Measure & Integration
Prof. Inder K. Rana
Department of Mathematics
Indian Institute of Technology, Bombay

Lecture - 06 A
The Length Function and its Properties

Welcome to lecture 6 on measure and integration. If you recall in the previous lecture we had started looking at various properties of the length function. In today's lecture we will continue looking at the properties of the length function. And then we will try to characterize some other countably additive set functions on the class of all intervals in the real line.

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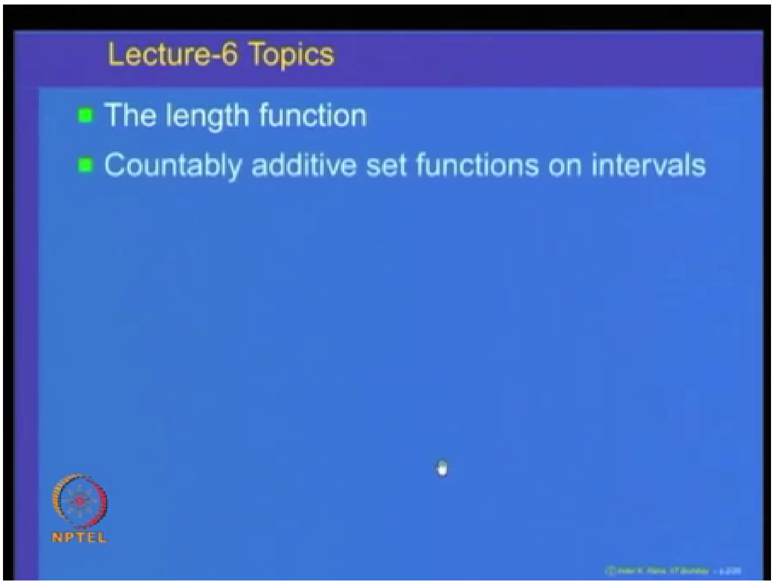

So, the first topic we will continue is the length function and it is properties, and then countably additive set functions on algebras. Let us just recall what are the properties of the length function that we have already proved.

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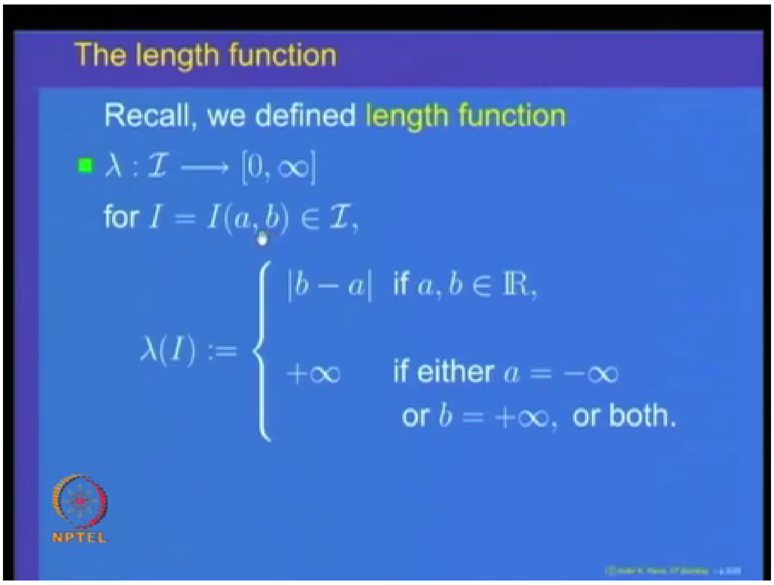

So, length function was defined on the class of all intervals that is I, and to every interval with endpoints left end point a and right end point b, or need not be left and right normally we will write the left end point a first and right end point b later.

So, for a interval with end points a and b, it is length lambda of I we defined as the absolute value of b minus a, if a and b are real numbers and in case either of it is plus infinity or minus infinity we will define the length to be equal to infinite. So, for all finite intervals the length is as usual concept of the difference between the values of the endpoints. So, that is absolute value of b minus a and the plus infinity if the interval is infinite.

