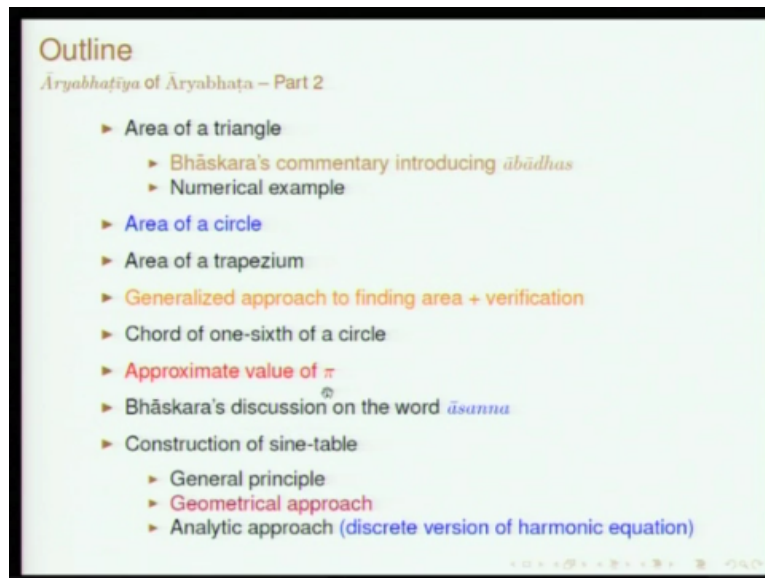


Mathematics in India: From Vedic Period To Modern Times
Prof. K. Ramasubramanian
Indian Institute of Technology-Bombay

Lecture-8
Aryabhatiya of Aryabhata-Part 2

So this is the second part of our lecture on Aryabhatiya, so in the earlier part so we saw so the algorithm presented by Aryabhata for extracting square root, cube root and so on.

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So in this lecture we will be starting with the formula given by Aryabhata of finding the area of various geometrical objects. So starting with triangle then proceeding towards trapezium circle and so on. So in the latter part, so we will see the approximation which has been given by Aryabhata for the value of Pi and the method which has been by him for generating the sin table. So in fact Aryabhata has given 2 different approaches for constructing sin table.

So one is the geometrical approach, the other is the analytic approach. So we will see both of them and then during our discussion on the verse which gives the value of Pi, so we also see an interesting note which has been given by the commentator Bhaskara on the use of the word asana. So this has been discussed at great length given by Nilakantha Somayaji which will be covered later.

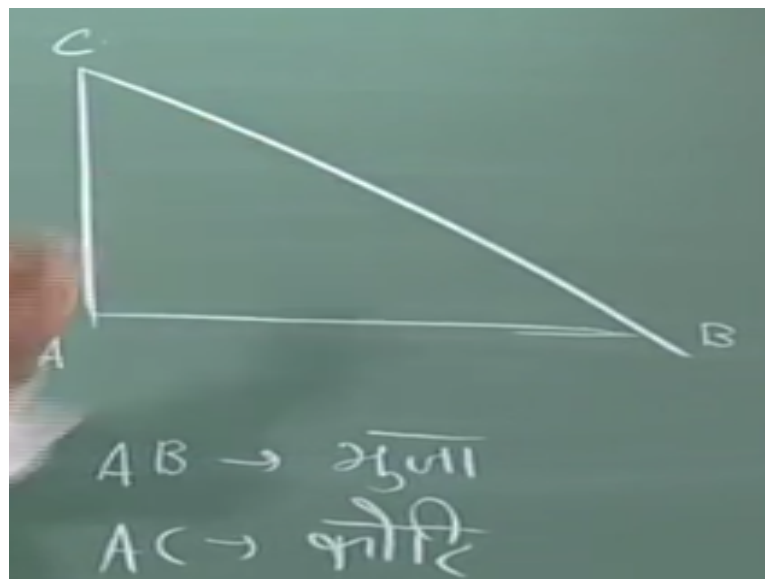
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Area of a triangle

- ▶ The formula for the area is presented in half *āryā*:
 त्रिभुजस्य फलशरीरं समदलकोटीभुजाधिसंवर्गः ।
 The product of the perpendicular [dropped from the vertex on the base] and half the base gives the measure of the area of a triangle.
- ▶ Bhāskara in his commentary observes:
 1. the term *samadalakoti* means 'the perpendicular dropped from the vertex on the base of a triangle', i.e., 'the altitude of a triangle'.
 2. it should not be interpreted as 'the upright which bisects the triangle into two equal parts'
 3. if we do so, the applicability of the above rule will be restricted to equilateral (*sama-tryaśru*) and isosceles triangles (*divisama-tryaśru*).

But here we will see the note which has been given by Bhaskara on used the word asana. So let us start with the verse which presents the formula for the area of a triangle. So Aryabhata says (FL) so normally he say half into this into height. So it is essentially same expression which has been presented (FL) is the word which has been employed by people to refer to the area. So here the word (FL) should be understood as (FL) the measure of the area.

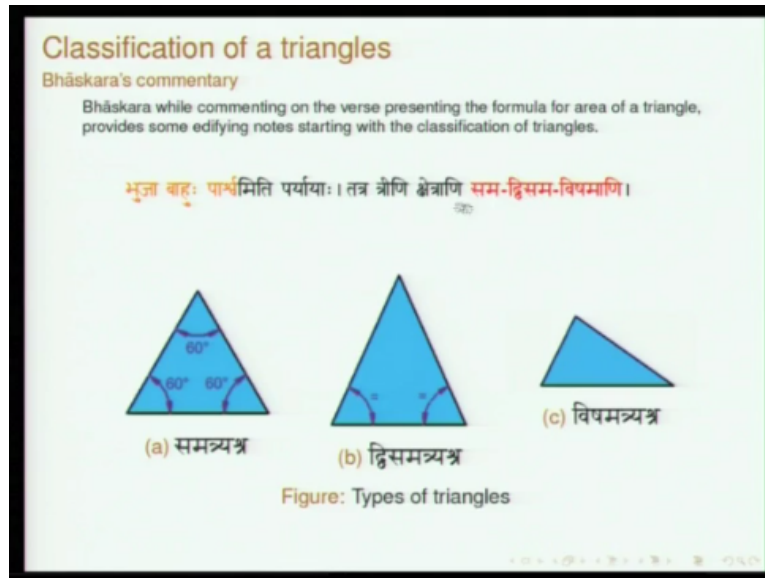
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(FL) so normally we are familiar with half into base into height, so here this (FL) can be taken to be this side is base and (FL) is the height ok. So in fact in the word (FL) has been analysed by the commentator Bhaskara, see in a triangle usually if you have a right angle triangle so let us say ABC, so if AB is (FL), then AC is (FL), and of course this is called karna. So if this (FL) can we interchange dividing up on the size of the angle.

