Mathematics in India: From Vedic Period to Modern Times
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Lecture - 37
Proofs in Indian Mathematics-Part 2

So, the topic on proofs Indian mathematics so is covered in three parts. So, the first part was covered by Prof. Srinivasan.

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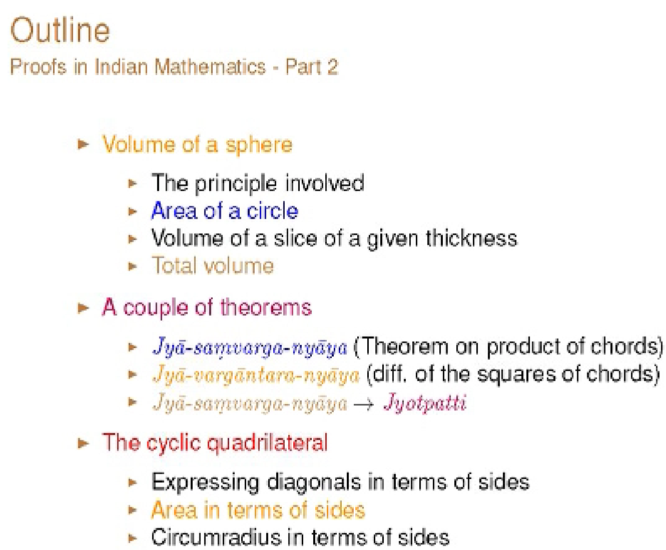

The second part in proofs Indian mathematics. So, I would be primarily discussing two topics one is finding the volume of a sphere, so how did Indians do so this is one topic which we will be discussing and then this will be followed by an interesting discussion on the cyclic quadrilaterals. So, cyclic quadrilaterals have been dealt with in great detail by Jyesthadeva in his Yukthibhasa.In fact both the proof that I am going to present are from Yukthibhasa.

Before starting the discussion, I would just like to give you an overview wherein I want to introduce these terms which are not that is not familiar, so one is this jya samvarga nyaya. so the term jya has been introduced as sign okay. So, samvarga is product, so this has been repeatedly used jya samvarga product of science. Okay this is one thing which will be discussing this will be one theorem and jya vargantara nyaya jya varga is sign square kind of a thing.

So antara is difference so jya vargantara nyaya refers to the difference in this squares of science so basically there will be two important formulas that will be presented. And based on those formulae some very interesting results have been presented in cyclic quadrilateral, the results have been primarily to express the diagonals in terms of sights as well as the area in terms of the sights, this is very interesting as you will see.

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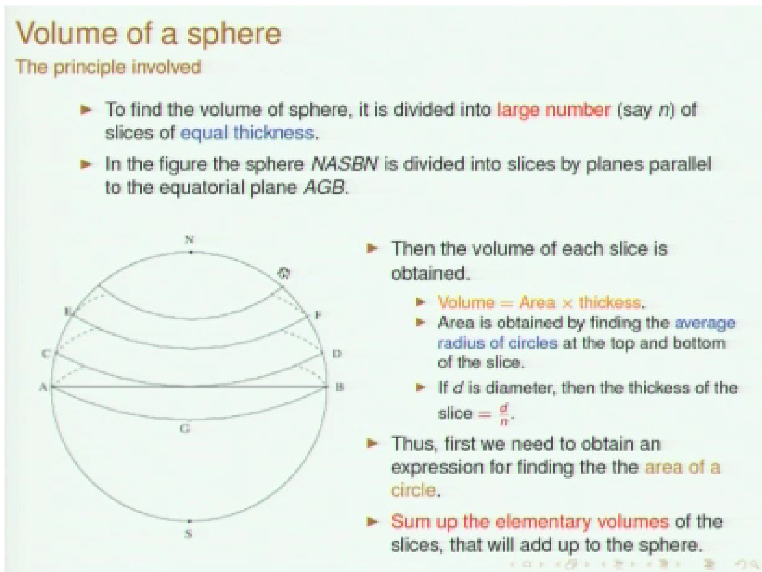

So, I will move on to the first topic volume of a sphere so to find out the volume of a sphere so what is done is primarily a sphere is taken and then it is sliced it into n bits. So, you just think of the sphere of diameter D what is done is you slice into n bits of equal thickness and so this is the first thing that one needs to conceive off so conceive of a sphere which is divided into n parts all equal thickness n slices then the idea is to find out so the volume all this slice first.

So, that is what is done then you just find out what is going to be the sum of this so this the general principle that is adopted fine? So sum of these elementary slices between given the volume of the sphere so this is the overall procedure so how do they do that.

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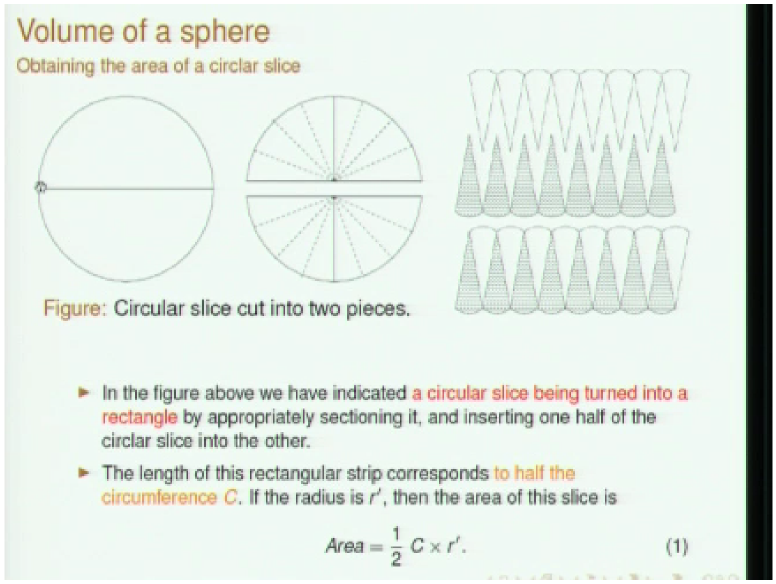

So, far doing so you conceive of one slice and then divide that slice into 1/2 so then you just cut it into this kind of almost similar triangular kind of a thing fine they are unfolded. So, you will have one piece like this and you will have the other piece so just insert them and what you will be getting is primarily a rectangular kind of a figure. So, if the number of bits in to which it is sliced is large then you will almost have a rectangle.

