Measure & Integration
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Lecture - 05 B
Set Functions

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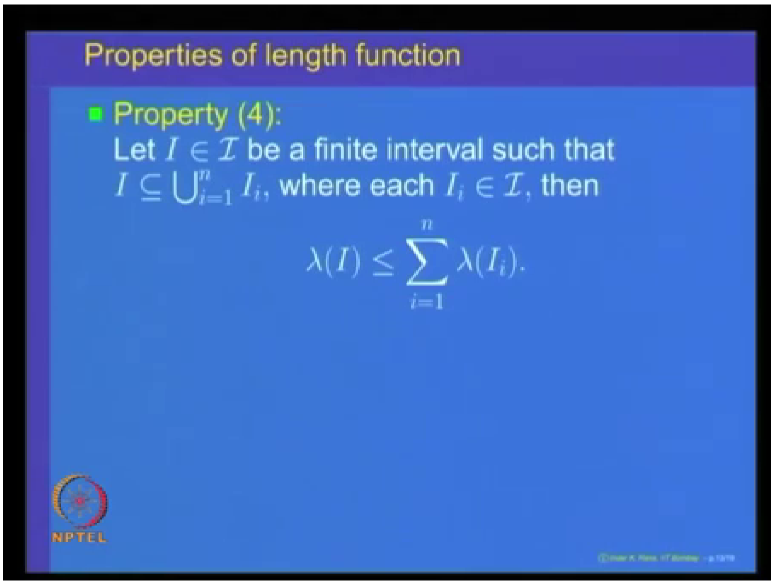

Next let us look at another property. Supposing I is a finite interval such that I is contained in union 1 to n, I i's where a finite union of the intervals, but we are not longer saying that they are disjoint. Then the claim is that length of I must be less than or equal to summation length of this intervals ai s. So, if you drop the condition that these are pair wise disjoint I is a subset of. So, we are saying if a internal I is covered by a finite union of intervals then the length of I must be equal less than or equal to summation of length of this intervals Ii's. Let us look at a proof of this the proof of this is once again similar to the earlier property.

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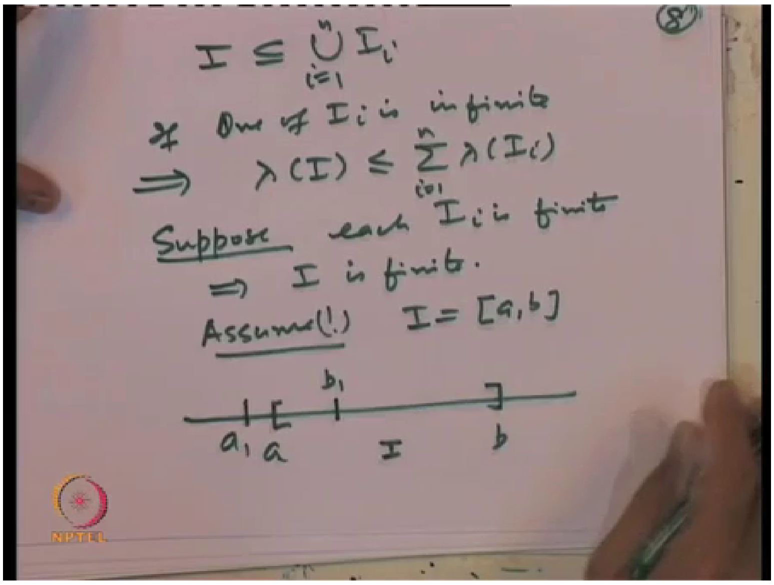

So, let us say I we are saying I is contained in union of Ii's, I equal to 1 to n; obviously, if one of Ii is infinite, then clearly this implies length of I is less than or equal to summation length of Ii's that is obvious because one of this one of these terms on the right hand side in the summation is plus infinity, which is always greater than or equal to length of I whatever be i.