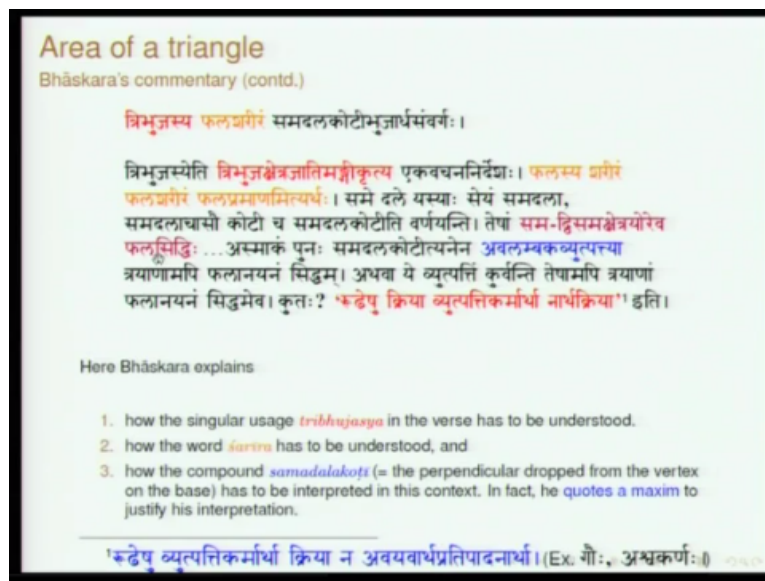
So one is (FL) and the third is karna, now if we consider a triangle, so this is the part which we are calling as (FL). So (FL) and this is (FL) so this is the base, this is height half of it base is the area of the triangle. In fact the word gala is generally used to refer to half splitting ok. So (FL) means so (FL) so generally is taken to be equal division ok, so when you thought of divide so this is how the meaning as such comes from that.

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But then if we use this then if not be applicable for all the triangles in fact Bhaskara, so comment is discussion by classifying the triangles into three types, he says (FL) they are all synonyms and there are 3 type of triangles.

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Further continuing the discussion he says (FL) in fact we should have 3 types of triangles (FL) this is applicable to all the triangle or is it something which is specific to one of the three

types or any of the 3 types. So he says this (FL) means so the group of all the class, so it is applicable across the class of the triangles (FL) so then he goes on to discuss about the word (FL). So if you say (FL) so if you use this kind of derivation of the word.

Then it is confined to only these two types ok. So (FL) so if you draw perpendicular then the perpendicular device the base into two equal half only in these two cases, (FL) scaling type. But this formula which has been presented for area is applicable to all types, then how do you understand the word (FL). So this is the question which has been raised by Bhaskar and he actually quotes very interesting matching I would say (FL).

So the word (FL) should be considered as a (FL), so (FL) in the sense, so if you not so try to find the derived meaning of the word and it has to be just for instance, so in the case of the word (FL) in Sanskrit, So (FL) is generally referred to as a goat. But if you go to the derived meaning that which is not known, so it never applicable to goat ok. So when you consider this (FL) so you just have to accept whatever is the meaning which has been accepted by the society at that point.

That is what he is trying to convey, so (FL) if at all you want to provide some kind of a (FL) is not for the purposes of (FL), but this is the very interesting matching, so which can be applied in various disciplines various contacts. Here the categorically states (FL) and therefore this derivation should not be taken in the literal sense here. So (FL) should be simply understood as the perpendicular that is dropped on the base from the vertex.,

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Area of a triangle

► The formula encoded in the āryā

त्रिभुजस्य फलप्रमाणं समदलकोटिभुजार्धसंघर्षः । 6a ।

may be expressed as

$$\begin{aligned} \text{tribhujaphala} &= \text{bhujardha} \times \text{samadalkoti} \\ \text{Area triangle} &= \frac{1}{2} \text{base} \times \text{height (altitude)} \\ &= \frac{1}{2} b \times h \end{aligned}$$

So this is what the sutra Aryabhata sutra means (FL) so the area of the triangle is (FL) so half of the base into (FL) height, so this is all it is, so it is applicable to all the three types of triangles. So this verse basically tells you that the area can be calculated using this formula. But then if you do not know the height, so you only know the sides of the triangle, then how do you go about finding the area.

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Area of a triangle

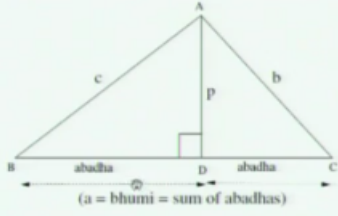
- ▶ The formula ($\frac{1}{2} \text{base} \times \text{height}$) as given by Āryabhaṭa can be employed **only when the altitude is known.**
- ▶ If **altitude is not known** and **only three sides are known** then how do we find the area?
- ▶ Bhāskara in his commentary presents a method by which we find segments of the base (BD & CD)—called *ābādhas*—and hence the altitude (p).

The diff. in the *ābādhas* is given by

$$A_{diff} = BD - CD$$

$$= \frac{c^2 - b^2}{a} \quad (b, c : \text{karṇas})$$

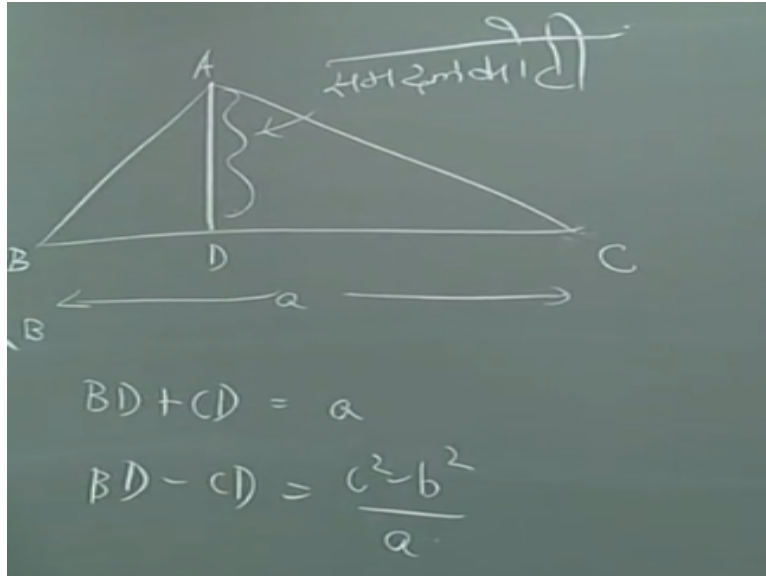
$$\text{ābādhas} = \frac{1}{2}(\text{bhumi} \pm A_{diff})$$