So, what is the area of this rectangle piece? So, area is half times the circumference * r prime. So, where r prime refers to this radius so why did I write r prime so in fact the slice if it is of significant thickness so then so you have to take this sort of average so of the top and bottom so that will be the r.

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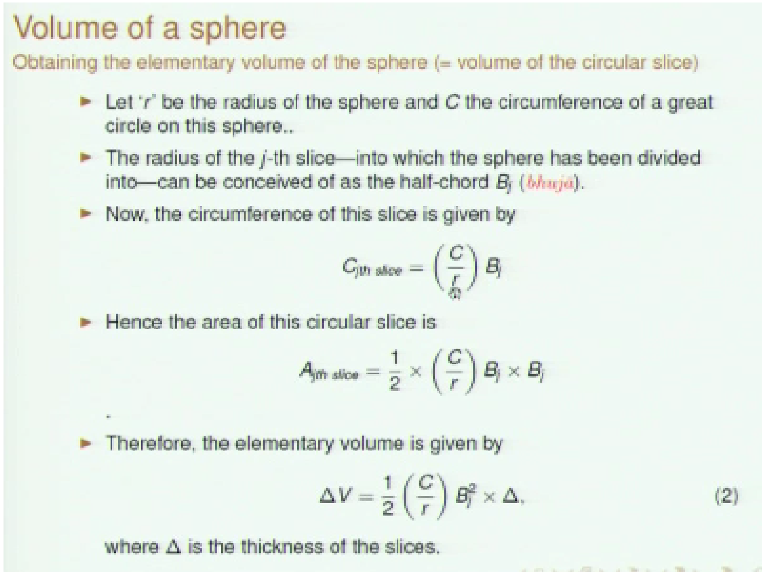

To obtain the elementary volume so all that that needs to be done is so this C/r, so always they express c/r basically today we know as 2 pi kind of a thing okay. So, if you want to relate so you can keep it that way but this is how they have been expressing. So, consider say for instance maybe in the next slide.

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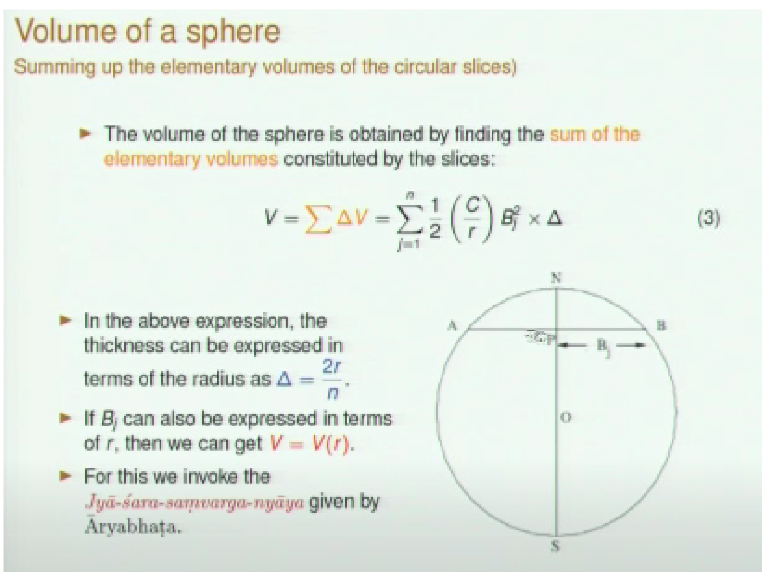

So, you have chopped off so let us see that this this is what is the radius of the slice. Okay so that is what I am referring to as Bj, I use the notation B because they use bhuja. So, if you think of this so OB is karna and PB is bhuja. So, if you consider the triangle OPB then PB is bhuja so that is basically the r sign. So, the circumference of this slice is C/r times this bhuja and the area of the slice so is 1/2 times*Bj is this clear? So this is what in the area of the slice is.

Then the elementary volume is obtained by simply multiplying this area*the thickness of it so then what do you do. So, you have to just sum these elementary volumes that gives you 1/2 of c/r*Bj square *delta. So, you have to find out Bj Square in terms of radius that is all the problem is. So, if you look at this right hand side of this equation so all that you have so C/r times delta, so delta is 2r/n so that is the thickness so you have to find an expression Bj square.

So, if Bj is also expressed in terms of r then you get an expression for the volume in terms of r. So, for this we invoke this Jya sara samvarga nyaya which we discussed when we discussed Aryabhata.

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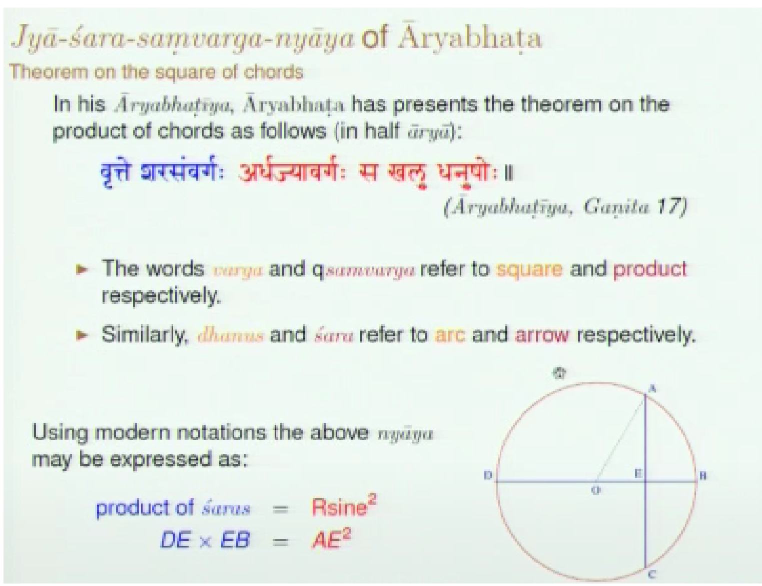

So, basically we need to get an expression for Bj so recall this nyaya (FL). So AE is b the bhuja so AE square=DE*EB fine. So, let us what is this EB so EB in this problem.

