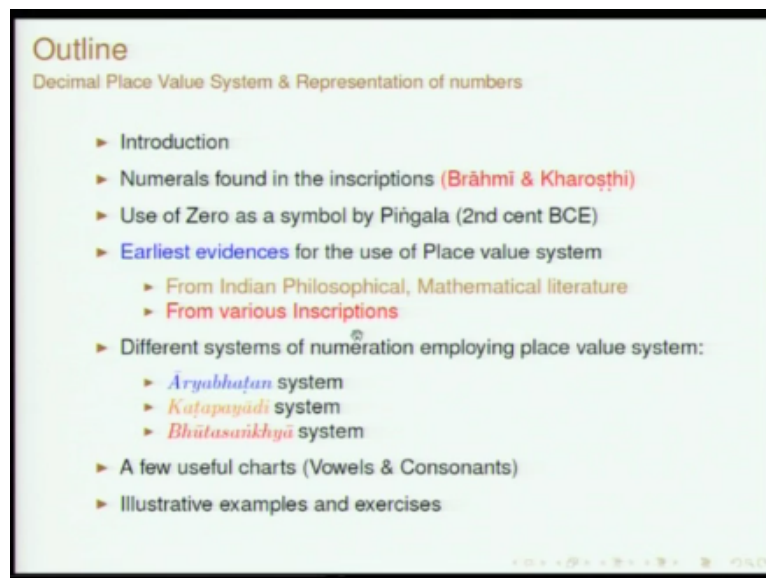


Mathematics in India: From Vedic Period To Modern Times
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Lecture-6
Decimal place value system

When this lecture will be discussing the decimal place value system, so its origin, how it has been used in the Indian mathematical and astronomical text, not necessarily mathematical astronomical. So that discussion about this decimal place value system as technology even philosophical literature. So I will be giving certain citations from the commentary of (FL) also.

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Wherein he draw sinologist from this and then tries to project a certain philosophical view point which he wants to project. a. normally we do not have opportunity to learn, so what is origin of a certain system. So because the history of science is never a part of our educational curriculum, anyway so we will try to find out so this one of the most fundamental discoveries.

And it has been fairly will recognise that Indians where the discoveries of this decimal place value system. The earliest concrete reference so in terms of the representation has to be paste from some of these are inscriptions. So we have this Brahmi and Kharosthi etc. at the earliest evidence can be traced even philosophical literature as I was mentioning after the introduction and the reference circum of philosophical as well as mathematical literature.

I will move on to describe three different representations which have been used by Indian astronomers to refer to numbers, aryabhatan system, katapatadi system and bhutasamkhyā system. So these are 3 different representation which have been employed to represent huge numbers which occur in the astronomical works. So the sequence in which I have ordered them does not necessarily represent the sequence in which these they were used in the society.

But it is for certain different convenience I have used the this ordering here. So Bhutasankhya seems to be the most earlier thing and based on the evidences that are available in literature this seems to be the earliest. So katapayadi when is not very sure as to when it was discovered but certainly (FL) has a employee and (FL) wherein the longitude are represented in some sentences particularly in the occasion of what is called upakarma.

So this is (FL) also rendered as per the various other rendering which is done (FL) and so on. Anyway so these are all different representation which have been employed by people and we will show certain charts, so wherein the vowels and consonants, so how are the numbers assigned to these vowels in different representations and then we will have a few example. So this is the outline of the present talk on decimal place value system.

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Introduction
Origin of numbers and Place value system

- ▶ Generally most of us do not get to know or have opportunities to get to know answers to questions like
 - ▶ When did we start counting?
 - ▶ Were there other systems of counting?,
 - ▶ What are the different ways of representing numbers? etc.,
- ▶ As we keep using decimal system of numeration right from our childhood, we are so familiar with that, that we tend to think that it has been there for ever.
- ▶ It is indeed pretty old. But how old?
- ▶ One of the most ancient literature *Rig-veda* presents the number 3339 using word numeration:
श्रीणि जता श्रीमहम्नाप्यमिं त्रिंशस देवा नव यामपर्यन्।
औक्षन् घृतैरमृणन् वरिहिरम्मा... [Rig-veda 3.9.9]
- ▶ From the above quote it is evident that decimal place value system has been in vogue amongst the Vedic seers.

As I was mentioning so many of us do not force the question when did we start counting, what are the different systems of counting or what are the different ways of representing numbers. So we have one way of representing (FL) script is different but even the way of representation, normally two things are known, that is what I learnt, so one is this the so-

called Arabic notation and rather is Roman notation. So these are the 2 things which are generally thought.

So we are so familiar that we do not even ask this question that whether they could be other representation and when did this way of using notation started, obviously we will be initially using words to represent number. So after all notation comes much later whatever be the discipline. So initially the language, so we have the word to represent a particular number 2,3, 10, 100 and so on.

And then comes in the notation, so we will briefly see how did this particular way of representing started in the course of our lecture, so as I am mentioning obviously pretty old, but how old so the as I was coaching yesterday, so this (FL) so where the number 3, 3, 9, as mean represented in word numerous ok. So we have word (FL) 300 and so on.

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Introduction
Ingenuity of the advent of Place value system & Zero

► Laplace¹ while describing the contribution of Indians to mathematics observes:

The ingenious method of expressing every possible number using a set of ten symbols (each symbol having a place value and an absolute value) emerged in India. The idea seems so simple nowadays that its significance and profound importance is no longer appreciated. Its simplicity lies in the way it facilitated calculation and placed arithmetic foremost amongst useful inventions. The importance of this invention is more readily appreciated when one considers that it was beyond the two greatest men of Antiquity, Archimedes and Appolonius.

¹A renowned French Scientist of the 18th-19th century who made phenomenal contributions to the fields of mathematics and astronomy

Talking about the ingenuity of the decimal place value system and mention of 0, so one of the famous scientist of recent pass the French mathematician Laplace so observes the following. The ingenious method of expressing every possible number using a set of ten symbols each symbol having a place value and an absolute value ok the place value and absolute value emerged in India.