$$\text{altitude} = \sqrt{\text{karṇa}^2 - \text{ābādha}}$$


So then there should be a way to find out the height of the triangle. So that is what is discussed by Bhaskara in his commentary. So the word (FL) has been employed to refer to the two parts which are generated by dropping the perpendicular from the vertex, ok to drop the perpendicular which is referred to as (FL) so Bhaskar essentially gives a certain method by which you force to calculate the (FL).

And then from that you find out the (FL), so if you know only the sides, so you know the base, so that is taken for granted and for finding height so based on the 3 size for the presence of formula. This is how it goes, so first he gives an expression for the difference in the Abacus and the difference in (FL) so BD-CD, BD is (FL) CD is (FL), so the difference is c square-b square/Armstrong

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So this can be easily seen and we have an expression for the difference in (FL) in terms of the sides of this ok. So if you have this basically, so ABC, so this is B, so $BD + CD = a$ which is known and $BD - CD$ if given to be $c^2 - b^2$ in this. So once we have these 2 we have the expression for so just add this and subtract this. So you have the expression for the 2 (FL). So this is stated to be half of (FL) is the word which is used to refer to the base.

So (FL) difference so this gives you the expression for both the (FL), so having known this (FL) so it since you see you can calculate P either from this (FL) or from the other (FL) so this is basically (FL), so the karna can be either c or b depending upon either of the triangle that we choose. SO this is what ((FL) .

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Area of a triangle

Bhāskara's commentary (contd.)

भुजयोर्वर्गविशेषः तयोर्वा समासविशेषाभ्यासः त्रिभुजक्षेत्रे आबाधान्तरसमासविशेषाभ्यासो भवति। भूम्या आबाधान्तरसमासप्रमाणया विभज्य लब्धं भूमावेव सङ्क्रमणम् ।
 *अन्तरयुक्तं हीनं दलितमिति। अनेन क्रमेण आबाधान्तरप्रमाणे लभ्यते। ताभ्यां आबाधान्तरप्रमाणाभ्यां विषमत्रिभुजस्य समदलकोटपानयनम्।

Let the sum and difference of the *abadhas* be defined as:
 $A_{sum} = BD + CD$ and $A_{diff} = BD - CD$. Then it is said that,

In a triangle $c^2 - b^2 = (c + b)(c - b) = A_{sum} \times A_{diff}$

Hence $A_{diff} = \frac{(c + b)(c - b)}{A_{sum}}$
 $= \frac{(c + b)(c - b)}{a}$

Now *abadhās* = $\frac{1}{2}(bhumi \pm A_{diff})$

From the *abadhās*, the *samadalakoti* (altitude, p) has to be obtained.

(a = bhumi = sum of abadhas)

(FL) in fact is a very interesting to read the way he present this (FL) c square-b square, (FL) is putting together essentially means sum and (FL) is difference, so he says bsquare-c square+c=b and there is also equal to (FL) so the sum of the two (FL) in the difference to the 2 (FL). So this is straight away seen from the conservation of these two triangles ok. So you get this, so this is the first relation at he says.

So then so a from this expression we see that so a difference is so csquare-b square, so which what he stated in the earliest slide and therefore (FL) is this. So this is how Bhaskara proceeds. In fact the sides verse (FL) so whenever you have something of this form this is called (FL) and this is called (FL), so you have you have to just sub this then find the difference.

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Area of a triangle
Numerical example in Bhāskara's commentary

कर्णस्त्रयोदश स्यात् पञ्चदशान्ये मही द्विसतेव।
विषमत्रिभुजस्य सखे फलसङ्का का भवेदस्य ॥

In a scalene triangle [one of] the hypotenuse is thirteen, and the other is fifteen. The base is two times seven only. O my friend, [please tell me] what would be the measure of the area of this [triangle]?

Having posed this problem, Bhāskara presents the solution in prose as follows:

... पञ्चदशकेन कर्णेन नवप्रमाणेन च आधाधन्तरेण लब्धा समदलकोटी १२। त्रयोदशप्रमाणेन कर्णेन पञ्चप्रमाणेन च आधाधन्तरेण लब्धा सैव समदलकोटी १२। फले 'समदलकोटीभुजार्धसवर्गः', भुजा भूमिः, तस्य अर्ध ७, समदलकोटीभुजार्धसवर्गः इति फलमागतम् ८४।

Now, $samadakoti = \sqrt{15^2 - 9^2} = 12$. Also, $\sqrt{13^2 - 5^2} = 12$: The area is $\frac{1}{2} \times 14 \times 12 = 84$.

In fact this is how demonstrated with the formula example (FL) so this is the Bhaskara from several examples on various occasions wherever he has to illustrate the use of a particular formula. So I will skip this basically the presents how one finds the area of a triangle. So given the three sides and you have to calculate the height and from (FL) you have to get the areas.

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Area of a circle

► The formula given in the following *āryā*

समपरिणाहस्यार्धं विष्कम्भार्धतमेव वृत्तफलम् । 7a ।

may be expressed as

phala = *parināhārḍha* × *viṣkambhārḍha*

Area = $\frac{1}{2}$ circumference × semi-diameter = πr^2 .

► Commenting on the usage of the word 'eva' Bhāskara observes:

एवकारकरणम् आर्यापरिणार्धं प्रतिपत्तव्यम् । अथवा एवकारकरणेन
 उपायनियमः क्रियते । समपरिणाहस्यार्धं विष्कम्भार्धतमेव वृत्तफलम्,
 नान्यदुपायान्तरमिति । नैतदस्ति, उपायान्तरश्रवणात् अन्यत्र
 'व्यासार्धकृतिस्त्रिसङ्कुणा गणितम्' इति । नैतदुपायान्तरं सूक्ष्मं, किन्तु
 व्यावहारिकमिति ।

... Or, [to be more appropriate] the use of the word *eva* is to indicate that this is the **only means** to obtain the area of a circle. ... there is no other means. This is not true since $3 \times r^2$... This alternative method is not for obtaining accurate value ...

