Mathematics in India: From Vedic Period to Modern Times
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Lecture – 34
Trigonometry and Spherical Trigonometry 2

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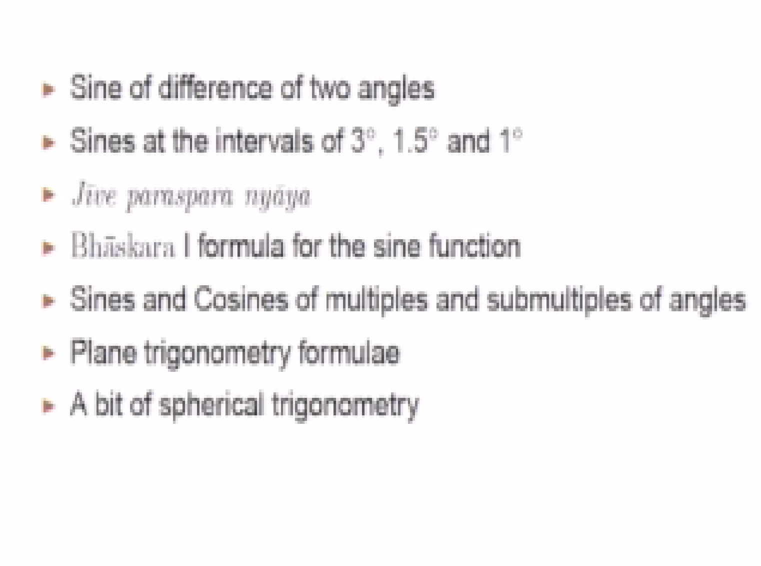

So, these are second lecture on trigonometry and spherical trigonometry, so I will commence with sine of difference of 2 angles, which is needed for a generation in the sine tables for way with various divisions of the quadrant, then specifically I will take up the sines at the interval of 3 degrees, 1.5 degrees and 1 degrees, in jyotpatti; these 2 are jyotpatti, so then a very important result called Jive paraspara nyaya; sin of A +; - B you know that formula for that.

Then and also cos of A + - B, then a little bit, we will talk about Bhaskara one formula for the sine function, which we have already seen. Then sines and cosines of multiples and submultiples of angles and some plane trigonometry formulae, which are important and which appear in even Brahmagupta (FL) and a bit of spherical trigonometry at the end so, which I will spherical trigonometry, I will discuss in detail in the next lecture.

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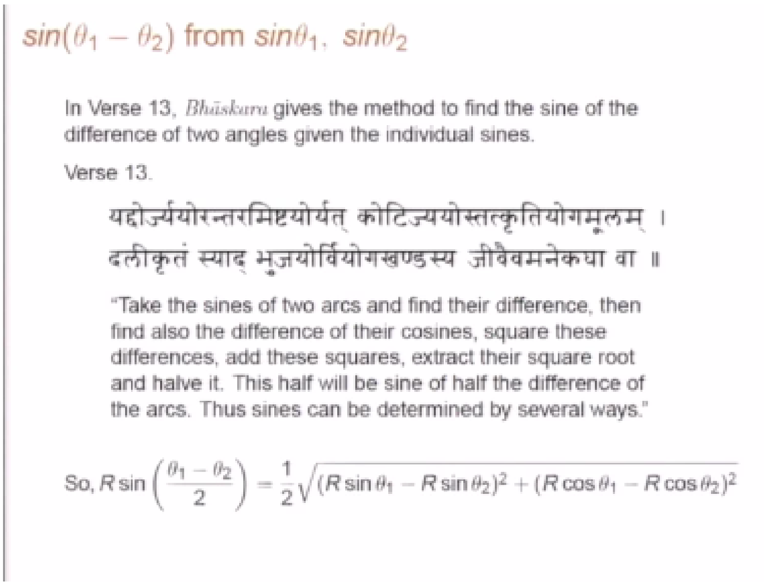

So, one can find out; if we know sine theta 1 and sine theta 2, one can find sine of theta 1 – theta 2, so in the verse 13 of jyotpatti; I told you, jyotpatti is the last part of this (FL) of Baskhara 2, so he gives the method to find the sine of difference of 2 angles given the individual sines, so, (FL) take the sines of 2 arcs and find their difference, then find also the difference of their cosines, square these differences and add these squares extract their square root and halve it.

This half will be the sin of half the difference of the arcs, thus sines can be determined by several ways. So, he had done it earlier for many other ways of finding sines, so this is also one important formula that is what he is trying to say. So, it is sin of theta 1 - theta 2/2; is this, sin theta 1 - sin theta 2 whole square + cos theta 1 - cos theta 2 whole square, so this can be shown easily.

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Proof

Proof:

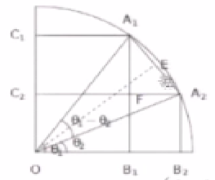

Diagram to find $\sin\left(\frac{\theta_1-\theta_2}{2}\right)$

We drop R.

In the figure,

$$A_1\hat{O}B_1 = \theta_1, A_1B_1 = \sin\theta_1, \quad A_1C_1 = OB_1 = \cos\theta_1,$$
$$A_2\hat{O}B_2 = \theta_2, A_2B_2 = \sin\theta_2, \quad A_2C_2 = OB_2 = \cos\theta_2.$$

Let OE bisect the angle $A_1\hat{O}A_2 = \theta_1 - \theta_2$.

Hence $E\hat{O}A_2 = \frac{\theta_1-\theta_2}{2}$. $A_1A_2 = 2EA_2 = 2\sin\left(\frac{\theta_1-\theta_2}{2}\right)$.

Say, for instance suppose, you have this figure okay, so this angle A1O B1, so that is theta 1, then A2 OB2, so that is theta 2, the difference of this is A1 O A2, so these are the difference of the 2 angles, so join this chord A1 A2 and draw a perpendicular from O to that chord, so that is E, okay. So, now you see OB1, so that is essentially R cos theta 1 and A1 B1 is R sine theta 1, so similarly OB2 is R cos theta 2 and A2 B2 is a R sine theta 2, so that is what I written here.

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Getting $\sin(\frac{\theta_1-\theta_2}{2})$ from $\sin\theta_1$, $\sin\theta_2$

Now, $A_1F = A_1B_1 - FB_1 = A_1B_1 - A_2B_2 = \sin\theta_1 - \sin\theta_2$

$FA_2 = C_2A_2 - FC_2 = OB_2 - OB_1 = \cos\theta_2 - \cos\theta_1$.

In the right triangle A_1FA_2,

$$A_1A_2^2 = A_1F^2 + FA_2^2$$

$$\therefore \left[2\sin\left(\frac{\theta_1-\theta_2}{2}\right)\right]^2 = [\sin\theta_1 - \sin\theta_2]^2 + [\cos\theta_1 - \cos\theta_2]^2$$

$$\text{or } \sin\left(\frac{\theta_1-\theta_2}{2}\right) = \frac{1}{2}\sqrt{(\sin\theta_1 - \sin\theta_2)^2 + (\cos\theta_1 - \cos\theta_2)^2},$$

which is the desired result.

$$\text{From } \sin\left(\frac{\theta_1-\theta_2}{2}\right) \to \cos(\theta_1-\theta_2) \to \sin(\theta_1-\theta_2).$$

So, if OE e bisects this angle; OA1 A2 is = theta 1 – theta 2/2, so this E O A2; sorry, E O A2, so that is theta 1 – theta 2/ 2 is bisecting, so A1 A2 is 2 *; so A1 A2 is 2 * EA2, so which is 2 * sine theta 1 – theta 2/2, so this one. So, your A1 F1; A1 F; so your A1 F, this one; this A1 F, so it is clearly the difference of this A1 B1 – FB1, which is A2 B2, so it essentially sin of theta 1 – sin theta 2.

So, similarly FA2 is clear that it is cos of theta 2 - cos of theta 1, so if you add these things, if you take these things into account, so then the A1 A2 whole square is = 2 * EA 2 whole square; 2 * EA 2 whole square, so you will get into 2 * sin theta 1 - sin theta 2 whole squared is sin theta 1 – theta 2 whole squared + cos of this thing; sorry, this must be sin of theta 1 - sin of theta 2, I am sorry, these are sin of theta 1 - sin of theta 2.

So, the cos of theta 1 - cos of theta 2, so therefore R; sin of theta 1 - theta 2/ 2 is = 1 /2 of the square root. So, if we keep the desired result, so from sin of theta 1 - theta 2/ 2, you are essentially getting from this, from sin theta 1 and sin theta 2, you can find sin of theta 1 – theta 2/ 2, then from sin of theta 1 – theta 2/ 2, you can get cos of theta 1 - theta 2, right because cos of 2 theta is = 1 – 2 sin squared theta.

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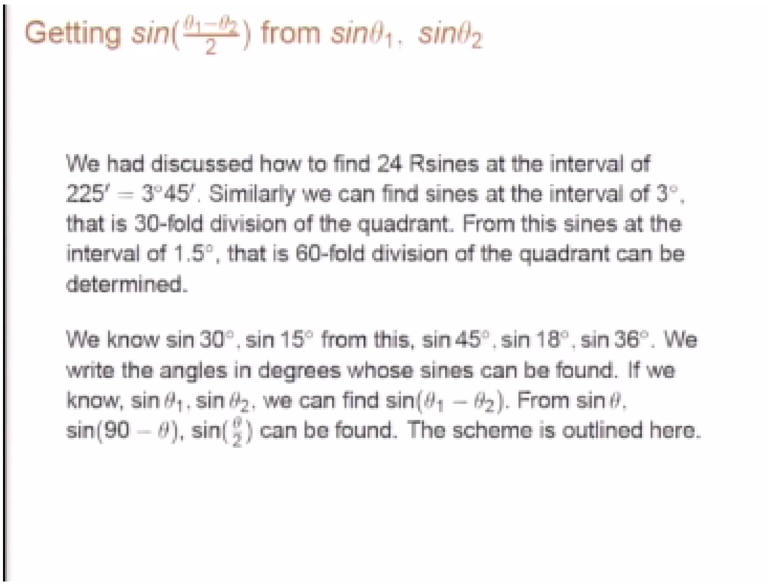

So, from that; and that is stated here in a slightly different form Varahamahira and Brahmagupta and from cos of theta 1 - theta 2, one can find out sin of theta 1 – theta 2, so that; so that is how it is So, from sin theta 1 and sin theta 2, you have found sin of theta 1 - theta 2, so that is the important things. So, now we had discussed how to find 24 hour sines at the interval of 225 minutes, so which is 3 degree 45 minutes.

