Mathematics in India: From Vedic Period To Modern Times
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Lecture-9
Aryabhatiya of Aryabhata-Part 3

So we had 2 lectures in Aryabhattiya, now I will be leaving the third part of our discussion on Aryabhatiya. So if you recall in the second part, so our lecture more or less ended with the discussion on the geometrical approach to finding the sin table. So I will start with that to recapitulate and then will proceed with the analytic approach.

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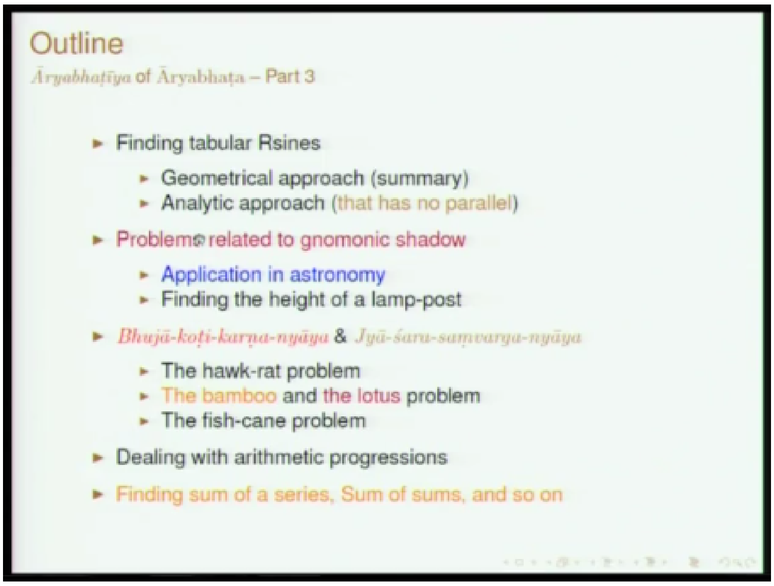

And this has been presented by Aryabhata as impact 2 parallel (FL) till even up to 15 century because what he has finally done he is the discrete version of what we know of harmonic equation today. So that is what it seems to be, so it is very interesting thing. So we will have a more detailed analysis on this lecture exclusively devoted for discussion on how the sin values have been improved upon over a period of time.

Then I will move on to certain interesting problems which have been discussed by Aryabhata. So in connection with certain formulae which he presents with this simple tool which we have been referring to us (FL), so that is what referred to us gnomonic shadow. So in samaskritham we call (FL). So based on that what are the things that can be found. We will discuss an application in Astronomy and then even day to day basis suppose if a lamp, so if you want to find out the height of the lamp, if you want to find out the distance of the lamp.

So how does this device tell so in finding all these quantities which will be of practical use. So we will do that. then we will move on to the discussion of the famous (FL) of course it has been discussed in greater detail, but what I am going to do is basically choose some very interesting problems which have been given by Bhaskara in connection with that. So here also notice what I have written is Jya-sara-samvarga-nyaya.

So Jua as I said is the card, semi card, so (FL) refers to the burner, so (FL) is product. So here it is basically the product of the card, he says since we know in a circle, so what are the applications of this rule. So that is what I will be dealing with that in great detail and in this connection Bhaskara has presented very interesting problem as illustration. So those will also be dealt with.

So the hawk-rat problem hawk-rat problem with bamboo problem, the lotus problem, the fish-cane problem, so all that will be highlighted today. Then I will move on to the arithmetic progression and sum of these.

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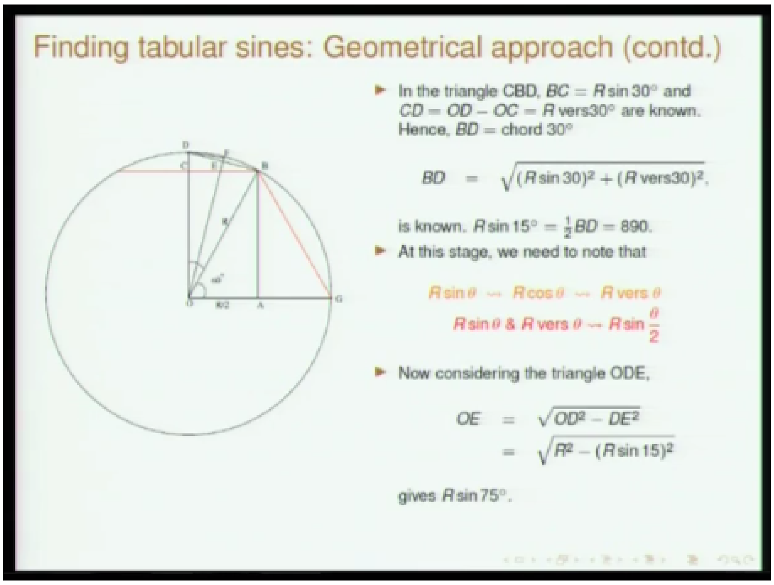

So to recapture what we did so this geometric approach essentially had a simple observation that to the card length of one sixth of the circumference is going to be radius. So with this the entire table was constructed. Once you know that the I told you that the radius, so is determined based on the value of pi which is given by, so once that is known 3438, so then we know r sin30, so rsin 30 is known.

Therefore this CD is also known or card circle is known and then r-r cos 30 is also known. So BC is known, CD is known and therefore this hypotenuse BD is known and from BD you get sin 15 degree, so sin 30 is known, sin 15 is known and the general principle sin theta to cos theta, cos theta to verse theta and from this to you get sin theta/2.