The idea seem so simples nowadays that its significance and profound important is no longer appreciated. Its simplest lies in the way it facilitated Calculation and placed arithmetic foremost amongst useful inventions. The importance of this invention is more readily

appreciated when considered it was beyond the 2 greatest men of Antiquity Archimedes and Apollonius. So I will not describe the implication of the last sentence.

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Origin of numerals: Which hypothesis to choose?

- ▶ Several hypotheses have been put forth on the origin of the so-called modern/(Arabic?) numerals.
- ▶ Commenting on it the French G. Beaujouan observes:²
the origin of the so-called 'Arabic' numerals have been written about so often that **every view on the question seems plausible**, and the only way of choosing between them is by **personal conviction**
- ▶ Generally **origins are traced through citations**. Though citations are testimonies, it is quite possible that it could be marred due to **gaps in perceptions** and **unavoidable distortions** of human memory.
- ▶ Hence, collating several evidences becomes essential.

²cited by George Ifrah in *The Universal History of Numbers II*, p. 1., Penguin Books India, 2005.

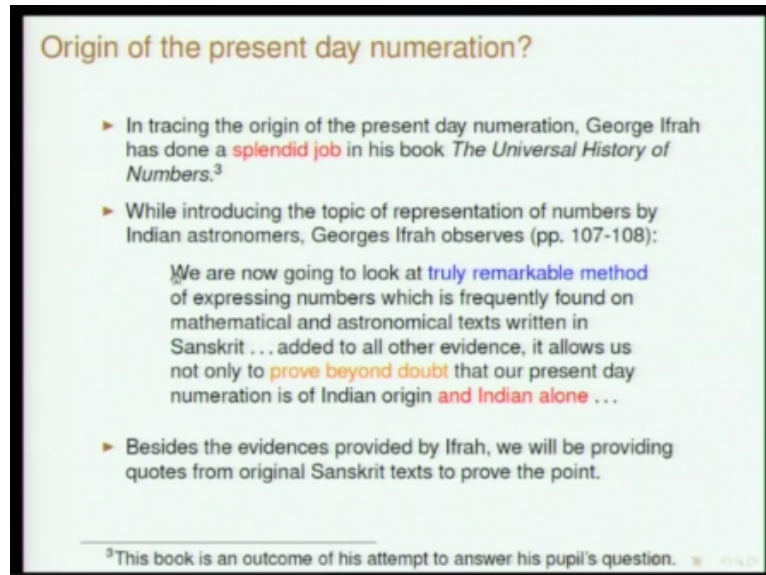
But anyway there is not much time but he is for you to founded over. So when historian try to trace back so what would be the origin and so on and so forth, obviously several hypothesis will be put forth. Many of the hypothesis so would seem to be true, so that has been very nicely heard, the origin of the so called Arabic numerals have been written about so often that every view on the question seems possible and the only way of choosing between them is by personal conviction.

So why does (()) (07:25) us so we go by certain citations and testimonies and over a period of time took loss of memory due to one soon way of understating what we convey, so it quite possible that there are gaps in process and this caution to take place and it is essential that 1 police several evidences to come to a certain conclusion and in this regard to the interesting book. So which I would recommend all of you to go through.

And this book is by Georges Ifrah. So the book title Universal History of Numbers, so I would take a couple of minutes to just to say how this book are originated. So apparently Georges Ifrah is a school teacher and he was teaching numbers, so to young kids, so one of the students asked so where did these numbers come from. So how do we understand where it got originated and the present day we of the presentation this is the only representation.

And so on and so forth, so Ifrah could not answer the question raised by the student, it is a certain conviction in his own mind and therefore he decided that the next few years tracing to find out evidences to answer this question. So he resigned the job apparently and then it is stated that this man so went around from one place to other place and he was even lying on road side not having enough resources to stay in.

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Origin of the present day numeration?

- ▶ In tracing the origin of the present day numeration, George Ifrah has done a **splendid job** in his book *The Universal History of Numbers*.³
- ▶ While introducing the topic of representation of numbers by Indian astronomers, Georges Ifrah observes (pp. 107-108):
We are now going to look at truly remarkable method of expressing numbers which is frequently found on mathematical and astronomical texts written in Sanskrit . . . added to all other evidence, it allows us not only to **prove beyond doubt that our present day numeration is of Indian origin **and Indian alone** . . .**
- ▶ Besides the evidences provided by Ifrah, we will be providing quotes from original Sanskrit texts to prove the point.

³This book is an outcome of his attempt to answer his pupil's question.

And he that narrates the seriousness with which you have gone ahead and written this book that is what has been stated, after doing research for decades, so he finally says with reference to the origin we are now going to look at it is truly remarkable method of expressing numbers which is frequently found on mathematical and astronomical texts written in Sanskrit, added to all other evidence.

It shows us not only to prove beyond doubt that our present numeration is of Indian Origin and Indian alone. So this is following on to convey through this, so beyond what Ifrah has said so Ifrah obviously is not so conversion with some the literature we will also go 8 o'clock evidence of literature, we just give a flavour as to how different interpretations can be given for it origin. I just have it photographs here.

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Various hypotheses on the origin of 'Arabic' Numerals

- ▶ The various hypotheses that have been put forward regarding the origin of the so-called Arabic numerals include:
 - ▶ the number of angles each numeral possesses
 - ▶ the number of lines the numeral possesses
 - ▶ the number of points, and so on.

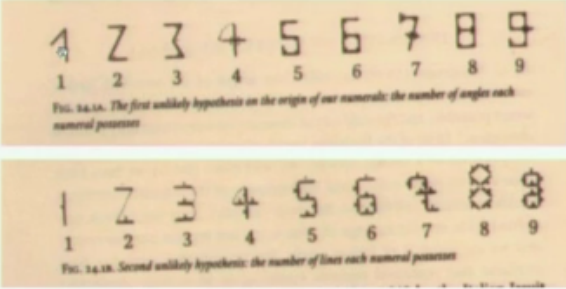


FIG. 14.16. The first widely held hypothesis on the origin of our numerals: the number of angles each numeral possesses

FIG. 14.18. Second widely held hypothesis: the number of lines each numeral possesses

So it says see the numbers, the number of angles see if you look at 1, so only one angle, it say 2 so only 2 angles, 3 there are 3 angles, 4 and so on. So this is one, rather present says so it has nothing to do angles it has to do the lines, so if you look at this, so only one line there are 2 lines he says, there are 3 lines 1, 2, 3, 4 lines 5 lines as was the different people can obviously come up with different hypothesis.