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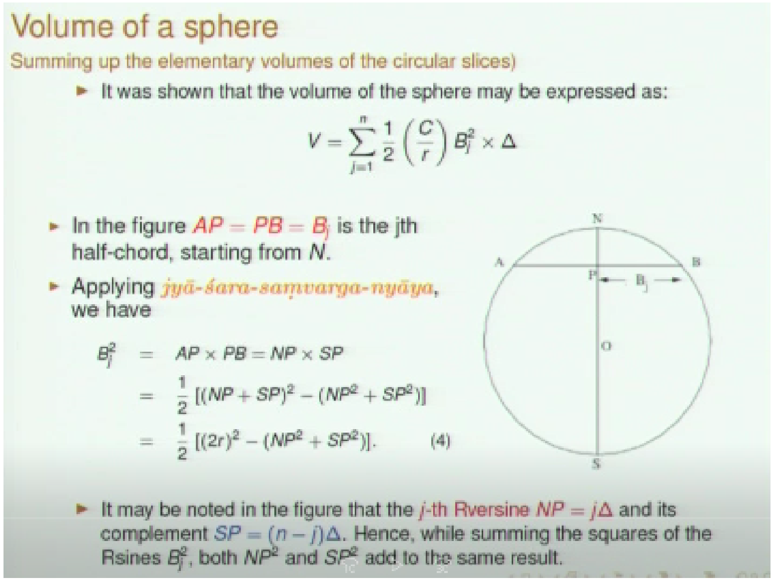

Since we have sort of sliced into various segments of equal thickness delta so if this is the j th slice from this point N so what you would have is NP. So, basically what you get is so this PB square is NP times this P right. NP+SP so these basically turns out to be 2r so that is fine. So, the other part NP if you look at so NP=j times this and SP=n-j times the slice so when you want to sum up for this as well as this will add up to the same right.

So, here you will have so NP square is 1 times delta+2 times so you will have 2 times delta+3 times delta so the whole Square and here it will be N-1 and N-2 so basically they sum up to 1 at the same thing.so what we have is.

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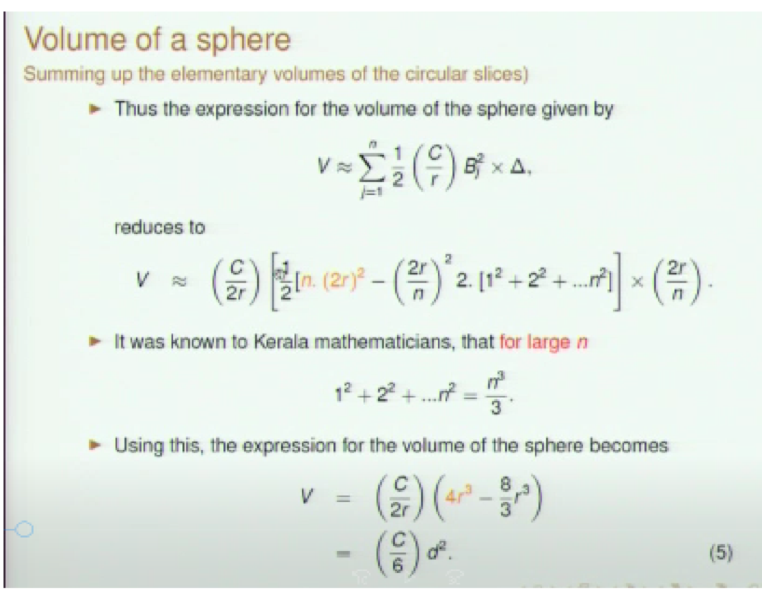

So, this Bj square is N times 2r square. So, if you look at this expression so you are going to sum up from j=1 to n **"Professor-student conversation starts"** No I am just saying so if you have to think r prime so you have to just take the mean value so that is why I just said is very decided so if you think of infinitesimal sort of thickness then I think you can just use it as r **"Professor-student conversation ends".**

So, here so here Bj square is made up of two parts ton times 2r square, so since I is 1 to n so you will have n times this contribution. So, n times 2r square and this basically reduces to 1+2 square+3 square+n square this sum in fact maybe we will be discuss for Mr. Srinivas in this third Kerala mathematician we have discussed so sum of all powers of n for large n . So, in fact this is a very important risk.

Arriving at the infinite series for various expressions trigonometric function as well as for pi. So, this summation of 1 square+2 square+ and so on for n square, so for large n it goes as n cube/3, so this result was known so once you plug it in so what you will have is 4r cube-8/3 r cube and this will turn out to be 4/3 pi r cube as we understand pretty well. So, that is a very neat proof which has been presented by Jyesthadeva in his Yukthibhasa.