So, similarly we can find Rsines at the interval of 3 degrees that is 30-fold division of the quadrant, you see and from these sines; from this, we can find sines at the interval of 1.5 degrees because if you know sin theta, you can find out sin of theta/2 from previous formulae, so that is 60-fold division at the quadrant can also be determined. So, now we know see a sin 30 degree, sin 50 degrees and from this, I know sin 45 degrees we can get.

Otherwise, also we knew but anyway; sorry sin 30 degrees and sin 25 degrees, you can get from sin 30 degrees and then sin 45 degrees we know, then sin 18 degrees we know that he had given the formula Bhaskara, so then sine of 36 degrees also we had found, right in the previous lecture, so we write the angles in degrees whose sines can be found, so if we know sin theta 1, sin theta 2, we can find sin of theta 1 - theta 2 that is what I did just know.

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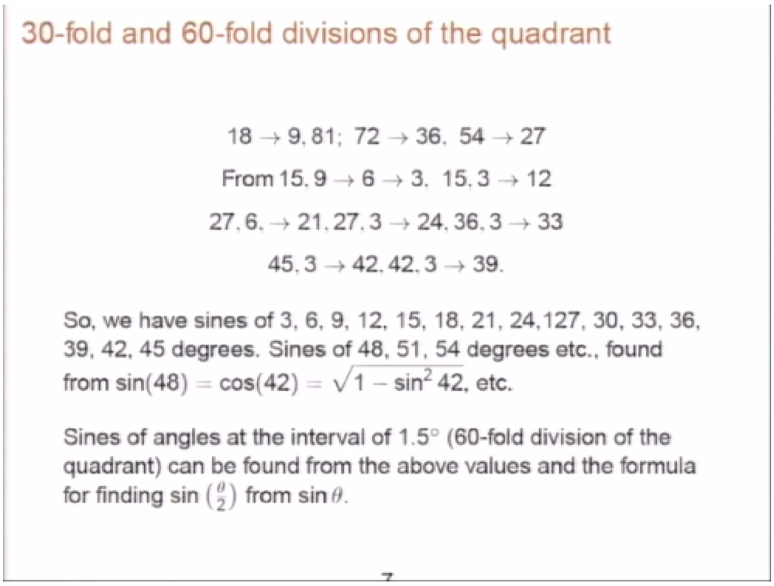

And from sin theta, sine of 90 - theta can be found and sin of theta 2/ 2 can be found, so the scheme is outlined here. So from 18, I mean 18 degrees, these not 18 sine, from 18 degrees, you can find sin of 9 degree and also 81 degrees and then from 72 degrees, from 18, you can get 72 degrees and from that you can get the sin of 36 degrees and 54 degrees, from 54, you can get sin of 27 degree, then from 15, you can get 15 and 9.

So, now from this difference these thing you know, sin of 15, you know, sin of 9, you know, so from this you can find sin of 6 degrees and from that you can get sin of 3 degrees and if you know say sin of 15 and sin of 3, then you can get the sin of difference, so that is 12 and then from 27, which you have got here, you get 27 and 6, you can get 21 and 27 and 3 and from 3, you can get 24, from 27 and 3, you can get there 24.

And from 36 and 3 degrees, you can get 33, and 45 and 3 degrees you can get, 42 and 42 and 3 degrees, you can get 39, I mean here we have to apply all these formula, you see that is all I am saying and we are given the; all the weapons are ready okay. So, we have sines of 3 degrees, 6 degrees, 9 degree etc. up to 45 then sin of 48 is nothing but cos of 42, which is square root of 1 - sine squared/ 42.

So, the whole thing 48, 51 etc. can be got from this, you see, there are complements and sines of angles at the interval of 1.5 degrees that is 60-fold division of the quadrant can be found from the above values and the formula for finding sin theta/2 from sin theta. So, you can; 24-fold division we had and then we can 30-fold division and then 60-fold division, we can; I mean as I told you, it is a lot of tedious work will be there nothing you know difficulty in principle.

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"*Jīve paraspara nyāya*"

Now, Bhāskara goes on to find the sines at the interval of 1°, using a different method. For that he needs to use

$$R\sin(A \pm B) = \frac{R\sin A\cos B}{R} \pm \frac{R\cos R\sin B}{R}$$

This is the famous "*Jīve paraspara nyāya*" in Indian trigonometry. This is the composition law. With the positive sign, it is the *Samāsa-bhāvanā* and with the negative sign, it is *Antara-bhavanā*. This is how, Bhāskara states it in Verses 21 and 22 of (*Jyotpatti*):

चापयोरिष्टयोर्दोर्ज्ये मिथःकोटिज्यकाहते।
त्रिज्याभक्ते तयोरैक्यं स्याच्चापैक्यस्य दोर्ज्यका॥ २१ ॥
चापान्तरस्य जीवा स्यात् तयोरन्तरसम्मिता।
अन्यज्यासाधने सम्यगियं ज्याभावनोदिता॥ २२॥

"If the sines of any two arcs of a quadrant be multiplied by their cosines reciprocally and the products divided by the radius then the quotients, will when added together, be the sine of the sum of the two arcs, and the difference of these quotients will be the sine of their difference. This excellent rule called *Jyā-Bhāvanā* has been prescribed for ascertaining the other sines."

Because various kinds of square roots will be there and you know; you know sum of square roots and using the square root also you may get square root all kinds of things, so it is in principle you can do but computationally, it may not be always not that you know, simple. Now, Bhaskara is a very famous formula, which is; which we all know that sin of A + - B is = sine S cos B + or -; the sorry this must be cos A, not cos R; cos A sin B.

Of course, always they will talk of Rsines, so it is written as R sine of A + - B, this Rsine A/R * cos B and Rsine A *; What am i doing yeah; It must be R cos B; R sin A */ R * R cos B and R cos B; R cos A/ R * R sin B, so that is how it is, so these are so called famous Jive paraspara nyaya in Indian trigonometry, so these are composition law, so they would have known it; you know, even earlier Baskhara is in 12th century before that so many things were known.

I mean it is; you know it could be shockingly Brahmagupta did not known; you would know it but you are not written in this form explicitly, so Bhaskara has written this and Bhaskara also does not use a proof of this in this work but you would have now known it, proof will come later and with the positive signs, it is called a Samasa bhavana and with a negative sign, it is called a Antara bhavana, so these how Bhaskara states it in verses 21 and 22 of jyotpatti.

(FL) If the sines of any 2 arcs of a quadrant be multiplied by their cosines reciprocally and the products divided by the radius, then the quotients will when added together be the sin of the sum of the two arcs and the difference of these quotients will be the sine of their difference. This excellent rule called Jya bhavana has been prescribed for ascertaining the other things, so that is he realized that was an important thing is and of course it is very important.

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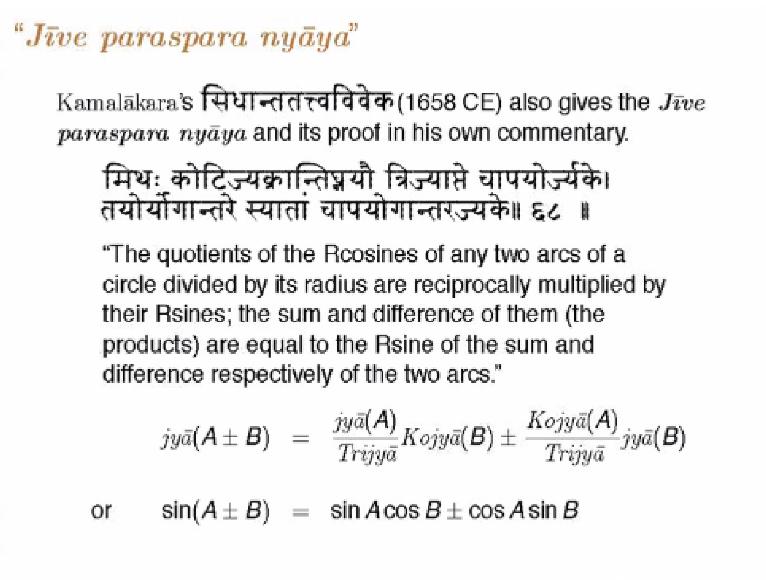

Various, even for finding differentials of sines and all that, this will be useful in, there are work called Siddhanta **(FL)** by Kamalakara, who are settled in Varanasi, so in 17th century, so he also used Jive paraspara nyaya and its proof also in his commentary, so this is what he says and it is possible that it might have appeared earlier also but this is certainly (FL) has been in a published and people have are aware of the contents, so the contents is here.

And goes the proof in detail, so it will be very useful you know; if you look at the proof or some stage, so it is formulated like this; (FL) the quotients of the Rsines of any two arcs of a circle divided by its radius or reciprocally multiplied by the Rsines, the sum and difference of them, the products are equal to the Rsine of the sum and difference respectively of the two arcs. So, essentially he is saying the same thing.

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Cosine of sum and difference of angles

दोर्ज्यो: कोटिमौर्व्योश्च घातौ त्रिज्योद्धृतौ तयोः।
वियोगयोगौ जीवे स्तः चापैक्यान्तरकोटिजे॥ ६९ ॥

"The product of the Rcosines of and of the Rsines of the two arcs of a circle are divided by its radius; the difference and sum of them (the quotients) are equal to the Rcosine of the sum and difference (respectively) of the two arcs."

$$Kojy\bar{a}(A \pm B) = \frac{Kojy\bar{a}(A)}{Trijy\bar{a}} Kojy\bar{a}(B) \pm \frac{jy\bar{a}(A)}{Trijy\bar{a}} jy\bar{a}(B)$$

or $$\cos(A \pm B) = \cos A \cos B - \sin A \sin B.$$

Jya of A + -B is = Jya A / Trijya, so that quotient is multiplied by Kojya of B, + or – Kojya of A divided by Trijya * Jya B or in modern notation, sin of A + - B is equal to sin A cos B + or – cos A sin B. So, then the cosine of sum and difference of angles, so that also it is surprising how Bhaskara did not state it but various scholars are; you know of the opinion that you would have definitely known it but somehow it has not been stated.

But Kamalakara states it explicitly, (FL) the product of the Rcosines and of the Rsines of the 2 arcs of the circle are divided by its radius, the difference and sum of them the quotients are = the R cosine of the sum and difference respectively of the 2 arcs. So, Kojya of A + - B is = Kojya of A * Kojya of B/ Trijya and then Jya of A/ Trijya * Jya of B + or - in between, so cos of A + - B is = cos A cos B, this must be - or +; cos A cos B - or + sin A sin B.

