Mathematics in India: From Vedic Period to Modern Times
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Lecture – 33
Trigonometry and Spherical Trigonometry 1

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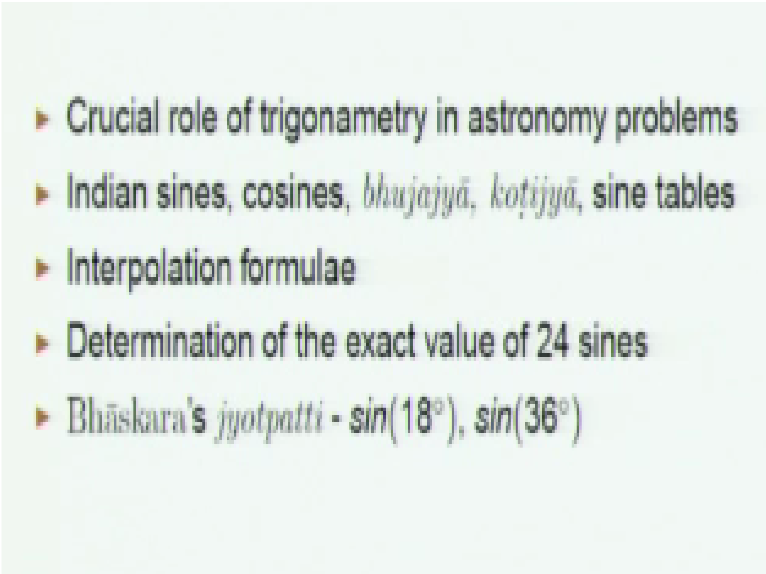

Okay, so these are first lecture on trigonometry in physical trigonometry in Indian works, so these are outline, so where I will first deal with the crucial role of trigonometry in astronomy problems, then Indian sines, cosines that is bhujajya and kotijya and sine tables, some interpolation formula which are needed for finding the sine and cosine at an arbitrary angle, then determination of the exact values of the 24 hour sines.

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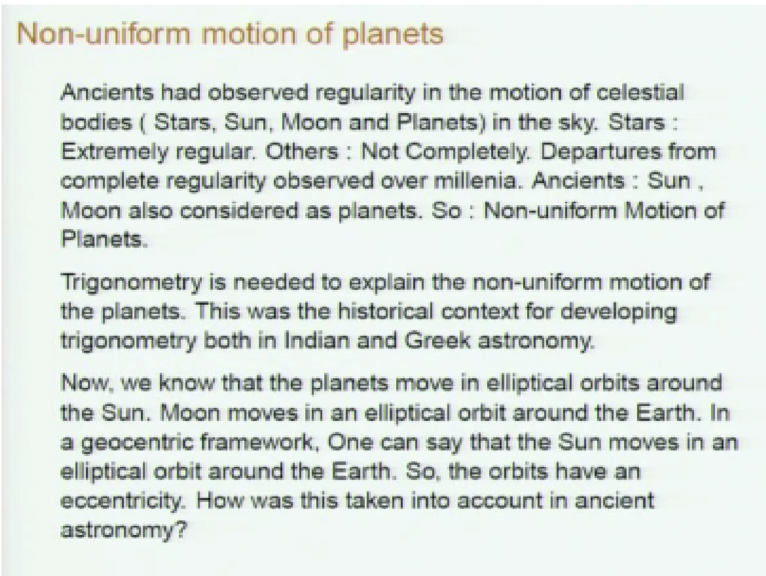

And Baskaras, some aspects of Baskaras jyotpatti, will be discussed in this lecture and the rest later. There is; inevitably there will be some overlap between this lecture and the previous lecture, okay. Now, the ancients had observed regularity in the motion of celestial body, so by celestial bodies I mean Stars, Sun, Moon and planets in the sky and actually in the very olden days, when things were uncertain; far more uncertain and life were more difficult.