So now I move on to the area of circle, so which has been presented by Aryabhata. This is the very interesting, so in fact it is half verse wherein he has used the word (FL) has lot of significant. (FL) refers to circumference, so (FL) is diameter, so (FL) is half of it, so (FL) so it refers to half the circumference. So the arc has to be absolutely significant thus far it means, so (FL) means multiply by the radius semi diameter.

So this is the (FL) fine, so this (FL) pi*r square fine, so (FL) commenting on the usage of the word (FL) so which is quite edifying, so he says (FL) so why did Aryabhata used the word (FL) so this is meaningless or can be assign from sense to that. So this is the analysis which it has. (FL) taking a very simplistic one can say, so in certain cases so we have to fulfil the metrical compulsions.

And therefore so one add some word which need not be necessarily conveying something which is very meaningful. So this is one way of saying that (FL) has been used in just to feel the metre of the bus. So (FL) putting a certain constraint on that means which you can involve, so finding the area, so is there any other means **no** no there is no that you only this way. So that is the way this has to be understood (FL).

So (FL) there is no other means, so this is the only mean to obtain the exact area of a circle. So (FL) what should he say, so if there are different path you reach the place then you can say choose only this path, there is only one path then there is no point in saying that you have to go only with path. So then he stars he ask this question no no this does not seem to be appropriate (FL) that also means.

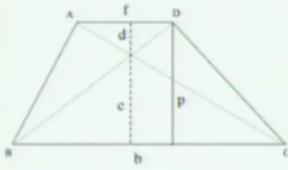
So what is (FL) so somewhere it stated so (FL) radius, (FL) is square , so (FL) r square multiplied by 3. So which means the value of pi is approximately getting to be 3 and for various practical purposes people having employing this. So in order to see that those things are not taken to be the right formula to be employed for finding the area of a triangle. So this (FL) Aryabhata has given a certain guide line that this is the (FL) to obtain the exact area.

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Area of a trapezium

► The latter half of the following verse in *Āryabhaṭīya* gives the formula for obtaining the area of a trapezium.

आयामगणे पार्श्वं तदोन्नहते स्वपातरेखे ते ।
विस्तरयोगार्धगुणे ज्ञेयं क्षेत्रफलं आयामे ॥



In this verse, as per the comm. of Bhāskara, the word *āyāma* means 'breadth'. The terms *pārśva* and *visṭara* refer to the 'length' (base and face) of the trapezium.

► The formulae presented here are:

$$c = \frac{fp}{f+b}, \quad d = \frac{bp}{f+b}, \quad \text{Area} = \frac{1}{2}(f+b) \times p.$$

ॐ: एकं पार्श्वं, मुखं इतरम् । आयामगणे भुवदने इत्यर्थः ।

So (FL) where you do not require that kind of accuracy. So next we move on to the area of trapezium (FL) see in fact the latter half of the word actually presents the area of a trapezium and the earlier half of the verse has been used to define something else. (FL) is a certain geometrical figure, so that is the sense which has been used. This (FL) is the area of this khetra.

(FL) here is used to refer to the base and the face ok. In this figure BC is base and AD is the face, (FL) is multiplying, so (FL) what is the other term (FL) ok so (FL) is used to refer to this perpendicular distance the distance of separation between the base and the face okay. So the area of trapezium is halftime (FL). So this is the formula which is used, so what is the earlier part of the verse.

So here he says (FL) distance I said, so which is mark as P here (FL) is multiplied by the (FL) so (FL) is also basically the site ok. So (FL) so this BC and AD are also referred to as 2 part, so (FL) so this F*p and d*p, so (FL) yoga is again (FL) refers to the some of the base and

face that 2 part was (FL) refers to c and b, so it consider 2 diagonals drawn. So then this is the formula that has been presented.

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The slide is titled "Generalized approach to finding area". It contains the following text:

सर्वेषां क्षेत्राणां प्रमाथ्य पार्श्वे फलं तदभ्यासः ।
In the case of all plane figures, having determined the two sides, the area is [obtained by finding] their product.

► Here Bhāskara points out that the purpose of the verse is: to present a certain alternative approach by which we can verify the areas of all planar figures—trilateral, quadrilateral and circle—for which Aryabhata has already given formulae.

► How is it possible to get the area of any planar figure?

प्रमाथ्य पार्श्वे । 'प्र'-शब्दः प्रकृष्टवाचो, प्रकर्षण पार्श्वे साधयित्वा इति ।
कठ तयोः पार्श्वयोः प्रकर्षः ? उच्यते -
Having obtained the two sides. The word 'pra' expresses some speciality; Thus it means finding the the two sides specially (prakarsena). What is the speciality of the two sides? It is said -

In fact the next was a very interesting verse wherein a analysis has been done and these analysis has been done by Bhaskara which is quite verifying when we look at his commentary. He says (FL) generalise approach to finding area. So initially started triangle anymore on trapezium, when it discussion about circle. So when he says either a way general way of understanding how we compute area.

When we see some kind of a similarity or can we do something so there can be general formula prescription can be given and you can fit in different cases into this by and considered as a special case then only thing that is kind of analysis which has been done by Bhaskara while coming up on this verse. So (FL) geometrical figure (FL) geometrical figure (FL) so you have to just find the side of appropriate value of side take a product of this.

(FL) so area is product of two things, so with this in mind this (FL) has to be different appropriately for various objects. So that is the import of this is what is conveyed by Bhaskara. So (FL) is area (FL) is the product, after all having given expression independently for all this. So why is Aryabhata give one more word, so this seem redundant. So do not need, so this is the kind of question that he rises.

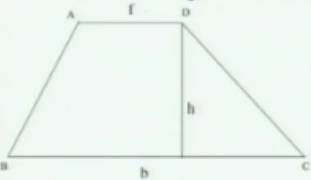
And then he says, so this can be considered as a different way of arriving at the same result, sort of cross check one with the other ok. So that is the purpose of this, and if the

commentary goes like this (FL) basically computing, so (FL) so this (FL) depending upon the case which you are parentally handling depending on the circle, trapezium, this whatever, so you have to appropriately find the size.

