Mathematics in India: from Vedic Period to Modern Times
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Lecture - 38
Proofs in Indian Mathematics - 03

So, this is the third lecture on proofs in Indian mathematics. So, we should concentrate in the large part of this lecture on the proofs of the results that sort of pertain to calculus. The discoveries are Madhava, so where really one has to take this limit of some quantities per large n. So, how does Yuktibhasa handle these kinds of proofs that we will be discussing in this lecture and in the end, we will have some general comments about Upapatti and proof and some comments on history of mathematics.

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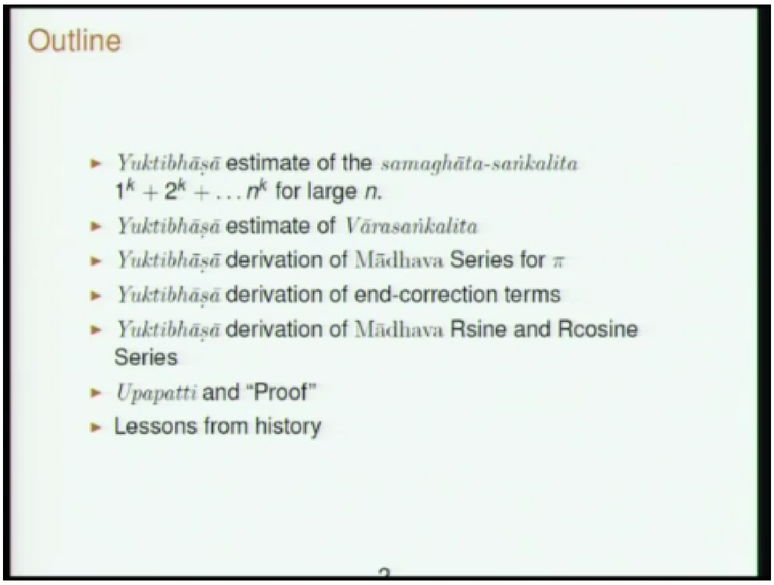

So, as we have repeatedly said the most detailed exposition of proofs or Upapattis in Indian mathematics is found in the Malayalam text Yuktibhasa or Jyesthadeva. Yuktibhasa sets out or states that its purpose is to give the rationale of the results and procedures presented in Nilakantha's Tantrasangraha.

This book is in Malayalam, so many of these rationales have also been presented mostly in the form of Sanskrit versus by Sankara Variyar (FL) of Nilakantha in two commentaries Kriyakramakari on Lilavati and Yuktidipika on Tantrasangraha.

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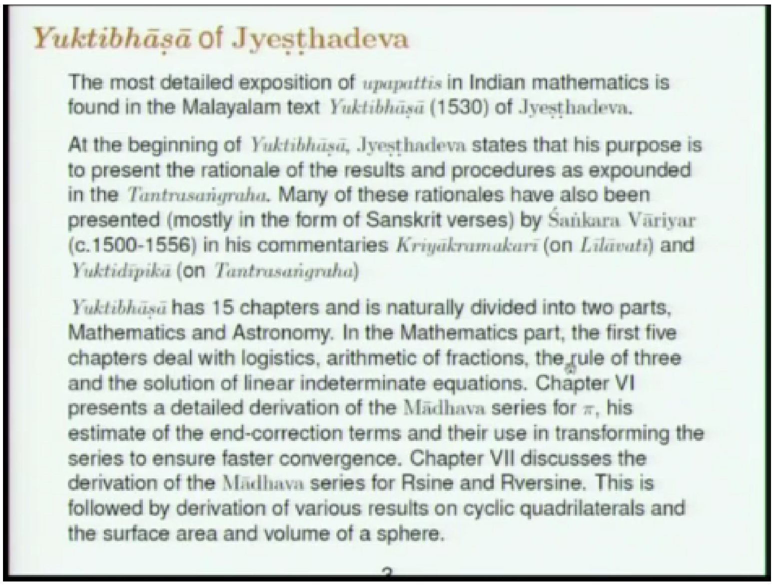

Yuktibhasa is 15 chapters. Its chapter 6 which deals with the Madhava series for pi is the chapter on the Paridhi-Vyasa sambandha. Chapter 7 is the chapter on Jyanayana which deals with Madhava series for the Rsine and Rversine. So, let us straight away look at how Yuktibhasa tries to estimate this sum, this is called Samaghata-Sankalita, sankalita is the sum, ghata is a product, Samaghata means equal powers, so sum of natural integers of the same powers.

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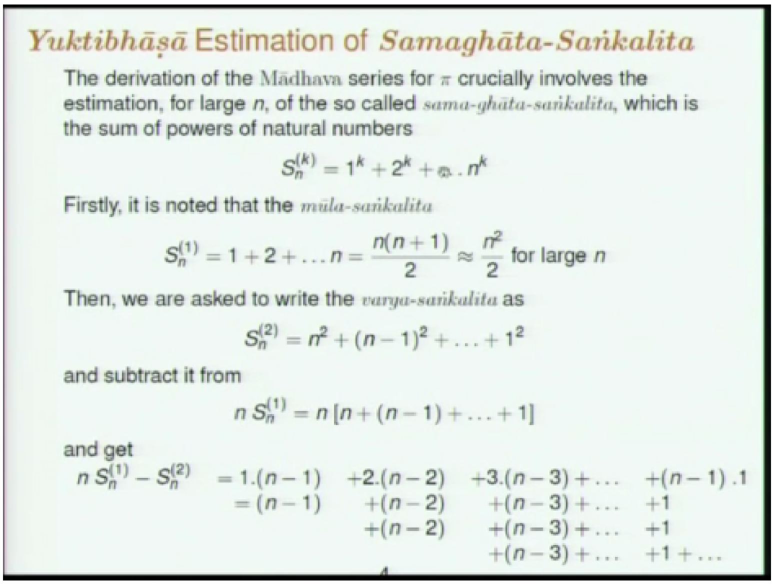

We have to find out how this unbehaves for large n, we have to asymptotic estimate of this sum and this is at the heart of development of calculus, estimating this was a major effort in whole of 17th century efforts in Europe, which arrived ultimately at the development of calculus. Yuktibhasa has a statement of the result and a proof of it, so we shall discuss the proof. First, it is noted that the sum of the first product, we have Aryabhatta's exact results, N * n+1/2.

And obviously for large n, you can neglect the n/2 term and you can say this goes like n square/2. So, it is this capable separation of the orders, in which has n becomes large. What is the significant term or in 1/n has n becomes large, what are the terms that can be neglected, this is where the heart of this limiting operation talks. Now, we are saying to this varga-sankalita. Of course, we know the exact formula is available.

So, we can again look at it and write down the asymptotic behavior for large n, but Yuktibhasa choose us to start writing down the proof in the way it will do for the general case, so that you understand the method of proofing the general case. So, it says write this Sn to in this way, n square + n - 1 square, etc + 1 square. Subtract from Sn of 2, n times Sn of 1. N times the mula-sankalita, 1 + 2 + etc. n.

And when you subtract nSn1 – Sn2 becomes 1*n-1 + 2*n-2 + 3*n-3 + etc., + n-1*1. Now, remember Yuktibhasa does not have any symbols, any diagrams, all these are set out in terms of sentences, all these equations are explained in terms of sentences, all diagrams are explained in terms of sentences, but to write diagrams, they will tell you start from the east corner, go to the north-west corner, then look at the place where this line cuts the circle.

So, they may very nice way of describing the geometrical diagrams. In the same way, this algebraic results are stated in verbal form, so n times Sn-1 – Sn2, therefore can be written as n-1, then these 2 n-2 can be written this way, the 3 n-3 can be written that way, the n-1 ones can also, so now you re-sum them that is now you look at this sum this way, you look this sum this way. So, this is mula-sankalita up to n-1, this is mula-sankalita up to n-2 etc.

So what we have, so nSn1 – Sn of 2 is Sn-1 1, Sn-2 1, Sn-3 1 that is mula-sankalita going up to n-1, mula-sankalita going up to n-2, so this is the, not here, we can look at it once again Sn2 - nSn1, we write like this and then rewrite it and sum it horizontally now and therefore, we have this ending. So this is at the basis of the Yuktibhasa proof actually. Later on, generalize this identity and prove the result by induction.

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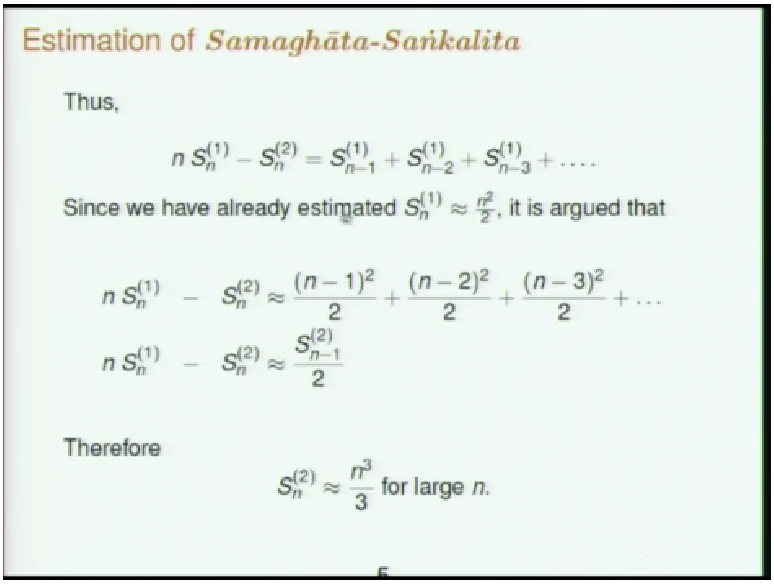

So, since we have already estimated the mula-sankalita summations of the first powers of natural integers go like n square by 2, so we substitute here, so Sn-1 we write it as n-1 whole square by 2, Sn-2 1 is written as this, Sn-3 is approximated by this and so on and now, we again behold that this is sum of the squares of integers, of course for large n, the later terms are not of significant and so this whole sum can be written as the square of the first n-1 integer.

The sum of the squares of first n-1 integers divided by 2. So, you have an equation like this. Now, this Sn-1 of 2 and Sn of 2 are almost the same. When n is very large, there is no difference between the sum of the squares of the first n integers and the sum of the squares of first n-1 integers, at least for the purpose of the large n behavior, which we are estimating now. So, these two can be taken to be the same.

Therefore, taking this to the right hand side, this will become Sn of 2, 3/2 of it and here already nSn1 is there, therefore finally you will get Sn of 2 goes like n cube - 3, in case this step is not clear, what we have shown is nSn of 1 - Sn of 2 goes like Sn-1 of 2/2, right, this what we have shown. Now, you take this to the right hand side, so you just get nSn of 1 approximately = Sn of 2, 3 times, right. These two are the same and this has already been estimated.