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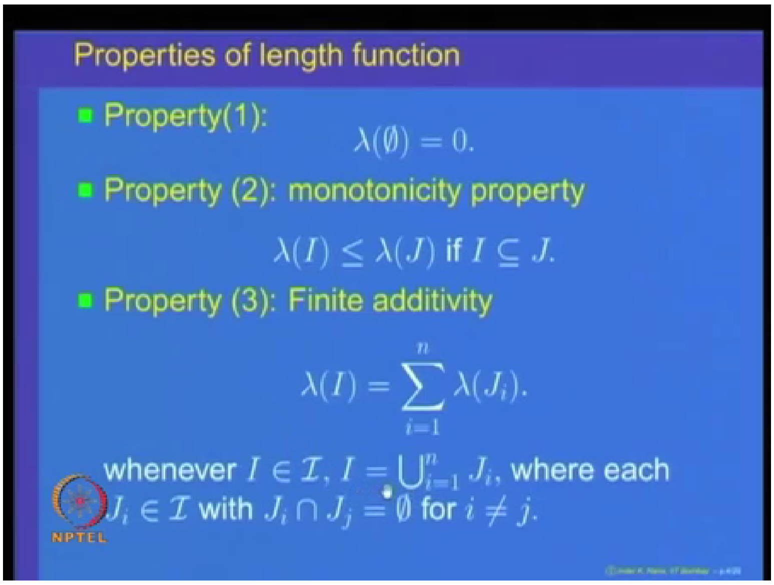

So, we proved the properties that the length of the empty set that is a interval is 0, then we proved the monotone property of the length function namely length of I is less than length of J, if I is a interval which is inside the interval J and then we proved finite additivity property namely if a interval I can be written as a finite disjoint union of the intervals J i, i equal to 1 to n then the length of the interval I is same as summation of lengths of individual intervals.

So, if I is a finite disjoint union of intervals the length of I is summation length of J is. And then we looked at a straight extension of this property namely if I is a finite or a infinite interval actually we will look at that.

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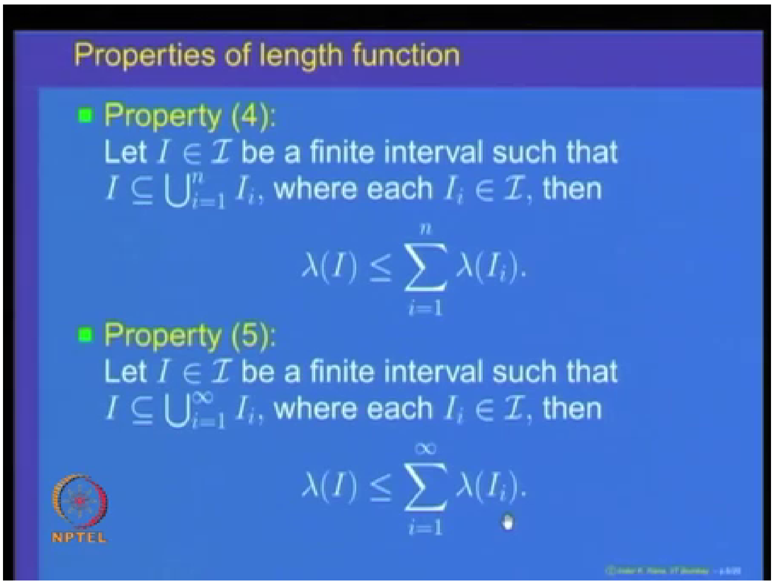

And it is contained in a union of intervals i is that is i is covered by a finite union of intervals which need not be disjoined then length of I is less than or equal to summation length of the intervals I is one to n the finite number of them. And then so, this was called for the finite I is covered by a finite union. And now then we extended this property to the arbitrary countable union. So, if I is a interval which is covered by a countable union of intervals I i which need not be disjoint. Then we proved that lambda of I is less than or equal to summation of length of the individual intervals. And if you recall this property used what is called the (Refer Time: 03:33) property on real line.