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'Arabic' Numerals & its spread in Western Asia

- ▶ Georges Ifrah, presenting several such hypotheses, finally dismisses all of them as merely fanciful explanations.
- ▶ He authentically quotes from dozens of works authored by Arabo-Muslim scholars as well as European scholars – spanning over almost a millenia (9–19th cent.) – to prove the point that discovery of decimal place value system was made by Indians.
- ▶ The Syrian Monophysite Bishop Serus Sebokht in 662 CE⁴ notes:

“the science of the Indians,” including “their subtle discoveries in astronomy, discoveries that are more ingenious than those of the Greeks and the Babylonians, and of their valuable methods of calculation which surpass description,” which he described as “done by means of nine signs” presumably ignoring th zero.

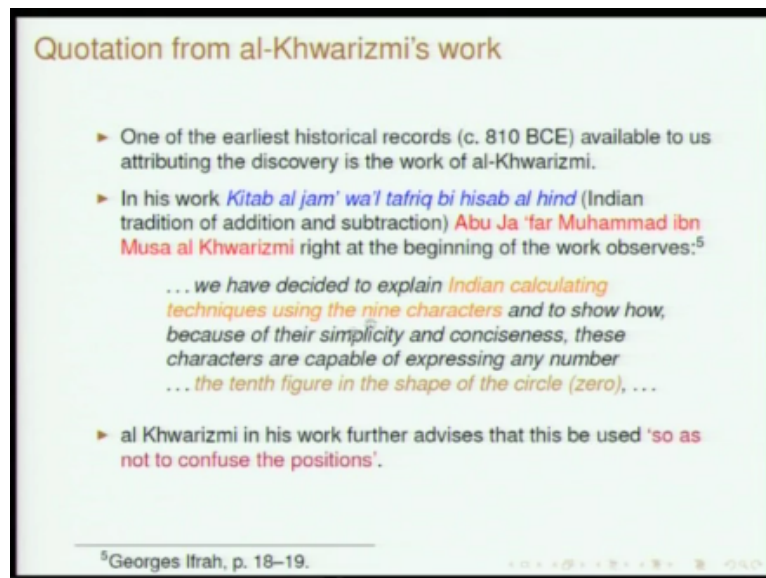
⁴Datta and Singh pp. 95-6, Kim Plofker p. 255.

It is just for the fake, just I want to show this slide there is nothing more to read here. So finally Ifrah says that most of the presentation that find that all scientific location, and therefore we need to trace back and not to rely on these types of information. So having coated from dozens of works so both from Islamic astronomer as well as European scholars, see finally come to the conclusion as they was seeing before that is (FL) origin.

This coat from the Bishop in 7 century also founds certain evidence that the science of the Indians including the subtle discoveries in astronomy, discoveries that are more ingenious than those of Greeks and the Babylonians and of their valuable methods of calculation which surpass description, I mean he has been in high price of what has been done here which he describe as done by means of 9 science.

In fact this facility is a lot, so that is what we need to understand from this quotation, so the kind of facilitation that it offers in the present in this as we will see from the astronomical work. So if one where to make a simple comparison I mean the way this Roman script, so one has to write at the addition subtraction the kind of dislocation multiplication all the offer that is something which is really amazing.

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Quotation from al-Khwarizmi's work

- ▶ One of the earliest historical records (c. 810 BCE) available to us attributing the discovery is the work of al-Khwarizmi.
- ▶ In his work *Kitab al jam' wa'l tafriq bi hisab al hind* (Indian tradition of addition and subtraction) **Abu Ja 'far Muhammad ibn Musa al Khwarizmi** right at the beginning of the work observes:⁵
 - ... we have decided to explain *Indian calculating techniques using the nine characters* and to show how, because of their simplicity and conciseness, these characters are capable of expressing any number
 - ... *the tenth figure in the shape of the circle (zero)*, ...
- ▶ al Khwarizmi in his work further advises that this be used 'so as not to confuse the positions'.

⁵Georges Ifrah, p. 18–19.

And that is why even the class 5 please with the reference to the advent of this decimal place value system. This (FL) also in his work he says we have decided to explain Indian calculating techniques using the 9 characters and to show how because our simplicity and conciseness capable, so these characters are capable of expressing any number. So be there all in high place of this decimal place value system.

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Numerals found in the Inscriptions (Brāhmi)

► Some of the **rock edicts of Asoka** (Mauryan dynasty) dating back to 3rd cent. BCE contain numerals in Brahmi script.

1	2	3	4	5	6	7	8	9	0	Ref.
I	II	+	E	E	E	E	E	E	E	EI, III p. 134 IA, VI, pp. 155 ff. IA, X, pp. 106 ff. Indrājī, JBRAS XII
			E	E	E	E	E	E	E	

Date: third century BCE.
Source: edicts of Asoka written in Brahmi, in various regions of the Empire of the Mauryas, from the regions of Shahbazgarhi, Manshera, Kalsi, Girnar and Sopara (north of Bombay) to Tosali and Jaugada in Kalinga (Orissa), Yerragudi in Karnataka, Rampurwa and Lauriya-Araraj in the north of Bihar, Toprah and Mirath north of Delhi, and Kumbhinder and Nigihva in Nepal (Fig. 24.27).

FIG. 24.29. Numerals of the original Brahmi style of writing; our present-day 0 is already recognizable

So think of multiplication which is representing so in Roman and if you had represent something like particular 20000 or something which is 1000 that, so it is all extremely confused ok. So the evidence in the form of inscriptions is the 3rd Century BC, you have something like this is the kind of representation that finds in the rock edicts of Ashoka.

