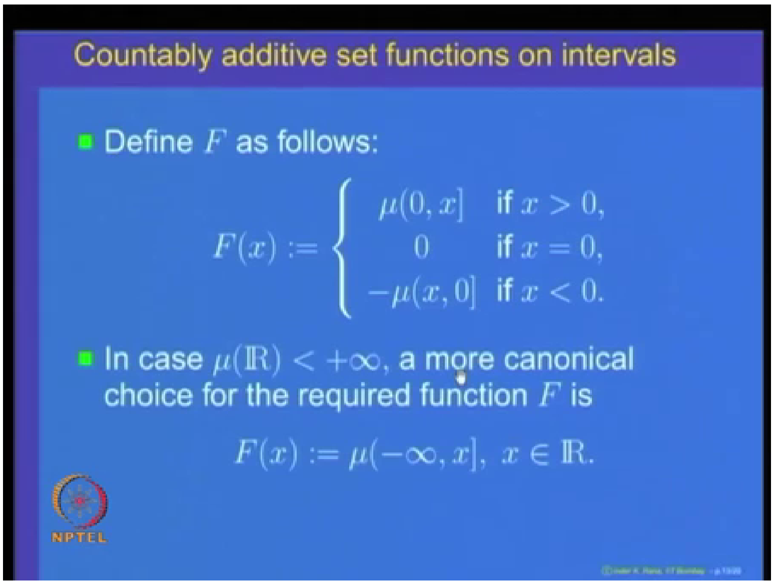

So, let us define our function F from the real line. So, F at any point x in the real line is defined as mu of the interval open interval 0 closed at x. So, left open right closed interval 0 x size or mu of that if x is bigger than 0, and it is defined as 0 if x is equal to 0. Because that will remain mu of the empty set is equal 0 it is countably additive. And mu of F of x to be equal to minus of mu of x of 0 if x is less than 0, because F x is less than 0 then this point x is going to be on the left side of 0.

So, left open right closed interval this. So, with this definition of mu we want to claim that this function has the required properties, namely first property that the first property namely. So, let us check these properties of this function. So, first this satisfies the required equation. So, we want to check that for a interval a b mu of a to b is equal to F b minus F of a.

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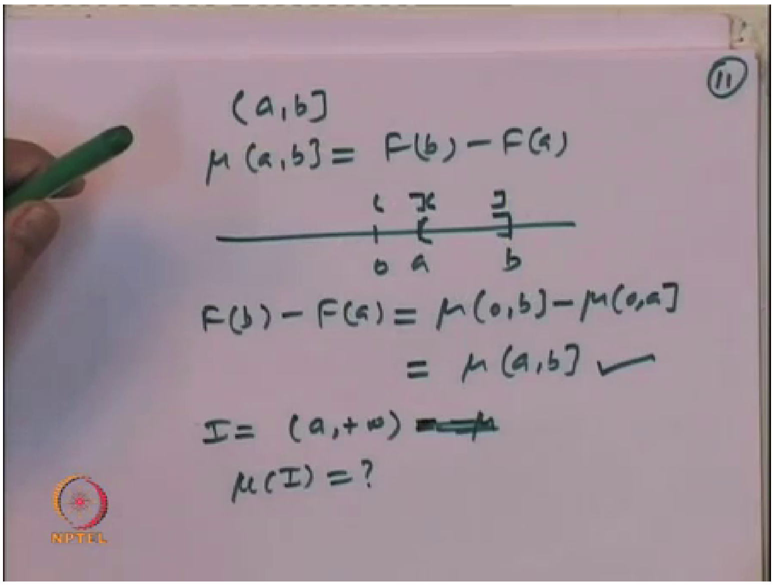

So, to check that let us take this is the point 0. So, if a and b are both finite numbers real numbers. So, let us say this is the interval a to b, then mu of then F of b minus F of a is equal to mu of 0 b minus mu of 0 to a right. And this let us observed that mu is given to be finitely additive. So, this is same as mu of a to b, because I can write 0 to b as union of 0 to a and union of 0 to be disjoint intervals so, disjoint peaces.

So, using that fact this is just mu of a b. So, that proves it and similarly if it is infinite. So, supposing the interval I is a to plus infinity then I can write it has, and then I can write this as equal to where to this is equal to mu of is finite. Now let us observe one thing that we have not if the interval is I is infinite then what is mu of I equal to? We are not defined what is the relation between. So, between this and the function.

So, keep in mind we have defined F of x is equal to mu of 0 to F x is equal to finite and this is equal to if this is finite. So, we want to check that this satisfies the required property, namely if I is equal to a 2 plus infinity then I want to check that mu of I is mu of I is equal to F of for a. So, sorry this is only for finite intervals, I am sorry. We wanted to check that only for finite intervals these properties true. So, whenever a interval I is finite interval then we know this is finite and this property is true. So now, let us check the next property namely that F is monotonically increasing.