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Finding the Sine at the Interval of 1°

Bhāskara-II then gives the method for generating the sine table for a 90-fold division of the quadrant. That is sine of multiples of 1° (1°, 2°, $\cdots$, 90°). He tells us how to find $\sin(\theta + 1°)$ gives $\sin\theta$.

Verse 16,17:

स्वगोऽङ्गेषुषडंशेन वर्जिता भुजशिञ्जिनी।
कोटिज्या दशभिः क्षुण्णा त्रिसप्तेषु विभाजिता॥ १६ ॥
तदैक्यमग्रजीवा स्यादन्तरं पूर्वशिञ्जिनी।
प्रथमज्या भवेदेवं षष्टिरन्यास्ततस्ततः॥ १७ ॥
व्यासार्धेऽष्टगुणाब्ध्यग्नितुल्ये स्युर्नवतिर्ज्यकाः। १८a।

So, this is also we are aware of the thing, right. It is an important formula we use it all the time in trigonometry. So, now some of these results are used by Bhaskara in fact, as the Jive paraspara nyaya is used to generate the sine at the interval of 1 degree, okay so because that will be more accurate. If you are having the tabulated values at the interval of 1 degree, then obviously it will be for astronomical computations, so it will be useful okay.

And it will be more accurate also instead of doing in; doing you know interval of 3 degree 45 minutes and doing an interpolation, this will be more accurate. So, what is saying is he gives a method of generating the sine table for a 90 fold division of the quadrant and that is sine of multiples of 1 degree; 1 degree, 2 degree etc 90 degrees. So, he tells us how to find sine of theta + 1 degrees given sine theta. So, this is the; (FL) so that he is saying.

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Finding the Sine at the Interval of 1°

"Deduct from the sine of *bhujā* its $\frac{1}{6569}$ part and divide the tenfold of koti by 573. The sum of two results will give the following sine (i.e, the sine of *bhujā* one degree more than the original *bhujā* and the difference between the same results will give the preceding sine i.e.,the sine of bhuja one degree less than original *bhujā*). Here the first-sine, ie., the sine of 1° will be 60 and the sines of the remaining arcs may be successively found."

Verse 18 a: "The rule, however supposes that radius is 3438′. Thus the 90 sines (multiples of 1°) may be found."

The translation is deduct from the sine of bhuja, its 1/ 6569 part and divide the 10 fold of koti by 573. The sum of the 2 results will give the following sine that is a sine of bhuja 1 degree more than the original bhuja and the difference between the same results will do the preceding sine that is a sine of bhuja 1 degree less than original bhuja. Here the first sine that is a sine of 1 degree will be 60 and a sines of the remaining arcs may be successively found.

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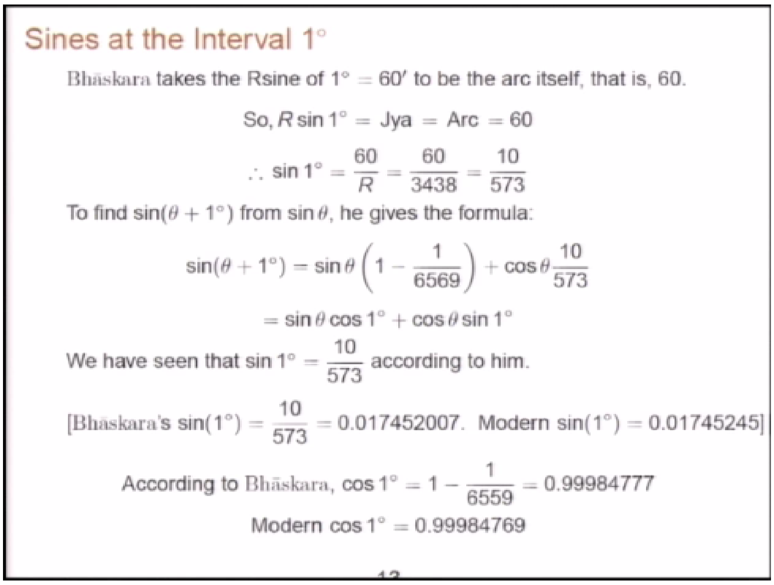

And then, he continues, he said that the rule however supposes that the radius is 3428, that the 90 sines may be found that is what he is saying. So, what he is saying is that the first of all he said that Rsine of 1 degree 60 minutes, so 1 degree 60 minutes, so Rsine of 1 degree, he takes 60 minutes to arc itself, so that is 60. So, Rsine 1 degree is = 60, so sin of 1 degree is 60/R and R is taken to be 3438, so which is from this is easily see that is 10/573, which is what he will be using, you see.

Now, to sine; find sin theta + on1e degree from sin theta, so R will be there in both sides we can cancel even in the modern notation, it will be sin of theta + 1 degrees. He is saying sin theta; you subtract the 1/6569 part, so it is sin theta * 1 - this thing + cos theta * 10/573 that is what he is saying, so it is sin theta cos 1 degree + cos theta sin 1 degree, we know that you know he has given the Jive paraspara nyaya.

So, now we have seen that sin 1 degrees is 10/ 573 according to him and so how good it is; how good is it? Sine of 1 degree is 10/ 573 according to Bhaskara, so which is if you calculate it, it will be 0.017452007, the modern sine one degree are calculated using a calculator, so

0.01745245, so it is slightly less but it is fairly accurate, see. So, now here is the thing is you know, that Aryabhata had taken sine of 225 minutes is = 225.

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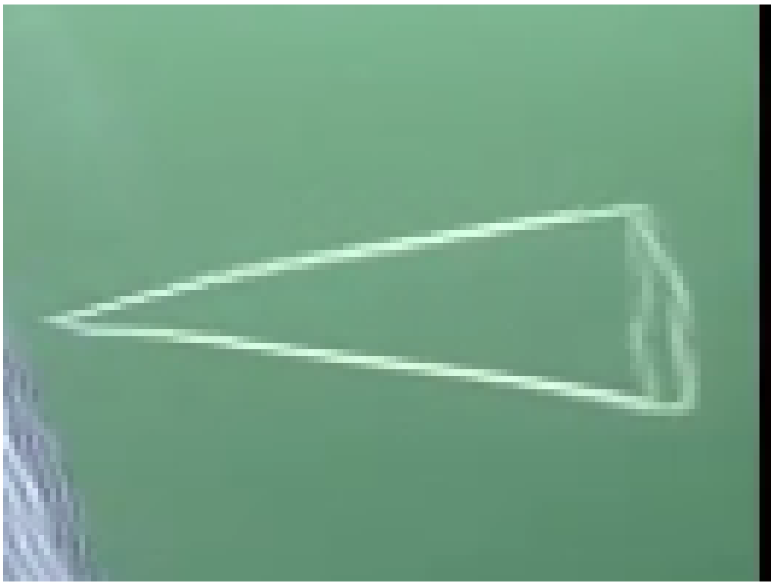

Here, he is saying sine of 1 degree is this thing you see, so it is more accurate; see smaller the arc, greater is the proximity of the sine and the arc, obviously you see say; (FL) is = this arc; (FL) is equal to jya, right. In fact, this is the sine and he is (FL), so if (FL) is very small then you know this will be equal to this basically because this will become smaller and smaller and they are identical.

So, if we take 225, sine of 225 = 225, see that is you know, that is how Aryabhata think that is somewhat you know for a large angle, you are assuming that sine theta is approximately = theta which is not this thing angle as large as 3 degree 45 minutes. It is not very bad but not that accurate. If we take sine of 1 degree, as a sine of 60 is = 60, that is obviously more accurate and in fact Bhaskara, will give what he gets for sine 25 to 225 later, it will be slightly less than 225.

We saw that Nilaknata gave a sine of 225 is 224 minutes 50 seconds, so he will give some other value, we will see that and according to Bhaskara, cos of 1 degree is 1 – 6569, sorry, what is the value sorry; 6569, and you can see how good it is; the cos of 1 degree is this whereas, the modern value is this. So, 5 decimal places is okay, you know 6 also; it is 6 here and 7, 7 here, you see almost 6 decimal places is fairly accurate.

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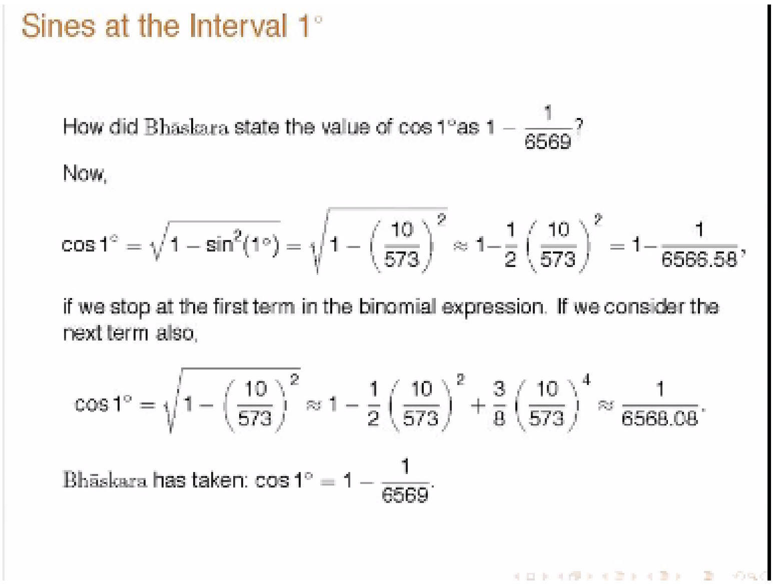

So, how did Bhaskara state the value of cos 1 degree as 1 – 1/6569, so I just took this here that stated how it is this thing but perhaps this is how it is. So, now cos of 1 degree is square root of 1 - sin squared 1 degree and you know that sin of 1 degree he has taken it to be 10/ 573, right. I mean Rsine is 60, so sine of 1 degree is 60/ 3438, which is 10/573, so on this thing. So, now using the binomial expansion, so this will be remembered that square root of 1 - x is approximately = 1 – 1/2x you see.

So, if we use that I will get 1 – 6566.58, you know you can easily see this. If we stop at the first term in the binomial expansion, if you continue the next term also okay, so suppose I take this and take the next term, so 1 - square root of 1 - x is 1 - 1 /2 x squared + 3/8 x to the power of 4, so that formula you know, the binomial expansion of you know 1 – x whole to the power of 1/2. Then, I get 1 over 6568. 08 and Bhaskara has taken cos of 1 degree is = 1- 1/ 6569, you see.