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Numerals found in Buddhist Inscriptions (2nd cent. BCE)

► Buddhist inscriptions on the walls of the **grottoes of Nana Ghat** contain numerical symbols in Brahmi script.

1	2	3	4	5	6	7	8	9	0	Ref.
~	=	F	Y	7	7	7	7	7	7	Datta and Singh Indrājī, JBRAS XII Smith and Karpinski
~	=	F	Y	7	7	7	7	7	7	

Date: second century BCE.
Source: the caves of Nana Ghat (central India, Maharashtra, c. 150 km from Poona). Buddhist inscriptions written for a sovereign named Vudhisi which mainly concern various presents offered during religious ceremonies.

FIG. 24.30. Numerals of the intermediary notation of the Shunga; we can already see the prefiguration of our numerals 4, 6, 7 and 9

So as I then proceed further there are Buddhist inscription so about 3, 4, so second century BC so we have this representation.

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Evolution of Numerals: Brahmi → Modern

Brahmi	↓	—	=	≡	+	μ	④	7	5	7
Hindu	↓	०	१	२	३	४	५	६	७	८
Arabic	↓	۰	۱	۲	۳	۴	۵	۶	۷	۸
Medieval	↓	०	1	2	3	4	5	6	7	8
Modern		0	1	2	3	4	5	6	7	8

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- ▶ It has taken **more than 18 centuries** (3rd BCE – 15th CE) for the numerical notation to acquire the present form.
- ▶ The present form seems to have got adopted 'permanently' with the advent of printing press in Europe. However, there are as many as **15 different scripts used in India** even today (Nagari, Bengali, Tamil (Grantha), Punjabi, Malayalam, etc.).

And we proceed further and in short so it seems to have started something like this and today view this. But almost taken about 18 centuries for the numeral notation which we use today. So starting from brahmi to modern. So the modern is estimated around 15 century, 16 century when the printing started. So up we have different scripts in different languages that is a different thing but the place value thing is a different things.

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Use of zero as a symbol by Pingala (2nd cent CE)

- ▶ Pāṇini's *Aṣṭādhyāyī* (c. 500 BCE) has the notion of *lopa* which functions as a null-morpheme.
अदर्शनं लोपः। (1.1.60).
- ▶ *Śūnya* appears as a symbol in Piṅgala's *Chandaḥ-sūtra* (c. 300 BCE). In Chapter VIII, while enunciating an algorithm for evaluating any positive integral power of 2 in terms of an optional number of squaring and multiplication (duplication) operations, *śūnya* is used as a marker.
रूपे शून्यम्। द्विः शून्ये। (8.29-30).
- ▶ Different schools of Indian philosophy have related notions such as the notion of *abhāva* in Nyāya School, and the *śūnyavāda* of the Bauddhas.

So with reference to 0 as number and how it is represented, so it is too difficult to precisely fix, but we find places of it in the Panini's work on grammar as well as Pingala from sutra, so we have (FL) so this was the word (FL) also represents certain understanding of what this physical world is with reference to do this the Philosopher's, this word (FL) is does not necessarily represent the number 0 when it is used in philosophical basis.

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Description of decimal place value system
 Indian philosophical literature

- In *Vyāsa-bhāṣya* on the *Yogasūtra* of Patañjali, we find an interesting description of the place value system:

यथैका रेखा शतस्थाने शतं दशस्थाने दश एका च एकस्थाने;
Just as the same line in the hundreds place [means] a hundred, in the tens place ten, and one in the ones place;
- In the same vein, Śaṅkara in his BSSB (2.2.17) observes:

यथा एकोऽपि सन् देवदत्तः लोके स्वरूपं सम्बन्धिरूपं च अपेक्ष्य अनेकशब्दप्रत्ययभाग्भवति - मनुष्यः, ब्राह्मणः, श्रोत्रियः, वदान्यः, बालः, युवा, स्थविरः, पिता, पुत्रः, पौत्रः, भ्राता, जामाता इति। यथा च एकापि सती रेखा (अङ्कः) स्थानान्यत्वेन निविशमाना एक-दश-शत-सहस्रादि शब्दप्रत्ययभेदम् अनुभवति, तथा सम्बन्धिनेरेव ...

But in this chandahsutra there is nothing to do with the philosophical implications but it is use as a marker, now I give a couple of quotations from commentaries on Patanjali yoga sutra and (FL). So this form a very very concrete evidence (FL) is a certain line ok so which you draw, so he says if it is placed in (FL) so obviously (FL) a certain place value (FL) 100, if you just shifted so it represent 10.

So (FL) the same, so it has a certain place value, it has a certain value to represent a number, so both of them and this is more graphic portrayal to convey a certain philosophy Shankara says (FL) we call him a boy, we call husband, will call directory, we call him grandfather whatever it is, so depending up on the relation that you want to associate with another individual, it is describe by different terms.

(FL) similarly he says (FL) the same line, so (FL) depending upon where it is placed (FL) by Shankar as an example to convey something else in philosophy, but the point is so we have a very very concrete evidence by Shankar science, so it has gained and such a understanding in the society, that examples are something which should be known for everybody, so the way did you not going to be certain example to convey something else which is more perform ok.

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Earliest explicit use of decimal place value system

Indian mathematical and astronomical texts

- ▶ The earliest comprehensive astronomical/mathematical work that is available to us today is *Āryabhaṭīya* (499 CE).
- ▶ The degree of sophistication with which Āryabhaṭa has presented the number of revolutions made by the planets etc., clearly points to the fact that they had **perfect knowledge** of zero and the place value system.
- ▶ Moreover, his **algorithms for finding square-root, cube-root** etc. are also based on this.
- ▶ The system developed by Āryabhaṭa is indeed unique in the **whole history** of written numeration.
- ▶ Not only unique but also quite **ingenious and sophisticated**. Numbers of the order of 10^{16} can be represented by a **single character**.
- ▶ However, it was not made use of by anybody other than Āryabhaṭa — perhaps **luckily** as it is **too complicated to read!**

The explicit use of decimal place value system obviously is found in the mathematical and astronomical literature. So the earliest text that is available for us today is aryabhatiya or Aryabhata. So this was composed in 499 degree. So the degree of sophistication with which Aryabhata have been able to handle is something which is quite amazing. He has introduced his own way of representing numbers.