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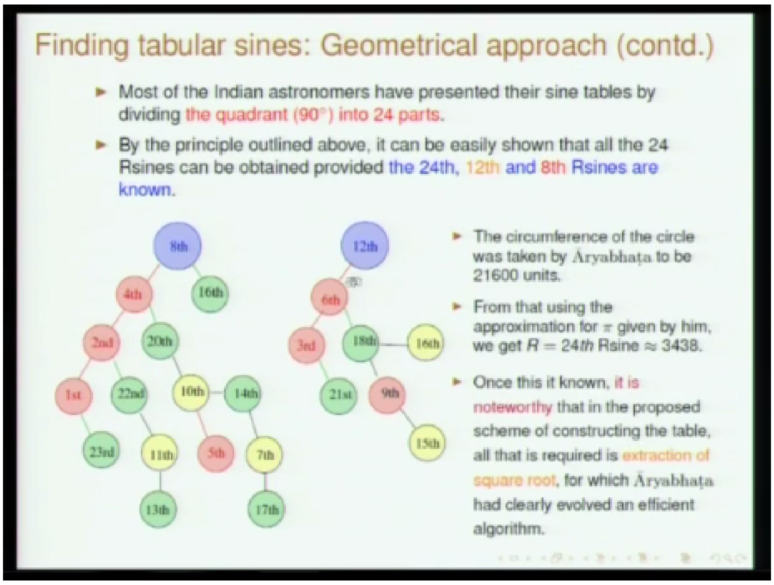

So this principal and so as I showed you so the table goes like this. So once you know r sin 90 and then with sin 30 we will be able to get almost 15 values ok. So this is what these 3 is and this 3 present 8 values of 15+323 and sin90, this si will all the 24 values of sin. So what is interesting to note here is, so in geometric progression mean you measure, so it is in some sense so you observe the geometry and then you get it.

So that is what it is all about and you do not do any measurement taking a rope and then determine the sin value that does not mean that. All that is required in this basically the technique for obtaining square and square root which has been thoroughly discussed by Aryabhatta and so you will be able to generate the sin table. So this is what it is.

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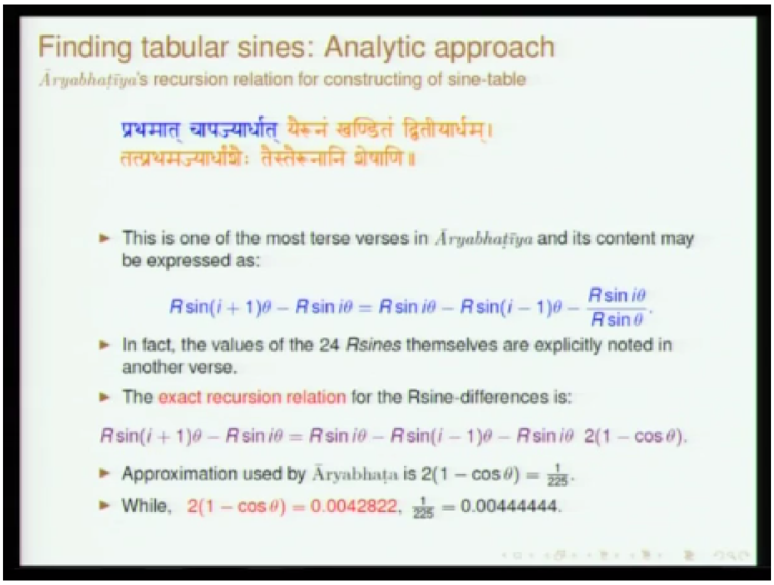

Now I move onto then another verse which presents the analytic expression with which it is a sort of recursion relation. So based on which will be able to get the entire sin table constructor. The verse goes like this (FL) in that this is one of the most thus verse we can be found in Aryabhatiya and 16 difficult for figuring out of it but for the help of the commentator.

So in fact this verse has been slightly differently interpreted by different commentator. So Bhaskara has given a certain understanding to us and Neelkanth slightly different presence this for ultimately all of them (FL) how is this relation to be extracted out of this verse, this has been slightly different approaches. So now I am not going to discuss that in great detail. All that I want to say is this verse basically translate into this equation.

So if you look at suppose I know value of sin theta, so theta here is the certain unit and this unit as I mention, so if you divide the quadron 24 this happens to be 225 minutes. So all that you do is sin theta is theta you just take as a approximation, so you take sin 225 minutes is same as 225 ok, so I will just take this. Then how do you generate the entire table. So the relation with has been presented by Aryabhatta amounts to this.

So Rsin theta is known here and for any value you will be able to generate once the first value is known using this relation. If you take I=1 and you want to get I+1 right, this is how the recursion relation is used. So you know for first value now you have to study, suppose it is I is 1, so then what is I+1 theta sin 2 theta, so we will be. So in this relation so this is 0 right. So and we have sin theta here, so the sin theta-sin theta/sin theta, so -1.

So the second value will be 224, the first value is 225, second value will be 224. So once more the second value you use this recursion relation, you will get the entire sin table. So this is the method which has been method which has been presented. So the method which has been presented by Aryabhatta, so is to essentially get the find difference table. So what he does is from the previous value.

See so this is previous value these 2 terms put together. So you have this previous value of sin theta, so divide this, this factory same, so this is 1/225 is what I am but I have chosen. The exact recursion relation, this is easily obtained, one can show that the factor which we will have here is 2*1-cos theta. So later astronomer, so this I will discuss in greater detail in a separate lecture.

So they have chosen the value see 1/225 amounts to this and the other astronomers have given this value is exact value is this an as improvement so this has been defined instead of 1/225. So the astronomers in the Kerala school have you chosen the constant to be 233 and half ok, that will be very very close to this almost 6 to 7 decimal place accurate, so how did they do extra will become clear when we move on to the Kerala school.