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Generalized approach to finding area
 In the case of triangle, rectangle and trapezium

- ▶ *Bhāskara* in his commentary suggested that the rule given by Āryabhaṭa is a **generalized rule** that can be applied for finding area of **any planar figure**.
- ▶ In the case of a triangle or rectangle or trapezium, its area can be found using the formula:



$$\begin{aligned} \text{phala} &= \text{vistarārdha} \times \text{āyāma} \\ \text{Area} &= \frac{1}{2}(\text{base} + \text{face}) \times \text{height} \\ &= \frac{1}{2}(b + f) \times p \end{aligned}$$


- ▶ In the case of a triangle (equilateral, isosceles or scalene) we get the area by putting $f = 0$.
- ▶ In the case of a rectangle, we obtain the area by putting $f = b$.
- ▶ How do we get the area of a circle?

(FL) in fact he goes on, let us consider this figure, so in this (FL) ok (FL) is basically can be defined as product of 3 things, so what is that, (FL) half of this. So in the case of the triangle so what happens to the face so I just collapses to 0 right, so (FL) 0 in the case of rectangle, so (FL) same as b ok and how do we go about for circle.

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Generalized approach to finding area
 The case of a circle

- ▶ In the case of a circle it is said:



वृत्तक्षेत्रे विष्कम्भार्धः विस्तारः परिध्यर्धं आयामः, तेदेव आयतचतुरश्रक्षेत्रम्।
 In the case of a circle, the **semi-diameter is the length**, [and] the **semi-circumference is the breadth or height**, that constitutes the rectangle (āyatacaturśraḥṣeṭram)

$$\begin{aligned} \text{Area} &= \text{viśkambhārdha} \times \text{paridhyardha} \\ &= \text{semi-diameter} \times \text{semi-circumference} \\ &= r \times \pi r \end{aligned}$$

- ▶ In his commentary *Bhāskara* also takes up more complex cases like drum, tusk of an elephant, etc. and demonstrates how to find the area of those shapes as well.

He says (FL) is the kind of distance separation between the faces, so in the case of a circle what you have to conceive is (FL) is to be taken as (FL) so this is the kind of (FL) visually fit

to be a rectangle even this circle can be conceived of (FL) this was has been explained by Bhaskara.

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Purpose of the verse: Computation or verification?

अथ कथमेकेनैव यत्नेन फलानयनं प्रत्ययकरणं च प्रसाध्यते? अथेदं प्रत्ययकरणार्थं प्रकृतम्, स कथं फलानयनाय भवति? अथ फलानयनार्थं, कथं प्रत्ययकरणाय? नैष दोषः। अन्यार्थं प्रकृतम् अन्यार्थसाधकं दृष्टम्। तद्वथा - शाल्यार्थं कुल्याः प्रणीयन्ते। ताभ्यश्च पानीयं पीयते, उपस्मृश्यते च।³ एवमिहापि।

But how does the computation of the area and the verification happen in one stroke? If this rule is meant for verification, then how does it help in computing the area? And [conversely], if this rule is meant for computation of area, then how can it be used for verification? There is nothing wrong with that. What has been designed for one purpose, is found to be serving another purpose. That is as in the following case - *Canals are constructed for the sake of rice paddies. And from these canals water is drunk and used for cleansing.* It is similar in this case too.

³अष्टाध्यायी, १-१-२२, पातञ्जलभाष्यम्

In fact further there is a very interesting discussion, so I want to just code this a certain maximus which has been employed by Bhaskara to justify the presence of this verse here ok. So (FL) in Tamil they say if you throw one stone you get 2 mangoes, oru kalula rendu manga, so something like this, so (FL) in one effort.

(FL) (FL) finding the area and prathikaram is verification ok (FL) for a particular purpose in mind, but it can also serve certain other purposes. So this quotation is quite interesting (FL) means water canal ok (FL) so it is a is basically form of rice, ok (FL) water canals are created to water the fields (FL) so incidentally somebody can go and use the water for some other purpose so nothing is lost.

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Chord of one-sixth of the circumference of a circle

परिधिः षड्भागज्या विक्रमार्धेन सा तुल्या।
The chord of one-sixth of the circumference [of a circle] is equal to the semi-diameter.

It is evident from the figure

$BC = \text{chord } 60^\circ = R,$

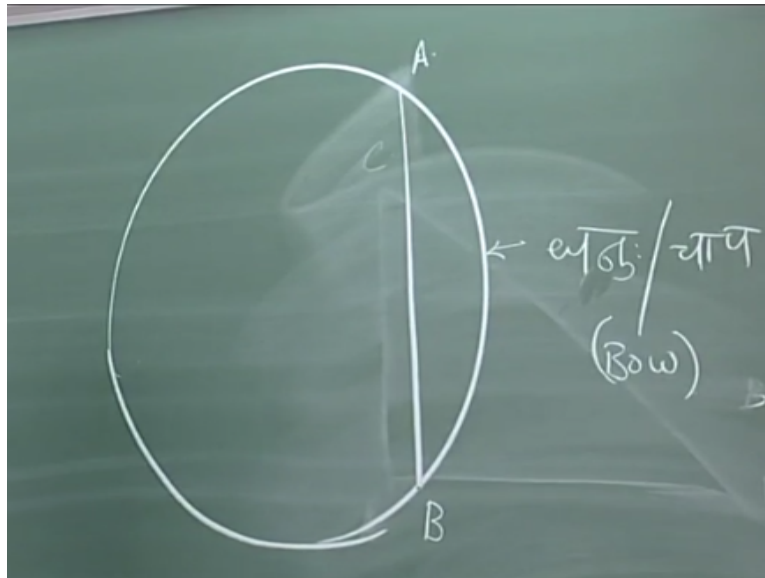
since OBC is an equilateral triangle.
 This is what is given in the verse.

► It is pointed out by Bhāskara that the purpose of the verse will become clear from the later verse '*samavṛtta-paridhipādaṃ*' :

प्रयोजनं चास्य षड्भागज्याप्रदर्शनस्य 'समवृत्त-परिधियादं छिन्दात्' इत्यस्यां कारिकायां वक्ष्यति।

So it is in this sense so we can understand that this sloka, so which has been composed by Aryabhata to cancel several purposes. So now I move on to get another interesting topic which actually forms the basis for one of the construction methods which has been described by Aryabhata for finding the time table. So this half verse (FL) this is the statement (FL) is circumference (FL) half of the circumference.

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(FL) refers to the god ok, so in fact (FL) suppose you consider circle AB is a part, so which is referred to as (FL). So this portion of the circle, so this is referred to as (FL) which actually means bow, so this looks like arch looks like bow and any arc length is referred to as (FL) or any synonym for that. So this card A B is refer to as (FL) various terms which have been used. So this is out of conceived as bow and this is considered as string and they all mean string.