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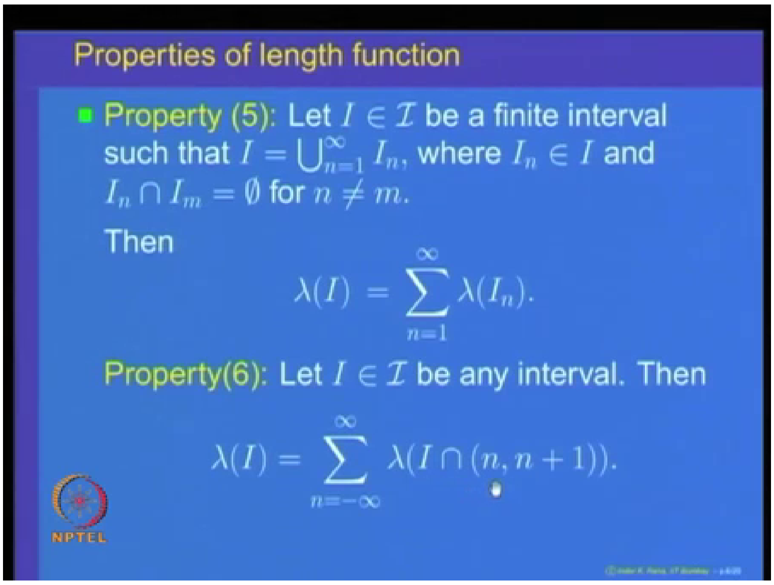

Let us continue our study. So, next thing we want to prove is the following that if I is a interval which is of is a any interval which is a the finite interval say and it is a union of intervals I n’s n equal to pair wise disjoint intervals then the length of I is equal to sigma length of I n’s. This property In fact, we had proved it. So, let us prove it to once again. So, let us look at this property.

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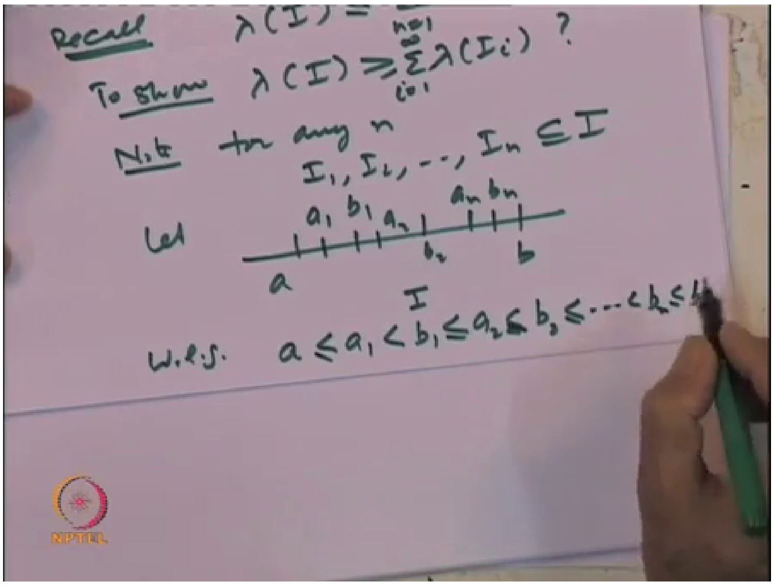

So, if I is a interval which is written as a union of intervals I n's n equal to 1 to infinity I is finite. So, keep in mind we are keeping I as a finite interval, and I n intersection I m is equal to empty then that implies length of I is equal to sigma length of I n's 1 to infinity.

So, recall we have already shown that length of I is less than or equal to sigma length of I n's. That is because of the property that just now proved I is covered by a union of interval. So, length of I must be less than or equal to prove the other way around. So, to show length of I is bigger than or equal to length of length of sigma i equal to 1 to infinity length of I is this is what is to be shown. So, let us note that for any m I 1 up to I n these are the intervals which are contained in I right, and I is finite. So, let us say this is the interval with end points a and b that is I, and a 1 I 1 is an interval which is inside the a b. So, it has end point say a 1 b 1 I to has end points a to b to and I n has end points a n b n.

But these being finite numbers and they are disjoint. So, we can arrange the intervals like a 1 here b 1 here may be a 2 here b 2 here and so on, and A m A n here and b n here. So, what we are saying is we can assume. So, without loss of generality we can say that a is less than or equal to a 1 is less than b 1 less than or equal to a 2 less than less than a 2 less than strictly less than b 2 and less or equal to and so on, and less than or equal to b n less than or equal to b.