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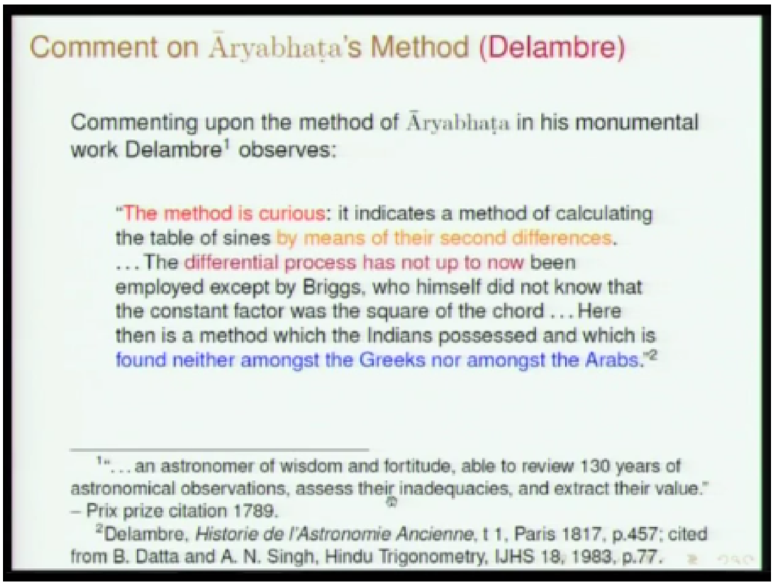

Our discussion on Kerala school will make everything clear, so with this I just want to make one observation. As regards this method which has been presented by Aryabhatta, this analytic approach to get the sin table. So there is an interesting observation which has been made by Dalamnre, he says the method which is curious he refers to the difference sin difference table which has been obtain just now, discuss now about this.

He says this method is curious, it indicates a method of calculating the table of sin by means of their second differences. So if you note this, so this is basically first difference and first difference, so you find the difference of these two first difference all that you note is it is proportional to sin. So this is what I meant by saying that this amounts to the difference equation second order differential equation ok.

That is what he has in his mind when he says that by means of second differences, the differential process has not approve been employed except by bricks, so he is talking about in 16 or 17 century and who himself did not know when the constant factor was the square of the chord, here then is the method with Indians possessed and which is found neither amongst the Greeks nor amongst the Arabs.

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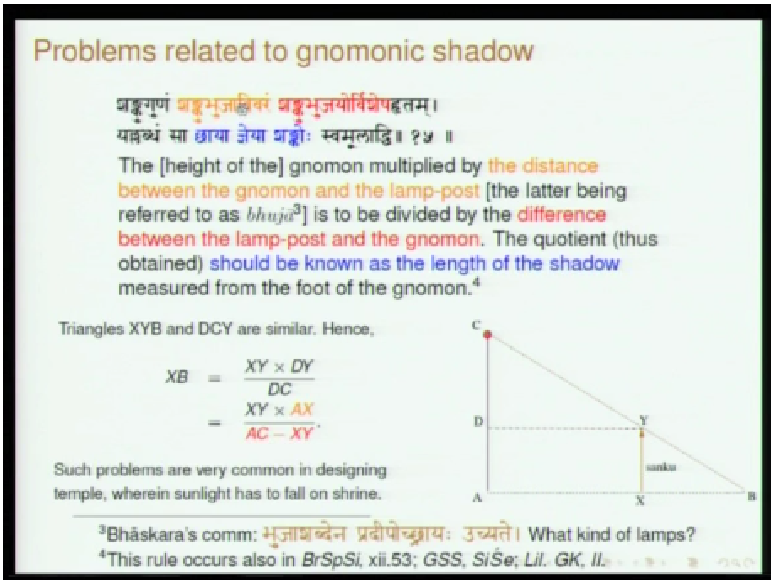

So this is something which I need, now I move on to problem s please quit Mini now I move on to problems related to this Shanku (FL) so this si the verse, he present the certain formula so which has to do with the shadow, how to obtain the shadow length and so on. So in this figure if you note xy is (FL), ok this is our device with which we carry on the experiment and here AC represents the lamp post.

So here this (FL) the term (FL) has to be understood, just keep in mind because we will have an occasion to recall this and therefore I have given the note which has been given by

Bhaskara, here he says (FL) is a lamp post (FL) height of it, so this (FL) of course it can be 1 seconds in a right angle triangle is this can be chosen as good as this can be chosen as (FL) does not matter.

But here so they have created a certain connection and disconnection has been made clear by Bhaskara in his commentary. Because this should (FL) is this and (FL) is this and (FL) basically means distance or separation. What is stated in verse is (FL) is xy, (FL) is multiplication, (FL) take a product, product of distance of separation Ax (FL) is what (FL) is used to find the difference between 2 quantities.

So (FL) normally means speciality, but in the context of mathematical text many times you will find the (FL) to be used to refer to the difference. So (FL) basically is the difference in the height of (FL) lam post, so this is what it is AC is (FL) so the height of that and Fi (FL) is the difference between 2, (FL) is division fine. So (FL) whatever obtain is (FL) so he says this is basically the shadow of the (FL).

So this verse has been stated by Aryabhatta in order to obtain the shadow that is cast by (FL) ok the length of the shadow, so which means if you know these two things xy and ac. So then you will be able to get this, but this has a certain astronomical application and that is why he has given this relation. So otherwise I am in this is just based on so two similar triangles and you will be able to get this relation.

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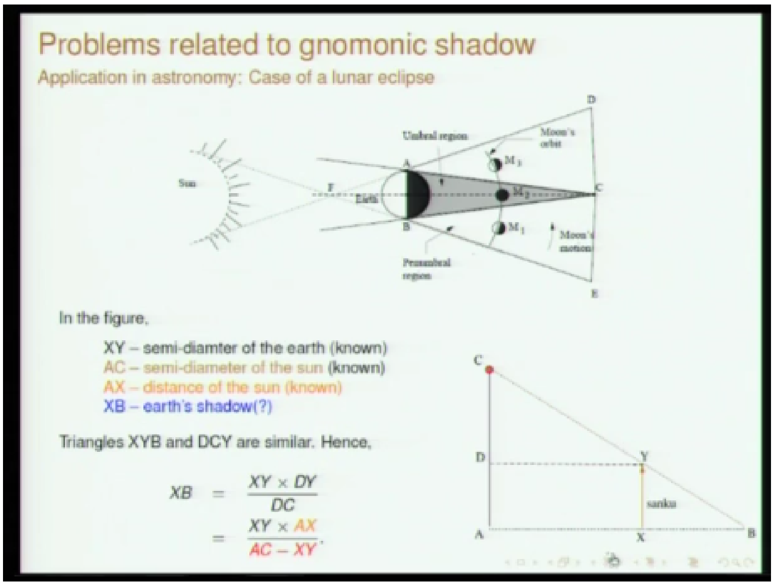

So (FL) formulas from the base of the (FL) where is going to be the tip of the shadow. So this is what this gives. So this is a simple straight forward application of considering two similar triangles. So the application comes here in the case of lunar eclipse. So you can think of you just see this so this is basically a depiction of what is happening in a lunar eclipse. So the moon enters into the shadow.