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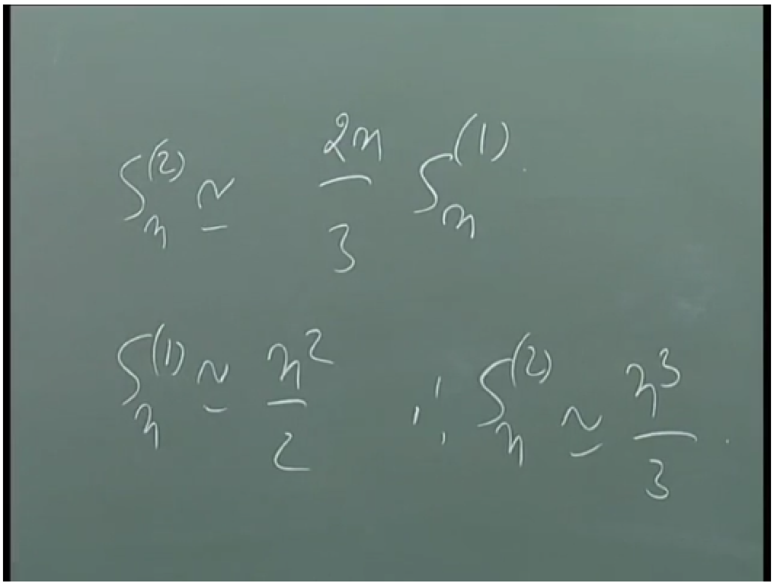

So, Sn of 2 goes like 2n/3 Sn of 1 what is Sn of 1, it goes like n square/2, we have already estimated that for large n, therefore we have Sn of 2 goes like n cube by 3. All this is set out verbally in a Yuktibhasa or in Kriyakramakari commentary on Lilavati. Similarly, it is shown that the sum of the third powers of integers goes like n to the power 4/3 for large n. This argument is also explained in detail in Yuktibhasa, same kind of argument.

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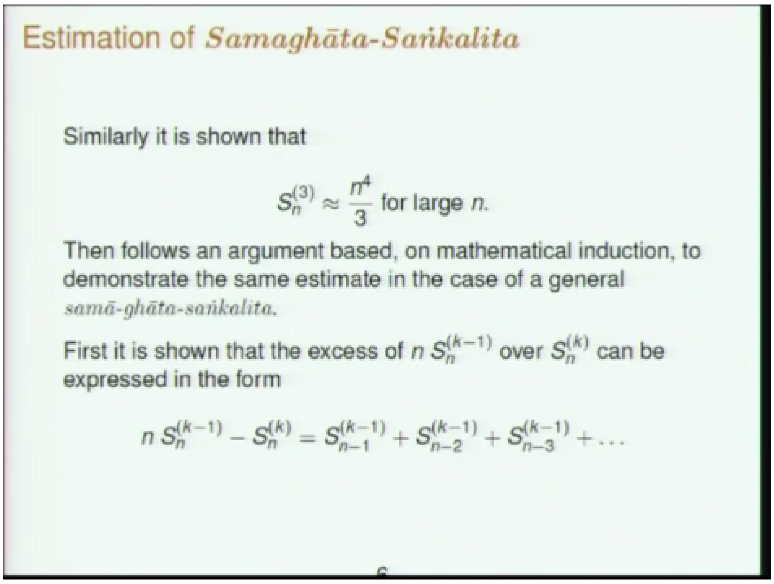

Now follows the argument by mathematical or using mathematical induction to demonstrate the same estimate for a general Samaghata-Sankalita. So, he is saying suppose you are interested in summing the Samaghatas where it goes up to **(FL)** some number. For that, you take first n times the Samaghata-Sankalita the lower order and subtract the Samaghata-Sankalita of the given order.

So, same thing that we did for nSn1 - Sn2, then this argument is given by recombining these terms that this is actually = the repeated summation of Sn-1 of k-1, the varasankalita of the lower order Samaghata-Sankalita that how it is explained. So, this relation is just like this relation. Only this is k-1 and this is k, the argument is very similar to prove this. So, this is the basic relation and this relation there is no estimate involved, it is exact.

Now, we argue what happens when n becomes large. So, in all of Yuktibhasa up to a point, the terms of all n are kept and its exact results are written and then the final limit and arguments are made. So, this result is exact that nSn k-1 - Sn of k is this. Now, if we have already estimated that is how a mathematical induction argument works. First you write down a relation, then assume something for n and then show it for the n+1 and then you have shown it for all n, if you have already shown it for n = 1.

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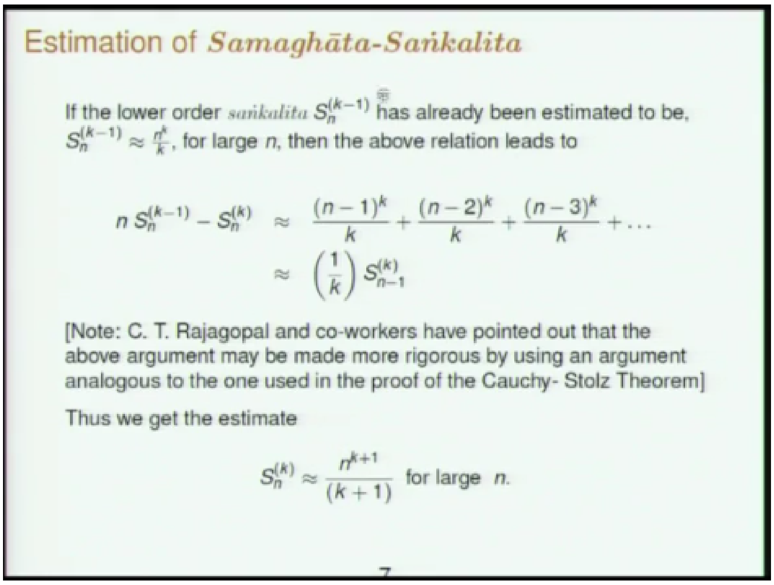

So, assume that the k-1 called as Samaghata-Sankalita has already been estimated and it goes like n to the power k/k. Once you assume that and put that into this equation, so this has already been estimated. So each of this go like n-1 to the power k/k-1, n-2 to the power k/k, each of them goes like this, n-1 to the power k/k, n-2 to the power k/k, n-3 to the power k/k. So, this is looking of course terms which are very small in the other end do not matter.

And this is clearly looking like the sum of the kth power r integers from 1 to n-1, so it is nothing but Sn-1 of k by 1/k for large n. So the previous one was an exact relation from that for the large n, we have this. Of course, there is some argument about finally this is going like 1 to power k/k kind of term is also coming there, so is it really true, is it really correct, so more technical argument needs to be made either the small sort of the epsilon delta argument that needs to be made to justify this.

This justification points since has been indicated in the papers of C. T. Rajagopal, who else one can use the same sort of arguments used in the proof of something called the Cauchy-Stolz Theorem to justify this step. So once this has come, the argument is just the same, Sn-1 of k is almost the same as the Sn of k, so if you use the already known estimate of Sn of k-1 for this, you will obtain Sn of k, n to the power k+1/k+1.

This is one of the beautiful proofs in Yuktibhasa for the Samaghata-Sankalita for large n. Now, the estimate of varasankalita, so for a varasankalita, what is varasankalita? Repeated summations. Not summation of powers, but summations repeated, repeated, repeated. So, first summation is 1 to n that is n*n-1/2. Second summation is sum of this n*n+1/2, you sum it from n = 1 to n = n. How will it look?

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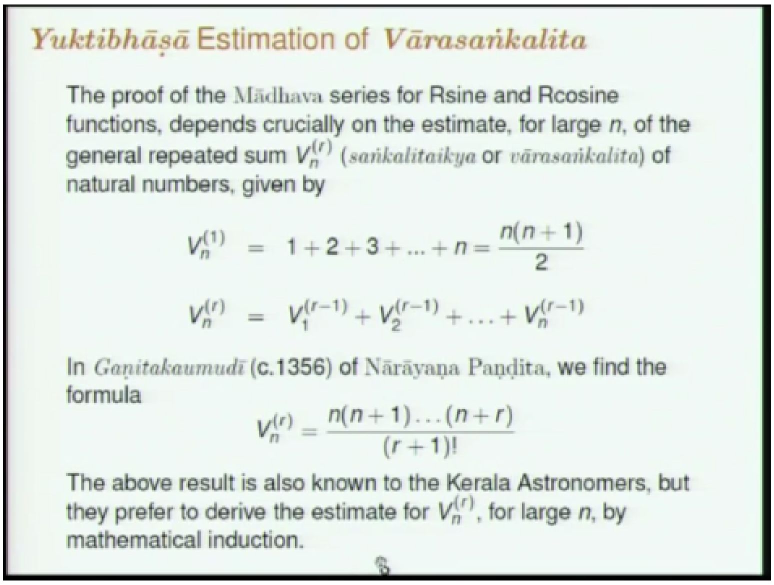

So, in general the r called a repeated summation is the summation of the r-1 (()) (12:42) from 1 to n that is the definition of the varasankalita and has been said repeatedly this was clearly the result for this was enunciated in Ganitakaumudi of Narayana Pandita that the r called a summation is this and show how does it behave for large n, it goes like n to the power r+1/r+1 factorial.

There is one n factor in each of this, so the highest power of n that occurs in numerator is n to the power r+1, so the way Vn of r goes for large n is n to the power r+1/r+1 factorial. So once we have Narayana's formula, you have this n * n+1 etc., * n + r/r + 1 factorial, so immediately you can see that it goes like n to the power r+1, that r+1 factorial for large n, right. In each of them, there is an nth term and so the largest power that will occur in the numerator is n to the power r+1.

The next term is of the order n to the power r, which can be neglect that in comparison to n to the power r+1, n goes to infinity, but Yuktibhasa knows this results, the authors of Jyesthadeva knows this result, but he does not want to use this to write down the proof. He wants to write down an argument by mathematical induction in the same way as he did for the summation of powers of n.

So, an argument like this he wants to do, so that the thing at we will give now. So, the first order summation we already know that result given by Aryabhatta n * n+1/2, it goes like n square/2. Now, say let us write Vn of 2, we already know the asymptotic behavior of Vn of 1 that goes like n square/2, so the same thing for n-1 and go on down the line. Therefore, asymptotically for large n, Vn of 2 bears in the same way as Sn of 2/2.

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Estimation of *Vārasaṅkalita*

Now,

$$V_n^{(1)} = \frac{n(n+1)}{2} \approx \frac{n^2}{2} \text{ for large } n.$$

We can express $V_n^{(2)}$ in the form

$$V_n^{(2)} = V_n^{(1)} + V_{n-1}^{(1)} + \dots \approx \frac{n^2}{2} + \frac{(n-1)^2}{2} + \dots = \frac{S_n^{(2)}}{2}$$

Using the estimate

$$S_n^{(2)} \approx \frac{n^2}{3},$$

we get

$$V_n^{(2)} \approx \frac{n^3}{6}$$

Then, we have already known Sn of 2 goes like n square/3, so Vn of 2 will go like n cube/6. So, restating it in the general case, Yuktibhasa says Vn of r goes like = Vnr-1, Vn-1 r-1, the r-1 sum, we have already estimated goes like n to the power r/r factorial. So, each of these terms can be replaced by n to the power r/r factorial, n-1 to the power r/r factorial etc. Again, the lower order terms do not matter for large n.