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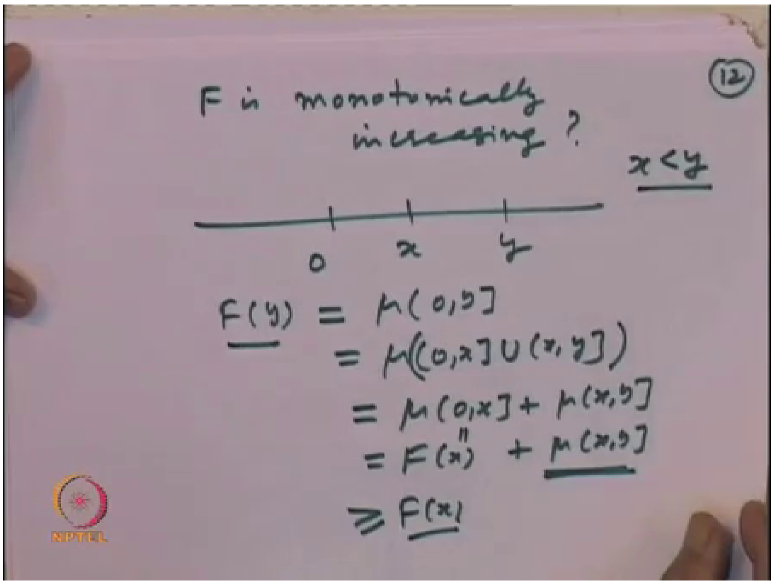

So, let us take the property that this. So, let us take 2 points. So, let us take the case here is 0 here is x say here is y. So, we have got x less than y we want to check F of y. So, we want to calculate F of y. So, what is F of y? By definition is mu of 0 to y. And that I can write as mu of 0 to x using finite additive property I can write 0 to x union of x to y mu of that and that by finite additive property is mu of 0 to x plus mu of x to y. And now this is equal to F of x by definition. So, this is equal to F of x plus mu of x to y and this is some non negative quantity. So, I can write this is bigger than or equal to F of x.

So, if x is less than y then F of x is less than F of y. So, that proves it is monotonically increasing.

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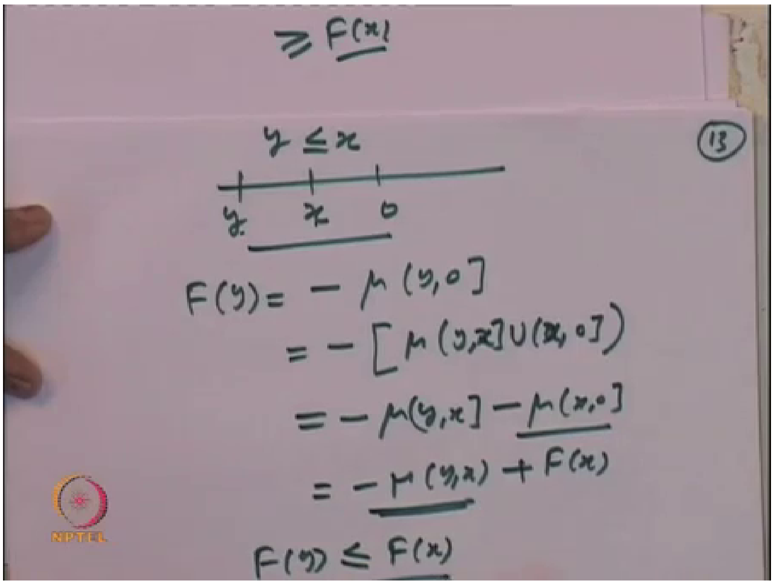

In the case when both x and y on the right side of it and same proof will work if they are both or on the left side of 0. So, here is y and here is x right. So, we have got y less than or equal to x. So, we want to look at what is F of y. So, that is equal to minus mu of y to 0 by definition. So, that is equal to minus y to 0. So, this I can write it as mu of y to 0 union 0 to y to x union x to 0. And that by again by additive property this minus mu of y to x minus mu of x to 0, and this is equal to minus this is F of x. So, we have got this is plus F of x because F of x is defined as minus of this. And this is a negative quantity; that means, F of y is less than or equal to F of x.

So, once again that property is true and the third case when the third case being Let us take it is 0 here, and y here and x here.