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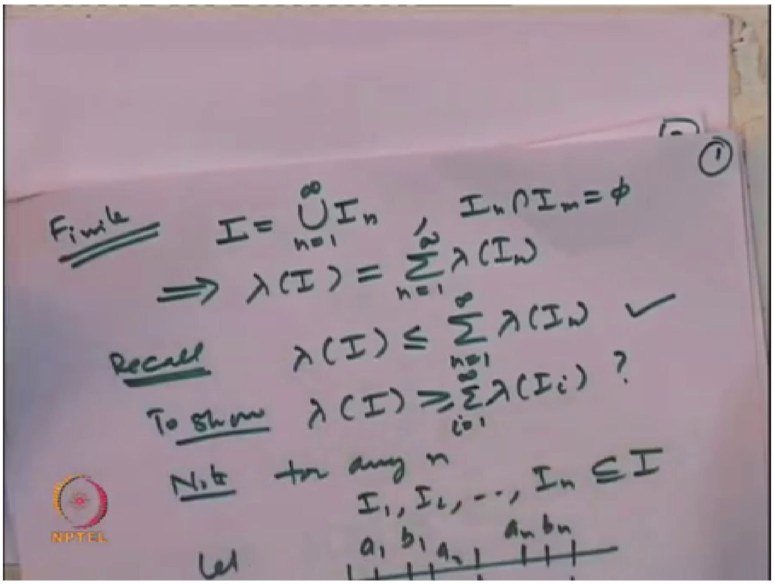

And once that property is true we can you So, immediate that b minus a is bigger than or equal to b n minus a 1, which is bigger than or equal to now you can add and subtract consecutive terms. So, b n minus a n plus b n minus 1 minus a and minus 1 and so on plus b 1 minus a 1.

So, add and subtract terms which are subtract a bigger term add a smaller term and so on. So, which is equal to sigma i equal to 1 to n length of I is and this b minus a is length of I. So, what we have gotten is this is for true for every n. So, that implies length of I sorry this is bigger than or equal to. So, this is bigger than or equal to sigma i equal to 1 to infinity length of I is i equal to 1 to infinity. So, that proves the other way round inequality also. So, hence what we have shown is that the length function has the property whenever a finite interval is written as whenever a finite interval is written as a union a countable union of disjoint intervals then the length of the interval I is equal to summation of the lengths of the individual intervals.

We would like to extend this property not only to finite interval. In fact, to any interval. So, for that we need a result namely suppose I is a any interval then we want to claim then the length of I is equal to summation lengths of I intersection the interval n to n plus 1. Likely we should. So, this is the property we would like to prove. And effect one can have here the interval which is left open n and right close n plus 1 because end point is not going to matter. So, we want to prove that the length of an interval I is same as lengths of it is piece which lie inside the intervals n to n plus 1. So, to prove this property let us observe the following.

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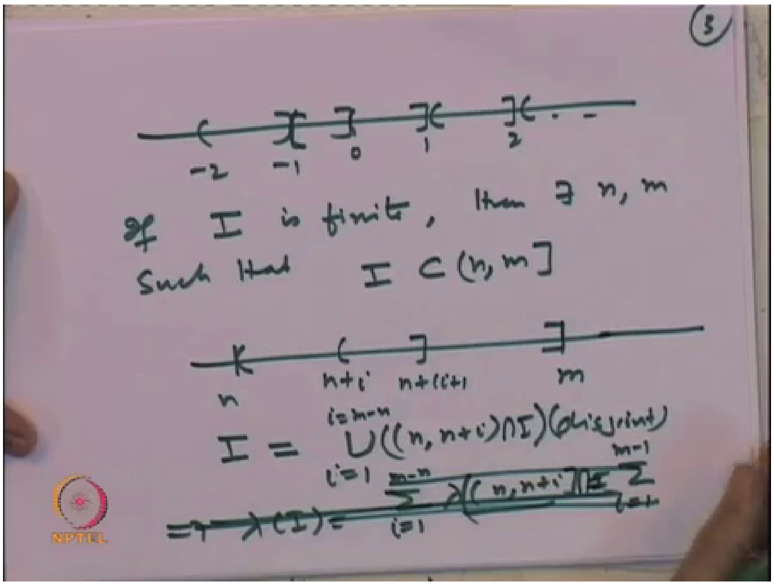

So, let us observe this is a real line. So, we can write it as the intervals. So, it here it is say 0 1 2 and so on and here is on the other side. So, here is minus 1 minus 2 and so on.

So, let us take a interval I. So, if possibly if I is finite it if it is a finite interval. Then obviously, it will lie between 2 some bounds. So, infinite then there exist some n and m such that I is inside n to m right. So, there will be some So, here is some n and here is some m. So, that I is inside this right. So now, let us look at the piece. So, let us inside let us look at piece of. So, this is n plus i, and this is n plus i plus 1. So, intersection with this side. So, what we are saying is this I can be written as union a union of n to n plus i i equal to. So, 1 to up to up to n plus i equal to m so, such that up i equal to n plus i equal to m minus n right and now let us observed that this piece these are disjoint union.