So what is stated in this work is (FL) so one sixth ok so if you conceive a circle so divided into 6 equal parts, so the card corresponding to one sixth of the circumference is referred to as a (FL) and he just states (FL) so this is same radius. So this is pretty evident so if you think of this triangle OBC, so which is an equilateral triangle, so we have this cards which is same as the radius fine. So card of 60 degree is same as the radius.

So what is the purpose of stating this, in fact he says (FL) state, so in fact Aryabhata is going to make use of this in yet another wherein he is going to present the sin table. So this is what (FL) the purpose of designing this can be understood with the verse beginning with (FL) so will come to that in a minute.

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Approximate value of π

- ▶ The *Śulba-sūtra*-s, give the value of π close to 3.088.
- ▶ Āryabhaṭa (499 AD) gives an approximation which is correct to four decimal places.

चतुरधिकं शतमष्टगुणं द्वापष्टिस्तथा सहस्राणाम् ।
अयुतद्वयविष्कम्भस्य 'आसन्नो' वृत्तपरिणाहः ॥

$$\pi = \frac{(100 + 4) \times 8 + 62000}{20000} = \frac{62832}{20000} = 3.1416$$

- ▶ The same value has been given in *Līlāvati*⁴ by removing a factor of 8 from the denominator and the numerator.

व्यासे भनन्दाग्निहते विभक्ते खर्वाणिसूर्येः परिधिः सुसूक्ष्मः ।
द्वाविंशतिप्रे विद्वतेऽथ शैलेः स्थूलोऽथवा स्याद् व्यवहारयोग्यः ॥

$$\pi = \frac{3927}{1250} = 3.1416 \quad \text{that's same as Āryabhaṭa's value.}$$

⁴*Līlāvati* of Bhāskarācārya, verse 199.

But before going to be procedure which has been delineated by Aryabhata for finding this time table. So we will discuss this very important verse which has been quoted in various forums in various ways and it is also been misinterpreted in various places. So (FL) see the first half of the verse basically presence the value 52832. So how does it go about (FL) so 4+100 (FL) multiply by 8 and (FL) is 1000 52000.

So this refers to this number and what is stated in the later half of the verse is (FL) so what are these numbers correspond to, so this is the number which has been employed in order to obtain something. So (FL) is 10000 (FL) is 20000. So if the (FL) diameter happens to be 20000 then this number which was stated happens to be the circumference. So what is stated is the ration of a circumference to be diameter.

The use of the word (FL) has been discussed in great length, so why did Aryabhata use the word (FL) you should simply said (FL) is something which is nearby close by which can be understood to be approximate. So we will see how this has been analysed, but before that I also wanted to give this verse as an example of the use of (FL) and also to say that this same value has been given by Bhaskara in a slightly different way.

So (FL) so it represent 27, (FL) are famous, so it refers to 9, agni is 3, so 3, 9, 27, so this is what it is, so (FL) divided, so divided by (FL) sum is 0, (FL) is 5, ok, the word (FL) has been employed to refer to number 5. So why is (FL) very good, so the names you have got it, but then the significant is the following. So (FL) arrow all the sense will be simultaneously attack, he will lose yourself completely.

So that is why he called (FL) ok, so (FL) this is fairly, this is (FL) in the sense there are various approximation which has been given, this is a far better approximations that is what we need to understand.

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Approximate value of π

Bhaskara's edifying discussion while analysing the meaning of the term 'āsanna'

आसन्नः निकटः। कस्यासन्नः? सूक्ष्मस्य परिणाहस्य। कथं विज्ञायते
सूक्ष्मस्य आसन्न इति, न पुनः व्यावहारिकस्य आसन्नः, यावता
अश्रुतपरिकल्पना⁵ सूक्ष्मव्यावहारिकयोः तुल्या। नैप दोषः, सन्देहमात्रमिदम्।
सर्वसन्देहेषु वा इदमवतिष्ठते - व्याख्यानतो विशेषप्रतिपत्तिः (न हि
सन्देहादलक्षणम्)। - इति। तस्मात् सूक्ष्मस्य आसन्नः इति व्याख्यास्यामः।

[The word] *āsanna* means close to or nearby. It is close to what? To the 'accurate' circumference. How does one understand that it is close to accurate value and not the one that is practically used (*vyāvahārikasya*), since the supposition of the unstated is equally valid in both the cases. There is nothing wrong [in the interpretation]. This is only a [valid] doubt. [But it must be remembered] that in all instances of doubts, the following principle gets invoked – **Clear understanding arises out of [traditional] explanation (just because there is a doubt ...)** Therefore we would interpret as ...

⁵In the printed edition, the text reads as श्रुतपरिकल्पना and it seems to be an editorial error while splitting सर्वर्णदीर्घसन्धिः।

So now I present the discussion which has been presented by Bhaskara to you on the use of the word asana, (FL) so close to what, so there are 2, 3 things which are used one is (FL) this one way of saying, so there is a (FL) which is very (FL) very accurate. So this is close to the very accurate value. This is one way of saying, but then he says (FL) is taken as 3, so this can be close to this or close to that also.

So (FL) is something which is common to both (FL) in fact the edition it was a problem so the (FL) is what is found in edition. So I was just breaking my head so how this (FL) something with you try to create a in your own mind (FL) is something which is your imagination. So (FL) is does not make much sense, when something is hurt then you have to imagine. So that is why we should be.

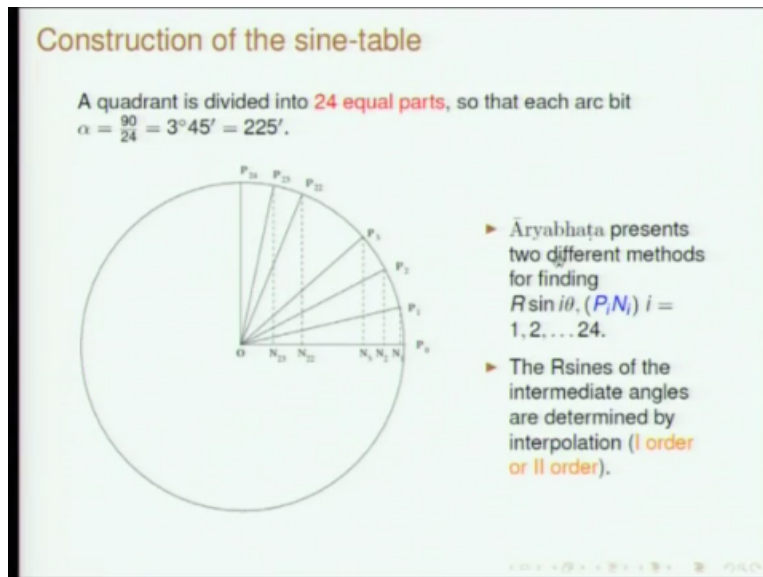
So this (FL) is just I believe because of this (FL) so this is split it as (FL) that is my guess. So (FL) then he says (FL) it is true that your objection is valid then he says (FL) general maths in which is very important to understand, so (FL) this will be coded very frequently in this study of philosophy (FL) all that. So he says (FL) whenever there is a sort of doubt which arises then you have to resort to the (FL).