So, let us suppose. So, let us suppose that each Ii is finite and this is a finite union. So, that implies I is finite. So, without as before without loss of generality assume without any loss of generality that I is equal to a comma b. So, here is once again the same picture here is a and here is b. Now the point a belongs to the interval i. So, this is my interval i. So, a belongs to i; that means, it belongs to this union. So, it will belong to at least one of the intervals Ii 's let us name any one of them, which contains the point a to be I 1 and let us say the end points of that is a 1 b one. So, the point a belongs to one of the intervals Ii's because it is in the union.

So, it will belong to one of them say I 1 and let us say the end points of I 1 are a 1 b one. So, here is the end point a 1 here is the end point b 1 now the possibility is this b 1 is on the right side of b right. So, one possibility is it is the right side of b.

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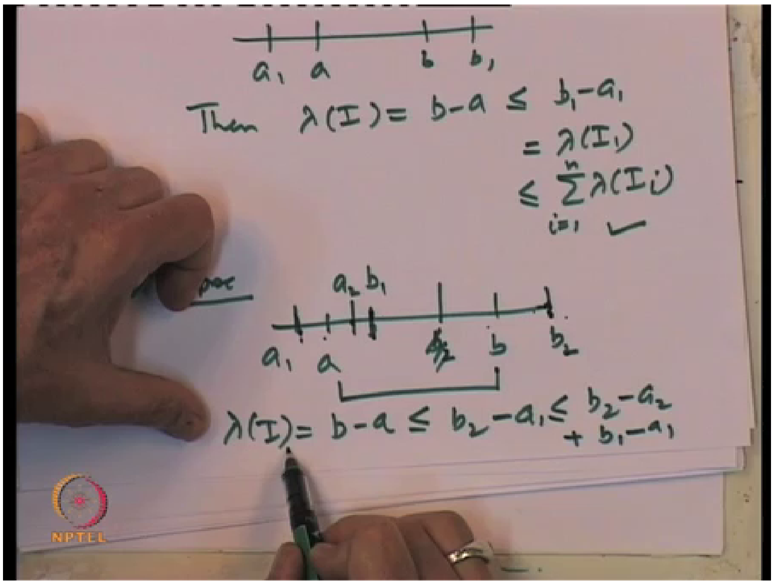

So, either. So, let us as write either b 1 is bigger than or equal to b; that means, my picture looks like here is a 1 here is a here is b and here is b 1 then length of I which is equal to b minus a is less than or equal to b 1 minus a 1 which is equal to length of 1 one and which is; obviously, less than or equal to summation length of Ii's I equal to 1 to n.

So, in the case b 1 is on the right side where; obviously, through by this case. So, what is the another possibility case 2. So, suppose this is the picture. So, suppose this is the picture namely we have got a we have got b and here is a 1 and b 1 is on the not on the right side, but on the left side of a b. So, let us take that as the picture. So, in that case the point b 1 belongs to that union. So, b 1 is in the interval a b. So, it will belong to that union. So, it belongs. So, b 1 belongs to y. So, it belongs to union. So, it will belong to one of the intervals in the I I’s. So, let us call that has some interval I 2.

So, b 1 belongs to I 2 so; that means, a 2 must start here and b 2 either it will be somewhere here or it will be on the right side. So, and if it is on the right side of it; that means, what. So, let us say I is on the right side. So, here is b 2 instead of here let us say b 2 is here then the length of the interval i. So, length of the interval, I which is equal to b minus a this is b minus a this is less than or equal to b 2 minus a 1 b 2 minus a 1 which is less than or equal to b 2 minus a 2 plus b 1 minus a 1. So, b 2 minus a 1 is less than or equal to b 2 minus a 2 plus b 1 we are adding something bigger and then a 1.

So, that is length of i.