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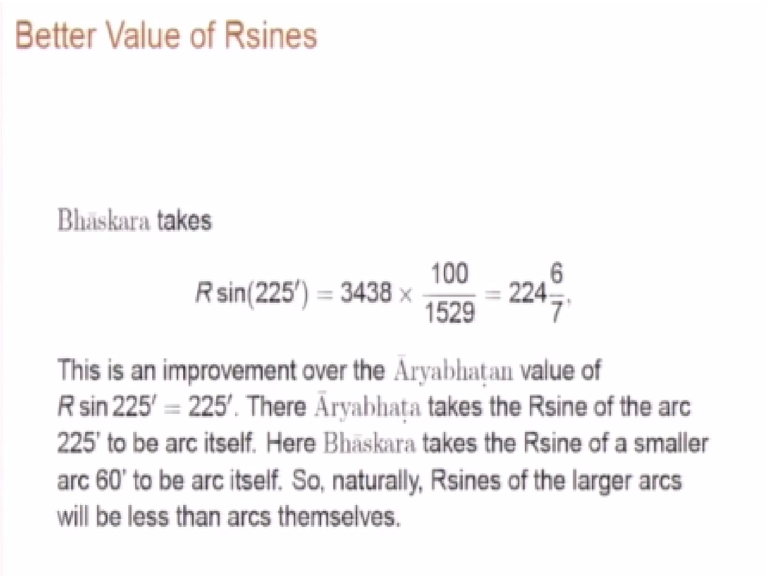

So, this is good; very good this thing, something like that you would have done but he has not stated you know that whether he has done binomial theorem and Bhaskara takes sine of 225 minutes is = 3438 * 100/ 1529, so that is 224 + 6/7; instead of 225 as in Aryabhatia, so it is an improvement over the Aryabhatan value of Rsine 225 minute is = 225 so there Aryabhata takes the Rsine of the arc 225 to be arc itself.

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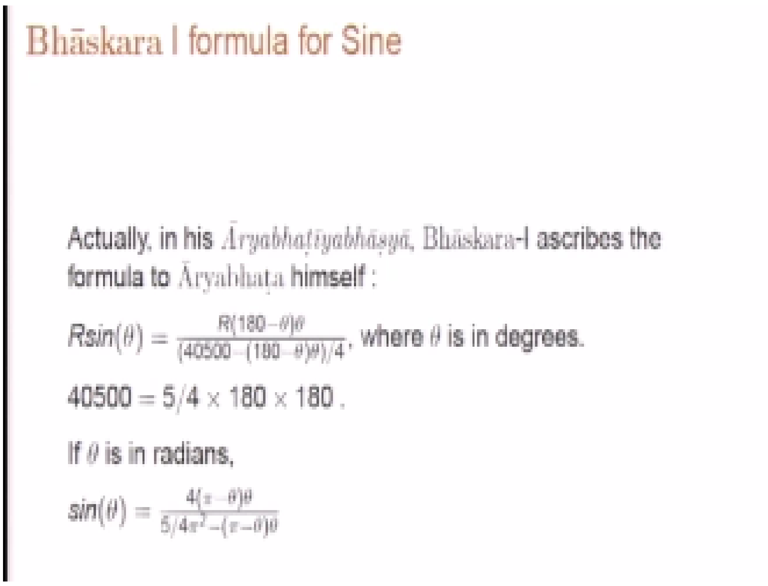

So, here Bhaskara takes the Rsine of a smaller arc to be the arc itself, so naturally Rsines of larger arcs will be less than the arc themselves, so it is better. Then, I will discuss briefly Bhaskara 1 formula for sine of course, in the first lecture today, it discussed elaborately what is the formula? and how it could have been possibly arrived at and so on, so this is the formula and Bhaskara 1 seems to ascribe the formula Aryabhata himself.

It comes in his Aryabhatiyabasya or (FL) one of these; right, both it comes in both but he Basya, he said that it is you know Aryabhata I mean, what he is saying seem to imply that Aryabhata himself knew it, so R sine theta is a remarkable formula. So, this is the formula when theta is in degrees and if you see that 405000 is = 5/4 * 180 * 180 and if you take theta is in radians, so then you will get this formula you know, you can write it like this.

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Comparison of actual Sine and Bhāskara formula

$\frac{\theta}{\pi/2}$	$Sin(\theta)$	Bhāskara formula	Percent Error
0.1	.156434	.158004	+1
0.2	.309017	.310345	+.43
0.3	.453990	.454343	+.04
0.4	.587785	.587156	-.11
0.5	.707107	.705882	-.17
0.6	.809017	.807692	-.16
0.7	.891007	.889976	-.12
0.8	.951057	.950495	-.06
0.9	.987688	.987531	-.01
1.0	1.0	1.0	0

4 * pi - theta * theta/ 5/4 pi squared - pi - theta * theta, I mean the symmetry and all that has been explained earlier you see, if I replace theta/ 180 - theta it will be invariant, so obviously this means that they know that you know, I mean sin of 180 - theta is sine theta and so this tabular, you know if you can; I am just taken theta divided by pi/2 and find out sine theta so that is 0.1 * pi/2, 0.2 * pi/2 etc., calculated sine theta.

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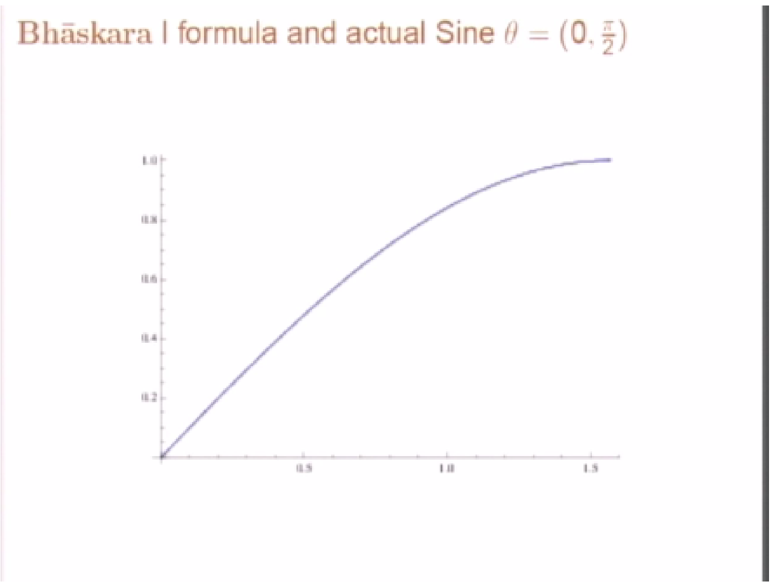

So, this is the modern sine theta; modern value using a calculator and Bhaskara formula gives this, so only for this is =1 % error, you see all the rest is you know much; so typically it has a 0.2% or so, remarkably accurate formula okay. So, anyway I mean and this is how you get even the 2 graphs are there here; the sine obtained from the modern, actually using computing in which high precision and a Bhaskara formula, you see.

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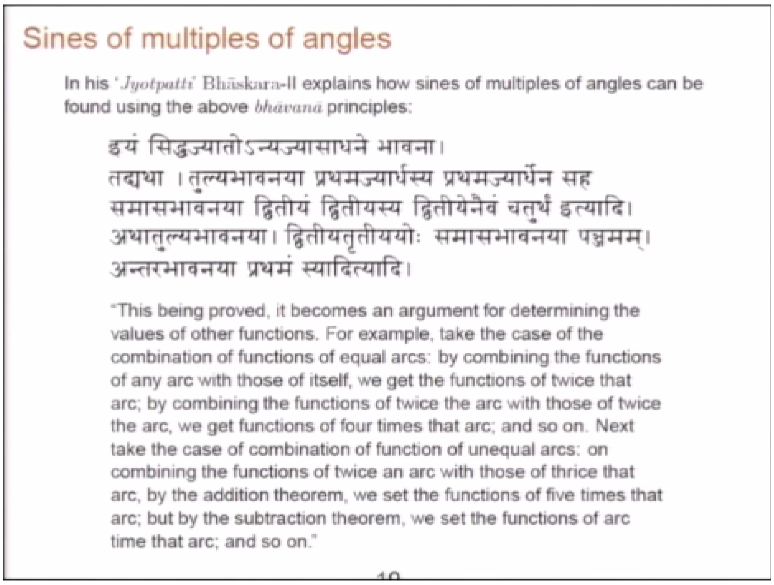

So, the resolution is said that you know is that you cannot differentiate the 2 graphs, so if you already have some high resolution and then take a small part, then it will be this thing, okay. So, this is the bhaskara one formula; I will not comment more on that, it has been much admired elsewhere and here already, so okay. So, then sines of multiples of angles is jyotpatti, Bhaskara to explains how sines of multiples of angles can be found using the above bhvanva principle.

So, he says that (FL) so what he is saying is that you know suppose, you know (FL) I mean the first sine, if you know the first sine, sine theta then you can find sine 2 theta by that Jive paraspara nyaya, sine of theta + theta is = sine theta * cos theta + cos theta * sin theta, so you can get sine 2 theta easily and from sine 2 theta, you can get sine 4 theta, that is what he says (FL) if you subtract; if you compose 2 theta and 2 theta, you get 4 theta (FL) but if they are different also one can use the same principle (FL).

So, that is if you know sine 2 theta and sine 2 theta and of course, you will know cos 3 theta if we know sine 3 theta, cos 3 theta will be known, sine of 5 theta is = sine 2 theta cos 3 theta + cos 2 theta sine 3 theta, so that is what he is saying. (FL) So, of course if you know sine 2 theta, 3 theta, you can find out sine theta from this thing, so that is what he is saying in this translation and this is how we can understand.