And earth is in between the sun and the moon and if you just think of cutting this into half, so basically what you have in this kind of a setup. So here the (FL) can be taken to be the radius of the earth, semi diameter of the earth and this object AC is semi diameter of the sun. So basically the ray from this stop, so as it moves and in goes then which as this point. So if you know this is (FL) height and this height then you will be able to get the shadow.

This is what basically Aryabhatta said and this is an application and wherein you know the semi diameter of earth, you know the semi diameter of sun. So all this will be specified and with this you will be able to and how do you know the distance between sun and earth. So that is see in terms of set another it is not the exact distance, but we will be able to get from the what is known as this (FL) ok. This is one application.

The other application is so privacy practical suppose you have lam post and if you want to find out the height of the lamp post, so you can actually measure the shadow, what is there was problem, so you know the shadow you measure the shadow and hence you can calculate any other quantity. So this is another this problem has been given as an illustration by Bhaskara which is (FL).

So found the length of the shadow to be 16, (FL) is the lamp post, (FL) is 72, 72 is the height of this post, (FL) say so what is this distance ray at, so you have to specify. This is just illustration, so you know one you get the other, this is ok.

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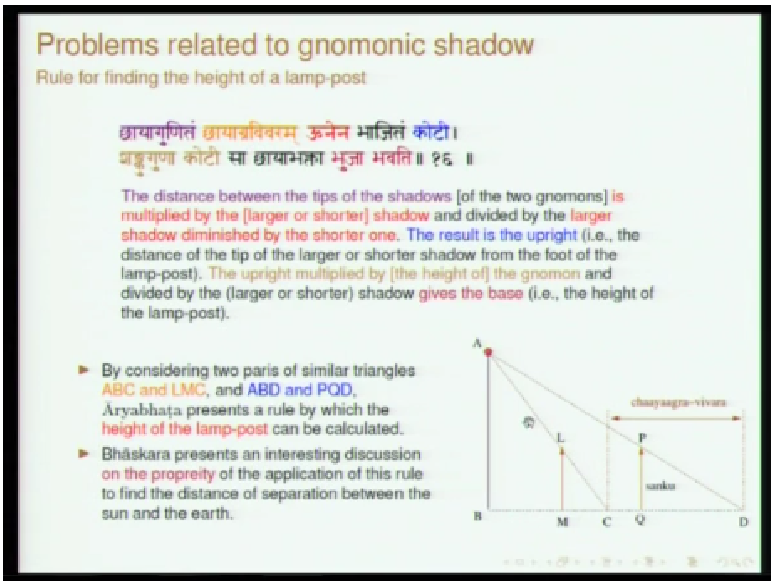

So we will move on to other interesting verse in Aryabhatiya and in fact the commentary to this verse by Bhaskara is something which is very fine I would say, I will discuss that (FL) so the problem here is to find out I mean one important application is to find out the height of a lamp post. This is consistent considered to be a general problem of this nature and here what he say you carry on the experiment twice.

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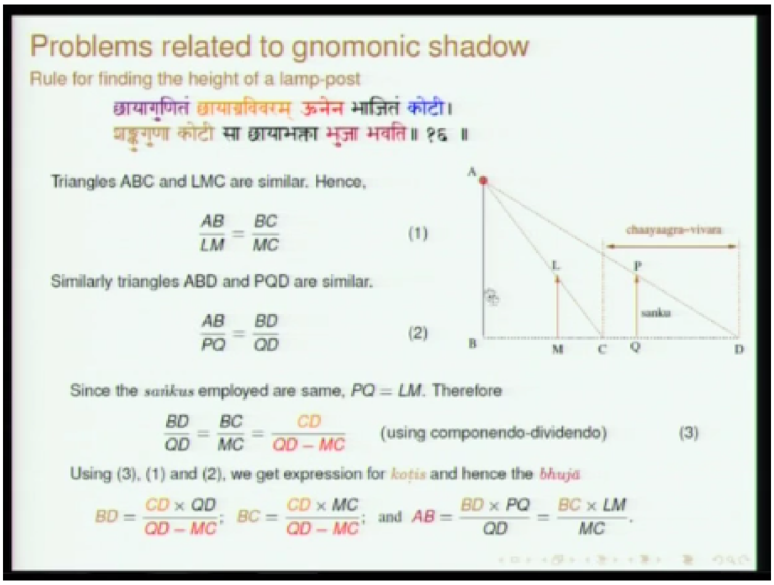

So once keep the (FL) in one place, move the (FL) to another place, so and then measure the shadows with cash. So from that, so what is it that (FL), so the formulation of the problem is this way (FL) let me explain this with equation, so (FL) is the length of the shadow ok, so (FL) so the length of the shadow, so he refers to MC as well as QD (FL) so both are (FL).

Since you are carrying our twice then (FL) is tip of the shadow (FL) is separation, so (FL) so if you consider these two shadows basically it represents CD. So c is 1 (FL) and D is another (FL), so (FL) basically is CD. (FL) is basically subtraction, but what (FL) so Aryabhatta has not stated, but this has to be understood and here so this happens to be the difference in the (FL) ok (FL) so you have to subtract, so (FL) is the remainder.

Ok so that is what is mentioned by Aryabhatta by the word (FL) divided. So what it gives (FL) actually dual usage (FL) and what does this (FL) mean see that is why I said you were remember the word (FL) was used to refer to the height of the lamp post, so that is (FL) then obviously the perpendicular is (FL) and the perpendicular here refers to the plane here ok, so this is what is (FL) and here there will be 2 (FL) and this is (FL).