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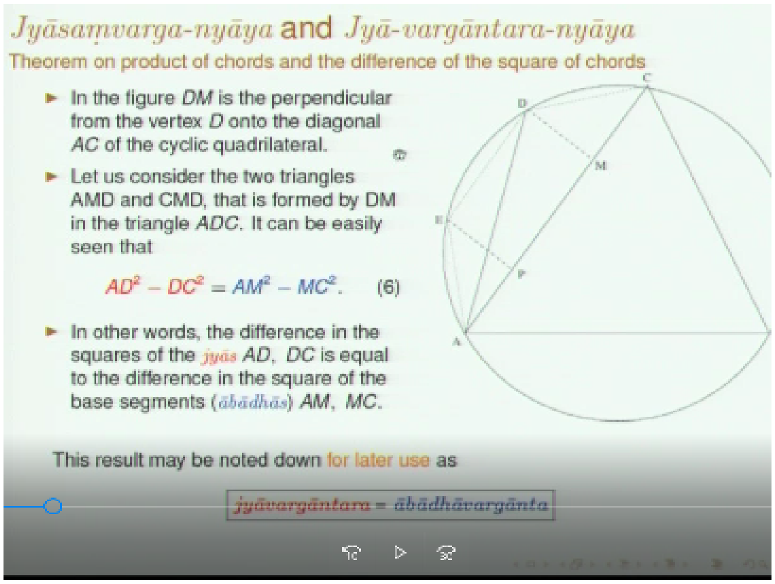

So, now I move on to the topic of cyclic quadrilateral so, before we take up the cyclic quadrilateral and the results some interesting results so I want to present certain important theorems so which will be to be frequently invoked in trying to prove the result. So, these results are general results which have to do in the chords in circle so it has nothing to do with specifically the cyclic quadrilateral for any chord you just consider in circle.

So, these results will be applicable and any segment obviously in a cyclic quadrilateral is going to be conceived as a chord, so that is why we are interested in the results in chord why are we are interested so this is precisely the reason. So, since all the 4 segments are going to be chords so if you know these are the results then you can so easily invoke this result to prove certain things with reference to cyclic quadrilateral.

So in this figure so conceive off this triangle ADC. So, this simple relation AD square-DC square=AM square-MC square. So, this is obtained by dropping this perpendicular DM in this triangle ADC, so DM square so is basically AD square-AM square and this is also=CD square-MC square and therefore you will get this result straight away equation 6 fine. So, in other words.

So, if you had seen earlier discussion on (FL) so basically this 2 segments AM and MC are called as abadhas so what you have is so this this AD and DC are jyas fine AD and DC are 2 jyas for chord full chord. So, AD square-DC square is jyavargantara so the difference in the vargas of the jyas=the difference in the abadhas. So, this is one result which will be used in various applications so that is why I wanted to mention this.

So, this result has to be kept in mind. So this has nothing to do with the cyclic quadrilateral this has to do with the chords. So, the difference in the jyas square of the jyas=to the difference in the abadhas.

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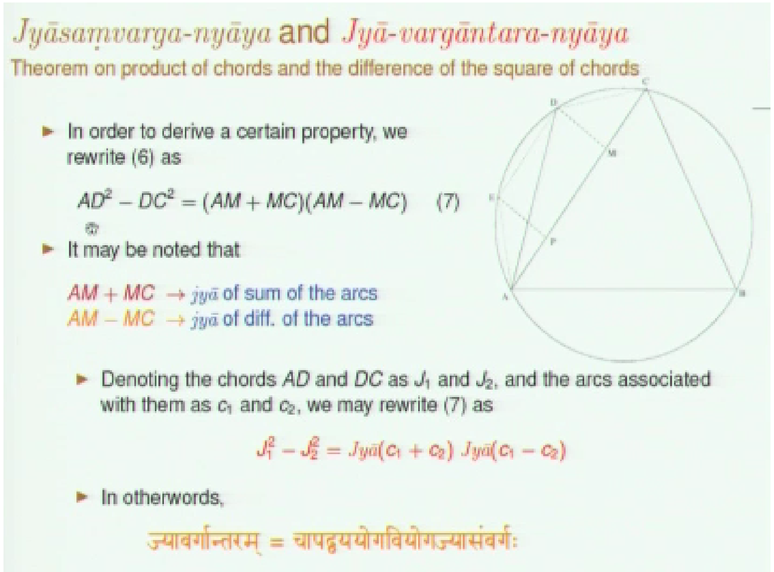

Now we are going to rewrite this equation at arrive at some other interesting result so AD square-DC square=AM square-MC, so you write it in this way so AM+MC*AM-MC, this AM+MC is the Jya of the sum of 2 arcs. Now we will speak in terms of arcs. So, this AD and DC are 2 arcs so you can this is chord and you can move this way this is arc, so I use the notation c for arc because (FL) understand,

So, AD the arc AD is denoted by C1 and arc DC is denoted by C2. So this AD is Jya so we use the notation J for that, So, AD square-DC square is Jya 1 square –jya2 square this is a very important result which will be very frequently using in a derivation which we will be dealing with soon. So, the difference in the squares of the jyas=so c1+c2 see AM+MC so Jya of C1+C2 *Jya of C1-C2, so this is the result and this we refer to as (FL).

So the difference in the squares of the jyas =jya of C1+C2+jya of C1-C2.

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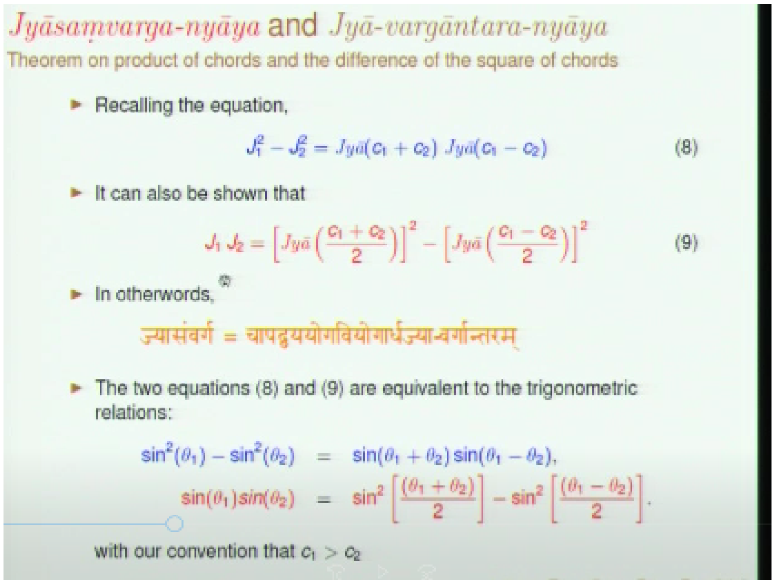

So, this another result, so which actually can also be shown is the following this is see the previous result has to do with the jyavarga jyavargantara, so this result is jyasamvarga, so this is product of 2 jyas, so these jyas will basically form the segments of the cyclic quadrilateral. So that is why we have this results in place before we move on to this so this of course one can prove this.