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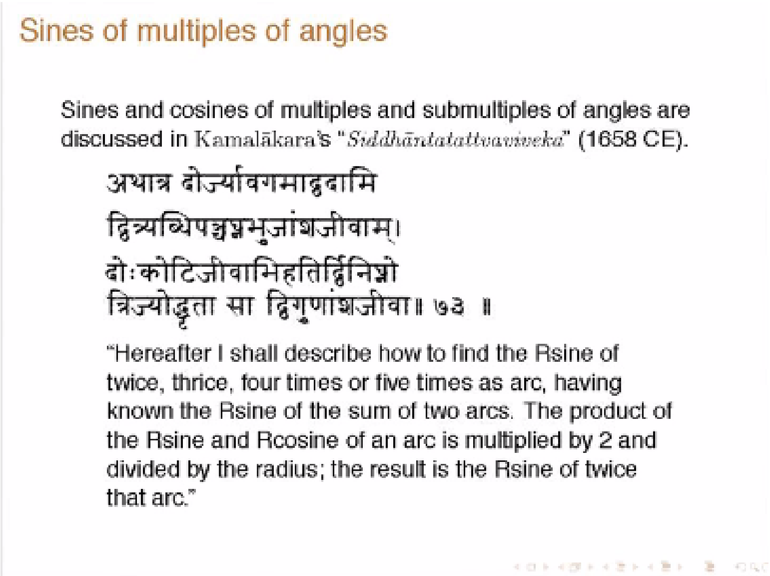

And sines and cosines have multiples and submultiples of angles are also discussing in Kamalakara Siddhantatattvaviveka, he says here after, I shall describe how to find Rsine of twice, thrice, four times or five times as arc having known the sum of sine of the sum of 2 arcs, the product of the R sine and cosine of an arc is multiplied by 2 and divided by radius, the result is Rsine of twice that arc.

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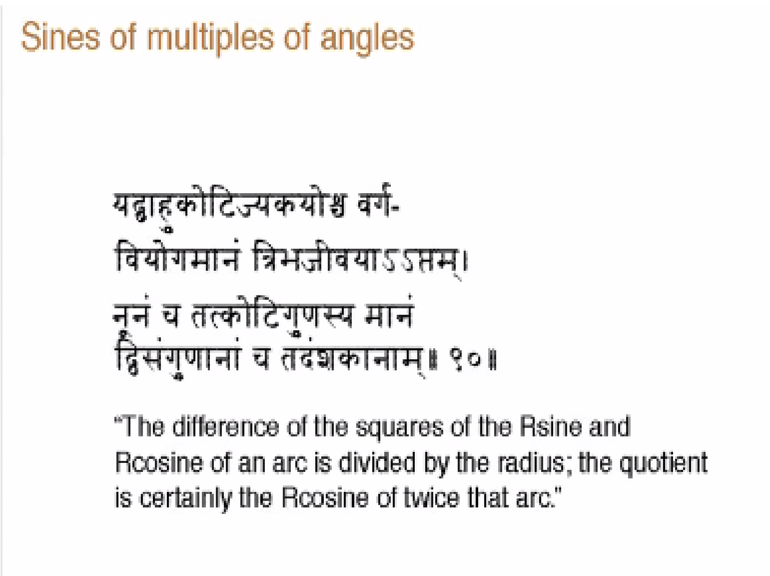

So, (FL) So, R sine of twice can be found out and again for a cos 2 theta also is giving (FL) the difference of the squares of the Rsine in the R cosine of an arc is divided by the radius, the quotient is certainly the R cosine of twice that arc, you see, so maybe one of a student express doubt, he said certainly it is so, do not have any doubt.

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Sines of multiples of angles

The relations explicitly stated by him are:

$$\begin{aligned}
\sin 2\theta &= 2\sin\theta\cos\theta; \quad \cos 2\theta = \cos^2\theta - \sin^2\theta \\
\sin 3\theta &= 3\sin\theta - 4\sin^3\theta; \quad \cos 3\theta = 4\cos^2\theta - 3\cos\theta \\
\sin 4\theta &= 4(\cos^3\theta\sin\theta - \sin^3\theta\cos\theta) \\
\cos 4\theta &= \cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta \\
\sin 5\theta &= \sin^5\theta - 10\sin^3\theta\cos^2\theta + 5\sin\theta\cos^2\theta \\
\cos 5\theta &= \cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta.
\end{aligned}$$

So, essentially he is saying that sine 2 theta is = 2 sine theta cos theta, then cos 2 theta is cos squared theta - sine squared theta, so again he goes on afterwards, so sine 3 theta is = 3 sine theta - 4 sine cube theta, cos theta is = 4 cos squared theta - 3 cos theta, so sine 4 theta cos 4 theta sine 5 theta cos 5 Theta etc., so these all these things are given, these are all you know we are doing all these things in BSC Loney trigonometry, all these for exercises right, so these are done.

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Sines of submultiples of angles

The following Sines of submultiples of angles are also stated by him:

$$\begin{aligned}
\sin\left(\frac{\theta}{2}\right) &= \sqrt{\frac{1}{2}(1 - \cos^2\theta)} \\
\sin\left(\frac{\theta}{3}\right) &= \frac{1}{3}\sin\theta + \frac{4}{81}\sin^3\theta \\
\sin\left(\frac{\theta}{4}\right) &= \frac{1}{2}\sqrt{2 - \frac{\sin\theta}{\sin(\theta/2)}} \\
\sin\left(\frac{\theta}{5}\right) &= \frac{1}{5}\sin\theta + \frac{4}{5}\left(\frac{\sin\theta}{5}\right)^3 - \frac{16}{5}\left(\frac{\sin\theta}{5}\right)^5
\end{aligned}$$

And seen similarly, he talks of sines of submultiples of angles, the following sines of sub multiples of angles are also stated by him; sin of theta/ 2 is square root of 1/2 of 1 - cos square theta, sin of theta/ 3 is 1/3 sin theta + 4/81 sine cube theta, sin of theta/4 is 1/2 of square root of 2 - sin theta/ sin theta/ 2, sin of theta/ 5 is; these are all very important you know, all these oscillation theories and all; that all these things will be used.

Harmonics; sub harmonics and all that you know these things will be very useful in analysing solutions of you know oscillator problems, sines and all this, so then signs of sines and cosines, you see, so did they know this the signs you know that we know, you know that now that sine you know in the first 2 quadrant sine is positive and in the third and fourth, it is negative.

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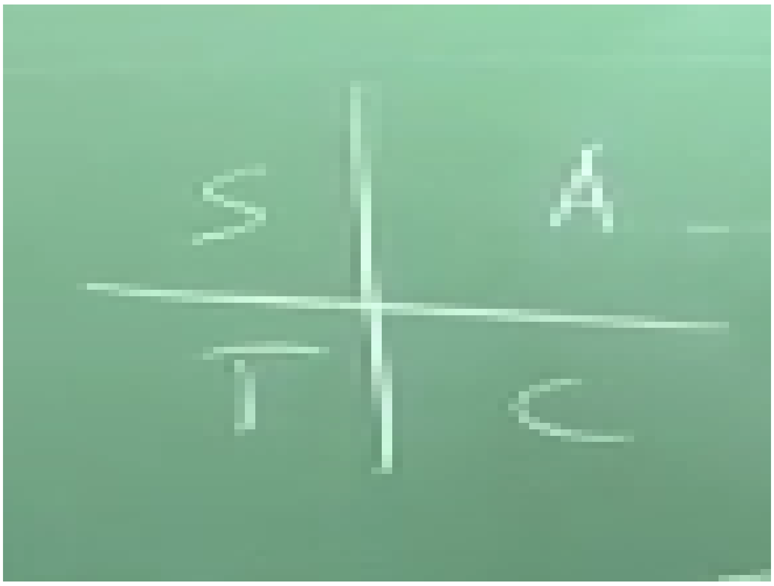

And in cosine, first and fourth it is positive and second and third, it is negative, right and tan, we know that you know and you have the same as this thing, right and all are positive here, sin is positive here, tan is positive here, cos is positive here, so all students take coffee or all silver tea cups whichever you want to remember.

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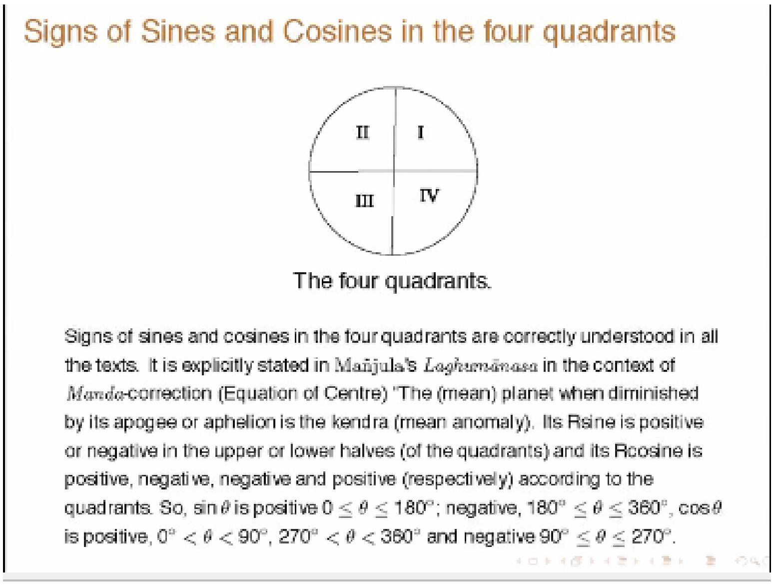

So, he is saying these things are realized signs of sines and cosines in 4 quadrants are correctly understood in all the texts, it is explicitly stated in Manjulas Laghumanasa in the context of Manda correction. He says that the mean planet when diminished by its Apogee or aphelion is the Kendra, its Rsines is positive or negative in the upper or lower half of the quadrants and R cosine is positive negative; negative and positive respectively according to quadrants.

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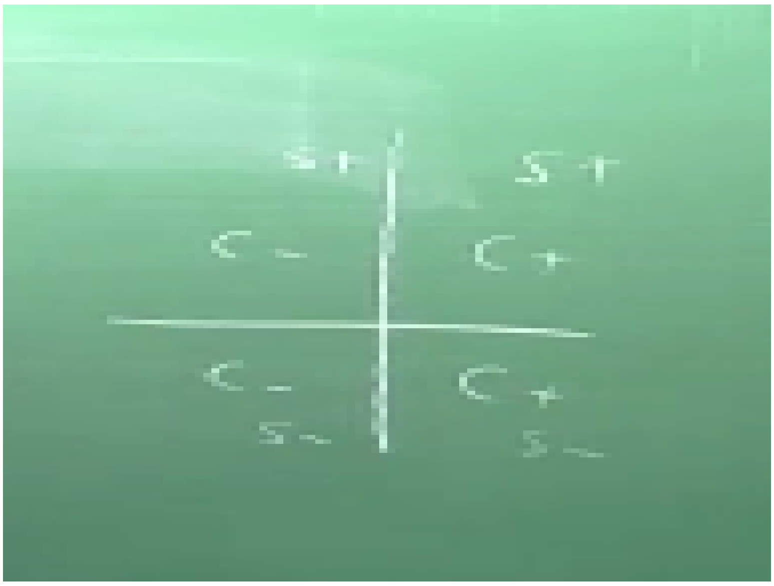

So, that is what he is saying essentially, so the sin is positive here and here and sin is positive, sine is negative here, whereas cos is positive here, positive here and cos is negative here so, that is this thing but of course, they will not; they will always mention the you know; the signs separately actually you know in any astronomy formula you see, suppose they are you know given; suppose you have, something like a + suppose the modern formula is this; a + sin theta, okay.