If you conduct the experiment by placing the (FL) at M then BC is 1 (FL) if you conduct the experiment by facing some quote Q then BD is (FL) both of them are (FL) depending on where you place the (FL) and that is why we have the dual usage. So (FL) so this formulation which Aryabhatta has given for the expression of (FL), so you get 2 expression for (FL) one is for BD, the other is BC.

So look at this now and connect it with the verse, so (FL) is QD (FL) this is one (FL) and this is another (FL) so then when he says you multiply this (FL) so PQ and LM in fact both are (FL) ok, we are at the same dimension obviously when you conduct this experiment (FL) divide by (FL) corresponding (FL), so if you then that will give (FL). So did I use this expression and use this expression and we will be able to get the height of the lamp post.

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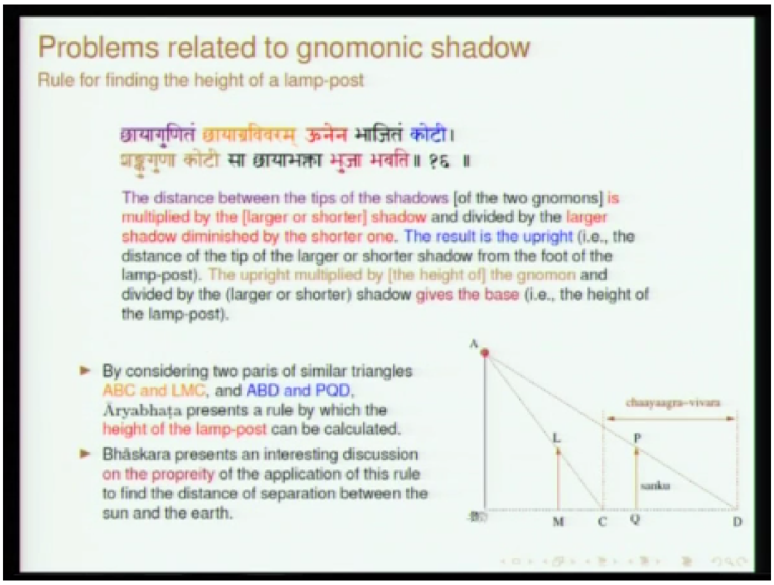

Bhaskara present some interesting discussion on the property of the application of this rule to find the distance of separation between sun and the earth. So one can think of sun be a sort of lamp, like lamp post showed is there, so we can use this, so you can do can we conduct an experiment and then get the distance of the sun from the earth, so this is what you, so (FL) means a certain day it is actually referred to as equinoctial day.

The sun actually moves from Equinox towards the north and then again towards the South. So this what we call as (FL) but suppose you conduct the experiment on equinoctial day, so (FL) so he says (FL) so imagine that the sun is exactly on the prime meridian, so sun on prime meridian, so the shadow will be exactly in the north south line. Ok so (FL) is so just now we had the discussion of finding the difference in the (FL) right.

So difference between the (FL) So (FL) you can find, so you have (FL) so all this (FL) so they try to do this, so (FL) so they thought that they can find out the distance of separation between the earth and the sun. So by conducting this kind of an experiment ok. So then he says (FL) means that is improper, (FL) in fact he says in a given to conceive of to speak of so that mean I can think of 2 spacing and 2 different spaces and then (FL) measured.

So this is not actually going to work out (FL) it just goes on, but the point is what we need to understand here is this is something which is of astronomical magnitude distance and the entire this like a point object, so we just think of placing a (FL) place it here or you place it in a in Mumbai or replace it elsewhere. So it is not going to make any sort of different, so this is not going to work out.

So that is what I think Bhaskara wants to imply that you cannot just think of using this experiment to find out the distance between the earth and the sun. This is something which is obviously possible for distances which is of terrestrial magnitude and not celestial magnitude. So this is what is convey.

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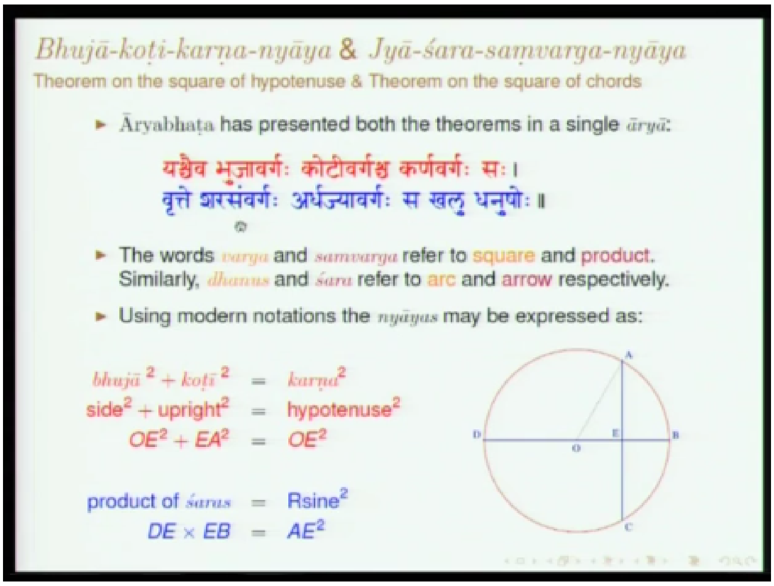

Then we have 2 (FL) so which is presented in one arya, (FL) is a certain theorem, we can take it via general principle is called (FL) so he says (FL) so this is something which is very well known. So (FL) so think of this, so just make the terms familiar once more I just think of this triangle OAE, AE is (FL), OE is (FL) and AO is karna. So he says OE is square b square OA square, ok. So this is what he says (FL).

Then in the later half of the verse (FL) which I have referred to as (FL) AE and EC, so the product of AE and EC means EC=AE therefore it is AE square so which is same as the product of DE and EB. (FL) is basically arrow, so if you conceive this part of the circle ABC then EB is consider the other part of the circle ADC then DE (FL) so all that he says is (FL) the product of (FL) product of chords is equal ok.