So, this is not very difficult to see but I will keep the proof of this but you can see that so in some sense so this is sort of related to this itself you can see that see jya product of jyas. Okay so you see jya square difference and here also what we do is jyavarga and if time permits I will try to prove this towards the end. So this is jyavarganthara, so the difference in the squares of the 2 jyas so product of J1 and J2 is this result.

So, jyasamvarga so is this (FL) C1+C2/2 and C1—C2 see in fact these results might have been taught even in schools so you can see that they are really close to the results which we generally study so sin of CDC1+D/2 and C-D/2 right, so sin square C-sin square D to be this. So, this is what it is done so this is a standard result so it is very important which we will be making use of in trying to prove.

Some other interesting results for getting the expression for diagonals in terms of the sides and so on. So, these are the two results so which are well known in some other terminology as we so here are basically jya is the sign and (FL)is we use theta and here they use (FL) arc length okay that is the so these are two known distance.

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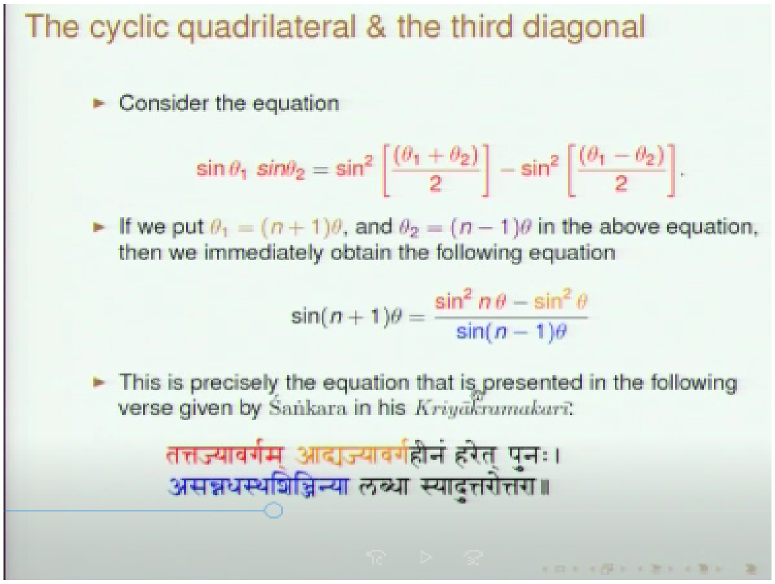

So, from this result an interesting way to obtain sin table has also been presented by Sankara Variyar in his commentary on leelavathy which is called Kriyakramakari. So, while he discusses the cyclic quadrilateral after doing that so he presents so we just recall this so sin of C* sin of D is this , so this is written as Jyasamvarga=this. So let us keep this in a more familiar forum.so if you put theta 1 in n+1*theta and theta 2=n-1*theta in the above equation.

Then it readily simplifies into this column so sin of n+1*theta so if you look at this equation, so if you know the previous value then you will be able to immediately get the next value of this sign. So, (FL) that is what he says in constructing the sin table okay. So this is one way of doing and this has been stated in this verse by Sankara Variyar see (FL), so (FL) refers to some sign of n theta, so (FL) so that particular (FL) is square so (FL) you divide by what.

So, there is a mistake here (FL) means one below this so if you consider n sin of n theta, so n-1 times theta so that will be the divisor and what you get is (FL) so you will get the successive jyas, so if you know the value of sin theta then you will be able to get the entire sin table constructor using this.

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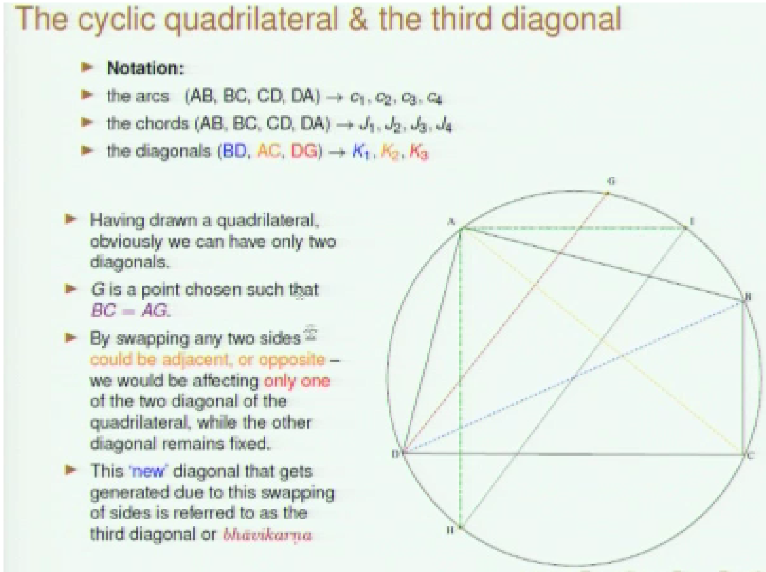

Okay so cyclic quadrilateral in fact yesterday someone was asking what do you mean by the third diagonal of a cyclic quadrilateral after all when you have a 4 sided figure you can have 2 diagonals so we will introduce that now. So, ABCD is the cyclic quadrilateral and as before we will have this Ab is the first side which I will denote as J1 so BC is J2, so CD isJ3 and this DA is J4.