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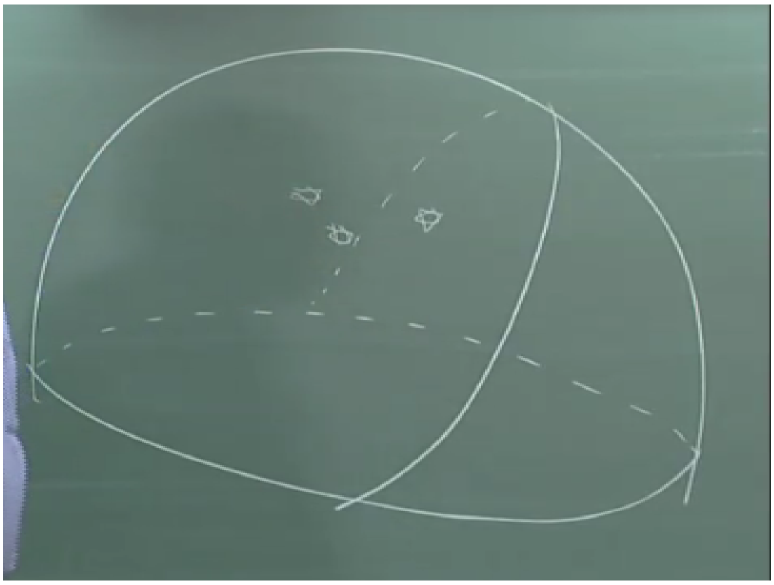

In fact, this is only regular thing which you would have seen you know; one of the most regular things would have been the motion of Sun and Moon and to a little; far lesser extent, the planets. The stars themselves have an extremely regular motion that is what I am trying to say the following. So, you observe the motion of celestial bodies in the sky okay, so far the celestial sphere, you see and we see that the objects; all objects will raise in the eastern portion of the sky, so then raise up go to the top and then come down okay.

And all of them, it will be seen that they will be moving parallel to what is known as a celestial equator which is parallel to or celestial equator itself, so all the objects will be moving from eastern portion of the side, rise and go up and then come down okay, that is what we see, everybody would have seen that. So, now this is the daily motion, okay but apart from that, what one has to see is the relative motion.

If you see the stars, okay; if you see the stars, if you have observed the motion of stars the relative positions of stars will be always fixed; the relative position of the stars will be fixed completely of course, now modern astronomy say that that also has a little bit of motion but we ignore that. The relative positions of stars in the sky will be fixed, suppose, they are moving but the relative positions that way, okay.

But whereas, Sun, Moon and other objects seem to be moving in the background of stars; in the background of stars, in the sky suppose, you know trace a path, they will be moving in the background of the stars and that will be from west to east, so eastward motion. So, that is what you know, is; I am considering a motion of these objects okay. Stars are extremely regular in the sense that you know, if you see some star in the top portion of the sky at the night, some way say, 12 o'clock in the night.

So, next day it will be exactly in the same position about 4 minutes earlier. Similarly, the second next day it will be about you know, same position about 8 minutes earlier and so on and always it will complete one in the circle in this distinct sky; I mean, if you observe the positions at some fixed time in the night and then again you know, so that is regularity, no question; extreme regularity will be there.

And that period is; as you know, it is slightly 4 minutes < than the day, so 23 hours 56 minutes that is called a sidereal day or nakshatra agadir. This sun, moon and all that will be moving in the background of the stars okay, so those that motion will be not so regular, not completely and departure some complete regularity has been observed over millennia okay not centuries millennia okay, 1000's of years; ancient.

By ancients, I mean not only in India, Babylonians Egyptians okay, Greeks so many, say Chinese all of them have you know observed this at whatever level and whatever level of sophistication of course earlier, there was not much of mathematical formulation only about for the past 2000 0r 2500 years, we have some mathematical formulation okay. So, now ancient considered Sun and Moon also as planets okay.

Whatever is moving, it is moving around the earth, from the earth we are see, so they also are moving, so they are considered as planets and what was seen was that over. Let us say I said millennia you know, for a long; after long observations over very long periods, they found that the motion is not so uniform with different you know, varying degrees of you know non-uniformity.

For instance, sun itself is fairly more uniform, with some small departure from uniformity, okay. So, that is if sun, there we also you know; in Indian terminology we say you know, which zodiacal sign it occupies you see, Mesha Rasi, Rishaba Rasi, so zodiacal sign, okay, so it completes one revolution, first they thought it was 360 days okay, it will complete; it will take 360 days, then more accurate observations told them that it is 365 days and some 366.

And more accurate things you know, even you know, 3000, 4000 years back, it would have known that it is not 365 but 365 and a quarter, see so that is the thing and of course, later some few thousand years more, they founded that also is not accurate but slightly < 265, 365.2 okay. So, even in that it is not fully uniform but slight departure for uniformity way but that is small. For moon, it is even somewhat more irregular, okay.

If you would see the relative motion of the Sun and the moon okay, so that is suppose, this you know from; let us say, new moon day is when Sun and Moon are in conjunction, so you take the observation from one new moon day to another that is called a month that is the Masa or the month okay. So, that the average is about 29 and 1/2 days; 29.5309 days but it varies. Sometime, the motion may be or the month may be just 28 days.