This has been illustrated with very interesting problems by Bhaskara. So I thought I will just spend a few minutes on that. This application of this has been illustrated in various contacts, so which look quite apparently different, but then the principal which has to be employed is one and the same. So that has been very beautifully brought out by Bhaskara by giving very interesting examples.

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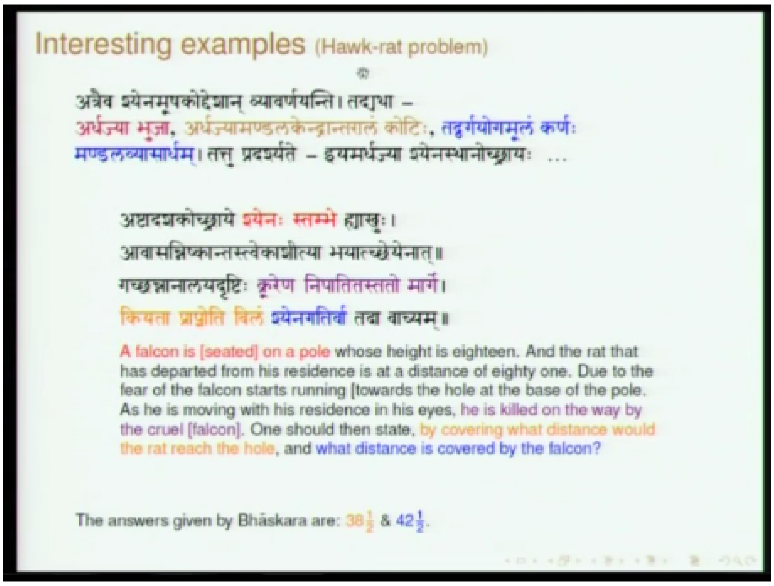

The first problem is this, this Hawk-rat problem, so Bhaskara says (FL) means in this connection while trying to explain this principle and its applications (FL) discuss in great detail, (FL) so first as an introduction he just says (FL) if you look at this is (FL) is centre O, (FL) is this so this is (FL) so keeping this problem which is going to be stated. So he says (FL) is this ok (FL) is the height of that.

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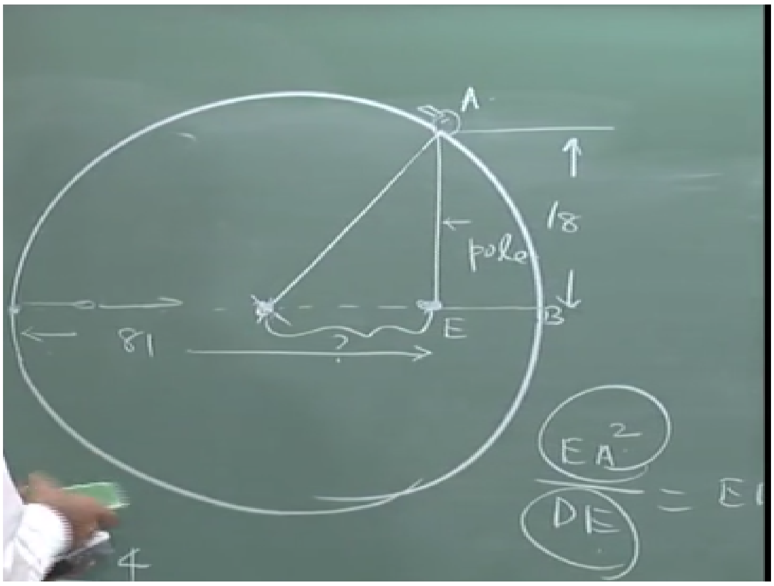

So (FL) the height of the pool, so now the problem is this, (FL) ok the problem is the following, so let us take this to be the pole, so this (FL) is sitting here (FL) and (FL) so this is given to be 18. So here is a hole at the bottom of this pole and then he says some rat which came out to the hole, so it is somewhere here. So this distance is he state it as I think 81, (FL) so the circle I just want to draw only for the application of this direct application of this principle.

So he says so this rat observed this half and then it got scared and it wanted to just run into the hole, ok this what he says, so (FL) which has come out (FL) out of here. So it want to get into (FL) get into (FL) is what is called (FL) so it just starts moving here, so this rat, but the moment we saw it also flew down and then grab the rat here and it finished it. So now the question is (FL) the distance travelled by (FL) means to how much it has to cover in order to reach the hole.

So **so** what was the distance, so basically the question is what is this and what is this assuming that both of them travel at the same speed. So this is the problem and this one can let me see that it is your application of this (FL) which has been discussed. So this distance and this distance on the same because it travel at the same speed. So we have this suppose you see same location that we use here. So EA so let us take EA so EA square/call the later d, so d is (FL) EB.

So EA is known, DE so that is all, you will be able to solve this problem ok this si a direct application of this. And very different problems more or less they use the same principle in fact the answer are 38 and half, and so this is basically so this you know so this (FL) able to solve this problem and what you will basically get a this 81+4. So this will be 4, to do that, so 81+4/2 so and this will be the radius and 81-4/2, so that will be this is what to ask ok.

So this is the answer, then we have this bamboo problem, so (FL) think of a bamboo and bamboo (FL) so the height is 16, (FL) from the base, (FL) the distance is 8, (FL) so tell me so a given height and he tells you so the distance of the tip from the base. So (FL) tell me this is a problem, so this also very very similar to this identical in fact. So one should think of this to be the so this is our bamboo.

And this distance so it stated to be 8 I think, so the tip of the bamboo, so this bamboo is like this. So this broke and top like this, so this is specified and this gives this, you should be able to get the desired answers. The answers are 10 and 6 ok. So 10 to the top and 6 from so the total high was stated to be 16, and the third problem is Lotus. So (FL) there is a lot of discussion which goes on as to how to make mathematics learning clear.