So, the corresponding chords, so these are the sorry these are the arcs, ABCD these are basically the Jyas, they are the chords they form the segments of the cyclic quadrilateral, So, the arcs AB BC CD and they are denoted as C1 C2 C3 C4 the (FL) and DB is a diagonal, AC is a diagonal so these diagonals we denote as K1 and K2 fine. Now I choose a point G such that BC=AG, I choose the point G such that the arc length BC=arc length AG.

And so which basically amounts to swapping two sides so this b is transported to this point so if you move this B to this then basically the segments the length of the segment will inter change right. So, in doing this so suppose you think of AG instead of B so AG C and D this will be the new quadrilateral but this is basically of the same area so this will not have anything different all that we have done is swapping.

In fact, the expression for the area of the quadrilateral will be in terms of this diagonals, so diagonals we denote as K1, K2, K3 any kind of swapping can be done any segment can we swapped so you can think of transporting this to other side other to this side so you do anything so 1 diagonal will remain invariant the other diagonal will change. So, this third diagonal will be referred to as bhavikarna in the text right.

So, this DG that you obtain is the third diagonal, so by changing inter changing the sides. So, either you swap or you just take it up opposite whatever you do whatever you get is the third diagonal and this is generally are referred to as bhavikarna, so the karnas K1 K2 and K3 are referred to as one is isthakarna so the other is itharakarna and the bhavikarna. Any kind of change that you do so you will get this diagonal as the third diagonal.

The length will be same so this much has to be clear **"Professor-Student Conversion starts"** **Y**es bhavikarna is a (FL) so when you inter change the sides so 1 diagonal will be invariant the other diagonal that you get is bhavi means it will come, it will come in the sense of after interchange whatever you get it is referred to as bhavikarna fine **"Professor –student conversation ends".**

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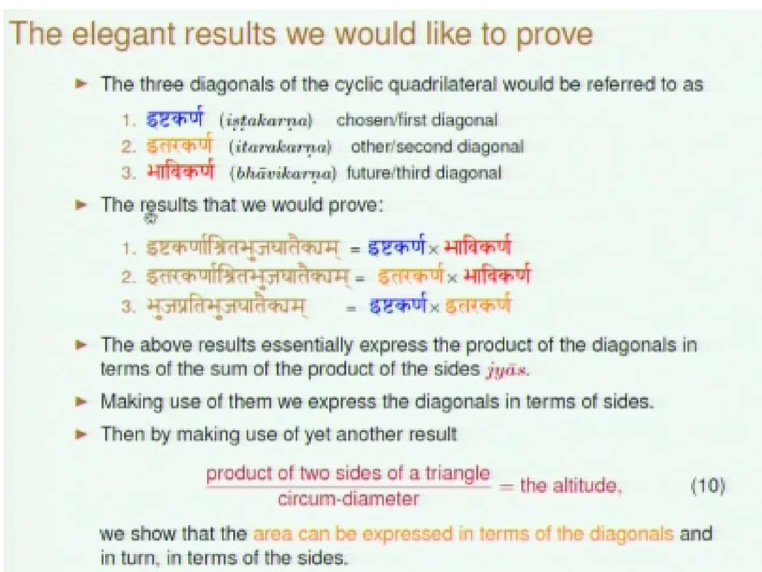

So, the result that we are going to prove is just summarized in this slide so I am start working on the board in fact so istakarna is the chosen diagonal or you can call it first diagonal so itarakarna you call it second diagonal. So, given a certain quadrilateral figure and the third which is obtained by swapping the sides is referred to as bhavikarna. So, what are we going to prove so in fact has been nicely stated so istakarna*bhavikarna is (FL).

As you can see in the right hand side what you got is the product of 2 karnas, the product of two diagonals, the left hand side is (FL) so given certain karna, see you can imagine this way so suppose if I chose this diagonal DB, the end point of this karna are tagged to 2 segments, so this end point 2 the other end point is other so what it says is so bhuja is side (FL) is product (FL) is sum.

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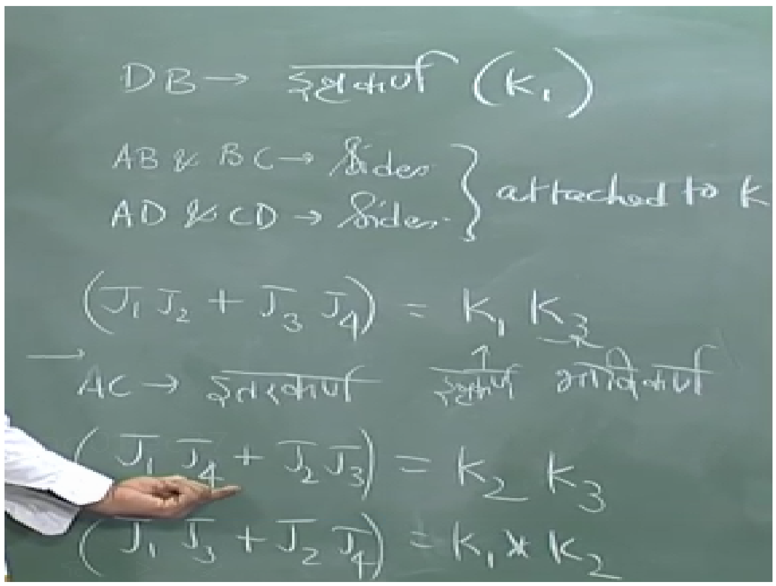

So in this diagram so what does it mean so DB is istakarna which we call it as K1 denote so this (FL) are AB, so AB and DC are the bhujas sides attached to one end of this karna, the other 2 are AD and CD are the other two sides attached to K1. So, all that it says is (FL) so is J1 J2 so (FL) is product and J3, J4 the sum of them, so (FL) so both of them (FL) is add so this is=K1*K3 so that is istakarna and this is bhavikarna.