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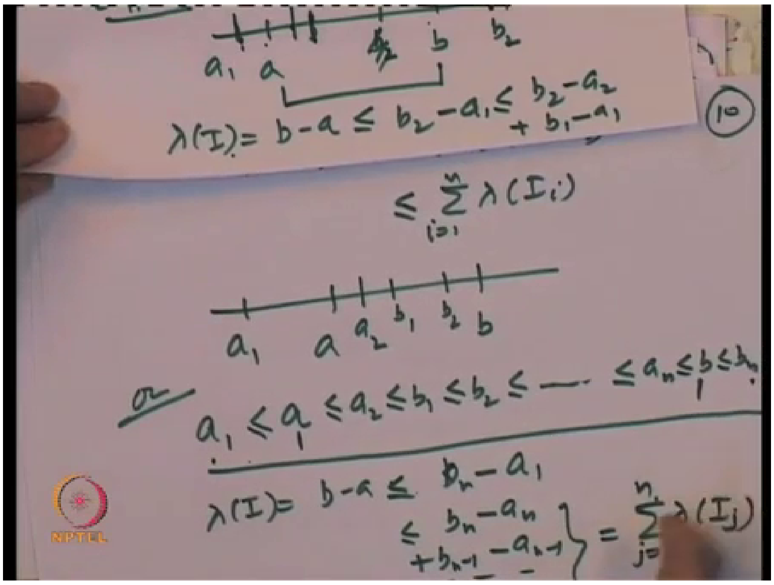

So, in that case length of I will be less than or equal to length of I 1 plus length of I 2 and so that is any way less than or equal summation length of Ii's, I equal to 1 to n. So, if you go on repeating this process. So, what does it mean? So, in the next stage, what is another possibility that b 1 is inside so; that means, here is here is a here is b, if it is not outside that must be inside; that means, there is a 1 here is here was our b 1 here is a 2 and somewhere here is b 2 it is not on the right side you will the left side. So, once again b 2 belongs and then we can proceed in the same way. So, either at some stage will be through or eventually f naught then we will have a 1 is less than equal to a is less than or equal to a 2 less the or equal to b 1 less than or equal to be 2 less than or equal to. So, on less than or equal to a n less than or equal to b less than or equal to b n.

So, what we are saying is either will be through at some finite stage or we can rearrange eventually after end stage is the end points in that way in that case again lambda of I which is equal to b minus a. So, here is a here is b is less than or equal to same idea b n minus less than or equal to b m minus a 1. So, go on adding and subtracting less than equal to b n minus a n plus b n minus 1minus a n minus 1and soon and plus b 1 minus a 1 and that is equal to sigma lambda of I j j equal to 1 to n. So, it is just whenever we were infinite stage the end points can be rearrange nicely and we get the property namely the length function is having the property whenever a interval I is covered by a finite union of intervals then the length of I is less than or equal to summation length of I I’s.

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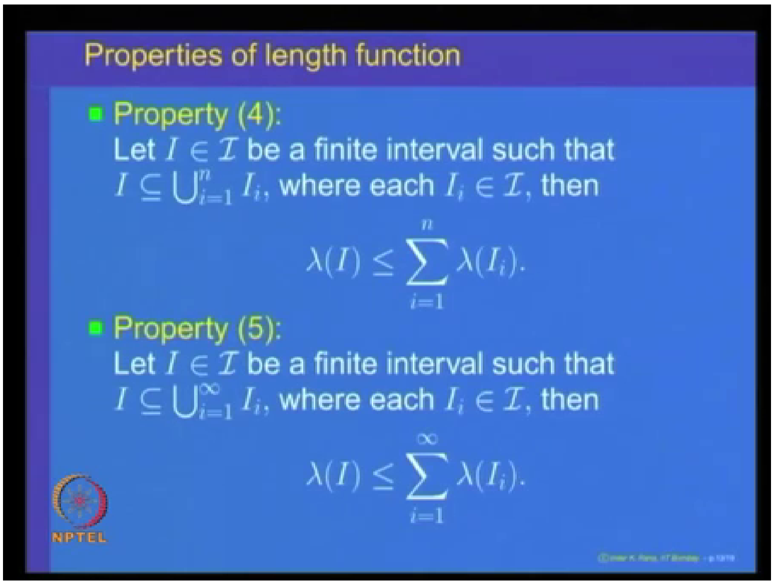