So that are very interesting problem which one can find in this. So illustrate very place principle and application of principal in a wide variety of problems. So this is again a completely different kind of a problem that you will see that it also (FL) today. So the problem that he stated here is the following.

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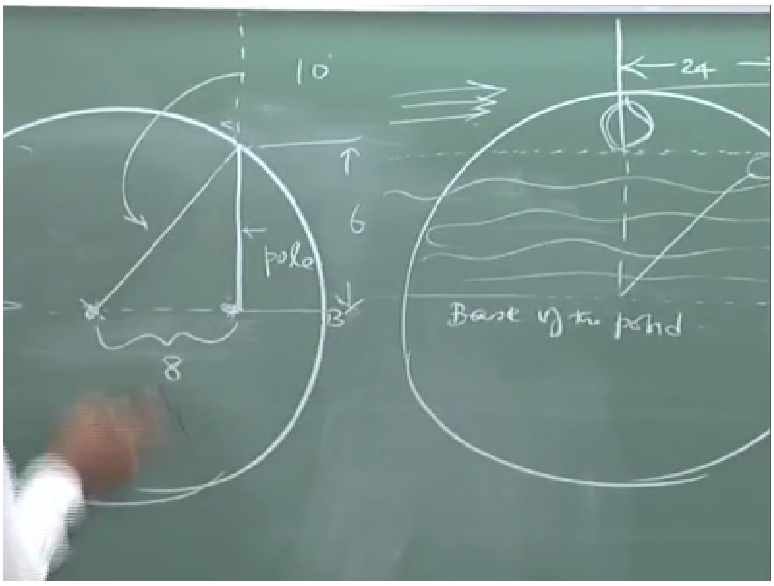

So he says that somebody noted the kamala kind of a lotus and this is all water, if he says (FL) so this is based upon, so I do once again the circle only to show that we make use of the same principle, so all that he says is there was a wind which was flowing mildly, so in this direction and this sort of (FL) so this flower got from here and it is like. So this all he says, (FL) tell me what is the height of the kamala and what is the height of the water.

So what is the data that has been given, so the data given is this is 8, and this distance so he specify it as (FL) is basically 24 ok, so (FL) is 24, so this once again you can see that it is the same principal (FL) we can see that so if imagine this problem is essentially. So this is stated so which is like (FL) gives this sometime he gives this distance, so in the case of Hwk-rat problem. so this distance was given.

Now he is giving this distance and in another problem he gave this distance. so it is all problems with very different the principal is one and the same. So the moment use this now I will be able to get solution to this problems. a. (FL) no no when it was vertical so it is 8 above the water yeah, no no it is (FL) so here refer to (FL) from the original. So (FL) this distance is 24, (FL) what is height of kamala and what is the height of water.

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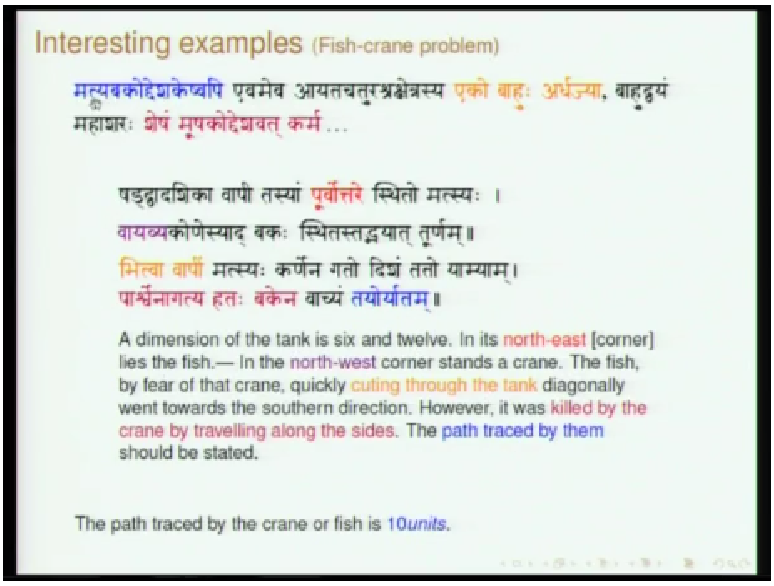

So the answer you can easily see the 40 and 32 everything is used on the same principle ok finally quickly so let us even more problem and then I will proceed further. So this is small error here this is (FL) not (FL) another kind of problem fish and crane, (FL) so this problem is stated like this, so (FL) is a slight variation of this problem not exactly identical but he presents this way, I use the same figure.

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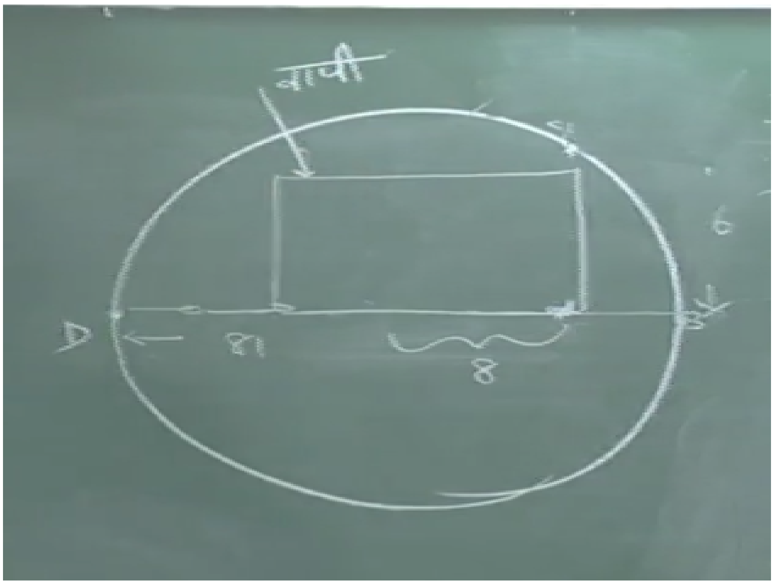

So we should imagine a certain pond, so this is what is referred to as ((FL) so the dimension of this so this is (FL) the dimension of (FL) is stated to be 6 and 12 (FL) suppose normally in all this kind of descriptions where you have to have geometrical figures drawn, so the always specification direction ok, so we just take it to be this suppose to east and this is north ok. So he says (FL) is this fishes, so the fish is here in this corner of this pond.