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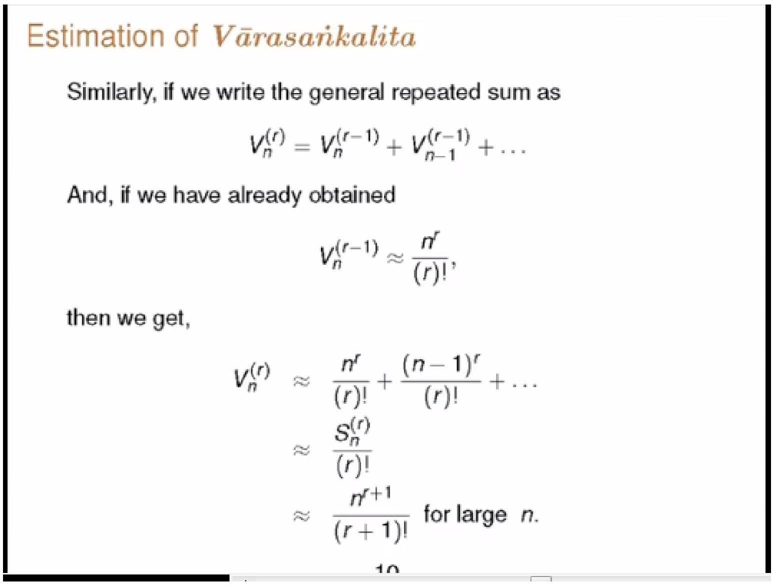

So this is nothing like the sum, but the sum of rth power of integers that is Sn of r divided by r factorial. Sn of r is n to the power r+1 over r+1, therefore the n of r goes like n to the power r+1/r+1 factor. So, the same induction argument, assuming this Samaghata-Sankalita, they are writing it by the varasankalita. Its in fact even more interesting one could from this to this. One could have used this to prove Sn of k goes like n to the power k+1 by k+1.

In fact, that is the root Pascal or Fermat took in 17th century. They conducted this result, Pascal actually proved this result. Fermat I think conducted this result and from this, one could get the asymptotic form and from that, one could get the Vedic Samaghata-Sankalita, so it is either way and as I told you, this result was already implicit in the Varahamihira's table **(FL)** for calculating the combinatorial coefficients in the 6th century.

The sum of integers, sum of sum of integers, sum of sum of integers is the number of combinations saying that it is a ncr kind of term like this, okay. So, this is about one of the most important results in mathematics for development of calculus, which is the estimate of this. You see the first order sum and the second order sum were known to Aryabhatta, they were known to the Greek's also and in some of his, what is called the quadrates of parabola etc, Archimedes use the asymptotic behavior of the sum of squares.

Of course, he worked out an argument in the standard Greek way by reductio ad absurdum further. The fourth powers were summed by I think Ibn al-Haytham in 11th century in West Asia, so here an estimate for the fourth powers, but the Samaghata-Sankalita with general order, the result like this was first given by Yuktibhasa along with the proof also. They had of course this estimate for the varasankalita also.

Both are used in the subsequent result that we are now going to discuss. So now, the derivation of the Madhava series for pi, so what is done, there is a first part is a purely geometrical argument converting the length of an arc into an algebraic expression through the use of geometry and as you can see how sophisticatedly Indians were arguing with geometrical magnitude you can see here how they converted it to algebraic expression.

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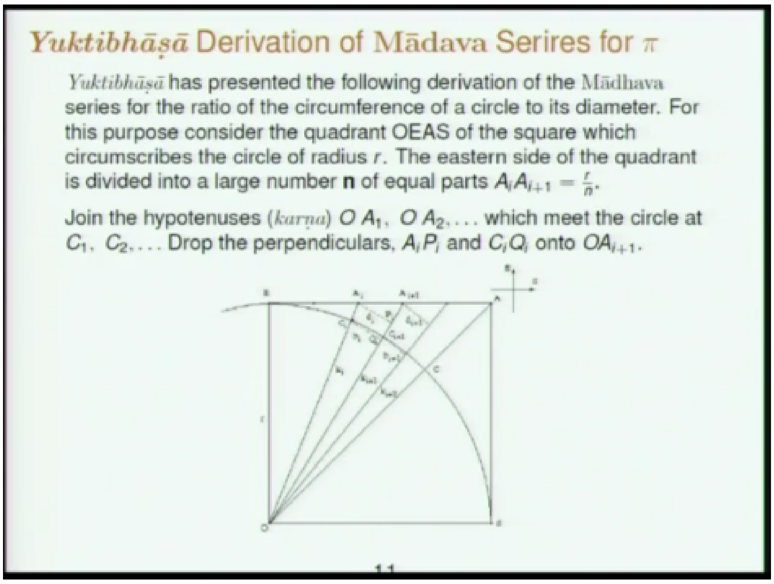

And then a limit of that algebraic expression is taken for large n, which is where the calculus comes in. So, what is the derivation of Madhava series? So, you take this quadrant of circle, the circle is of radius r, draw this square that that circumscribes the circle. Now for this quadrant, this side of this was EA is the same as the radius of the square. So, you have a square of radius r, now divided this radius into n equal parts.

We are not dividing the arc into n equal parts, we are dividing what we do it called as tangent into that is the beauty of this Madhavas derivation that makes gives you a nice formula that you can handle for the length of this curve. So, divide this tangent into n equal parts and join this hypotenuse at, so this may be nth bit of this. So, EA is divided into n equal parts. EA is of length r. So, each of these bits are of length r/n.

Join this OA1, see where they meet the circle, draw a perpendicular from there to the, so these karnas from these two points in one karna draw perpendicular to the next karna. So, this is the geometrical construction. What is the next thing? Of course, it is noted that the whole calculation let to be done for when n becomes very large and the bits become very **(FL)**. So, first thing is two similar triangles AiPiAi+1 is similar to OEAi+1, this is one.

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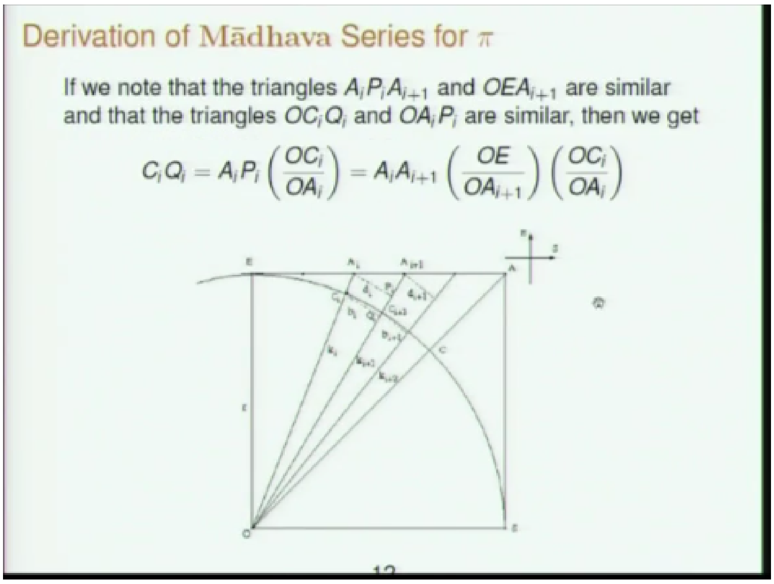

So to argue the similar triangle, it is argued that the sides of one triangle are perpendicular to the sides of the other triangle and the example of the Kerala temples intersecting those wooden sort of beams is what is mentioned in the Yuktibhasa, it is like those wooden beams there the one sides of one triangle are perpendicular to the sides. It is said that Madhava got this proof by sitting inside the Kerala temple and contemplate.

Then another thing is OCQJ, this triangle is same as OAiPi, both of them arise by drawing perpendiculars from one line to the other line, so they are similar. So, this argument from similar triangle immediately gives you an expression for the sine of this arc bit. CiCi+1 is an arc bit that is wedged between these two karnas where this AiAi+1 which are equal. So, the CiCi+1 are not equal, but they are small.

So the argument now is, this CiCi+1 when n is very large can be approximated by CiQi, the sine of the same arc bit and therefore, this one-eighth of the circumference that is the circumference from E to C because this is the last karna, this radius is divided into n equal part, you are estimating the CiCi+1 and each of them is written in terms of this OAi+1 and OAi, they are called Ki and Ki+1.

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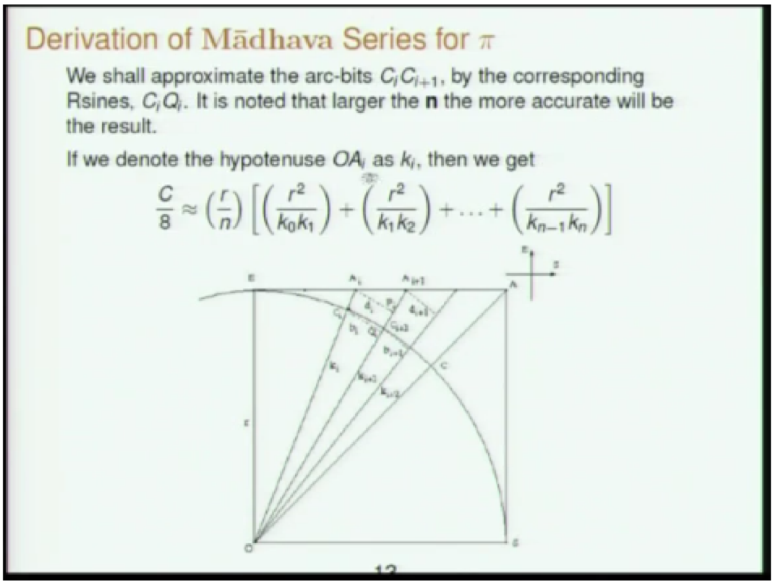

So immediately get this nice expression for the, so you are connecting the arc of a circle into a beautiful in the limit of course of large n in algebraic expression. Now, we are going to play with this one. Geometry is over, C/8 goes like r/n * r square/k0k1, r square/k1k2, r square/kn-1kn. Ki's are these karnas, OAiOAi+1. So, next is when n is very large, these Ki's are very close by this is argued out also.

And so the earlier sum, this sum is replaced by this sum and actually, their differences is actually shown to be negligible when n is fairly large and now, the next step is to realize from geometry that each of these karnas is actually EAi square + EO square and EAi is obtained by the after the i arc bits are over. Therefore, EAi's, i times r/n, OE is of course r itself and therefore, ki square is r square + ir/n square, finally we have obtained this expression.