Let us look at the an extension of this property namely exposing I is a finite interval such that I is covered by a union of intervals Ii's 1 to infinity; that means, the interval I is covered by union of a countable union of intervals Ii's then again the claim is length of I is less than or equal to summation length of I I’s. So, let us prove this property and keep in mind here we are assuming our interval I is a finite interval.

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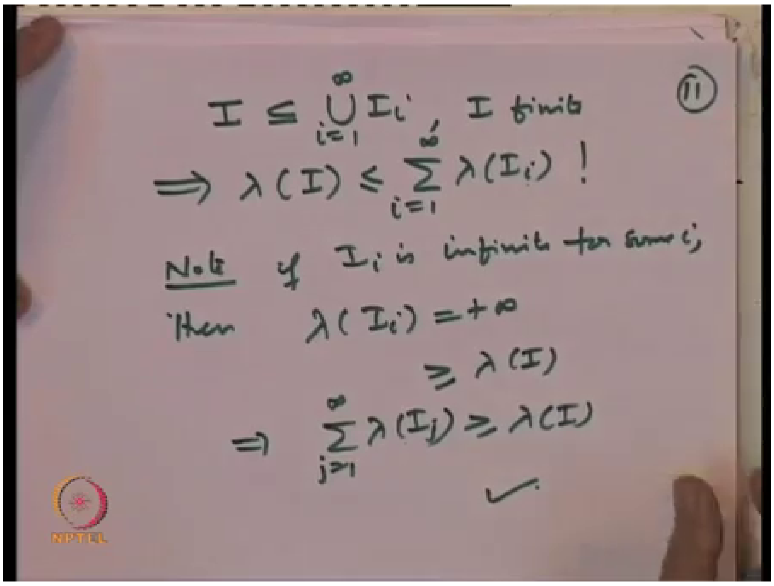

So, interval I is contained in union of intervals Ii's I equal to 1 to infinity these are intervals I finite this implies length of I is less than or equal to summation length of Ii's, I equal to 1 to infinity this is what we want to prove.

Obvious guess note, if any one of the terms on this side lambda of Ii's is infinite then, we are through. So, that is note if Ii is infinite for some I then what will happen length of Ii will be equal to plus infinity which is bigger than or equal to length of i, whatever it may be weather finite or infinite. So, that is. So, implies sigma length of I j j equal to 1 to infinity is also bigger than or equal to lambda of because one of them is infinite. So, that case is obvious. So, let us assume not only I is finite all the intervals Ii's are also finite and we want to check this property. So, what we want to check is the following.

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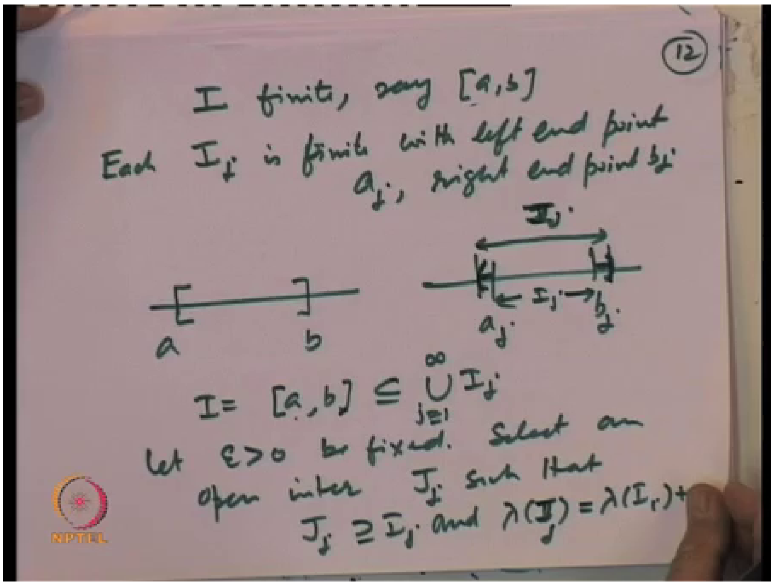