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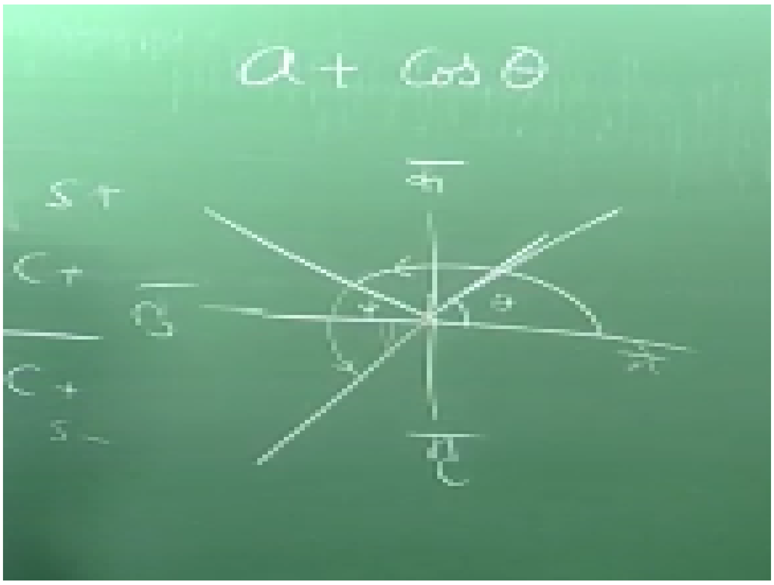

How they express it in this thing you know is that you find the sin of the angle but always sin of the angle means 0 to 90 degrees only you know according to them, you see always is 0 to 90 degrees, so that also is stated, so here the sin, if you want to find the sin, so this angle only take you have to find the sin, okay suppose, you have to take the sin of this angle okay, so then they say as you take the sin of this distinct angle, which is passed, which is yet to come, okay.

So, that is what they say and similarly, in this quadrant; this is third quadrant, they will say you know you take the sin of this angle okay, always because sin tables are all given for 0 to 90 degrees okay and similarly, in this quadrant like that but what they say is you know that they specifically mentioned okay, in the first 2 quadrants after finding the sin, the sin should be added to this quantity.

And in the third and fourth quadrant, it should be subtracted, okay so like that they will always say and similarly, they will say that you know, a + cos theta, so they will invariably say that if I find the cos and then they say that you know in the; in fact; they will not say like you know 0 and 90 and all that (FL) and that is what they are saying you see, so this is Mesha, you see Mesha this thing you know.

Then, this is from; and this is Capricorn, so these (FL) okay then be the beginning point of kataka and this is thula okay, so then always they will say that you know if you are you know manipulating with sines, they will say that you know from Mesha to Thula; beginning point

Mesha to Thula, you add this thing quantity and then to Thula to this thing; you subtract it and similarly for cosine, they say is you know (FL) you add and then (FL) to this thing.

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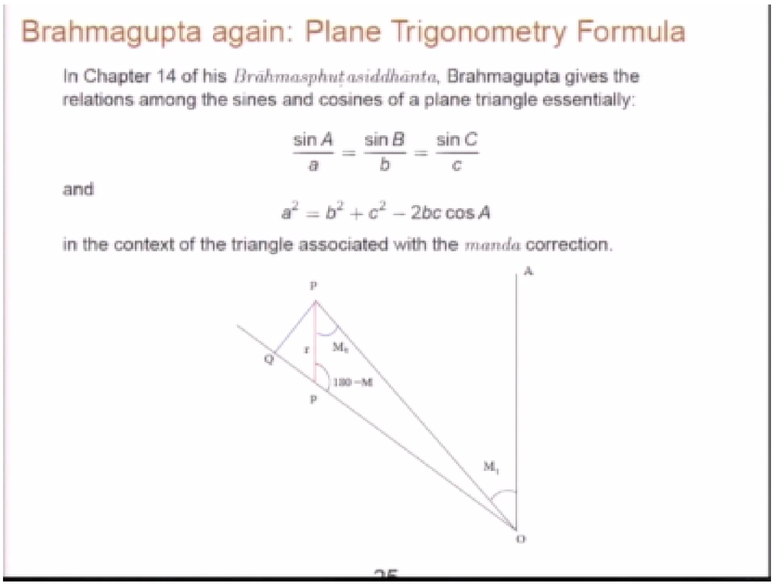

So, it is always clear, you can context you know because when they say they sign normally, know the magnitude but this is understood you know, this very clearly understood okay. So, now Brahmagupta again some plane trigonometry formula in chapter 14 of his Brahmasputasiddhanta Brahma Gupta gives the relations among sines and cosines of a plane triangle essentially.

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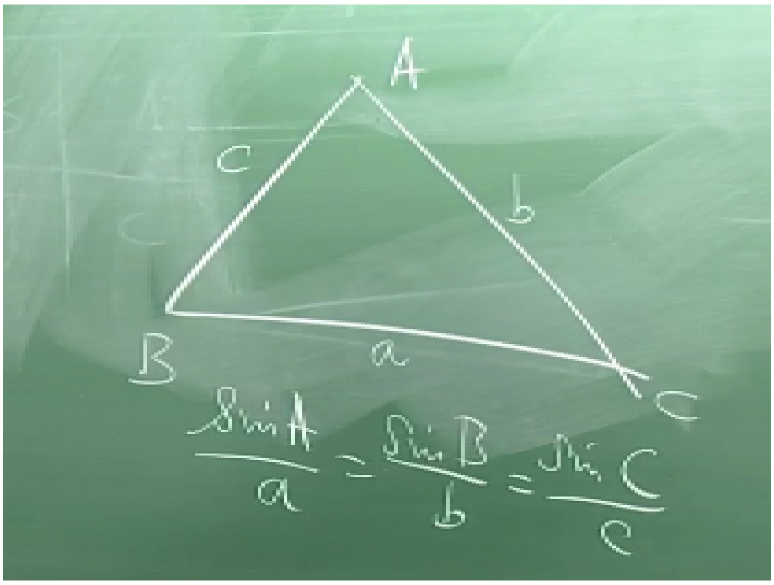

He will not try to tell you, so we know that in a triangle you see, if you have a plane triangle see suppose, have a plane triangle A, B, C so these are the ABC or the angles small a, small b, small c or the sides but then, we know that sin A/ A is =ine B/ B is sin C/ C of course, this is not stated in this form actually, he is not even discussing it in the sectional mathematics, he discussing in you know in this astronomy; one of the astronomy chapters.

So, he does it in the context of the manda correction, which I had talked in the morning and then a previous lecture you see this should be also be p0, so remember that this is some direction called Apogee okay and then this; so you see the true planet, D is the mean planet okay and this angle P P, this one is P0, P P0Q, so that is M okay and then so that it is equal to AOP0, okay so that is M, then that is called mean anomaly.

And then this is called true anomaly AOP is empty and is equal to this, so essentially see this 180 - M and this is empty okay, so in this context he will say you know how these sides and angles are related, sides and angles; I do not have the verses but he very clearly said that you know this called a; R is called on (FL) and so on, so what he is saying that you know this if we would divide this by the sin of this, then this by the sin of this and this by the sin of this, they are equal kind of a thing.

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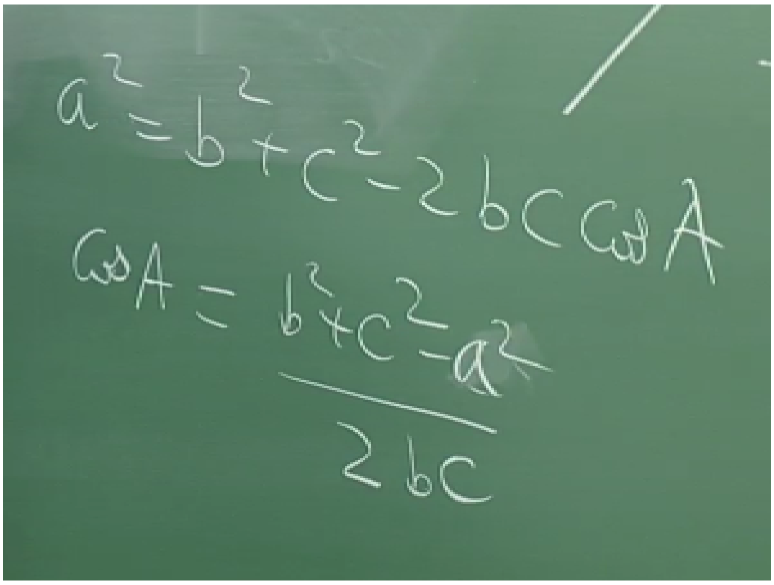

So, that is what he is saying which is equivalent to that okay, which is equivalent to that and then we also know that if this is a situation, you also know that you know a squared is = b squared + c squared - 2 bc cos A, is a modern reducing formula for a strained triangle right so this is the 3 sides, so this is this and this also is stated in that fashion you know, that he says that if you take this OOP, so then this PP0, this much it P0, I am sorry PP0 or P0.

You square them and then you know; you know what he is saying is it essentially you will say this cos of A; in fact, cos of A, you will say is = B squared + C square - c square/ 2bc, so that is what he is saying; so this formula for the; this is also there. Though not in a general context, he is talking of this but it is equivalent to that you know it obviously, he knows in how these sites and distances; sorry B squared + C squared – a square, sorry okay.