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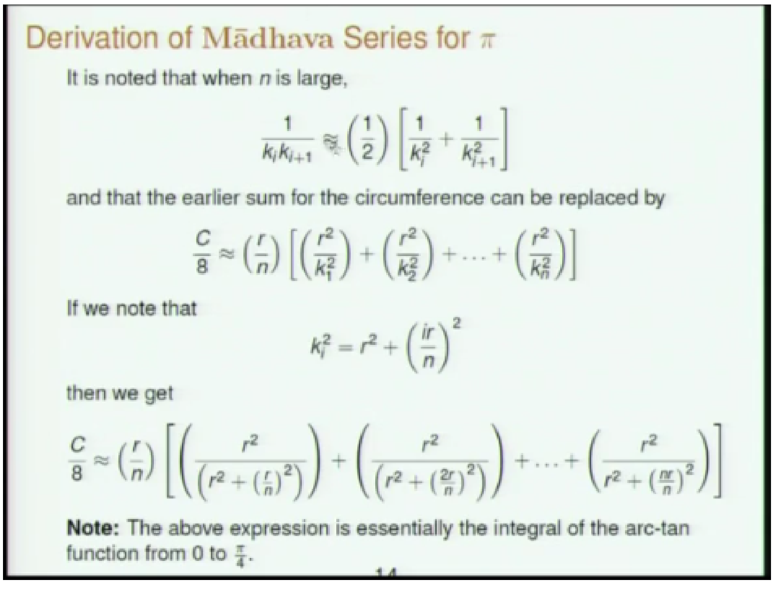

So except for the left hand side, where you are saying this is approximately = C/8, all this is there on the right hand side has been more or less exactly manipulated. No approximations are made except for this one, which is this one. Now, anyone who have done this calculus and has written an integral as a set of sums can straight away realize that you remove the r square as a common factor out.

This is nothing but integral 1 over 1 + x square dx from 0 to pi/4 that is the integral of the tangent inverse x function which will give you pi/4 as the answer. So now what is the next step, next step was 2, which have already been done. One is this denominator is expanded using binomial series which you have already done, so using binomial series expand each of these denominators, so that is the first step and group together terms involving same powers of n.

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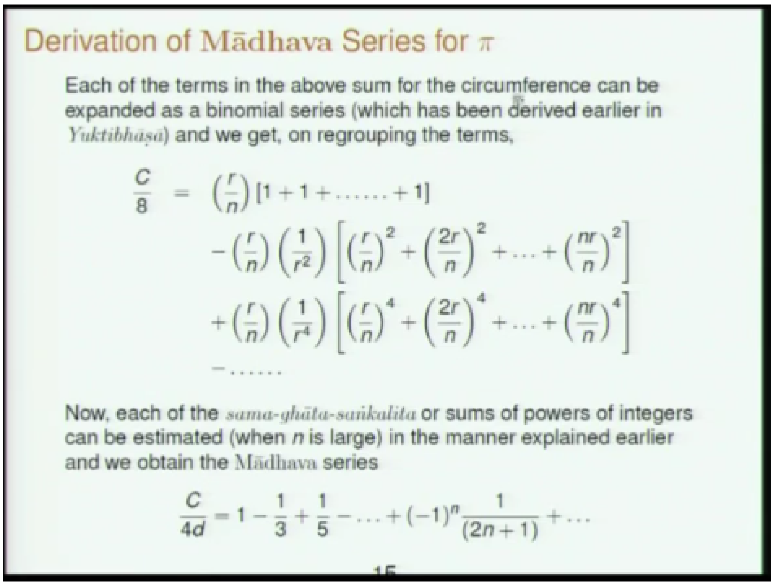

So you will have a term involving sums of integers, you have a number involving squares of sums of integers, you have a term involving the sums of integers taken to the fourth power, so use the second thing that we have done the estimate of the kth power of the Samaghata-Sankalita when n is very large and therefore, you will immediately have obtained.

When n is large, take out the r, so you have 1 to the power 4, 2 to the power 4 etc, n to the power 4 that goes like n to the power 5/5, this will go like n cube/3 and this will go like n, so that will cancel with that n and this will cancel with the n * n square n cubed here, so you have finally the series of Madhava, C/4d, 1-1/3+1/5- 1 to the power n, 1 over 2n + 1. Next is the derivation of the end-correction terms.

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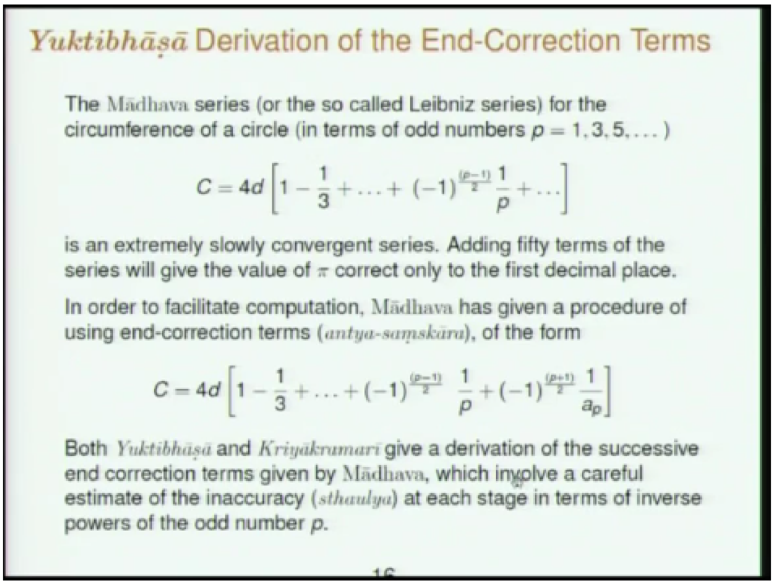

So, the Madhava series is written this way and as where you marked it is a slowly convergent series, so let us assume an end-correction term like this. Now, Yuktibhasa and Kriyakramakari, Madhava has actually given these end-correction terms, Yuktibhasa and Kriyakramakari will tell you how to derive. So for the derivation, all that you have to say is let us assuming that 1/ap gives the exact value. Let the end-correction be such that.

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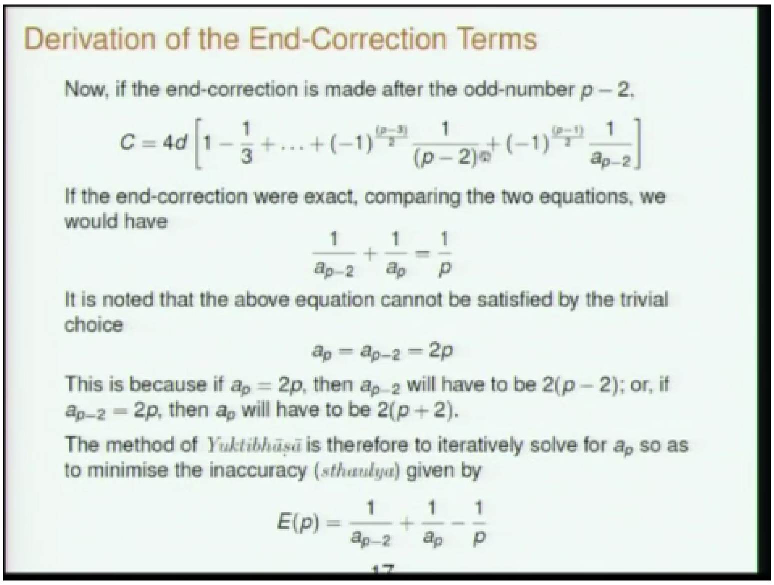

Then, suppose I terminate the term at series at p-2, then 1/ap-2 should also give me the exact value, so you have to equate the exact values which 1/ap here and 1/ap-2, immediately you get an equation like this. So, this equation is at the heart of the estimation of ap. We used this to estimate ap. Now looking at it, one can say trivially, take 1/ap as 1/2p and everything is done that is not correct because if you take ap as 2p, then ap-2 will have same dependence on p-2 as ap heart as on p.

So ap-2 will have to become 2 * p-2. Similarly, if you take ap-2 as 2p, then ap will have to be = 2p+2. So, there is no simple integral in that of condition, you have to have a fairly complex expression involving p, so it is a functional equation in p, which will be sort of complex, which has a fairly complex behavior, but what we do is, this quantity Ep is = 1/ap+2 + 1/ap - 1/p is called the inaccuracy sthaulya by Yuktibhasa.

We will successively approximate this sthaulya and obtained end-correction terms. So, the first approximation to the end-correction is ap = 2p + 2. If we put this, if this sthaulya = 0, the result is exact. I mean that end-correction divisor is giving you the exact value of the Madhava series, but sthaulya will get never be 0, so if you put ap = 2p + 2, you just algebraic, so this algebraic manipulation is very beautifully explained in Kriyakramakari putting various boxes and all that.

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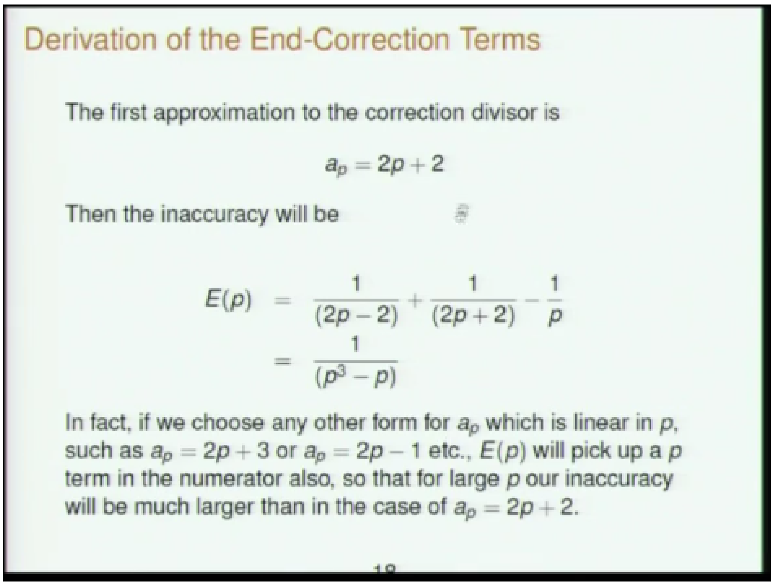

So how to keep the different polynomial coefficients in hand. You are going to have ratios of polynomials in p in general, whenever you do this kind of manipulation. So, you get 1/p cube - p. So, supposing I chose instead of 2p+2, 2p+3 or I chosen even 2p-1, this sthaulya would pick up that p term in the numerator also. What is the disadvantage? As p becomes very large, this goes to 0 much faster than the other one.

So, this is the best term for the inaccuracy to have that went for fairly large p, you have reasonably accurate result, whereas if you have chosen ap = 2p+3, it will only go by like 1/p square instead of 1/p cube and you can already see that in this, the Madhava transform series is also coming after all this expression for sthaulya is what Madhava manipulated in this equation itself to, and that procedure was called sthaulya parihara.

So, use the error corrections in the series itself to transform the series. So you are obtaining a p cube – p term by a correct choice of ap to first order, so this is the first order correction. Now of course our result is not exact, we still have Ep which is nonzero, so to make it better, we have to add a quantity which depends on p, but which is less than 1. So, let us take A over 2p + 2 as a next order correction to the correction-divisor.