I finite say end points say a b we can assume it is a closed interval because the length of I is not going to change each I j is finite, with left end point a j right end point b j.

We are not saying that we are assuming this I j's are open or close or anything we are just naming the end points the left endpoint of I j. So, we are saying I looks like an and b and each I j is a j, b j we are not saying that this end points are included and we are given that I which is a comma b is contained in union of I j I equal to 1 to infinity right. And if this was finite there will be already know how to manipulate that that we have already done earlier in the previous case. So, the idea is from that infinite union brings it to a finite union and here is closed bounded interval contained in a infinite union and you want to say this is going to be contained in a finite union.

So, somewhere the compactness property of the interval a to b is going to be used so, but for that we need the intervals to be open. So, let us make this intervals ideas open, but of course, the lengths will change. So, let us let us fix let epsilon greater than 0 b fix select an open interval say call it is j j, such that this j j includes the our interval I j and does not change the length much. So, length of I length of this j j is equal to say length of Ii plus epsilon. So, slightly increase. So, what were saying in this picture take a interval from here to here the open interval from here to here call that as j j of j. So, each I j I j which was some a j to here b j is enclosed in an open interval slightly bigger, but the length. So, this is the length portion that you had that at the most is equal to epsilon.

So, now what happens is the following.

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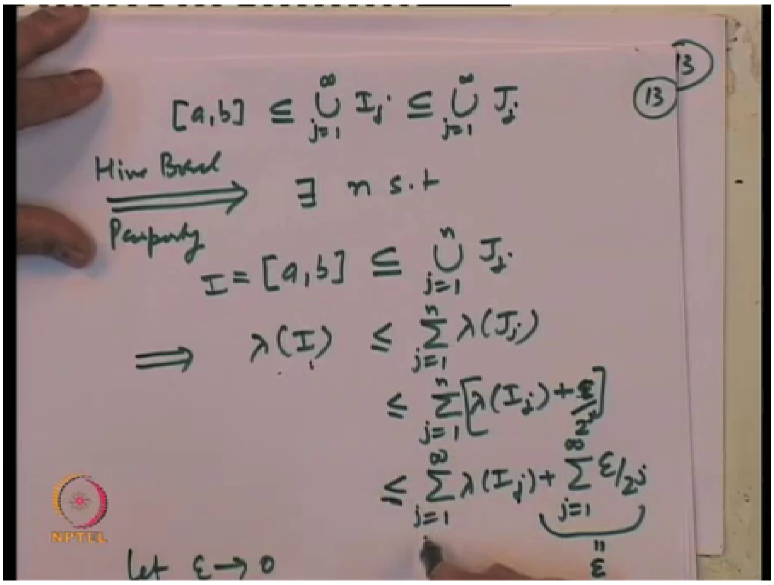

So, a b is contained in union of I j's. And each I j is contained in the union of j j's and each j j is a open interval. So, we are taken a open interval. So, we have got a open cover of the closed boundary interval a b. So, implies by Hine borel property of the real line which says whenever I closed boundary interval is covered by an collection of open intervals then the implies there exist some n such that finite number of them we will cover it. So, a b will be contained in union of j equal to 1 to n j j's finite number of them will cover it and now implies by our earlier case that length of i. So, this was my interval I length of I is less than or equal to sigma length of j j's 1 to n ok.