This union is a disjoint union and a finite number of them. So, this will imply length of I is equal to summation i equal to 1 to m minus n, lengths of n plus i. Now this interval I does not intersect with any other interval out with which is bigger than m and which is less than n. So, all those intervals this is in the intersection with I is empty. So, what we can write is this is same as sigma of i equal to 1 to m minus n this is the intersection of should I have written the intersection with the interval I so, the piece because the interval may start somewhere else. So, let us write this is intersection with the internal I. So, let me write this again. So, this is intersection with I. So, let us write this again.

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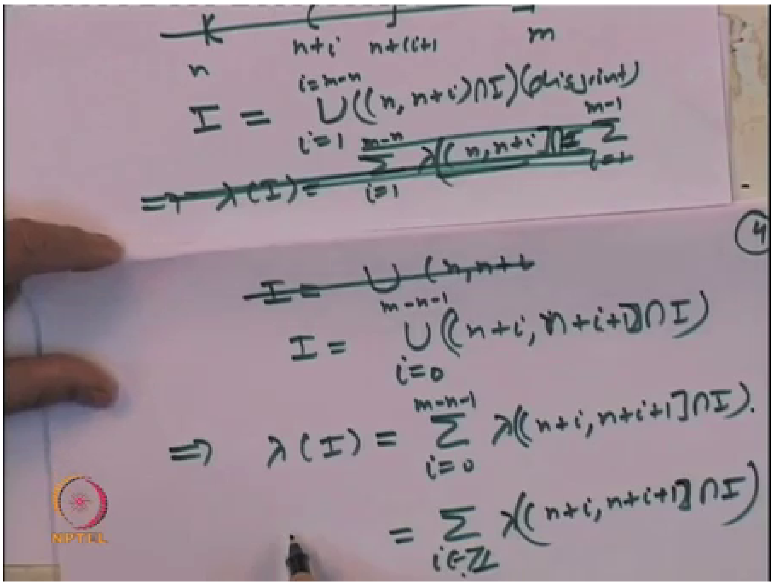

So, the I is written. So, I can be written as a union n to n plus from sorry, this is this is also not wrong. Let me just write n to I can be written as union n plus i to n plus i plus 1 intersection I i equal to starts with n. So, 0 and goes up to when n plus i plus 1 is equal to. So, that is m minus. So, we want n plus this is the equal to m. So, m minus n minus 1 so that implies length of I, because this is a finite disjoint union. So, this is equal to summation i equal to 0 to m minus n minus 1 length of n plus i to n plus i plus 1 intersection I.

And now for other parts the bigger be 0. So, I can write as sigma over I belonging to integers length of n plus i n plus i plus 1, intersection I over all I integers all integers I, because the intersection with the other intervals is going to be empty and that is going to be 0. So, this proves that whenever I is finite way are true. So, I finite case is.

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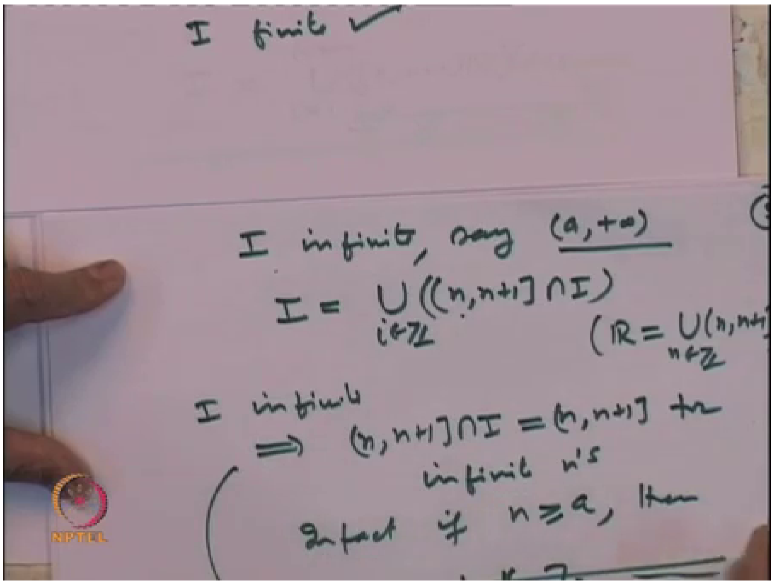

Now, let us prove the same thing when I is infinite, the let us take I infinite. Then I can write I is equal to union of same thing n to n plus 1 intersection I, i belonging to integers right. This is because keeping in mind that the real line is equal to union n to n plus 1, n belonging to integers. So, interval I is this interesting this.