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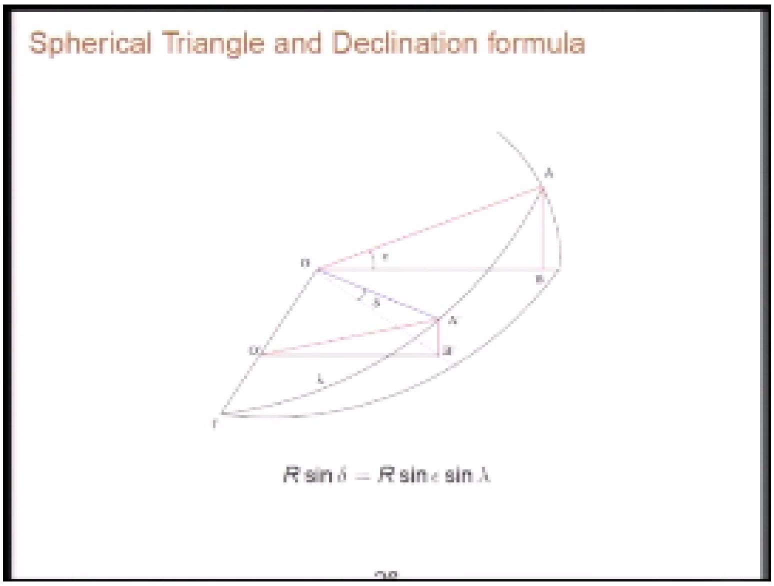

So, then a big tough spherical triangle and declination formula, so I will just introduce the topic so next lecture, I will discuss it in more detail okay. So, now I should define what is the spherical triangle, these are all plane triangles which we are considered planes, we are considered.

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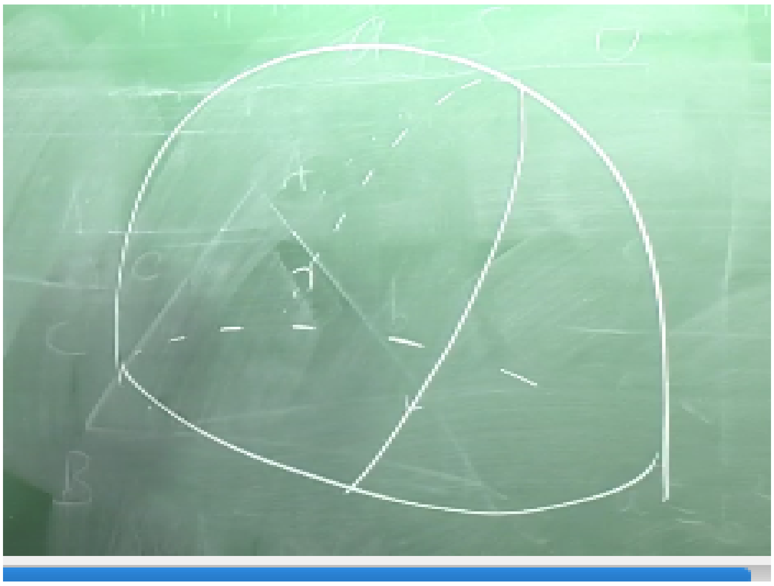

But we know that all the objects seem to be, as I told you in the motion of these celestial objects, you see, they all seem to be in the surface of a sphere of a large radius right, so they will raise and then to this thing, maximum uppermost point and then come down raise here, set here, this will be the typical path of a, this thing. So, they are all going in a; so spherical you know, so they are all you have to consider this here and various things.

So, now you have to define a spherical triangle but that is what will be useful for various quantities you know, you found to find out the time from shadow I gave one example and various other things you know, some declination from Meridian transit, so there are some technical things but which are quite simple really, okay. So, now you have to define what is known as spherical triangle, okay.

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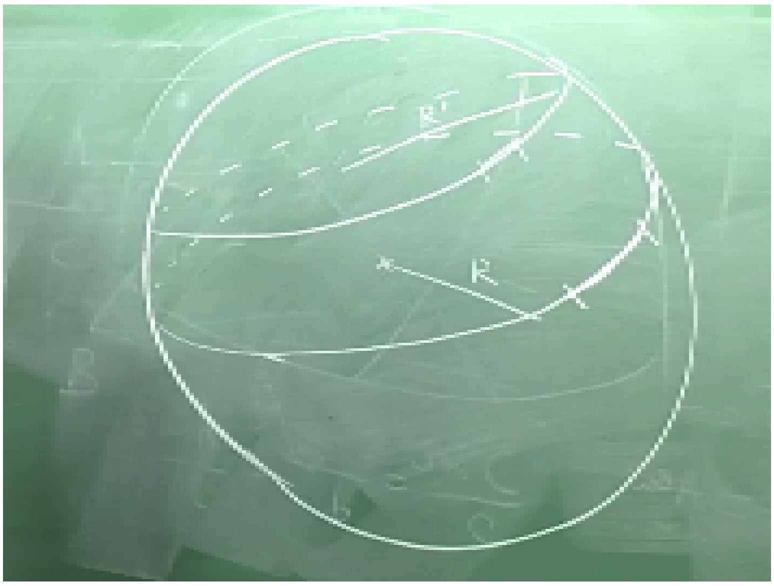

So, how that is defined we should this thing, so draw a sphere okay, draw a sphere okay, so these are centre okay, so now; so this spherical surface suppose, you some plane intersects this sphere okay and if that plane intersects; if the plane passes through the centre of the sphere okay, it will always intersect in a circle. Suppose, it is the plane passes through the centre, so this circle will have a radius of the sphere itself and that is called a great circle okay that is a great circle.

Not all circles may be great because you know usually, you can cut it like this right, you can cut it like this, so if you cut like this, so then the radius of the circle will be this, which is < R, which is < R okay, so it is called a small circle and either great circle okay. So, now a spherical triangle is something which is formed by great circle arc. So, if there is a, this thing you know if you have this arc you know of this all which is lies on the great circle that is called a great circle arc and this is a small circle arc okay.

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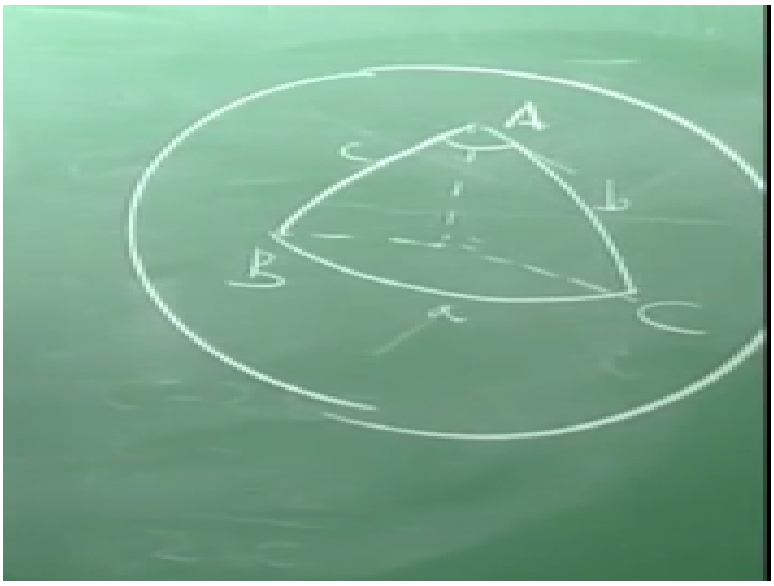

But invariably, all the you know very useful relations or which are somewhat simple relations or always between great circle arcs their lengths and angles okay. So, suppose you have a sphere, suppose you have 3 great circle arc like this okay, they are all great circles, okay so that means that; so they lie in a plane you see which is passes through the centre of the sphere okay. So, now these, so these great circle arc, they are called a sides of the spherical triangle A then B, and C.

And this angle is spherical angle okay, so you have to define it carefully what you have to do is, you have to find the tangent to this here and tangent it should find the angle between them, so that is called the spherical angle or alternatively, you take the plane you know this is a plane which is passing through this centre, so that you take and the other plane will be passing through this, so this angle will be the angle between these planes okay that is the spherical angle and so on.

So, these are the spherical angles and spherical these things and there are many useful relations among them, so one of them is you know that it is not a cone, No, no it is 3 spherical, you know it is a circle arcs that is not a cone, all of them are lying on the surface of the; so cone means it is tapering you see here is all the things are you know on the surface of the; **"Professor – student conversation starts"** BC is a circle of arc, see all of them are circular arcs yeah, all of them are circular arcs; **"Professor – student conversation ends"**.

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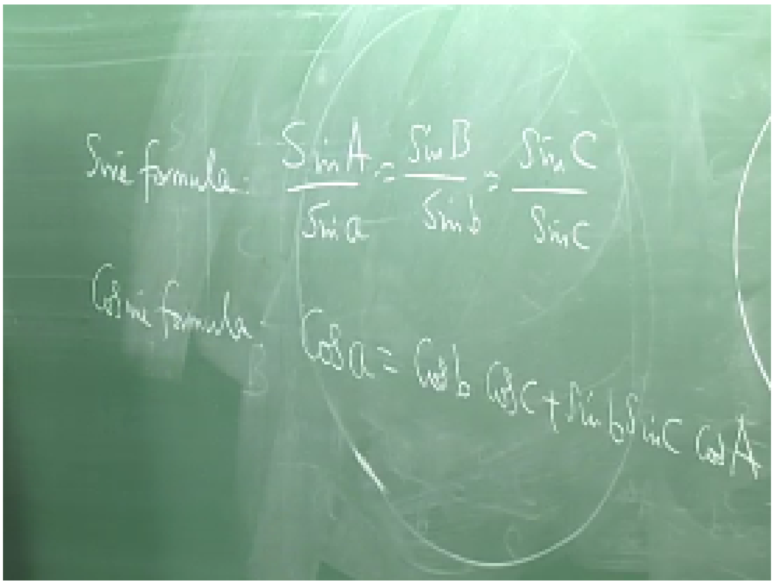

There are some important formula associated with these, so modern; in the modern notation, sin A/ sin a is = sin B/ sin b is = sin C divided by sine c, this is called a sine formula in a spherical triangle and it is what is known as a cosine formula; cos of A is = cos of b cost of c + sin of b sin of c cos A, this is called a cosine formula okay. So, in modern spherical trigonometry book, you know you will get all these; these are in the first few pages itself it will be there.

In fact, spherical trigonometry used to be a part of many; there are earlier many BSc courses had this astronomy okay; spherical astronomy, so in that this spherical trigonometry used to come, so it is BSc level stuff, you know not too complicated but one can derive all these things, it is not that difficult, I will not; you can see that you know when ABC are small you see, suppose, we will take a very small spherical triangle, all the sides are very small okay.

So, then the arc can be = this in the side itself, it will be some part looks like a straight line and sin A will be a and all that, you get the plane triangle relation, sin A/ small a is = sin B/ small b is it alright, so plane triangle result you will get. So, anyway so Indian unit; **"Professor – student conversation starts"** no, this is arc, everything is an arc, so sine of that arc, yeah; yeah so this angle, so that arc, right yeah.