Now, if you choose A as = 4, we will immediately get Madhava's first correction divisor. The first correction that Madhava gave you can recognize is this. So for that A will have to be 4. Again, the argument is let us calculate Ep with this ap. What is Ep? Ep is this, 1/ap-2 + 1/ap-p, this is our Ep, so let us calculate that with this expression for Ap, then we get Ep = -4/p to the power 5 + 4p, if I take this.

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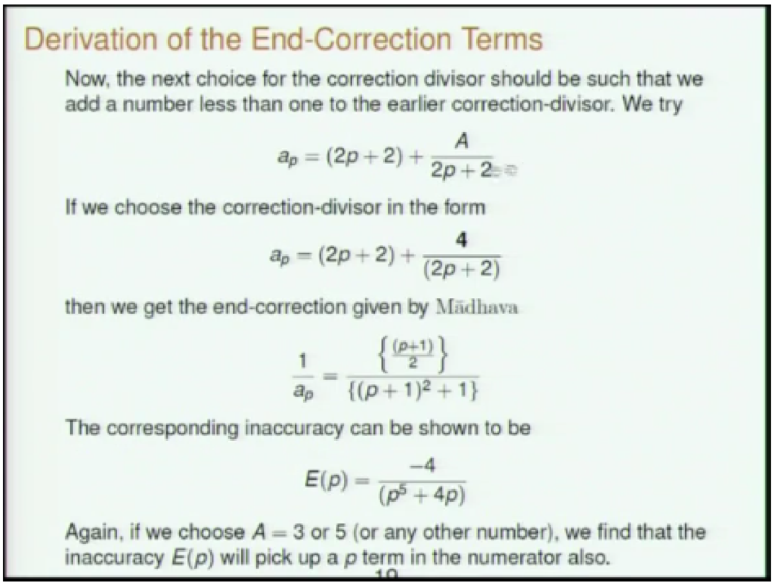

But if I take instead of this, any other value for A, 3 or 5, then we will find that Ep will pick up a p term in the numerator. So, the sthaulya will not go like 1/p to the power 5, it will go much more slowly like 1/p to the power 4, so we will not have a more fast decreasing sthaulya, therefore for better this thing inaccuracy, we need to choose this constant to be 4.

Next, so the finer by know the procedure is very familiar, there also thinking of a continue attraction.

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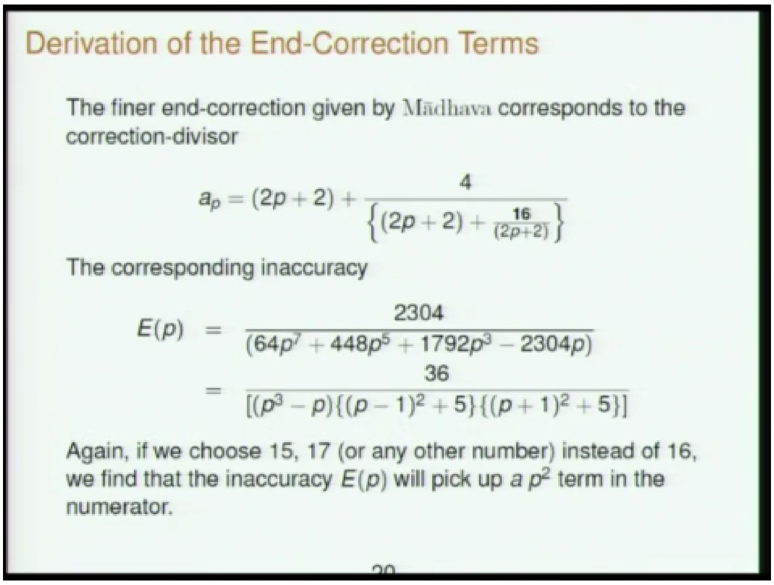

We are also wanting a continue attraction, so we are going to get that, so to be 2p + 2 + 4 over 2p + 2, now you say let me take 16 over 2p + 2, then the sthaulya will go like this. Instead of 16, if I choose 15 or 17, this Ep will pick up actually p square term, not even the p term, it will pick up a p square term in the numerator and in all this, you can discover the Madhava transform series is also coming here, here also terms like of the Madhava.

So, this derivation of the end-correction term, there is very brilliant one done in Yuktibhasa. Many people had misunderstood it that even till 1990s people were writing when Yuktibhasa has not analyzed paper saying, Madhava guess the value of pi as 355/133 or 113 or something like that and using it, he tried to estimate the end-correction term after knowing say 15 terms in the series or 20 terms in the series.

This is very general and an argument that we would be proud to be doing in any course on real analysis today in our colleges. So the general correction divisor of Madhava is of the form, 1 over ap, 1 over 2p + 2 + 2 square, of course this is not stated in Yuktibhasa. This was first stated in a paper by Rajagopal and Rangachari as a continue traction. I mean that this was pointed out by the great British historian of mathematic right side who has edited Thomas Newton's papers in 18 volumes.

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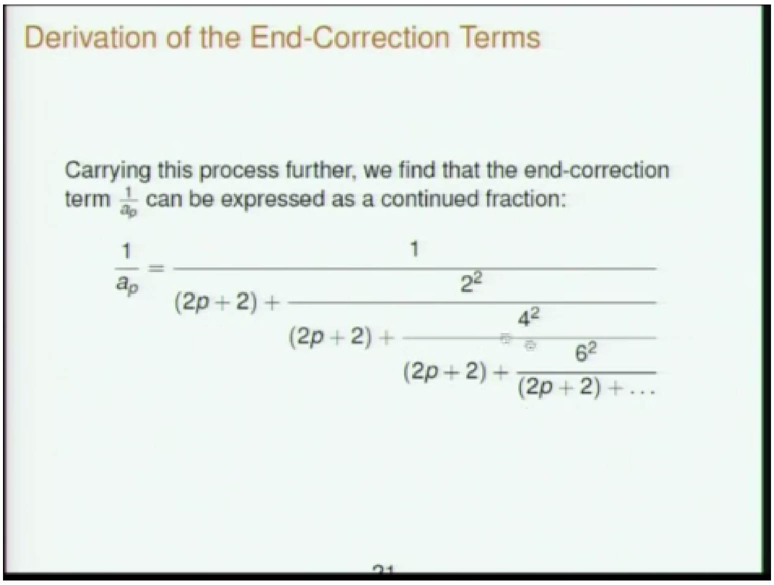

Now, we go to Yuktibhasa's derivation of the Madhava sine series. So, again this is the day of sine today, it is coming in every lecture. So, again the arc bit is somewhere here. Let us say we want to calculate the sine of this arc E2, see which is not marked in this diagram. Again divide that arc bit into n equal parts now. We are dividing the arc into n equal parts and CjCj+1 is the jth arc bit in that.

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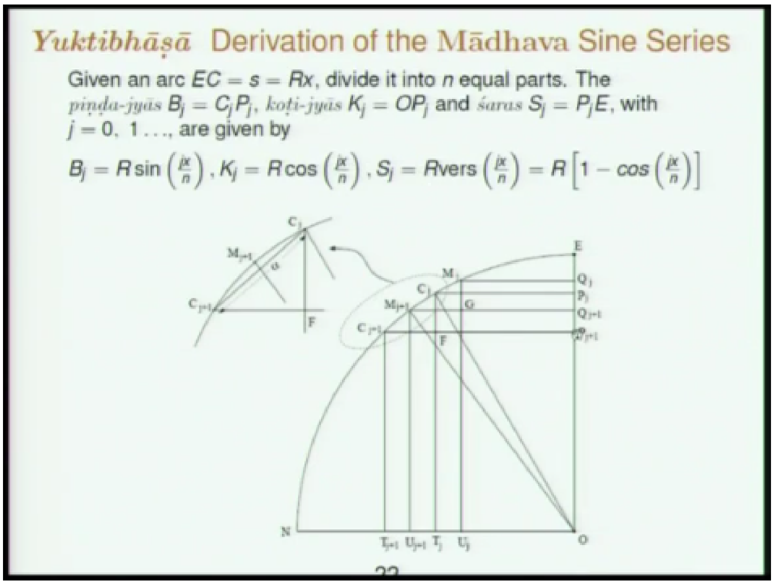

So, S is divided into n equal parts, this is the jth arc bit. So, at the jth arc bit, the Rsine is this, Rcosine is this, Rversine is that. CjPj is the **(FL)** that Bj, Rsine jx/n, OPj is the koti, which is kj Rcosine jx + 1, PjE is the saras Sj which is Rversine jx + 1. Now that bit of derivation which we did not do, we are again going into the famous Aryabhatta second order sine difference formula.

So that is dependent up on the similarity of this triangle, Cj+1 FCj, Cj+1Fcj and Mj+1GMj, Mj+1 or Mj are the mid points of the arc bit, CjCj+1 and Cj-1Cj. Similarly, OQj+1Mj+1 that OQj+1Mj+1 and OPCj, they are straight away the sides are parallel, so those are similar triangles, so using these similar triangles and we will denote the called associated with the arc bit CjCj+1 as alpha.

Alpha is the called associated with the arc CjCj+1, so if we do that we obtained Bj+1 – Bj as alpha/R Kj+1/2, Kj-1/2 – Kj+1/2 which is same as the saras difference Sj+1/2 – Sj-1/2 as alpha/R Bj. So, they are just using simple similar triangles in the example that we consider. So, subtracting Bj – Bj-1 – Bj+1 – Bj is a difference of saras and it = alpha/R square * Bj.

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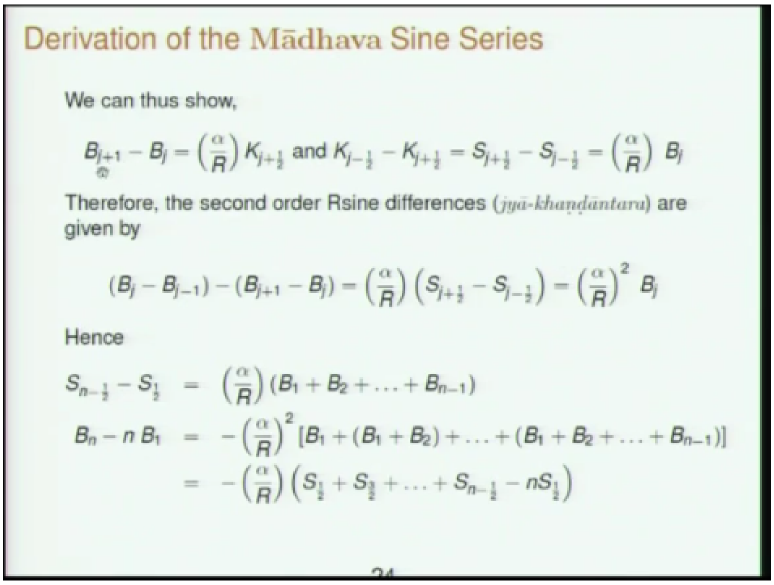

This is the famous Aryabhatta formula, only this alpha/R square is then to be written as delta 1 – delta 2, the first sine difference. Now, we just sum over these sine differences and then we will get Bn – n times B1 by an expression like this and already you can see a repeated summation coming in here. B1, B1 + 2, B1 + B2 + etc Bn-1 and some of the saras, the difference between the last sara and the first sara is of this form.