And now because I is infinite, let us say it looks like say something like a to plus infinity. So, in that case this intersection of So, I infinite implies that n to n plus 1 intersection I is equal to n to n plus 1 for infinite n’s. In fact, some stage onwards so In fact, if n is say bigger than or equal to a, then that is the interval. So, here is a and here is n then n to n plus 1 and so on they are all going to be non empty intersections with intersection being equal to that n plus 1. So, this implies that sigma length of n to n plus 1 intersection I is going to be equal to plus infinity right. Overall n belonging to z and that is same as length of I because I is a infinite interval.

So, that proves the property namely. So, this proves the property namely for any interval I if for any interval I the length of the interval can be written as the length of it is piece lengths of the piece I intersection n to n plus 1. So, this is an important property. So, it says length of any interval is a summation of lengths of it is piece. This property and note that length of each one of this piece being a finite interval is a finite number. So, we have this says that any interval can be written as a countable disjoint union of intervals each having finite a length. So, this is a important property which is going to be called as sigma finiteness property of the real numbers of the length function.

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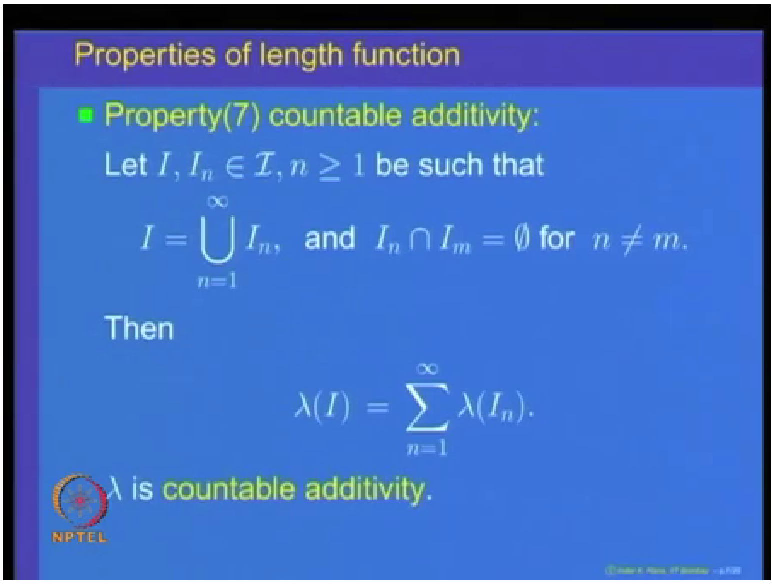

So, let us we are going to use this property to prove what is called countable additive property of the length function. And that says that the length function if a interval i is written as a countable disjoint union of a intervals I n's, then the length of the interval I is equal to summation lengths of I n’s.

So, to prove this property. So, let us start looking at the proof of this property. So, to prove this property let us write.

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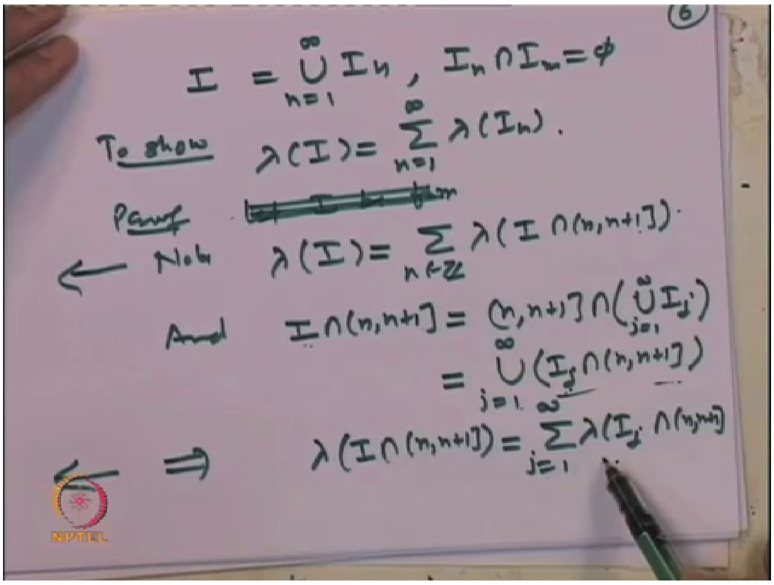