Obviously it is a decimal place value that will be very very evident not only from the representation of numbers but also some the procedures which Aryabhata has described to obtain square root, cube root etc. So the algorithms which will be discuss later we will make it extremely clear to you as to how they have been able to use this place value system in order to do lot of mathematical computation.

In fact quite ingenious and sophisticated that a single syllable can represent a number order of magnitude 10 to the power of 16. This is quite convenient to represent huge numbers but the same time this extremely difficult to pronounce as you will see with a few examples which I will show little later.

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Vowels employed in *Devanāgarī* script

► The chart below presents a summary of the vowels and used in the *Devanāgarī* script:⁶

	Short		Long		Diphthongs	
	Initial	Diacritic	Initial	Diacritic	Initial	Diacritic
Unrounded low central	अ	a	आ	ā	पा	pā
Unrounded high front	इ	i	ई	ī	पी	pī
Rounded high back	उ	u	ऊ	ū	पू	pū
Syllabic variants	ऋ	r̥	ॠ	r̄	ऌ	l̥
	ॡ	r̄	ॢ	r̄	ॣ	l̄
Secondary vowels	Unrounded front		ए	e	पे	pe
	Rounded back		ओ	o	पो	po
		०			औ	au
					पौ	pau

⁶<http://www.omniglot.com/writing/devanagari.htm>

For instance this 4320000 years is represent by (FL) by Aryabhata (FL) represent this particular number and what Aryabhata has done is so he has chosen the vowels to represent the place value, so we have a system of representing vowels and ingenuity lies in Aryabhata making use of the vowels to in other words I mean he has thought of vocalise to the place value I would say. So vowels are essential for pronouncing consonants.

And there is a will teach (FL) we do not say (FL) so what are essential for pronouncing consonants and Aryabhata has smartly using these vowels to pack a certain place value to that and the consonant will the person the number. So for those who are not familiar with vowels so I just thought I will just display this chart. So (FL) so this is all had a longer one (FL) so that is how sutra also (FL). So this refers to Panini, so give the set of vowels (FL).

So in Aryabhata systems once we reach (FL) with a certain function had tagged to that, so will be multiplying that consonant by 10 to the power of 16. So Aryabhata has further classified the consonant into 2 books.

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Consonants employed in *Devanāgarī* script

► The chart below presents a summary of the vowels and used in the *Devanāgarī* script:

	Voiceless plosives		Voiced plosives		Nasals
	unaspirated	aspirated	unaspirated	aspirated	
Velar	क ka	ख kha	ग ga	घ gha	ङ ṅa
Palatal	च ca	छ cha	ज ja	झ jha	ञ ña
Retroflex	ट ta	ठ tha	ड da	ढ dha	ण ṇa
Dental	त ta	थ tha	द da	ध dha	न na
Labial	प pa	फ pha	ब ba	भ bha	म ma

	Sonorants and fricatives			
	Palatal	Retroflex	Dental	Labial
Sonorants	य ya	र ra	ल la	व va
Sibilants	श śa	ष ṣa	स sa	

Other letters	
ह ha	ळ ḷa

So this start to help in understanding this kind of classification, so the matrix that you see here (FL) so this 5/5, so 25 he calls them as varga letters, so varga and (FL) the term has also the meaning of square, so the operation of square, squaring. So this is also referred to as varga, so but as far as this is concerned this set 5*5 I mean this 25 may be it is in this sense is called vargs, it falls into a certain class ok.

So then we have (FL) I do not think is employed by Aryabhata, but up to this certainly so he has associated certain numbers to all this consonants and the place values will be decided by the vowels ok. So this is extremely important because when a string is given to decide for one has to have a very clear understanding whether this false in (FL) so when we place these consonants with the vowels.

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Assigning numerical values to consonants

► Āryabhaṭa's scheme of assigning numerical values to the 33 consonants can be represented as:

gutturals	क ka = 1	ख kha = 2	ग ga = 3	घ gha = 4	ङ ṅa = 5
palatals	च ca = 6	छ cha = 7	ज ja = 8	झ jha = 9	ञ ña = 10
cerebrals	ट ṭa = 11	ठ ṭha = 12	ड ḍa = 13	ढ ḍha = 14	ण ṇa = 15
dentals	त ta = 16	थ tha = 17	द da = 18	ध dha = 19	न na = 20
labials	प pa = 21	फ pha = 22	ब ba = 23	भ bha = 24	म ma = 25
semivowels	य ya = 30	र ra = 40	ल la = 50	व va = 60	
sibilants	श śa = 70	ष ṣa = 80	स sa = 90		
spirants	ह ha = 100				

So it goes as 10 to the power of 2 in this sense it can be called as 10 decimal system. So here I would also like to show a chart wherein some error in fact I offered this course on mathematics in India to our student at IIT, I thought something and then for the student send me send me a mail. So looking at something in the web and then said so there is he seems to be discrepancy between what you said and what is there in the web.

So I said there is an error there, so I thought I will just show this to you also to see that so this year is not committed ok. So we have to be very careful when we take materials from the web. So the error is the following so (FL) stopped this and therefore there will be serious error in deciphering the number when we take this stop to be the (FL) ok.

(Refer Slide Time: 25:00)

Āryabhaṭa's representation: Illustrative examples

► Example 1: Decode the alpha-numeral **ख्युषु** (*khyaṣu*).

Category	V	A	V	A	V	A	V	A	V
Vowel		ṛ	ṛ	u	u	i	i	a	a
Given syllable	-	-	- gh	y	kh	-	-	-	-
Its value			4	3	2	0	0	0	0

The given number is: 43,20,000.

► **Note:** In this scheme NULL SYLLABLE ≡ ZERO VALUE.