So, these are the exact result and now, we start taking the limit, then n becomes very large, we start making various approximations and based on that we will obtain the expression for the Rsine series. So when n is very large, Bn is approximately = B. Bn is the Rsine associated with the nth big anyway that will be = to be even that is nothing further to say. Sn-1/2, the midpoint of the last arc-bit can be approximated by the sara, S1/2 which is 1 - cosine theta, then theta goes to 0, this goes to 0.

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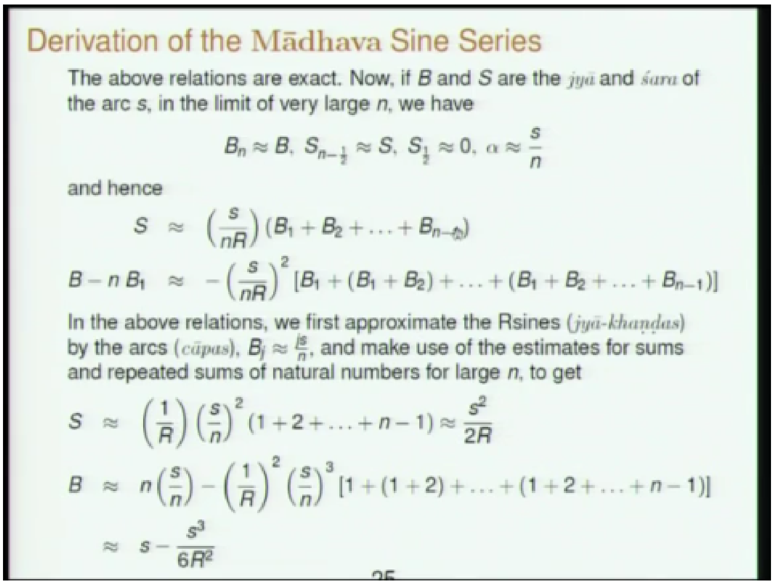

The called associated with the first arc bit when n becomes very large, we calling the arc bit can be equated to each other. So, these two equations are therefore replaced by these two equations. The sara associated with our arc S, the first approximation is this. The (FL) associated, the sine associated with our arc S. Now what is done is each of this (FL) khandas B1, B2, B3 etc successive approximations are made for them that is putting to both these equations.

And from that the new value for (FL) is calculated, from the new value for (FL), the (FL) are calculated again, they are flowed back into the equation and then again, new values for (BL) are calculated, we will see that. So to lowest order Bj, we are taking to be = arc itself, this is of course a very gross approximation, the Bj is as you can see this BjCj, this we are saying = arc ECj is a very, very gross approximation.

So to the first order, we take Bj to be the arc itself, so the jya-capa difference, the difference between the jya and the capa, here it taken as 0 to lowest order. Now, you put this in Bj, j goes from 1 to n, so we put this in these two equation, we get summation of numbers and summation of summation of numbers and varasankalita and the Sankalita estimates will come at the, we put those, so 1 + 2 + etc n-1 that goes like n square/2, 1 + 1+2 * 1+2 etc n-1 that also goes like n cube/6.

So we put those two, we get sara goes like S square/2R, (FL) goes like S – S cube/6R square. You are already seeing the pattern, you have generated the first term in the sine series and the versine series. Now what do we do, for each of the Bj's, we substitute from this equation. So, we take the (FL) != S, the first approximation was B = S, second approximation Bj is we take from this equation, so Bj is js/n – js/n whole cube/6 R square.

So, we plow that into these two equations for Bj and again, summation of integer, summation of summation of squares and cubes of integers will come and you will get the next approximation for sara, the next approximation for (FL) S – S cube/3 factorial + S to the power 5/5 factorial and sara = S square/2 factorial – S4/4 factorial, so again we plow this back and by now, we know the factor.

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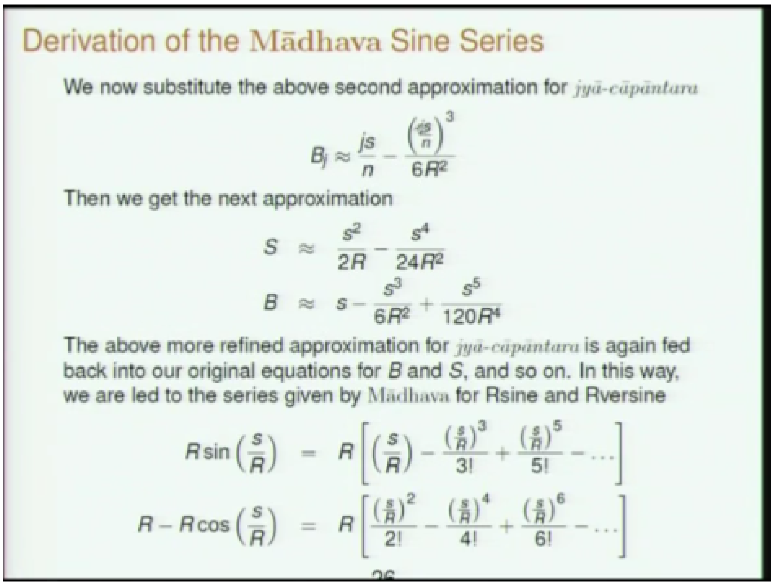

And therefore, we can conclude that by repetition, we will get the Madhava series for the sine and the Madhava series for the versine. So, the series for the cosine is 1 – x square/2 factorial + x to the power 4/4 factorial. So, this was the Madhava's proof of the sine series or this is the Yuktibhasa proof of the sine series.

So apart from this, the only other proof in Yuktibhasa, which involves using this large n limits is the proof of the surface area of the sphere and the volume of the sphere, which I think the volume of the sphere was covered in the ugly of lecture on proofs. So, these are the proofs that are contained in Yuktibhasa, which involved dividing an arc or a line into a large number of segments and working out the complex expression for the geometrical quantity that you want as an algebraic expression.

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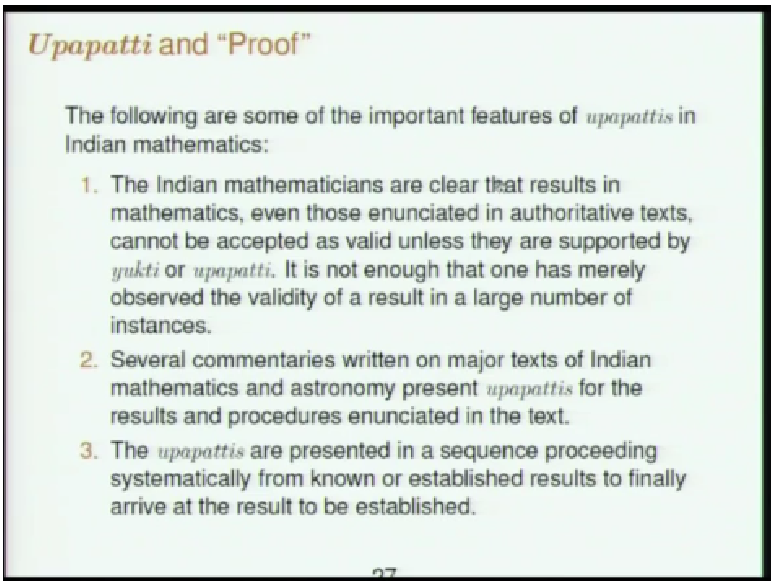

And when playing around with it and only in the end after sufficiently far away in the game, you start taking the limit, when n becomes very large and use the asymptotic forums that I have also been derived in Yuktibhasa and obtained the results and it is a very sophisticated way of doing things as we know. So, we are more or less come to the end of discussion of the various kinds of proofs that are found in Indian mathematical literature.

Starting from the simple proofs of the Pythagoras theorem to the fairly complex and sophisticated proofs of the infinite series for pi and the sine series of Madhava. Let us just summarize again and put these points in my inaugural or the interactive overview lecture also. At that time, it might have not been clear, what we were talking about. So, we go through these points somewhat more carefully now.

So, first point is that I am trying to compare the idea of Upapatti as found in Indian mathematics with the idea of proof that we commonly know or which goes back to the Greek tradition or the modern European tradition of doing mathematics. So, first is that it is clear that Indian text that given results of mathematics which even those enunciated in authoritative texts need some yukti or Upapatti.

And it is not enough that one has merely observed the validity of a result in a large number of instances and the right authorities taken to present these Upapattis by these various commentaries that are written on text. These Upapattis are presented in a sequence and you go from known results to unknown results anything like that and arrived at the result to be established.

Now, the purpose of Upapatti, they repeatedly emphasize is to make you clear what the result that you are discussing or the process that you are considering, remove doubts about that and to sort of make you understand how that in the community of mathematicians make them appreciate and accept the result that you are reposing. Crucially, Upapatti may involve observation or experimentation.

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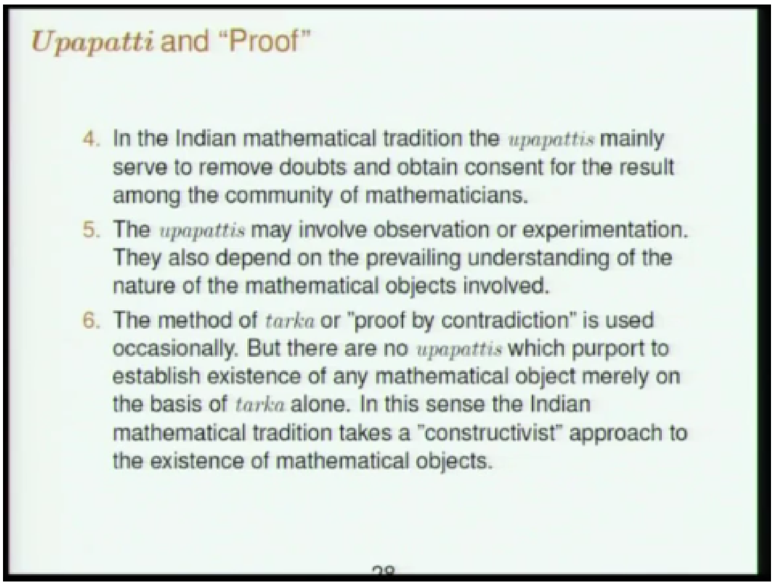

It also may depend up on the prevailing understanding of the nature of the mathematical objects involved. So, it is not a purely a formal or an abstract or a nonempirical exercise. So in that sense, mathematics was not thought as a nonempirical science in India. Mathematical results did not have any other extra level of validity than result in other disciplines. The results did depend upon observational validity of the result that was being enunciated.