So, I is a interval I is a interval which is written as a union of I n's n equal to 1 to infinity, where the intervals I n intersection I m is equal. To empty to show that length of I is equal to summation length of I n's n equal to 1 to infinity. So, let us look at a proof of this case 1. So, assume. So, let I be finite, likely finite or infinite is not important. So, let us let us take general case itself. So, note I length of I is equal to summation length of I intersection n to n plus 1. And this is because of the property that we have just now proved. And now also note and I intersection n to n plus 1 can be written as this is a finite interval and I is equal to So, this is n to n plus 1. Intersection this interval I is a countable disjoint union. So, it is a union of I j, j equal to 1 to infinity.

So, we can write this as union J equal to 1 to infinity of I j intersection n to n plus 1. And now this is equality for finite intervals only. Because I intersection n plus n to n plus 1 is a finite interval which is written as a because I j's are disjoined. So, these intervals are disjoint and they are finite. So, thus this implies by the additive property for finite intervals which are disjoint that lambda of I intersection n to n plus 1 is equal to summation J equal to 1 to infinity lambda of I j intersection n to n plus 1 right. So, the here we are using the fact that whenever a interval I, is a finite interval which is a countable disjoint union of intervals 1 to infinity then the length of I is equal to summation of length of this.

Now, look at this equation here and look at this equation here. So, length of I is equal to summation n over integers, length of I intersection n to n plus 1 and that is computed to be equal to this.

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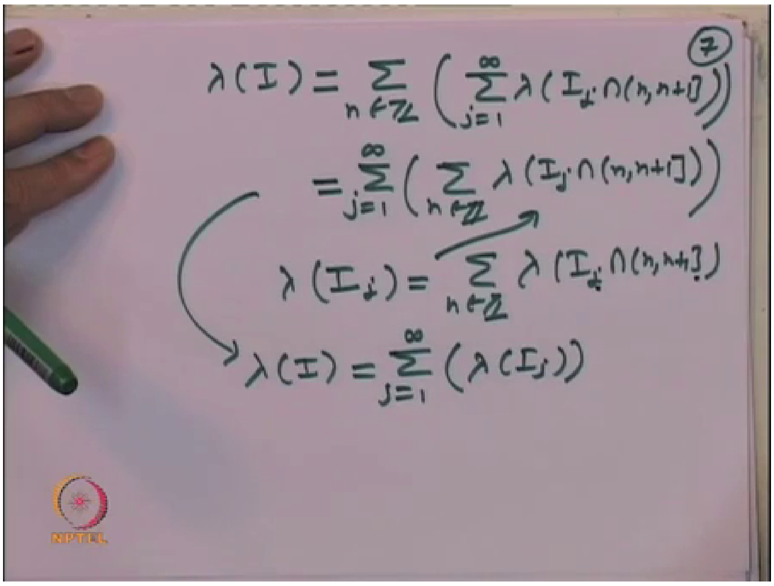

So, combining these 2 we get the property that length of I is equal to which is summation n belonging to z of length of I intersection n to n plus 1 and that property we are put going to put it here. So, summation J equal to 1 to infinity lambda of I j intersection n 2 n plus 1. And now keep in mind that all this is a double summation of series, and all of them are nonnegative. So, I can interchange the order of integration. So, I can write this as summation over J equal to 1 to infinity, summation over n belonging to integers of length I j intersection n to n plus 1 right.

And now once again I use the fact that the interval I j length of I j can be written as length of I j intersected with n to n plus 1, summation over n belonging to z. Just now we have proved that fact the sigma finiteness of the length function any interval length I is summation length of it is pieces inside n to n plus 1. So, this is here. So, that gives me the fact that. So, this gives me length of I is equal to summation J equal to 1 to infinity and this is length of I j. So, that proves the countable additive property of the length function namely if a interval I is written as a countable disjoint union of intervals. So, this proves the countability property that if a interval I is written as a countable disjoint union of intervals I n's, then length of I is equal to summation length of I n's whether I interval I is finite or infinite that does not matter.

So, the length function is countably additive that is what we have proved that is the important property of the length function. Let us extent this property to coverings which are not necessarily disjoint.