So, this is the thing, all of them are actually arc, okay yeah, so they are also really in a way, they are angles yeah. **"Professor – student conversation ends"**. So, how do we; this is the modern this thing of course, it is different from even Greeks also had they did not have this exactly but somewhat in a different manner but Indians, it in a different way, so maybe I will discuss that in detail tomorrow in the next lecture.

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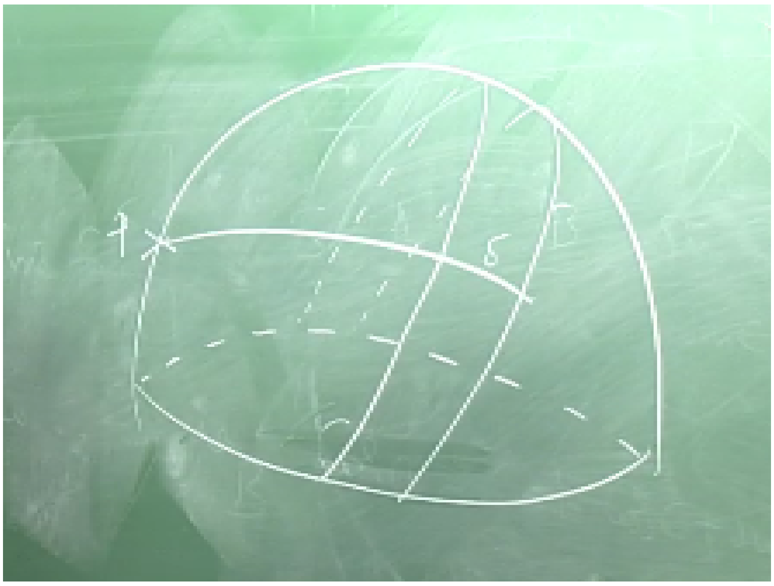

But I will just do one specific example okay, so there is what is known as the declination of the Sun okay. So, when the Sun is; to consider the part of the Sun in the sky, so this is called equator and Sun will be moving in a circle called a diurnal circle okay and this is the pole; North Pole direction and the distance between these and that is called a declination, how much away from the equator, the Sun is that is called a declination okay.

And it is called Kranti in Sanskrit, and this is called a (FL) and is called (FL) and so on. So, now declination is a very crucial thing which will determine various things you know, the duration of the day from sunrise to sunset, how much time is there, all this will depend upon the declination and the latitude of the place and so on okay. So, now the declination of the Sun depends upon the its longitude okay.

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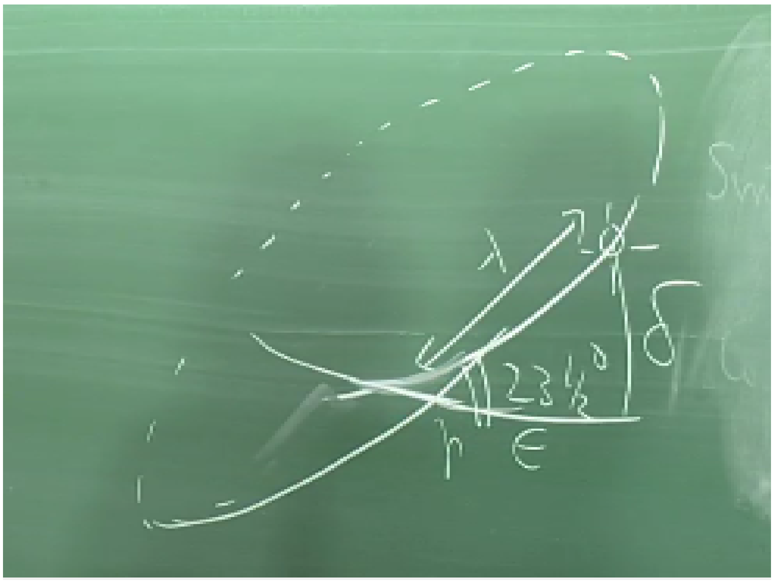

The Sun is essentially with respect to the earth, the Sun is moving as I told you in a circle called a ecliptic, it is moving in a circle called the ecliptic and these are equator and when it crosses these thing that is called a equinox, so that is on March 21st. So, March 21st and September 23rd these are the equinoctial days, when the duration of day and night are exactly the same throughout the earth okay for all places called equinoctial day okay.

So, then how much this is sun; sun is somewhere here, so this is called a longitude; so, this is a longitude of the Sun and the distance on the equator is called as declination and the angle between this is called a 23 and 1/2 degree essentially, so that is called the obliquity of ecliptic Epsilon; in sometimes it will be there, sometimes it is called (FL) it is called (FL) also.

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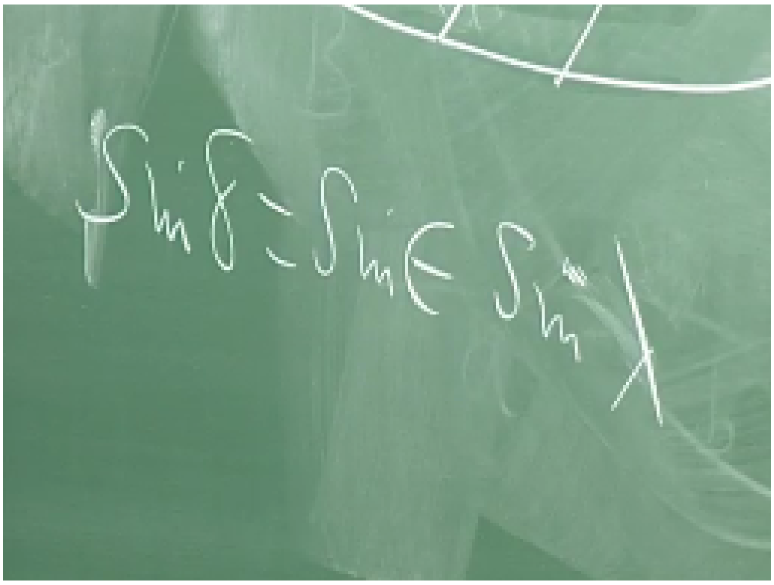

So, they are all you know a (FL), means it goes in the other direction, is a anti clockwise direction (FL), so that is why it is called a (FL), and so this is the (FL), the angle between them is this, so then this delta and this will be related lambda, so then that is what we get or sin delta is = sin epsilon * sin lambda, remember the 20 degree and 1/2 degrees if the latitude the Tropic of Cancer all of them related to this thing only you know this.

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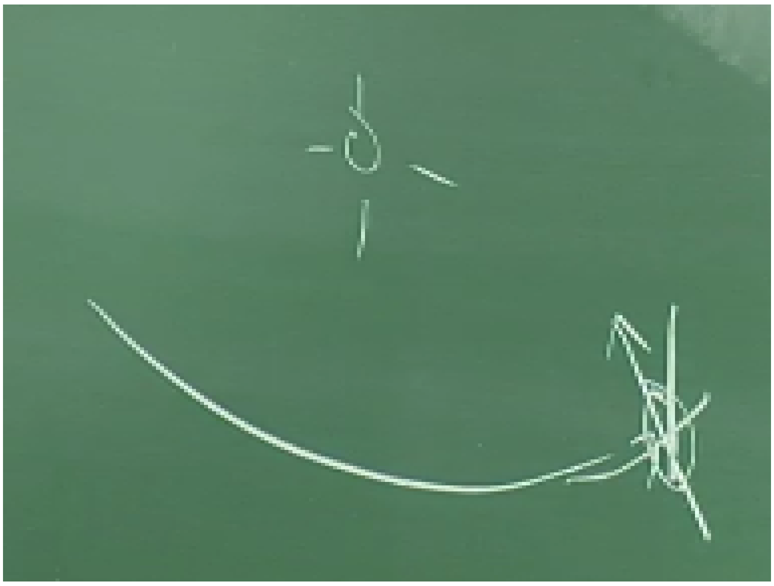

So, that I mean in modern language, it is Earth's axis is tilted, arc is moving around the Sun and Earth's axis is you know, if you take the perpendicular to the ecliptic the Earth's axis is tilted okay, so that is precisely what is happening here also. So, in the modern language essentially, Sun is here okay earth is moving around, earth is moving like this okay and these axis will be like this, it is tilted to this, so that is essentially.

One has to draw the figures and all that you know will be; you know the declination will be related to the longitude, so that is what you know, very simple this thing, which is; or they very critical use is made of this in Indian astronomy in fact, they will use this and the (FL), you see for getting all the spherical trigonometry formula in fact, they will never state these spherical trigonometry formulae in that way.

All the things are you know somehow reduced to that kind of a formulation that is suppose, they are 2 arcs are there okay, suppose, from one arc, suppose you drop a perpendicular what is the perpendicular distance; suppose that is A; A prime is a point on the torque OA, which is inclined to the other arc, so drop a perpendicular from A prime to the plane. other plane, so what is the length of that, so that is how it is formulated.

And for instance in this case, so this epsilon is the tilt, you know the angle between these 2 planes, so then your OA is R and one can show that this OA prime is R sin lambda, when it does travel distance lambda in that tilted arc okay, so then this OA; this OA prime you see; this is R sine lambda, this is r, radius of the whatever circle you are taking, these epsilon and this is the delta how much it is you know which is upper down from the equator, so that is called a declination.

And A prime, B prime see this is the R sine Delta, so then these 2 triangles are equal; OAB and OA prime, B prime are equal okay OA is R, OA prime is R sine lambda, this AB is R sine epsilon, A prime B prime is R sine Delta, so from the similarity of these 2 triangles you get sine delta is = sine epsilon sine lambda is called a formula for the declination; longitude yeah, sine of lambda is called (FL), in Indian astronomy takes.

“Professor – student conversation starts” gamma A prime; sorry gamma A prime, this lambda is the torque, you know gamma A prime or so that is R lambda is R okay capital R * lambda, so that is the arc and this is the OA prime is R sine lambda yeah, so typically Indians they take most of the text they take epsilon to be about 24 degrees, you say so but more or less okay, so slightly more than 25, that keeps varying also. “Professor – student conversation ends”

And the still this axis also will you know that rotate in a circle it is called precision of equinoxes, we will not get into all that, so this and this formula is you know important for many daily problems, which I will discuss tomorrow time from shadow and all that you know how they are doing, how they are considering, so these are all discussed in the chapter called (FL), in astronomy texts.

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So (FL) you know time direction and space (FL) okay some from shadows various things can be done, so all these will need the aid of spherical. The references are given here, okay, we will stop here, thank you.