Only that you have some more logical argumentation to support your result is all that perhaps extra. More crucially that is of course not a fairly important point in itself, but more crucially, this proof by contradiction is used only occasionally and there are no Upapattis that is we know in Indian mathematics which purport to establish existence of any mathematical object merely on the basis that nonexistence of that object would contradict whatever else that we know.

But, we have no way of establishing its existence by any other direct means of validation. So, this it to some extent, this approach is called the constructivist approach to doing mathematics. So, there is no plain in Indian mathematics that the Upapattis irrefutably they prove the absolute truth of the given proposition and there is no attempt made to write down a set of axioms once in for all and then obtain all result.

The writing down axioms once in for all is important in a mathematical system which depends on the reductio ad absurdum. So, because there the axioms are all put together and anything that contradicts that is always used to prove other results further and further. In a tradition way, reductio ad absurdum is not used in that way. More and more postulates keep coming on the way as you keep building up your mathematical framework.

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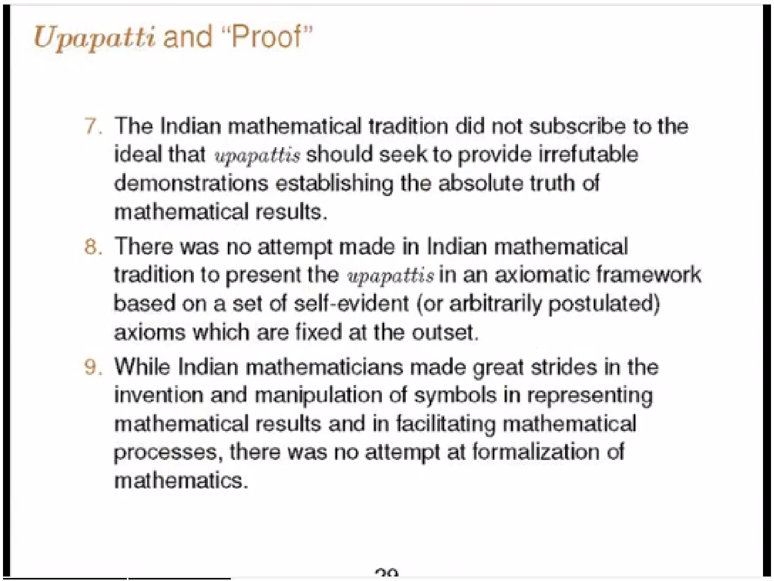

Axioms are not listed right at the beginning and the last thing is that though several symbolic and formal sort of approaches to discussing and formulating mathematical problems were evident starting from Panini in India. There was no idea that mathematics dealt with nonempirical quantities and there was no idea formulization of mathematics. So in this sense, Indian mathematics is got direct of a methodology.

But seems to be an instance of an approach involving a fairly different and a sophisticated methodology that is different from what we have been acquainted with, because we are mostly acquainted with either the Greek approach to mathematics or the way mathematics has been developed in the last couple of centuries in Europe, where Europe also went back to the Greek approach.

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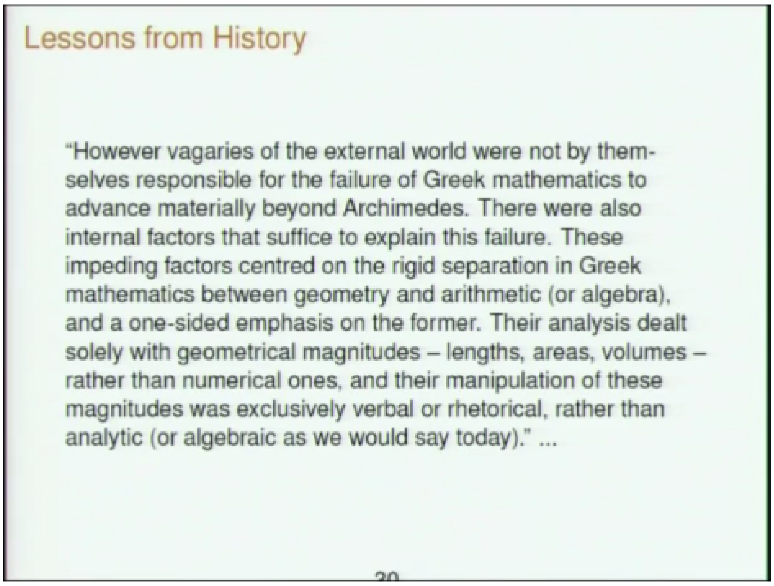

Now here is something that we see about, did really all of history of mathematics did it follow the Greek canon or the Greek method. The Greek method of mathematics starts with the text of Euclid that is, that is a first available text, which was written around 300BC called the elements and it almost ends, I mean Archimedes is the high point and it almost ends with Apollonius around first century BC and it is in Astronomy, it continues with up to **(FL)** in the first century AD.

So it is about 300-350 years that mathematics is done this way. After that, though very high praise is given to the Euclid and the methodology of mathematics that ought to be followed if we were following the Euclidian approach. Most of developmental mathematics occurs in spite of or independent of the Euclidian approach and that is what I think which we should really be very aware of.

Because mathematics is almost presented as they were going from proofs to proofs all these were from the time Euclid to today that is not there it goes. So, here is the statement from a very famous historian of calculus. This is a book written about 30 years ago, so he is telling us what is the lesson that we can understand from the history of calculus in the European tradition, however, vagaries of the external world were not by themselves responsible for the failure of Greek mathematics to advance materially beyond Archimedes.

There were also internal factors that because it sake that the Greek mathematics died because of the persecution of the Greek mathematicians. There were also internal factors that suffice to explain this failure. These impending factors centered on the rigid separation in Greek mathematics between geometry and arithmetic or algebra and a one-sided emphasis on the former.

Their analysis dealt solely with geometrical magnitudes – lengths, areas and volumes rather than numerical ones and their manipulation of these magnitudes was exclusively verbal or rhetorical, rather than analytic or algebraic as we would say. This is more (()) (47:00), but they would not think of the fourth power of a quantity because that cannot have a geometry representation. Indians did specialize in geometry.

But they also applied algebra in a way from the time of (FL) we are talking of na square which is n times another square. Now, the more important point, it is somewhat paradoxical that this principle shortcoming of Greek mathematics stemmed directly from its principle virtue – the insistence on absolute logical rigor. The Greek imposed on themselves standards of exact thought that prevented them from using and working with concept that they could not completely and precisely formulate.

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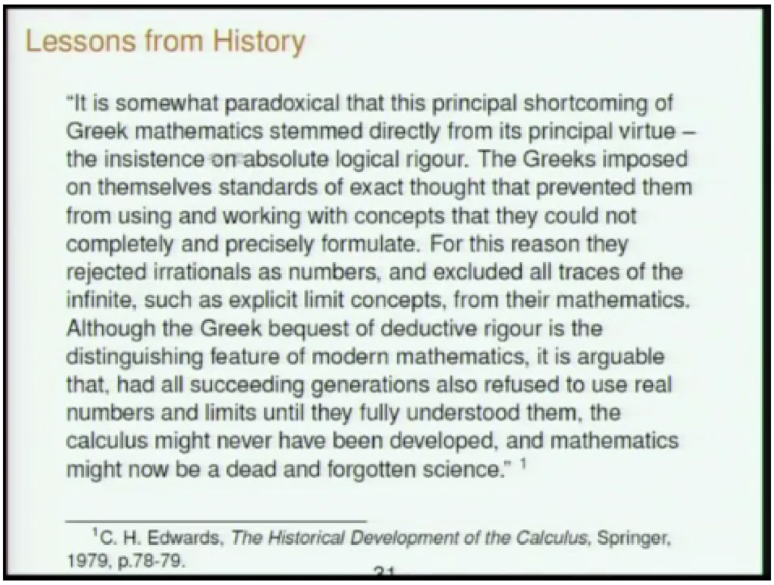

For this reason, they rejected irrationals as numbers, excluded all traces of the infinite and even 0 such as things like explicit limit concepts from their mathematics. Although, the Greek bequest of deductive rigor is the distinguishing feature of modern mathematics, it is arguable that, had all succeeding generations also refused to use real numbers and limits until they fully understood them, the calculus might never have been developed.

And mathematics might now be a dead and forgotten science. This is a very serious book on history of calculus. This is not a philosopher; this is not like George Berkeley objecting to Newton's use of infinites. This is more sophisticated understanding of, because all of the calculus that we know and we teach is the reformulation that has occurred in the last 100-150 years, say ever since the notion of real number as what we formulated by Dedekind of the notion of the limit was reformulated by Cauchy in 19th century.

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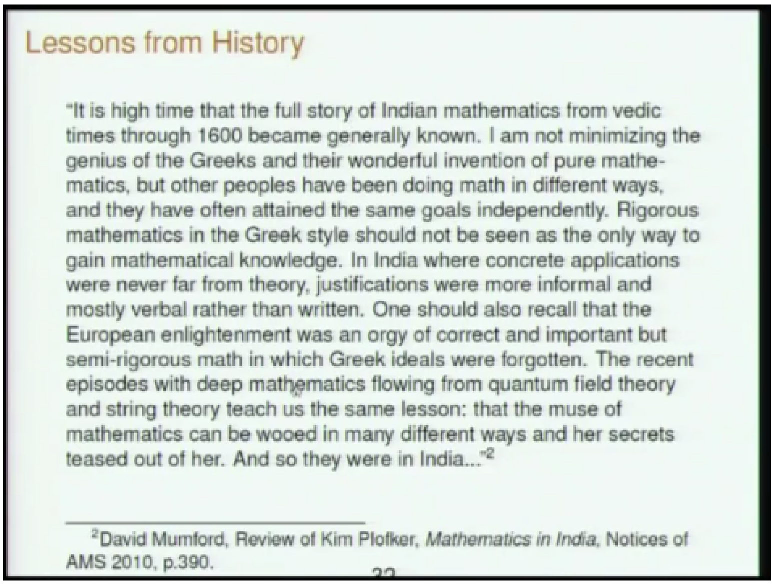

And much more so after the elements of (()) (48:50) which is the elements of the 20th century which reflects the elements of the 3rd century of Euclid, reformulating most of the mathematics inside a rigid beautiful axiomatic preamble and I am quoting David Mumford again to show the same point, not emphasize now by a historian, Edwards (()) (49:13) David Mumford is very eminent practitioner of mathematics.

But, he is aware of the Indian traditions somewhat deeply is the field (()) (49:23) some other contacts, but he wrote a review of this recent book on history of Indian mathematics and then again the main point that he is telling will you, is just Indians discovered several beautiful nice results and what it does convey is, that in all mathematics, need not be done in the Greek way, very beautiful and creative mathematics could be possible outside of Greek preamble.

And it has Indian weed sow in history and it often happens in the history of mathematics that beautiful mathematics does get done independent of the kind of rigorous (()) (49:56) that the Greeks have imposed on it from their times into. So, these are all statements which are sort of perform philosophical significance, but that is not very important even ordinary sense the fact that there has been an Indian tradition of mathematics where things were logically, rigorously derived.