Now, each one of them is less than or equal to sigma j equal to 1 to n length of I j plus epsilon. And now we want to separate out this summation and let it go to infinity and could want to infinity, but the problem will come because of the summation epsilon added n times that summation will tend to become very large, we do not want to happen that. So, what we go is we revise our construction. So, where we selected for a given epsilon fix select and open interval j j such that this is. So, instead of epsilon for the interval I j let us divided by 2 to the power j. So, instead of having this extra length to be equal to same length as epsilon for everyone interval I j for I j we want this extra length to be equal epsilon by 2 to the power j.

So, once we do that we are in a better shape because now this as estimate will be 2 to the power j so; that means, it is less than or equal to summation j equal to I can put it 1 to infinity because it is less than or equal to lambda of I j plus summation epsilon by 2 to the power j j equal to 1 to infinity, and now this series is convergent because it is a geometric series with common ratio one by 2 which is less than one. So, this term is equal to epsilon. So, what we are saying is length of I is less than or equal to summation length of I j's plus a number epsilon, but epsilon was a arbitrary. So, let epsilon go to0. So, we will get length of I is less than or equal to summation length of I j's. So, what we are saying is

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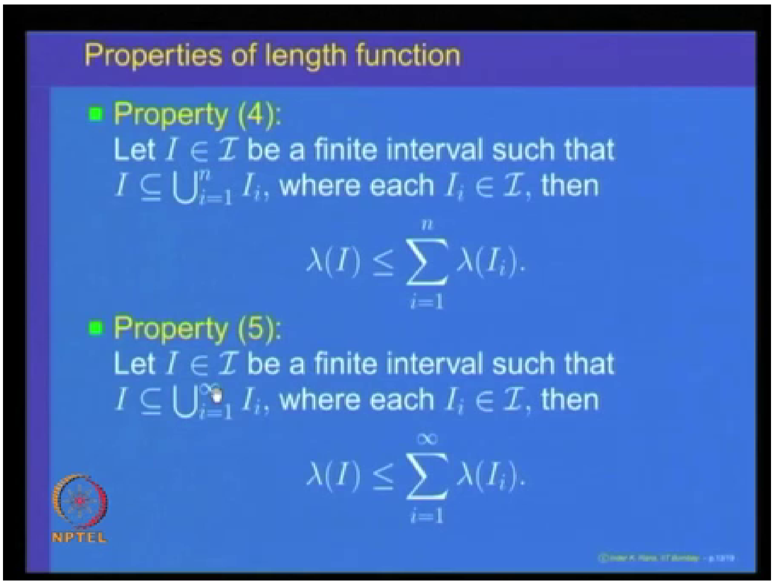

That the countable property that way you looked that namely length of I is less than equal to summation length of Ii's whenever a interval I which is finite is covered by any countable union then the length of I is less than or equal to length of I I’s.

So, this is we have extended that earlier property from whenever a finite covering is there we have go extended it to a countable infinite covering, but only for finite intervals. Now we would like to extend this to even arbitrary intervals which are not necessarily finite. So, for that we will have to do a little bit more work. So, let us look at the next property.

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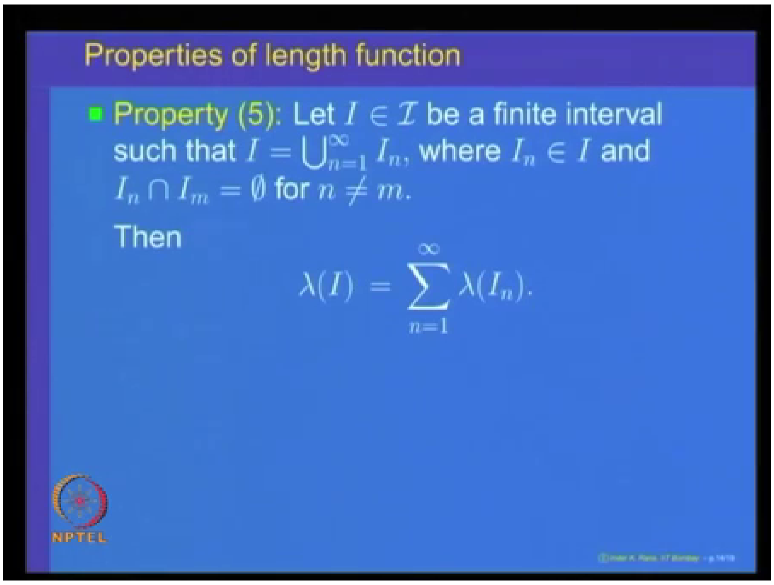