► The verse in *Āryabhaṭīya* that lays down the procedure is:

वर्गक्षराणि वर्गे अवर्गे अवर्गक्षराणि कात् षो यः ।
 खद्विनवके स्वराः नव वर्गे अवर्गे नवान्त्यवर्गे वा ॥

► In the above verse, we have to make आवृत्ति of the word वर्गक्षराणि.

1. वर्गक्षराणि वर्गे [स्थाने स्थाप्यानि]
 2. कात् वर्गक्षराणि [सङ्ख्यां भजन्ते] – the varga letters take values 1,2,3 ... 25 starting from 'k'.

► Also षो यः defines the value of ya. That is, $ya = n + ma$.

So let us see a couple of examples, so cute root, so this is a very important number and that is why I decided that we should stop with this example, so what is to be done before getting into this table. So you please say look at this word, so Aryabhatta says (FL) so here we use so term varga in 2 senses, when he uses the compound (FL) it refers to the varga letters, but when he uses the word (FL) separately.

So it means you have to place it in the (FL) we mean 10 to the power 0, 10 to the power of 2, 10 to the power of 4 and so on. So here whenever you are given a string, so when you find a varga letter, so then we place in the (FL) and if you encounter (FL) you have to place in the (FL). So now it will become pretty evident you, so you have to create a certain table, so it will be very convenient in the initial stages for us to decide for the number.

For every vowel you have to create two places one is (FL) so for all we have 2 for (FL) and so on and so forth for every vowel will create two places and whenever we get a varga letter we place in the varga and encounter avaraga consonant you place it under varga. So this is what the (FL) essentially detail and if we look at the last part of the first half of the word it says (FL) so the number starts ok (FL) assigning numbers to various functions 1-25.

And then he defines what (FL) represent, so very clearly he states (FL) so if you look at this table, so (FL) put together, so whatever be the value that is the value which is assign (FL) ok. So that is what it means (FL) so this incidentally will give you a flavour of how Aryabhata has written his work. So that is this style of composition in those days and the entire Aryabhata is just 108 versus, so which deals with all mathematics, all astronomy and so which will be just take in A4 sheet ok.

So that is how it is this and so this is what it is (FL) basically refers to vowels, (FL) arranges student want means, so (FL) so now we look into this example (FL) 3 vowels and 3 consonants that define here, and vowels are (FL) 2 vowels and 3 consonants, so (FL) is the second thing ok third, so the value of (FL) is 2 and we find (FL) also there. So that has the value 4 (FL) and we find (FL) now when we try to arrange it here so since (FL) does not have a vowels on its own.

So when you have this kind of a combination so then the vowel that is tagged to the consonant with succeeding it, so will be the vowel which is tagged with this previous consonant with vowel also. So this is the annotation and (FL) place this in varga place (FL) set below varga, fine then we move on to the next syllable. So the next syllable is (FL) is 4 and it has to be placed below (FL), so because the vowel tagged with (FL).

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Āryabhaṭa's representation: Examples & Exercises

► Ex. 2: Find the no. given by चयगियिङ्शुङ्गुलं (cayagiyiṅśuṅḡulaṅ)

Category	V	A	V	A	V	A	V	A	V
Vowel	ī	ṛ	ṛ	u	u	i	i	a	a
Given syllable		l	ch	ś	ṅ	y	g	y	c
Its value		5	7	7	5	3	3	3	6

The given number is: 5,77,53,336.

► Ex. 3: Find the no. given by दृङ्घिष्य (dṛiṅḡhiṣya).

Category	V	A	V	A	V	A	V
Vowel	ṛ	u	u	i	i	a	a
Given syllable			dh	v	ṅ	v	gh
Its value			14	6	5	6	4

The given number is: 1,46,564.

► Exercises: Find the numbers represented by
 निशिवृण्शुङ्गुलं (niśivṛiṅśuṅḡulaṅ) & रजुषिध (rjushidha)

So this actually represents 4 zeros followed here, so 4320000 so this is the representation which has been given by Aryabhata. Aryabhata give comprehensive numbers. So we will see one more example (FL) so that is why I said it has an advantage of representing huge numbers in very short form, but it is bit difficult to read (FL). So here we again notice that there are several vowels which have been employed in this string.

We have (FL) so if you look at this table (FL) so this basically represents the number of revolutions made by moon, so in fact to give you the significance of this number so this gives the number of revolutions made by the sun in a large period which is called (FL) so under this representation the number of revolutions of sun and this represents the number of revolutions made by the moon in the same period of 4320000 years.

So this are strings which have been provided by Aryabhata in order to tell you the number of revolutions made by the planets in a large period called (FL) and one more example (FL) so here we have (FL) without vowel and the following vowel is (FL) and therefore (FL) represent 4 and (FL) represent 6 ok. These things (FL) it is very very difficult to pronounce, but these all basically the revolution numbers which has been stated by Aryabhata.

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The sine-table of Āryabhaṭa
 (Algorithm for constructing this table will be discussed later)

► Verse 12 of *Gīṭikā-pāda* of *Āryabhaṭīya*⁷ gives the value of Rsine-differences (expressed in arc-minutes):

मखि भखि फखि धखि णखि ञखि
 डखि हस्ड्ग स्ककि किष्ण स्यकि किष्च ।
 घलकि किग्र हक्य धकि किच
 स्म श्द्व द्व ङ्ग म फ छ कलार्धज्याः ॥

225, 224, 222, 219, 215, 210, 205, 199, 191, 183, 174,
 164, 154, 143, 131, 119, 106, 93, 79, 65, 51, 37, 22,
 and 7—these are the Rsine-differences [at intervals of
 225' of arc] in terms of the minutes of arc.

► Recalling that in Āryabhaṭa's notation: म → 25 & खि → 200,
 the word *mukhi* represents $jyā\ 225 = Rsine\ 225 = 225$.

⁷This verse is perhaps one of the most terse verses in the entire Sanskrit literature. Only after several trials would it be ever possible to read the verse properly, let alone deciphering its content.