But they did not do so in the manner that we are all aware of in the way you create geometry is written, should show that there are in the different ways of formulating, teaching, doing mathematics and even expanding mathematical knowledge in a valid way. There is another dimension to it like this alternative approach to mathematics put indeed have been very truthful even for contemporary times, only somehow that got cut 2-3 centuries ago.

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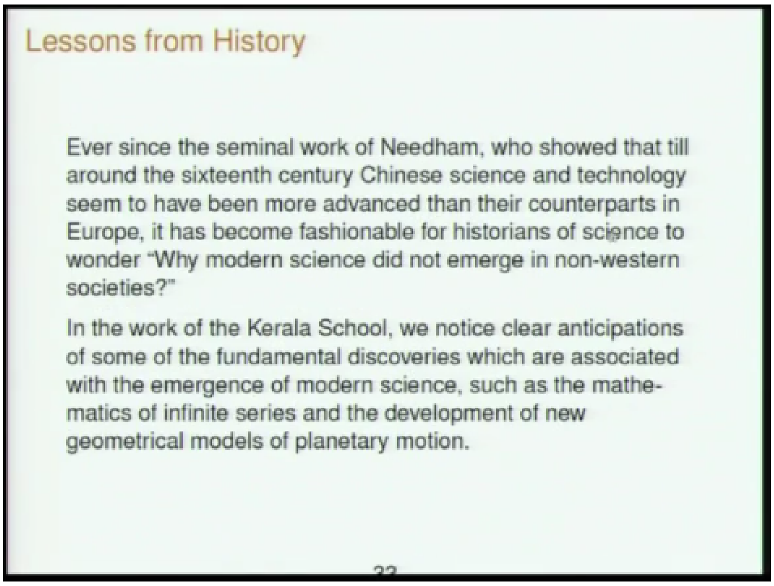

You see ever since the work of Needham, it has been recognized that till around 1600, the Chinese science and technology was much more advance especially the technology and also the science done what reviled in Europe there, but Needham after documented it, ended up with a question that why modern science did not emerge in non-western societies?

Now, in the work of Kerala school, at least we have discussed the mathematics part of it, it tells you some of the fundamental things that was rediscovered in most of 16, 17 and perhaps even 18th century Europe and some of them which were not discovered like the alternative Chakravala algorithm things like that and similarly, in the astronomy many things that were called hallmarks of emergence of modern science like a alternative model of planetary motion etc.

All these were there in Kerala school of mathematics, so one should really in fact wonder first why in non-western societies did not continue with their base of doing science say after

16th or 17th or 18th century that is more a historical question. It is not a question which is entirely internal to the history of the sciences. It is not merely because that these disciplines lacked certain methodological rigor or certain philosophical sophistication that modern science brought in.

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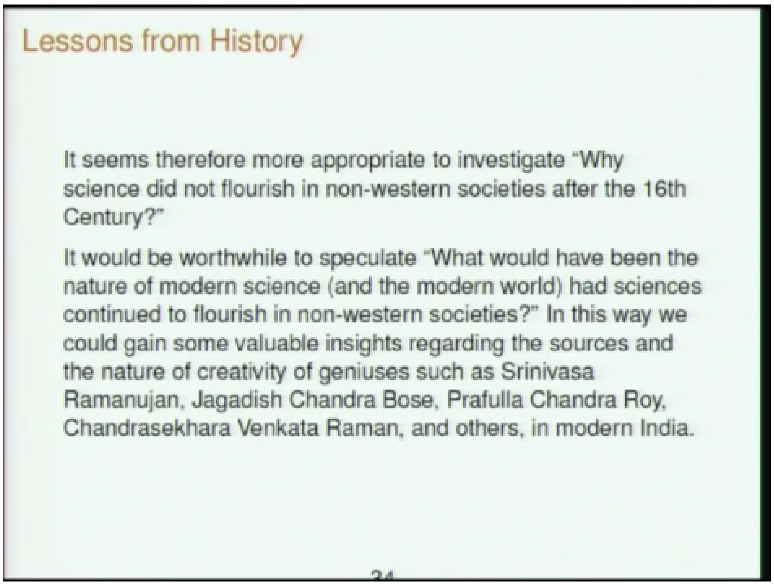

But much more so in a world that is now really becoming mix of various diver civilization, which even more important that we should speculate what would have been the future trajectory of science, it other civilizations continue to contribute significantly to science or we can sort of speculate what will be the future of science if different civilization now start contributing as it seems to be happening in a significant way.

And then only, we will be able to appreciate some of the creative geniuses in the modern Indian times such as Srinivasa Ramanujan, Jagadish Chandra Bose, or Prafulla Chandra Roy or Raman, many others. Now, saying it may be sounding somewhat outrageous but, I am just quoting now a very professional historian of science, whose name you have already heard half a dozen times, 'Takao Hayashi' whose is just scholar of history of Indian mathematics.

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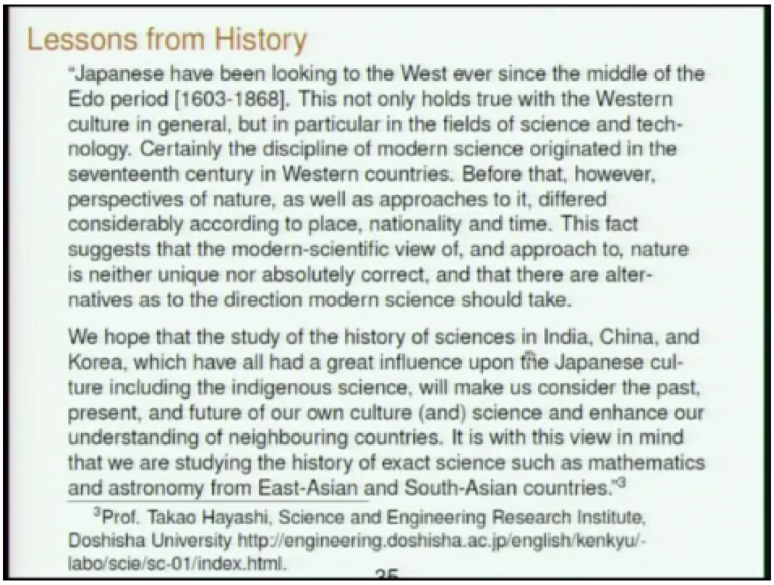

He is also scholar of history of Korean and Japanese mathematics. Basically, he is saying that since 1868 or since the middle of the Edo period, which would be 1700 to 1750, Japan has been following western knowledge, western culture, western technology. He says that the discipline of modern science originated in 17th century does. Before that however, perspectives of nature as well as approaches to it differed considerably according to the place nationality and time.

This fact suggests that the modern scientific view of and approach to nature is either unique nor absolutely correct. Something that should have been obvious to us, except for the fact that while we learn modern science, we learnt no other science of any other civilization and therefore we thought that that is the only way of doing things, I mean other than that there is no profound discovery here.

It arises because of a very deep awareness of a different tradition of mathematics or a different tradition of science. So, suggests that the modern-scientific view of and approach to nature is neither unique nor absolutely correct and that there are alternatives as to the direction modern science should take or could take and then he says we are studying history of India, China, Korea, and this put in his web page, okay.

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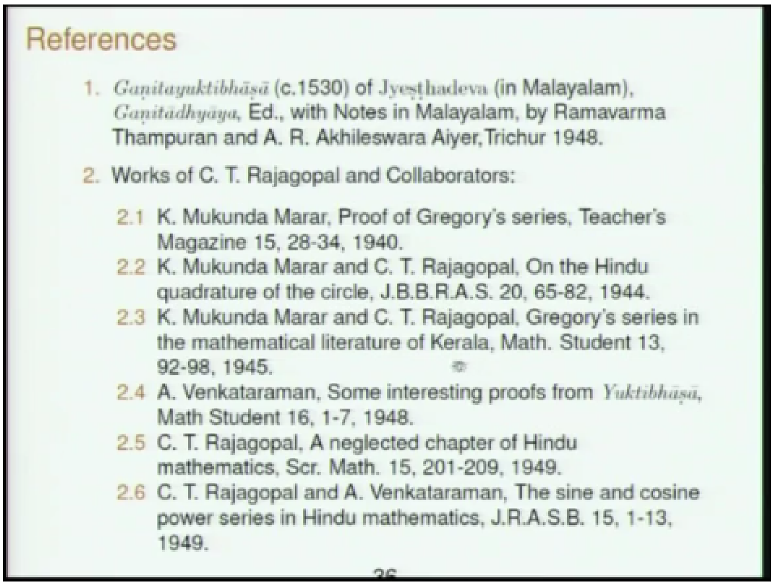

So with this sort of the technical aspects of proof and its philosophical implementation for the developmental mathematics in India more or less, we have had a lower view. There are many, many issues which are unclear and in what I have tried to present, maybe I have tried to show the picture somewhat in a very elongated way in somewhat in one direction, but analysis of many more texts, which contain proofs in Indian mathematics and astronomy.

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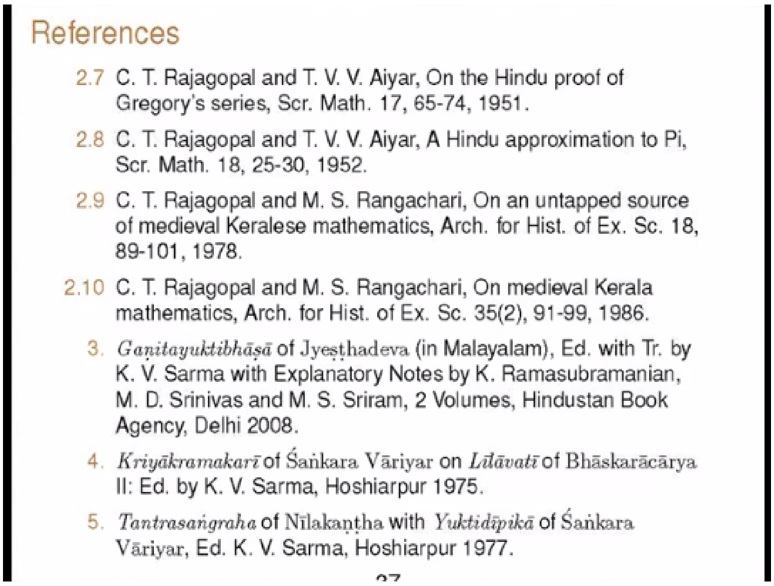

We will be able to give us a more balanced overall view. But, the view will not be that Indians were following Euclid or Indians were following (()) (55:19) or somebody. They were having their own methodology for doing mathematics, which was fairly sort of rigorous, accurate. It would have been restricted in its outcomes like any other approach to mathematics would have been.

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But, it was creative all through. It was not that mathematics topped with the Greeks and then had to sort of start on a different put in the renaissance times and get reformulated in the Greek where in 1950th century. The same idea of regard more or less persisted in the way Indian mathematics moved and I think with this comment, I will stop this lecture. Thank you.