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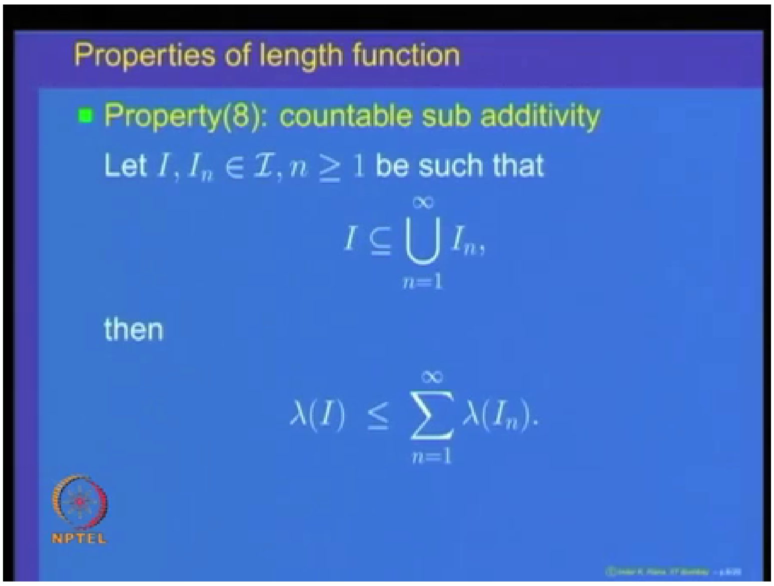

So, that is called countable sub additivity. So, that is says that if a interval I is such that I is contained in union of intervals I n's n equal to 1 to infinity which may not be disjoint then; obviously, we should expect that the length of I is less than or equal to summation length of I n's. And it is proof is very much similar to the earlier case. So, let us go through the proof again. So, that we understand how sigma finiteness of the length function is used.

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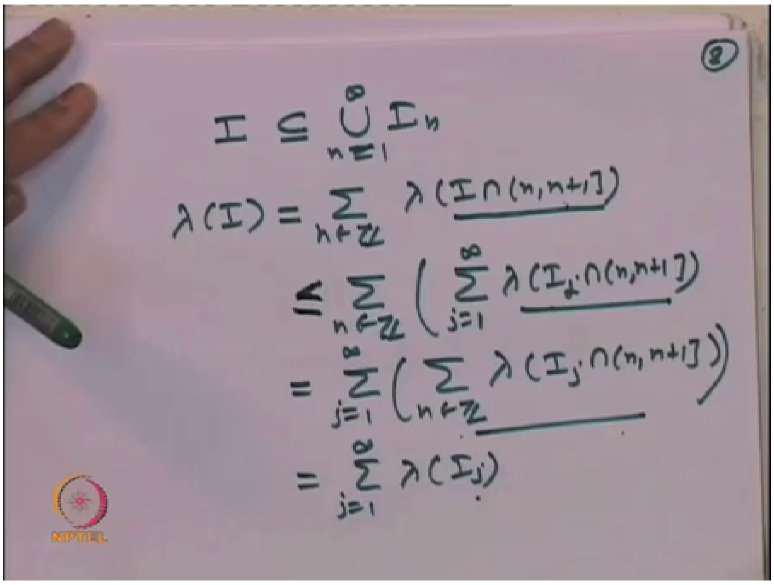

So, I is a interval which is contained in union of I n's n equal to 1 to infinity, this intervals I n's may not be disjoint.

Now, what we do is look at length of I, I can write this is equal to sigma n belonging to z length of I intersection n to n plus 1 that is sigma finiteness of the length function. And now the interval I intersection n 2 n plus 1 is inside because this I, I can write as union over I n’s. So, this is equal to. So, let me write this is less than or equal to sigma length over n. So, this thing is less than equal to sigma length of I j intersection n plus 1, J equal to 1 to infinity. So, here what we have use the fact that the interval I to this intersection this interval is covered by the union of these intervals. So, and this is the finite interval. So, length of this must be less than or equal to length of this. And now once again non negative series of non negative numbers I can interchange.

So, this is equal to sigma J equal to 1 to infinity sigma n integers length of I j intersection n to n plus 1 and that once again this once again is nothing but the length of the interval I j. By the fact that just now sigma finiteness of the length function, so length of I is less than or equal to length of sigma length of I j’s. So, this a property is called countable sub additive property of the length function.

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