Much more interesting thing is so Aryabhata actually present the entire sine table in numbers. So you may remember that in school days you would have referred to sine table classmate one page to page all that will be there, but this man has presented in simple words (FL) so this is how the, so all these basically represent these values of sine difference in fact we say sine table.

So the quadrant is divided into 24 parts will see later as to how Indian astronomers and mathematicians have found very efficient ways of computing accurate sine values for a certain period of time. So this Aryabhata what he has done is he has basically presented this sine table in one word (FL) because it is tagged with E, so you have two 25 25. So this two 25 basically represents the sine, so sine theta suppose theta is 3 degree and 45 minutes.

So sine theta is theta you know right, so it is presented in minutes and therefore so 225, so if you take 3 degree 25 minutes if you represent in minutes 225, so sine theta is theta, so what is the kind of thing, so what has increased the next value of 225 it chose so it will reduce will keep on reducing. So basically what Aryabhata has given is sine table in the form of differences, suppose you want to find out the sine of 10 degrees.

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Kaṭapayādi system

- ▶ The name 'Kaṭapayādi' stems from the fact that here the Sanskrit alphabets *ka, ṭa, pa, ya* etc. are used to denote the numbers.
- ▶ According to this system,
 - ▶ the vowels **standing alone**, represent the number zero.
 - ▶ the same vowels **in conjunction with the consonants** have no numerical significance.
 - ▶ the 33 consonants *k, kh, g, gh, . . . , ś, ṣ, s, h* are associated with numbers.
- ▶ The mapping of the consonants with numbers is as follows:

Number	1	2	3	4	5	6	7	8	9	0
Consonants used to represent numbers	<i>k</i>	<i>kh</i>	<i>g</i>	<i>gh</i>	<i>ṅ</i>	<i>c</i>	<i>ch</i>	<i>j</i>	<i>jh</i>	<i>ñ</i>
	<i>ṭ</i>	<i>ṭh</i>	<i>d</i>	<i>dḥ</i>	<i>ṇ</i>	<i>t</i>	<i>th</i>	<i>d</i>	<i>dḥ</i>	<i>n</i>
	<i>p</i>	<i>ph</i>	<i>b</i>	<i>bḥ</i>	<i>m</i>	—	—	—	—	—
	<i>y</i>	<i>r</i>	<i>l</i>	<i>v</i>	<i>ś</i>	<i>ṣ</i>	<i>s</i>	<i>h</i>	<i>l^ḅ</i>	—

^ḅThis is a special character—denoted in *Devanāgarī* script as **𑂣**—rarely employed by the Kerala astronomers to represent the number nine.

So you have to add this+this and then you had use some interpellation to get the value, so how this has represented ok. Now we move on the other system called which is called kattapayadi system. So the name kattapayadi stems from the fact that in this system (FL) so all of them represent 1. See if you look at this table so below so we have representation from 1-0. So here also it is mapping of consonant to a certain number.

But since we have 25 of them so in kattapayadi what has been done is (FL) and so on and (FL) so Aryabhata not assign the value to (FL) and that represents number 9, here there is no classification as varga and avarga that you have many to one mapping or other one to many mapping. So if you want to represent number 1 you can choose any of the 4 here ok. So 2 you can choose many of the 4 and so on.

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Kaṭapayādi system

- ▶ The following verse in *Sudratnamālī* of Śaṅkaravarman (c. 1830 CE) succinctly summarizes the system:

नत्रावचञ्च शून्यानि सङ्ख्याः कटपयादयः ।
 मिश्रे तुपान्तहल्संख्या न च चिन्त्यो हलः स्वरः ॥

[The letters] *n, ñ* and the vowels [when standing alone] denote zeros. [The consonants] commencing from *ka, ṭa, pa* and *ya* denote the numbers [1, 2, 3, . . .] in order. In the case of conjunct consonants (*mishre tu*) only the last consonant represents the number. The vowel suffixed to a consonant should not be counted.

- ▶ In this system, it is the **least significant decimal place that is given first**, and the highest the last.
- ▶ For example, the word **आयुरोग्यसौख्यं**, represents 1712210.

<i>ā</i>	<i>yu</i>	<i>rā</i>	<i>ro</i>	<i>gya</i>	<i>sau</i>	<i>khyam</i>
0	1	2	2	1	7	1

- ▶ Vararuci's *chandrvākya*s of unknown antiquity and Haridatta's *Mahāmārganibandha* (c. 600 CE) form the earliest examples.

So this is what basically katapayadi system is all about. What has been done here has been beautifully summarised in one single words by (FL) so the entire systems has been captured in this. So it says (FL) then it comes to a conjunct form of a syllable then how do you decide to (FL) here in Aryabhata system since the vowel was used to tag the place value, so we have problem here it is not that way.

So you find the string and you keep on it always start from so the least the place value and then you proceed to the higher, so when you have a certain string fun instances you look at this (FL) so in fact a famous work (FL) ends with string (FL) represent this number, so when you look at the definition which has been provided in this. So it says (FL) actually represents consonant ok.

So (FL) means any vowel tagged to the consonant once should not considered. So why is he stating this, so if a vowel occurs on its own without a consonant activate then it represents a number, but if it is tagged along with so only consonant has to be considered. So that is what (FL) means that which is close to so for instance in this string which has been given here (FL) is a vowel so all vowels where present number 0.

So (FL) so if you look at the definition so (FL) is a certain pneumonic which has been used to refer to all the vowels, so it has to prove Panini sutra (FL) which is used to refer to all the vowels. Similarly (FL) in fact if you look at the maheshwara sutras (FL) something which is used to refer to all the consonants. (FL) so we can see this example (FL) consonant, so what to do is you drop (FL) and you take the consonant which immediately preceded the vowel.

So it can be a consonant 2 consonants and 3 consonant fed with all those cases, the one which immediately proceed to vowel has to be considered, has to be counted and therefore you count only (FL) what is drop only is taken. So basically it is represent number 1,7, 1,2,2,1,0, commentator as said this actually represents the date on which this work (FL) got completed. So this is the another way of coding.

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Kaṭapayādi system

The table below presents a few illustrative examples chosen from the texts on Indian mathematics and astronomy.