So how it has been analyse, how it has been understood, so (FL) means so a certain explanation which has been upward in the tradition, so from that we understand, we understand certain special meaning (FL) therefore we say (FL) edition says that this is close to the accurate value and therefore we will say (FL) means only this in this context. (FL) he says (FL) let not take the word (FL) could be something which is close by.

(FL) residing close by, (FL) this the different way of look at that, is it if you let us not debate on that. So if you say (FL) already a gross value (FL) is closer to that, so which means (FL) we will get a much gross value for the circumference (FL) is going to make some effort in specifying some values saying that this si going to be much loser and therefore use this. So this does not make much sense.

And therefore why purely logical reasoning so we can arrive at the computer (FL) ahs to be understood only as something which is very close to the exact value. Then he asked question (FL) so this is the similar question which Nelakanta also ask and then beautifully explain what means by irrational and that which I think professor Srinivas will discuss the quotation later. So here Bhaskara says (FL) by which you will be able to exactly state the perimeter.

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So here is out of conclude the discussion on the use of the word of asana the method which has been described by Aryabhata to find out the tabular or science. So generally in almost every astronomical work on mathematical work we will see that a certain method has been described in which typically they will divide a quadrant into 24 parts. So as shown in the diagram, so you take the circle, if you take this quadrant (FL) p24 and divide this into 24 equal parts and the points are marked there P1 P2 P3 on so on.

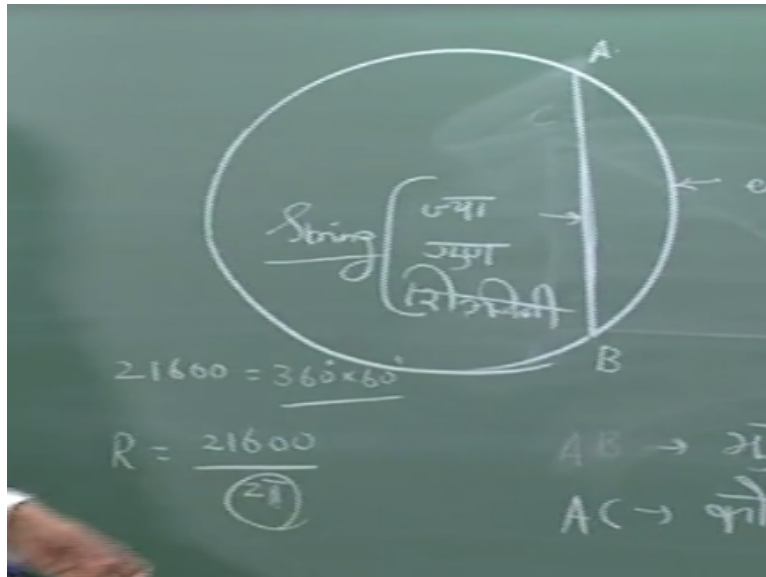
So what is it that we are interested in. So we are interested in the card links, so which is basically fine (FL) so the projections which have been shown here P1 and P2 are basically the science. We consider this triangle On2P2, so P2n2 is basically sin if you consider this as angle P2ON2 is angle and the sign is PQ. So these are refer to as (FL) and we denote them as $R \sin$ of i times theta. So theta is $90/24$, so which is 3 degree and 45 minutes to 25 minutes.

Here so the purpose is to determine all the $P_i N_i$, ok so this 24, the last value obviously is going to be OP_{24} which is same as the radius of the circle ok and once these values are known $P_i N_i$ which are referred to as in fact this could be precise they should be refer to as (FL) total card length ok, as I showed here so this is (FL) half of it which is what is sin, but for the process of convenient, so people have started using jaw itself to refer to (FL).

So this understood from the context ok, if we know $R \sin$ values corresponding to the multiples of theta so firmo 0-90 degree, then for any intermediate value so we use the position and usually this first order interpellation employed and if more precise values are required to

later second order interpolation formula has also been stated I think maybe by Sriram or myself will do it later.

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So what is the verse, so in fact I refers to eelieer (FL) so this specification of radius as equal to the value of $\cot 60$ was in connection with this work. So (FL) so this si the geometrical approach to construction of the sin table given by Aryabhata. (FL) is the circumference ok, so the circumference of the (FL) circle, (FL) refer to one fourth. So (FL) refer to one quadrant of a circle (FL) means may you divide may you split.

(FL) so if you conceive it as triangle and rectangle (FL) ok, so (FL) without the detailed commentary given by Bhaskara. So this verse as follows the next verse which has been presented by Aryabhata which is analytic approach is going to be almost impossible for us to understand. So the value of $\cos 60$ was said to be R and this value is basically 3438, so this is pretty evident. So because usually what people do is in almost all the astronomical there.

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Finding tabular sines: Geometrical approach

समवृत्तपरिधिपदं छिन्द्यात् त्रिभुजाद्यतर्भुजाद्यैव ।
 समचापज्यार्थानि तु विक्रममर्थं यथेष्टानि ॥
May the quadrant of the circumference of a circle be divided (into as many parts as desired). Then, from (right) triangles and rectangles, one can find as many Rsines of equal arcs as one likes, for any given radius.

- ▶ We know, chord $60^\circ = R = 3438$
- ▶ From the triangle OBC,

$$R \sin 30 = BC = R/2 = 1719$$
- ▶ Now consider the rectangle OABC. Here, $AB = R \sin 60^\circ = OC$.

$$OC = \sqrt{OB^2 - BC^2} = \sqrt{R^2 - \frac{R^2}{2}} = 2978$$
- ▶ Thus it may be noted that from 8th Rsine (30°) we immediately get 16th Rsine (60°)
- ▶ In other words, $R \sin \theta \leftrightarrow R \sin(90 - \theta)$.