So, what you see is the product of 2 karnas have been expressed as the sum of the product of the sides so (FL) so is istakarna*bhaikarna so similarly (FL) itharakarna*bhavikarna.So, if you look at this diagram so itharakarna is AC, so AC is ithrakarna, so there the product is so J1, J4+J2 J3 and this will be=K2*K3. So, you can easily write the third one so it is (FL) so take the opposite sides so J1, J3+J2, J4.

So, this is K1 times K2 so these are the 3 results that will be very quickly deriving now so this is what you stated the overall principle is so if you think of these 2 equations. So, you can take a product of these 2 and then you will be able to see that you get K1 square and K1 square * K2, K3 will be there so K2, K3 you can once again express in terms of the sides. So, you will be able to get expression for each of these karnas purely in terms of the sides of the cyclic quadrilateral.

So, they prove it very simple and nice proof and this proof is just based on the 2 relations that I stated before so I will just work it out so let us see this relation.

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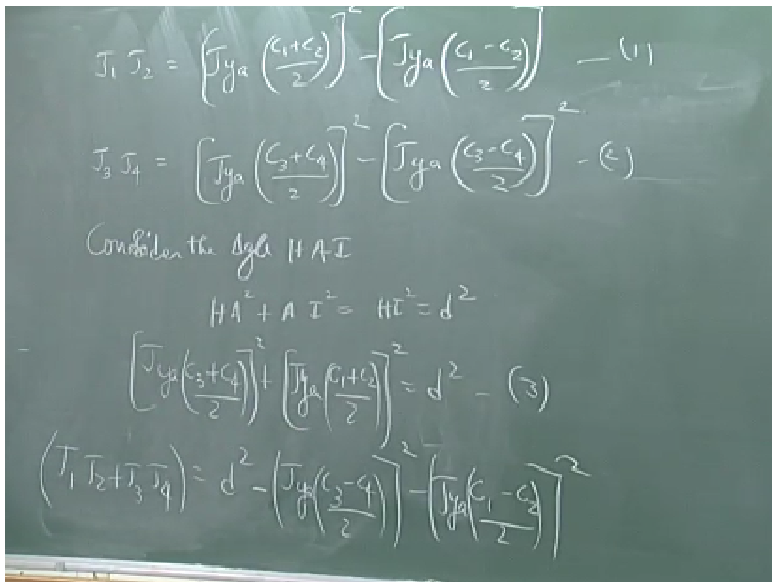

So J1,J2 Jya of C1+C2/2 the whole square-Jya of C1-C2/2 , so this is one relation and similarly I write J3.J4 Jya of C3+C4 to so the text actually said the sum of the products of the Jyas=the product of the karnas , so this is what you have to just prove. So, let us have this diagram in front of us. If you look at this so what you are basically looking at is this section so this AD is J4 and AB is J1 okay you consider the triangle HAI.

I will explain what this point I and H are, this point I is chosen such that so it is at the center of GB. So, which essentially means this point I is mean is basically the center point of the 2 arcs so C1 and C2 right. Similarly, on the other side you choose point H such that it is the center of this ADC, so you divide this into two things so cut across this diagonal AC, so the center point is AI and the center point is H.

So, this center essentially means AI=C1+C2/2, so similarly AH=C3+C4/2, so this is clear now consider this triangle so obviously HI is going to be the diameter. So, since the center of this point and the center of this is chosen any line drawn is going to be so diameter of the circle. So, this HI process through the center of the circle now consider the triangle HAI HA square+AI square is obviously this is that.

So, consider before I add this I just wanted to have this result in place so that you can just see immediately what you want to have consider the triangle HAI so we have HA square+AI square =HI square =D square what is HA and AI are, so HA can be written as Jya of C3+C4 +Jya of C1+C2* the whole square=d square. So, now we have these two expressions where in this occurs right so when I add these 2.

So, this is 1 and this is 2 and this is 3 so using 3 in 1 and 2 so we have J1, J2+J3, J4=D square – Jya of C3-C4/2-Jya of C1-C2. Now we are going to consider one more triangle, so we are going to express so D square so-this so I want to sort of eliminate so that is why I am just going to use this for this so D square-the difference in the Jya C3 C4 let us consider this triangle HDI, HDI okay, so this is obviously a right angle triangle.

So, what is HD, HD is Jya of C3-C4, so this is Ah is the sum of this and DH is the difference of the arcs, So C3-C4/2 the Jya so this if you consider the triangle HDI obviously it is a right angle triangle so this D square- Jya of C3-C4 square so that is going to be DI square so this gets eliminated.

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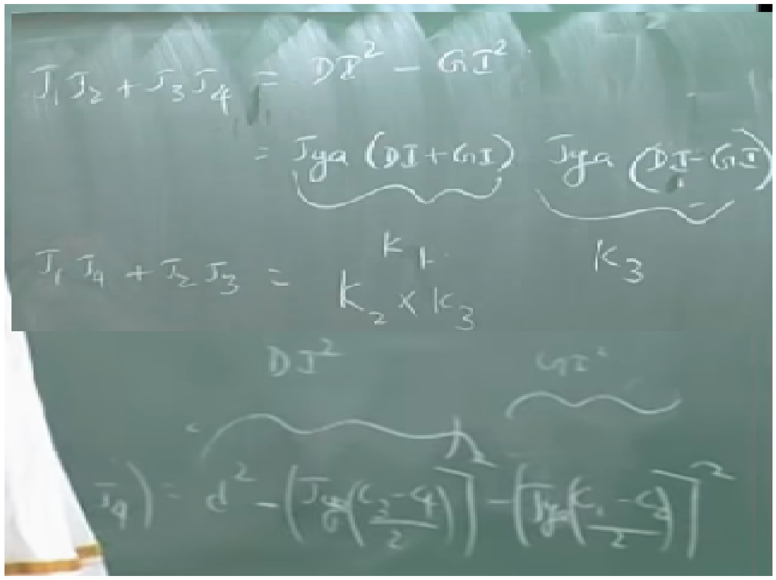

So, these two so they get eliminated with DI square and Jya of C1-C2 so this can be conceived of GI square, so we have J1 J2+J3 J4=DI square- that is all we are more or less true so now you invoke the other nyaya see we have J1 square-J2 square. So, they are basically chords, So, DI and GI are basically chords so all that it is says is the difference in the chord square can be expressed as the sum product 2 Jyas which actually sum up the arcs right.