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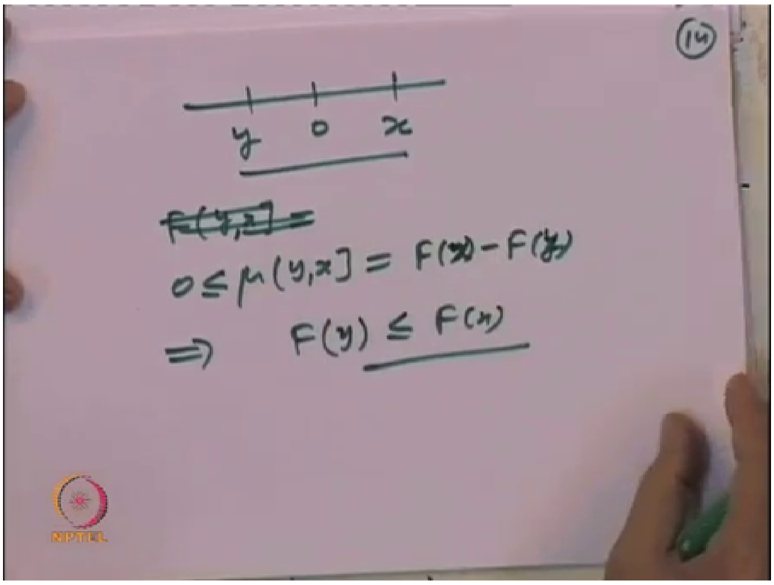

So, in that case what is F of y? Let us look at y to x that is equal sorry what is mu of y to x. From this I can write it as is equal to F of y minus F of x by definition. And this is bigger than or equal to 0. So, implies F of this is F of sorry, this is not F of this is F of x minus F of y right. So that means, F of x y is less than equal to F of x. So, once again in all possible cases we have checked that F as defined above is a monotonically increasing function.

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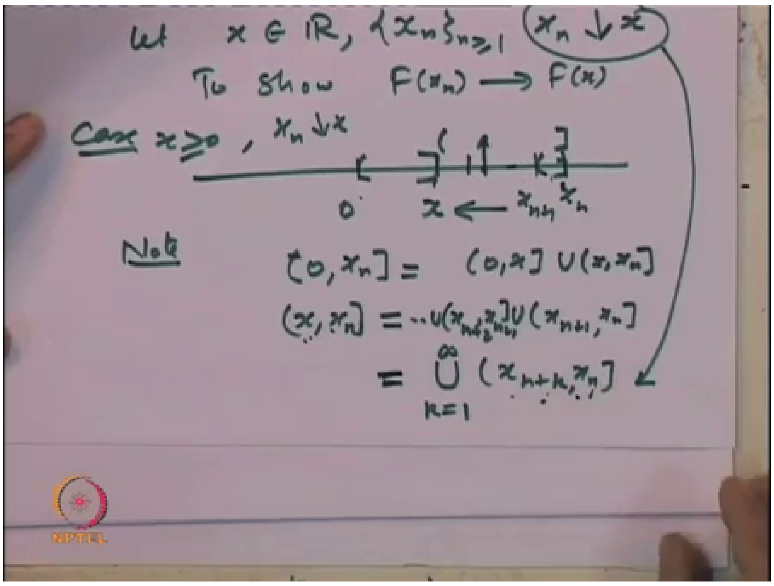

So, we want to check now that if mu is countably additive, then this implies F is right continuous, continuous from the right.

Finite additivity property gave us that F is monotonically increasing. And we are claiming that if mu is countably additive then F must be right continuous. So, what is right continuity? So, let us take a point let x belong to R and let us say take a sequence x n say that x n decreases to the point x right. To show that F of x n converges to F of x. So, that is what we have to show that F of x n converge is to F of x. So, let us try to look at a picture. So, this is 0 and let us look at the case when x is bigger than 0. So, this the case when x is bigger than or even equal to 0 we can take it. So, and here is the sequence x n x n decreasing to x; that means, here is x n here is x n plus 1 and so on, and that is converging to decreasing to x right.

So, let us observe in this case. So, note look at the interval which is 0 to x n. So, this interval 0 to x n I can split it as interval 0 to x left open right close 0 to x, and then x to union of x to x n union of this x to x n. Now I want to split this portion also. So, the interval x to x n plus 1 x n this interval is same as let us start. So, this part. So, that is x n plus 1 comma x n the next part. So, that will be x n plus 2 comma x n plus 1 and so on right. So, can I claim. So, I want to claim that this is union of x n plus k to x n k equal to 1 to infinity. So, and that is because if I take any point between x to x n. So, take any point here this x n converges. So, it is going to cross over this point any point inside the interval x to x n right.

So that means, that it is going to fall inside one of these intervals, and all this intervals are subsets of it. So, it is quite easy to check that x to x n this interval is a union of intervals x n plus k to x n k equal to 1 to infinity. And here we are using the fact that x n decreases to x. So, this fact is being used here. So now, realize that x to x n is countable disjoint union of this intervals. So, and mu is given to be countably additive. So, what we have is the following property.