So (FL) is northeast (FL) actually is north west ok, so (FL) is standing here, ok this is the bird (FL) and he thought that he should slightly escape from there, so he says (FL) cutting across (FL) so this is north, so what he says is so this fish (FL) try to come to the southern side, but that time (FL) also walk, so it noted that he is moving and this all went like this. So along the (FL). So now he is asking so (FL) so what is the distance travelled by these things.

So once again you will see that this application of the same thing so without. So he only thing is what is be understood here is this distance and this distance are one on the same, so (FL) shifted it and you will get the same problem and (FL) this of the different kinds of interesting problems which are presented by Bhaskara as illustration of these fundamental theorem ok. Then I will very quickly discuss to show how Aryabhatta has been able to present the formula in very interesting form of composition.

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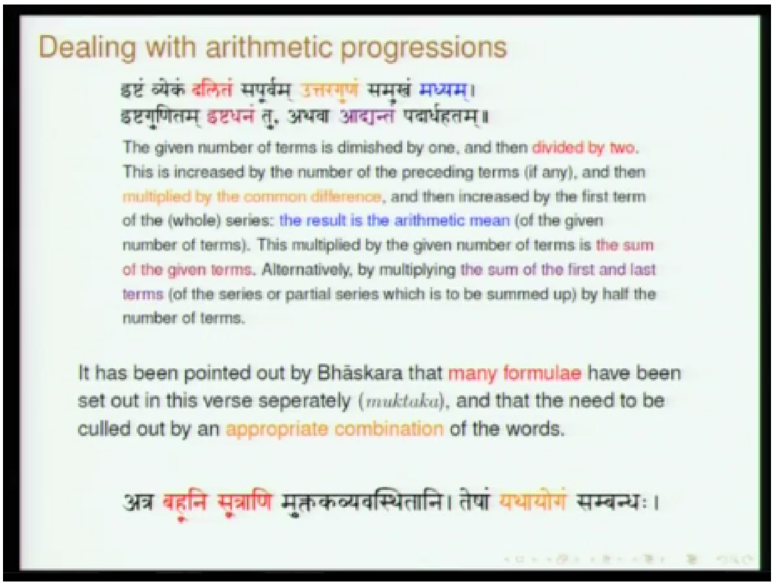

So this arithmetic progression problem. So this is all well known result but it is only the form of the languages with wanted to convey and how Bhaskara has interpreter this. Bhaskara says present 2, 3 expressions. So it is not necessarily a single expression, but you have to appropriately combined the words which has been presented in this to get different formulae. so this is what it is only to show what kind of thing I am just putting this version the commentary.

So Bhaskara in fact says before commenting up on the verse (FL) is a certain formula (FL) many formula have been given by Aryabhatta in this single verse, (FL) this is a certain style of composition, see in Sanskrit so supposed 2,3 versus which are combined then it is called (FL), so this 2 verse have to be read together in order to get the meaning, so in Kalidasa (FL) and then some 4, 5 verses will be there.

So all the surface that we put together in order to understand the only one verb, so which runs through all this process, but (FL) is opposite kind of a thing, so within a single thing. So it can combine independently (FL) means to be kind of a thing, so freely can order this word to get 1 formula you can freely order some other words and then you will get another formula. So that is why he is saying (FL) means they have been arranged in the (FL) format.

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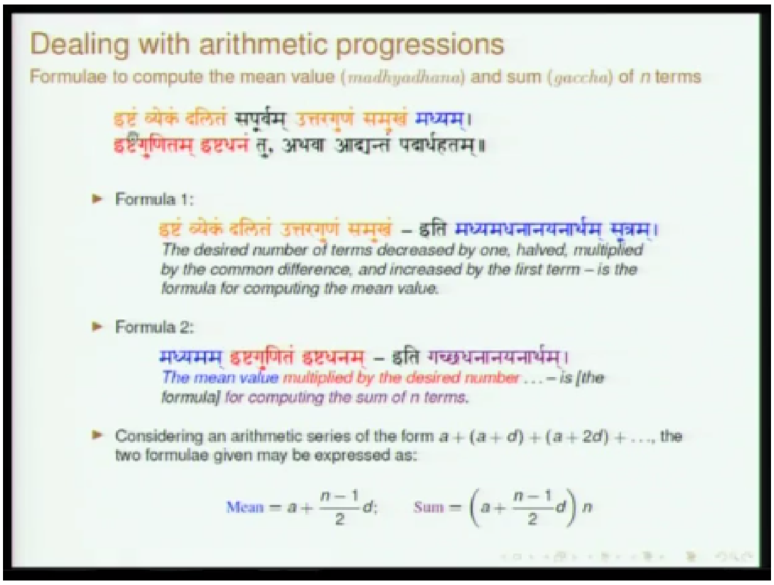

(FL) have to put them together to get the formulae, is just to give you a flavour I thought I should do this (FL) so this is with reference to an arithmetic series, so think of an arithmetic series a+ a+2d and + so on. so this formula he says is (FL) means one is removed from that. So (FL) basically is a number of terms in this is so (FL) remove one (FL) divided by two, so (FL) common difference so by factor which with increase (FL).

So (FL) is the first term for the first time, so add the first term, so what does we use, this give you the (FL) so he just have to combine this (FL) value of the series fine, so then he says (FL) so this (FL) so whatever you are saying so that you take to be the other part (FL) so it gives you the (FL) is sum of n terms ok. So this multiplied by n gives you (FL) the different things. So (FL) is the first term and (FL) is the last term. So (FL) which is divide by 2, so then also you get the so this are in fact much more interesting thing has been discussed by Bhaskara, but just I will stop here, thank you.

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