So, I will just write it as so Jya of DI+GI *Jya of DI-GI, so once again we will go to the figure so DI+GI , so DI+GI so this GI is same as BI so that is the midpoint and therefore what I have is , so the Jya corresponding to the arc so DI+BI so which is nothing but DB so which is one of the diagonals K2 , so this is K2 and you can see what the other is going to be so this is DI –GI.So, this is basically DG which is the third karna bhavikarna K3.

So, thus you have seen that the sum of the product of the sides so is basically the product of the diagonals so this will be true of all other karna and therefore so we have shown this particular case istakarna*bhavikarna is this (FL) and one can show that in a similar way that this is going to be the product of the other sides so (FL) will be K1 times so this is K2 itharakarna*Bhavikarna K3 fine we can just write it down.

And so itharakarna so in the diagram so itharakarna is AC. So, this (FL) is going to be J1 and J2, so for this karna at the point A these 2 are tied up and here these 2 are tied up, so it is J1.J4+J2, J3 and this is so this will be so (FL) so I think this should be K1 I made a mistake this this is K2*K3. So, that was basically K1, so K1, this is K2*K3 so you can see, so once you have this.

So, then so you know you take a product of these 2, you get K3 square and K1, K2 you have an expression so you will be able to get all the products so each of these karnas can be expressed in terms of the Jyas the sides of the cyclic quadrilateral. Now the other result so making use of them express the diagonals purely in terms of the sides then so we can express the area of the cyclic quadrilateral also in terms of karnas.

So, karnas can be expressed in terms of sides and areas can be expressed in karna so which itself so you can see that the expression for area can also be expressed in terms of the sides of this quadrilateral this is also nicely proved. So the area of the cyclic quadrilateral can be conceived of as the sum of the areas of the 2 triangles DAB and VCD and this is 1/2 times.

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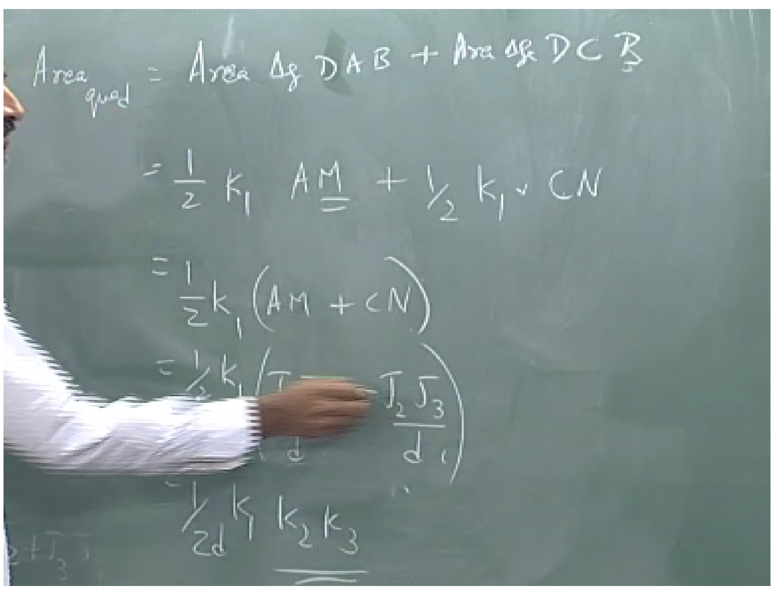

So area of the quadrilateral=area of triangle A DAB+area of triangle, so DCB yeah so this is 1/2 times so DB so is K1 okay so that is karna 1 so 1/2 times base * height, height is the perpendicular to be dropped from suppose if there is a point so for you drop the perpendicular from this point and let this meet at say M. So AM , where M is foot of the perpendicular on K1 and +1/2 times again K1*suppose if you drop a perpendicular from C to AB K1.

So, let it be N okay so CN CN so this is what we have, so this is 1/2 of K1*AM+CN. S this Am this result so we know that the product of the sides of the triangle so divided by the circum diameter gives the altitude. So, AM and BN are basically the altitudes and we use this result. So, the product of these two sites so in the in this case so suppose you are dropped a perpendicular from A to M.

So, the two sides are J1 and J4, so this is ½ K1* so J1 J4/D+the other 2 so J2 J3/D so this is 1/2 1/2D times see J1 J2 and J2 J3 so J1 J2 and J2 J3, so this are we right sorry AM is this J1 J4 this should be so this should be J1 J4 and J2 J3, so this is J1 J4 so this is itharakarna (FL) so this is itharakarna and so therefore what we will have is this is basically so K1 K2*K3, so this will be J1 J4 J2 J3 is K2*K3.

So, therefore area is the product of the 3 diagonals/circum diameter of the cyclic quadrilateral. These are interesting results that have been presented so in a very simple manner and this is just obtained from the 2 nyayas. So, one is the product of another Jyas and the other is the difference of the squares. So, this is in turn to the cyclic quadrilateral and there is also other discussion on this formula.

So root of S-A*S-B*S-C that is also discussed in this yukthibhasa which has been discussed in other text also. So thank you.