But for some so this circumference is taken to be 21600 which is basically 360x60, so 360 degrees*60 minutes, so the number of minutes in the circumference or rather the number of units in a circle is taken to be 21000 etc. Now that Aryabhata in his verse (FL) he has given the ration of the circumference the diameter which is essentially 5. Once you know that then you know what radius is. Radius is 21600/2pi.

So this was sort of stated by Aryabhata and this will be very close to 3438, this value, in fact Bhaskara so gives all this value, so while explaining in this verse in detail. So how do we proceed, so by geometrical construction, so all that require is so this value initially you know. So you have to just take with 3434 it is known, so since you 21600, the value of 5 is 1 and therefore the radius is known.

Ok we will start with them, then if you look at carefully so since this was stated to be radius consider this triangle OBC, so this is radius, so BC this angle is 30, and therefore the radius is known BC is known, so sin 30 degree is also known, so sin 30 degree is basically the 8 sin, if you conceive it to the 24 part, so 30 degree is going to be 8, now you know since you know the radius, you also know sin 30 ok r/2.

So this is basically r/2, this is basically r/BC, r/2 and therefore this is also known, no you consider this, OABC, so this is the kind of (FL) which can think of, so sin 60 is AB and is also equal to OC, but you know r you know r/2, so you do r square-r square 2, so you basically get this value OC, so you know sin 90, you know sin30, you know sin60, so what is to be understood in general is if you know sin theta.

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Finding tabular sines: Geometrical approach (contd.)

► In the triangle CBD, $BC = R \sin 30^\circ$ and $CD = OD - OC = R \text{ vers} 30^\circ$ are known. Hence, $BD = \text{chord } 30^\circ$

$$BD = \sqrt{(R \sin 30^\circ)^2 + (R \text{ vers} 30^\circ)^2}$$

is known. $R \sin 15^\circ = \frac{1}{2} BD = 890$.

► At this stage, we need to note that

$$R \sin \theta \rightsquigarrow R \cos \theta \rightsquigarrow R \text{ vers } \theta$$

$$R \sin \theta \text{ \& } R \text{ vers } \theta \rightsquigarrow R \sin \frac{\theta}{2}$$

► Now considering the triangle ODE,

$$OE = \sqrt{OD^2 - DE^2}$$

$$= \sqrt{R^2 - (R \sin 15^\circ)^2}$$

gives $R \sin 75^\circ$.

Then obviously you can get $\sin 90 - \theta$ ok, so this is known, ok then since 30 is known so you also know $\text{vers } 30$, so vers is basically so $r - \cos \theta$, so $\sin 90 - \theta$ is basically $\cos \theta$, so $r - \cos \theta$ in this diagram so OC is basically $\cos \theta$ (FL) and CD is the (FL) ok, so OD is r , so OC is $r \cos \theta$, so $r - \cos \theta$ is basically CD , so this is also known. So once it is known so you have $\sin 30$ and $\text{vers } 30$.

You know $r \cos \theta$, so $r - \cos \theta$ was 1 and so what 30 is known, we just do this, so BD if you see you know BC , you know CD , so now you can in this triangle BCD you know BD also, so what is BD , BD is basically the chord of 30 degree, we started with chord of 60 degree which is basically radius. So now you have come to chord of 30 degree, so half of it going to be $r \sin 15$ (FL) so you can keep on expanding this kind of an argument.

And you will be able to construct the entire sin table, so the principle is simple so $r \sin \theta$ gives $r \cos \theta$, and that gives (FL) which is $r - \cos \theta$, and from $\sin \theta$ and $\text{vers } \theta$ you will be able to this is what account here, we will be able to get $\theta/2$. So once θ is known $90 - \theta$ is known and $\theta/2$ is also known. So with this we will be able to construct the entire sin table.

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Finding tabular sines: Geometrical approach (contd.)

- ▶ Most of the Indian astronomers have presented their sine tables by dividing the quadrant (90°) into 24 parts.
- ▶ By the principle outlined above, it can be easily shown that all the 24 Rsines can be obtained provided the 24th, 12th and 8th Rsines are known.

- ▶ The circumference of the circle was taken by Aryabhata to be 21600 units.
- ▶ From that using the approximation for π given by him, we get $R = 24th \text{ Rsine} \approx 3438$.
- ▶ Once this is known, it is noteworthy that in the proposed scheme of constructing the table, all that is required is extraction of square root, for which Aryabhata had clearly evolved an efficient algorithm.

I am just going to show this how this seem to be work. So you know 8 sin, so once you know 8 sin it is 30, so it can get the 4 sin and the theta/2 and 90-theta, so 16 sin is known, from 4 we can go to second and on other side we can go to 90-theta is 20th, so you can follow this 3, and this will give you the most of the sin values, and you could start with again 12, so 45 degree, so 45 degree all that going to do is we have to just draw line GD.

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Finding tabular sines: Geometrical approach (contd.)

- ▶ In the triangle CBD, $BC = R \sin 30^\circ$ and $CD = OD - OC = R \text{ vers } 30^\circ$ are known. Hence, $BD = \text{chord } 30^\circ$

$$BD = \sqrt{(R \sin 30^\circ)^2 + (R \text{ vers } 30^\circ)^2}$$

is known. $R \sin 15^\circ = \frac{1}{2} BD = 890$.

- ▶ At this stage, we need to note that

$$R \sin \theta \leftrightarrow R \cos \theta \leftrightarrow R \text{ vers } \theta$$

$$R \sin \theta \ \& \ R \text{ vers } \theta \leftrightarrow R \sin \frac{\theta}{2}$$

- ▶ Now considering the triangle ODE,

$$OE = \sqrt{OD^2 - DE^2}$$

$$= \sqrt{R^2 - (R \sin 15^\circ)^2}$$

gives $R \sin 75^\circ$.

So for 45 degree all that you need to do is you have to draw line DG, see this GD gives you the card of 90, so half of it going to give you, so this GD how do you know, so you know r, so you find r as r root 2 r, so root 2 r is known. So then you know 12 sin, and then you follow the same 3, so 6 sin. So here if you go with 24.

There is not square, so from 6 we get 13 or 18 and so on. So you will see that in all the 24 sin values can be easily obtained by this geometrical approach. Ok, so with this now I stop the discussion on Aryabhattachiya, so we will continue in the next lecture, thank u.

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