Which says the following let, I b a finite interval such that I is equal to union 1 to infinity I n where I n's are pair wise disjoint then at least we can conclude that the length of I is equal to summation length of I n's. So, whenever a finite interval is a countable union of pair wise disjoint intervals then the length of I is equal to submission length of I n's. So, let us prove this property.

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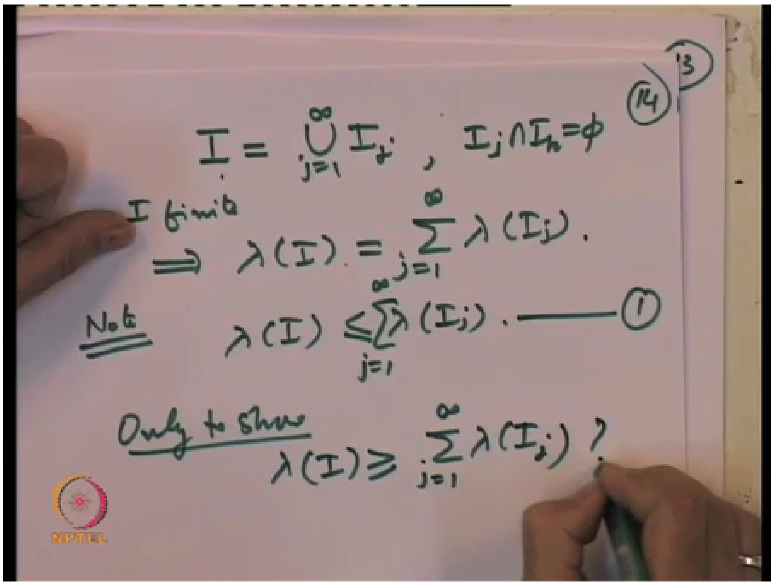

So, what we got I is equal to union of I j's j equal to 1 to infinity I j's are pair wise disjoint I finite implies length of I is equal to summation length of I j's

So, let us observe note we have already proved just now that if a interval is written as this a union of countable disjoint union note length of i, we have just now shown is less than or equal to length of I j's added up j equal to 1 to infinity call it 1. So, length of I is less than or equal to this, I have just now we have proved for finite levels. So, only to show only to show that length of I is bigger than or equal to summation j equal to 1 to infinity length of I j. So, only this is to be shown and for that. So, here is.

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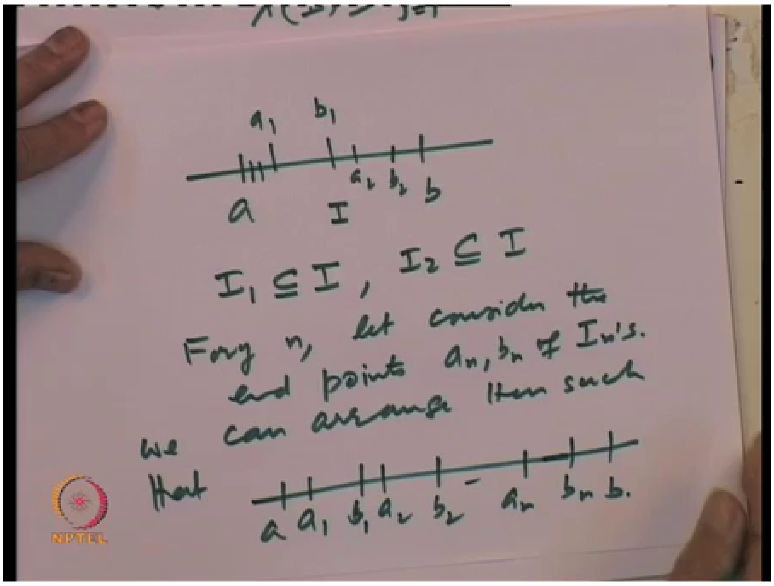