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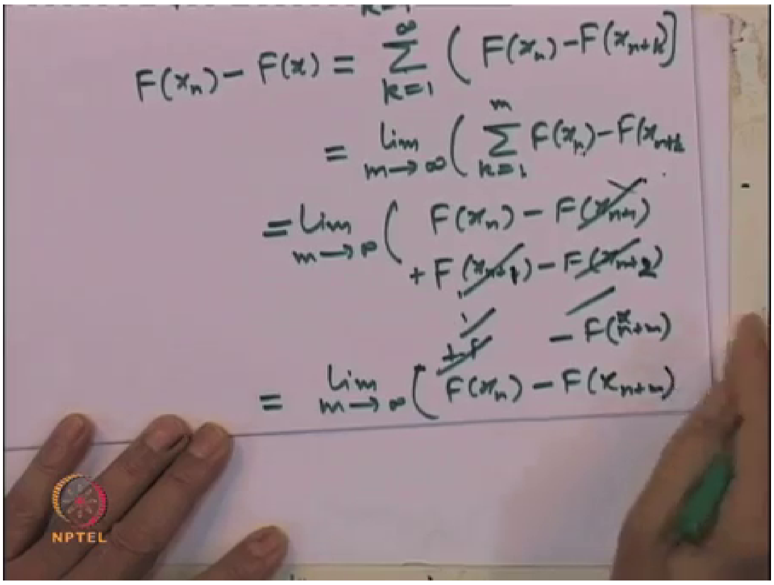

So, that says mu of x 2 x n is equal to summation k equal to 1 to infinity of mu x n plus k comma x n right. So now, let us write this in terms of F. So that means, F of x n minus F of x is equal to summation k equal to 1 to infinity of F of x n minus F of x n plus k.

Now, this is a series of non negative terms. So, I can write as limit of the partial sums. So, let us write limit of m going to infinity of summation k equal to 1 to m F of x n minus F of x n plus k. And this is equal to limit m going to infinity, now this is a partial sum right. So, what does it mean? So, this is k equal to 1 n to n plus k. So, n to n plus 1 n 2 2. So, this a sum where terms will cancel out. So, let us be just write it. So, this is nothing but. So, F of x n minus F of x n plus 1 the next term will be plus F of x n plus 2 minus F of x n plus 3 and so on. So, it will be going up to right. So, up to m. So, k equal to m. So, F of So, minus F of n plus m. So that means what? So, you note that n plus 1 so we starting with k equal to 1 n 2 n plus 1. So, n plus 2 k plus 1. So, what are the terms which are cancelling?

So, this says that n plus 1 n plus 2 n n 2. So, n plus sorry it should be n plus 1 minus n plus 2 right. So, when the next term comes n plus 1 n plus 2 and so on. So, this terms cancel right. So, what will be left with equal to limit m going to infinity of F of x n minus F of x sorry x of n plus m. So that means, what we get is the following and this is independent of m.

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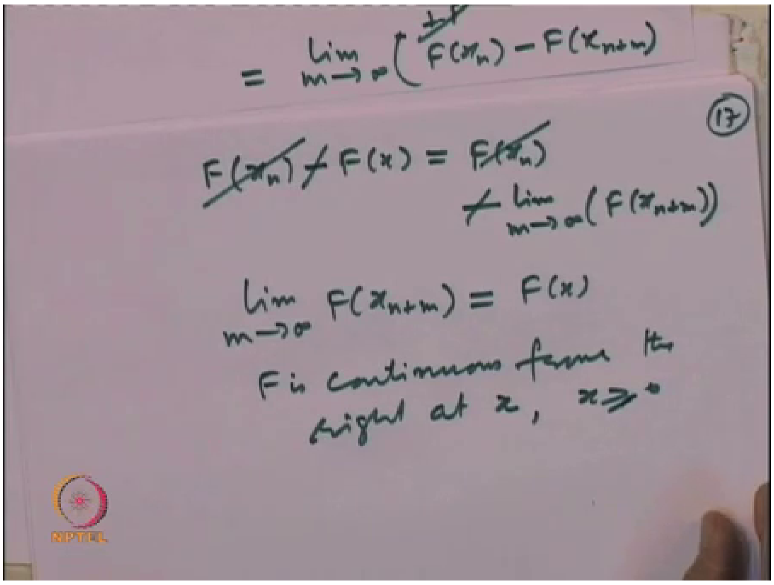

So, what we get is left hand side was F of x n minus F of x is equal to F of x n minus limit m going to infinity of F of x n plus m. And this cancel out negative sign cancels out we get limit m going to infinity F of x n plus m is equal to F of x. So, that is same as saying that F is continuous from the right, right at x and that we have proved when x is bigger than or equal to 0. So, when x is bigger than or equal to 0 this is right continuous. The other case when x is negative and still whenever x n converges to 0 or x n decreases to 0 we will show that F of x n converges to F of x, showing that F is right continuous at x when x is negative also. We will do this in the next lecture.

Thank you.