Word/Words	Number represented
विद्वान्	44
तुन्नवलः	3306
कवीशनिचयः	160541
सर्वार्थशीलस्थिरः	2735747
निर्विद्धाङ्गनरेन्द्ररुक्	22203940
भद्राङ्गभव्यासनः	714324
ऊनधनकृद्भरेव	42410900
धोगापाङ्गजळाङ्गस्त्री	23983139
नानाज्ञानतपोधरः	29160000
हे विष्णो निहितं कृत्स्नम्	1680548
लक्ष्मीशनिहितध्यानः	1680553

In fact there are several examples like this (FL) also it starts with the (FL) it represents date and (FL) appears in the last verses text and determiners this number if you see this the difference is only 5 days, apparently this person Nilakantha has composed the commentator say this entire work something about 430 verses and all in just 5 days. So this all exam.

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Bhūtasankhyā system

- ▶ The word *Bhūtasankhyā* is a compound word which has two constituents, namely *bhūta* and *sankhyā*—referring to a 'being' and a 'number' respectively.
- ▶ Here words commonly employed in Sanskrit such as:
 1. The physical entities such as Earth, Moon (1), planets, stars, ocean, mountain, fire, sky, direction etc.
 2. The parts of a human body such as eyes, ears(2), jaws, knees, hands, fingers, teeth, nails etc.
 3. The animals, such as serpent, horse, elephant etc.
 4. The names of the gods, such as Śiva, Indra etc. and sometimes historical figures such as Manu, Rāma (3) etc.
 5. The season, fortnight, month (12), week, etc.

whose meaning has the potential to evoke a certain number in the reader's mind, were used to denote specific numbers.

Quickly I will also mention about this bhutasankhya, in bhutasankhya system as the term itself gives an idiom, so bhuta is something which is existing, so sankhya is number. So you choose a certain elements in nature, so for instance eye, when you say eyes so they represent 2 ok. So when you say fingers it represent 10, when you say veda it represents 4. So based on the familiarity so with the bhutas.

Bhutas not necessarily the physical elements, we could be physical element it could be certain mythological think it could be referring to some literature which is quite common. So for instance when they say (FL) so moon the moon associated with the earth. So they all represent number 1. So any synonym of earth to represent 1, of moon will the present. So then for representing 0 the use (FL) space looks like a shadow empty in the sense.

So space is used to represent number 0, any synonym of (FL) is 0. So as I said eyes, ears, jaws, knees, hands, finger system, hands is 2. So sometimes they use (FL) there are 14 manmantras and therefore (FL) represents 14 ok and rama Ayodhya Rama, Balarama, Parasurama, so it represents 3. So guna when they used (FL) all this things, so this is how the bhutasankhya system is.

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Bhūtasankhyā system

Illustrative example: Approximation to π given by Māadhava

- ▶ The commentary *Kriyākramakārī* while presenting several values of π given by different *Ācāryas*, also lists the one due to Māadhava. We give this as an illustrative example of the use of *Bhūtasankhyā* system:

माधवाचार्यः पुनः अतोप्यासन्नतमां परिधिसङ्ग्रामुक्तवान् -
 विबुधनेत्रगुज्राहिहताशनत्रिगुणवेदभवारणबाहवः ।
 नवनिखर्वमिते वृतिविस्तरे परिधिमानमिदं जगद्विधाः १^९

- ▶ The values of π given by the above verse amounts to:

$$\pi = \frac{2827433388233}{9 \times 10^{11}} = 3.141592653592 \quad (\text{correct to 11 places})$$

- ▶ How did Māadhava arrive at this value? This will be covered in the later part of the Workshop!

^९Vibudha=33, Netra=2, Gaja=8, Ahi=8, Hutāsana=3, Triguna=3, Veda=4, Bha=27, Vāraṇa=8, Bāhu=2, Nava-nikharva=9 × 10¹¹. (The word *nikharva* represents 10¹¹).

When they use month there are 12 month, so one interesting example is Maathavas value of Pi, so what I thought just used herr, so (FL) so you look from here so (FL) basically represent (FL), so therefore it is 33, so it has to do with the description found in Puranas and (FL) is 2 then (FL) and we have 3 here, 3 also represents 3, represents 333 consecutively ok. (FL) 3 and then we have (FL) and so this first line basically represents this number.

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Bhūtasāṅkhyā system

► The table below presents a few examples.

Word	Number represented
खाद्विरामाशयः	3370
वेदवेदाङ्कचन्द्राः	1944
वेदचन्द्रद्विवेदाग्निनागाः	844214
भुजङ्गनन्दद्विनगाङ्गवाणषट्कतेन्दवः	146567298

► **Advantages**

- As the language is extremely rich in synonyms, an **author could choose any synonym that would suit the metre.**
- From the reader's view point, since the words are familiar it enormously **enhances the readability.**
- However, **lack of familiarity** with the connotation of a specific *bhūta* and/or the lack of knowledge of synonyms could pose a problem, not to mention **improper splitting** of the words.
- This of course has to do with the ignorance on the part of a reader, and is **no reason to blame the system!**

And the denominator is represent (FL) this actually gives you the value of pi which is correct to 11 decimal places. (FL) will discuss later. So these are all few other examples. So (FL) so to conclude, so Aryabhata system though quite ingenious is a bit difficult to use that systems to represent because it is too difficult to pronounce and therefore not many people in fact we do not know of any (FL) Aryabhata using his system to represent the number.

So it is very profound in its own way, as far as (FL) is familiar with the saturn system it is in fact it is very convenient to use both (FL) system for composition, so which is also see if you are familiar with certain terms it is quite convenient to read and therefore mostly astronomers have resorted to (FL) system. In fact that seems to have been the most deadliest system but then Aryabhata device own and (FL) had own format.

And in fact as I told you (FL) something else and it also represent the number in fact the entire (FL) system of astronomy is based on this (FL) when you decode it gives you if anything conclusively. So all these (FL) proved the facility with which people have been using decimal place value system. So with this we conclude here, thank you.