The interval a to b I is finite. So, there is a interval finite interval I now look at I 1 is a subset of i. So, it will be somewhere inside. So, somewhere is a 1 somewhere is b 1.

Similarly, I 2 is also inside i. So, somewhere it has to be either it has to be a to here, b 2 here or it could be here somewhere right and soon. So, for every n let us look at let us consider the endpoints the end points a n b n of I n's we can we can arrange we can arrange there only finitely many of them such that, here is a here is a 1 here is b 1 here is a 2 here is b 2 and soon here is a n and here is b n. And here is b.

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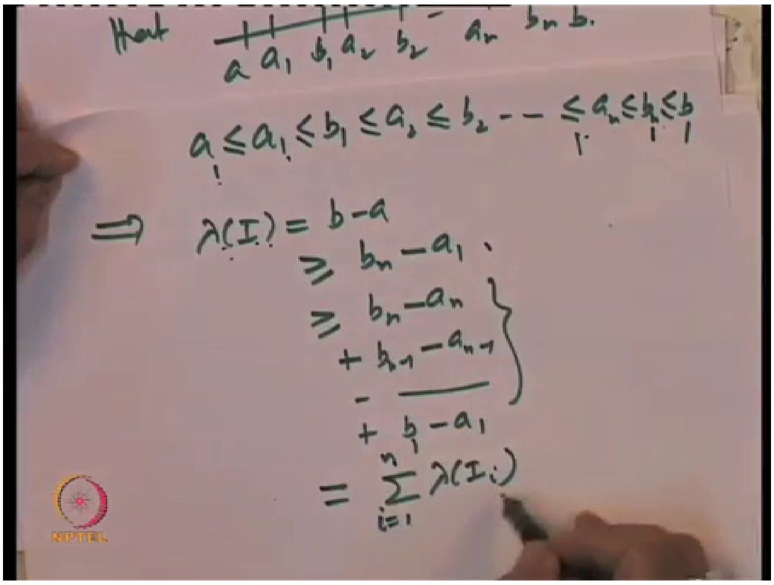

that means, we can arrange them in such a way that a is less than or equal to a 1 less than or equal to b 1 which is less than or equal to a 2 less than or equal to be 2. And soone less than or equal to a n less than equal to b n and which is less than equal to b once that is done.

So, this implies by a simple algebra; that means, length of I which is equal to b minus a now this is b this is ai am going to make it shorter b n and a 1. So, this is bigger than or equal to b n minus a 1. So, which is bigger than or equal to b n minus a n plus b n minus , 1the next one here minus a and minus 1and soon plus b 1 minus a 1. So, this put together is nothing, but which is equal to sigma I equal to 1 to n length of Ii. So, what we are saying is for every end the end points of the intervals I 1 I have to I 1 can be rearranged in this fashion and hence by looking at the ordering of this length of I is bigger than and this happens for every end.

So, that implies length of I is bigger than or equal to sigma I equal to 1 to infinity because which is happening for every end I can let it go to infinity length of i. So, the other around in you collate is also proved so; that means, we have proved that whenever I is a finite interval which is written as a countable union of pair wise disjoint intervals the length of I is equal to sigma length of I n's. So, with that we proved and important property of the length function for finite intervals. We will continue our study of the length function in the next lecture.

Thank you.