And sometimes, it may be 30 kind of thing, so there is a variation is about 2 and 1/2, so that is because moons motion is not that uniform, more less uniform than suns okay. Now, planets they do not, you know; have such an important role in various you know, religious activities and all that and also, they are not so conspicuous but even then, planets also they could see the regularities but these have a different nature.

You know; in fact, it is very difficult to see a pattern, if you just observe you know; start observing okay Sun, Moon and all that. Suppose a person has a; you know, very good extremely intelligent mind and he is very meticulous reservation, he can find a pattern between Sun and Moon, I know their motion; planetary motion, their rate of motion and so on but for planets, it will be very difficult to guess you know, what is the kind of a thing.

We know the reason now, you know that they move around the Sun, whereas we are observing around the earth but anyway still they could make out that you know that there was some kind of a regularity in this motion and there was non-uniformity. So, this non uniform motion of planets, you see that is at the (()) (08:35) you know lot of developments in astronomy okay, non-uniformity; you see, how they handled the non-uniformity.

And astronomy was the exact science in the earlier time in the before 15 century, so astronomy played the same role as physics in the recent times you know. Physics see some kind of a; you know; central signs is around, which most fundamental of the signs around which other things are built. So, astronomy had a similar role earlier okay and then so, I explained this new this thing; a new non uniform motion, trigonometry is required, that is what I am giving this background.

So, trigonometry is needed to explain the non-uniform motion of planets, so this was a historical context for developing trigonometry both in Indian and Greek astronomy, okay. Now, we know I mean of course, even a school boy or school girl will know that the planets moving elliptical orbits around the Sun and moves in a' moon moves in an elliptical orbit around the Earth.

And in a geocentric framework one can say that the Sun moves in an elliptical orbit around the Earth, you see those orbits you do not observe, what you see is you know variations you know; it is in the background of stars okay, and you have to construct the picture after observation over so many of these things. So, behind this simple sentence you know that it moves in an elliptical orbit, at the extreme hard work over millennia.

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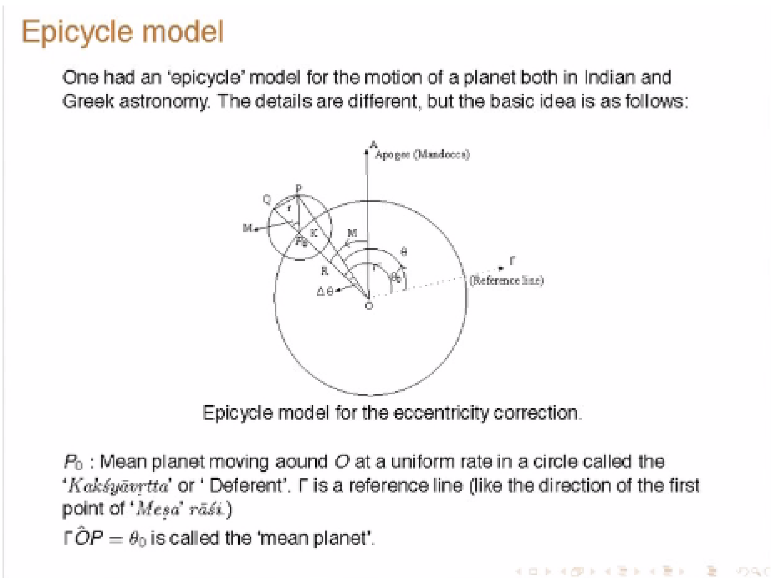

Now, the ellipticity will mean that there is some kind of a; you know eccentricity it is a departure from circularity and that will lead to non-uniform motion. So, how was this taken into account in ancient astronomy, we have to see that. So, one had an; so called epicycle model for the motion of a planet both in Indian and Greek astronomy. The details are different but the basic idea is as follows, okay.

So, essentially suppose; you assume that you know to the first approximation, the planet is moving uniformly around the earth okay, in the background of stars okay. So, P0 is the so called mean planet or the Madhyamagraha, okay, so this is moving at a uniform rate in a circle called Kaksyavrtta called Deferent. Now, this gamma; this is a reference line the direction of the first point of Mesa Rasi.

According to Indian normally, according to Indian convention, all of you would have heard of Mesha, Rishaba and all that, so there is a division of the zodiac 127 part of this thing. The beginning point of that is the this Meshhadi they say, Mesha Rasi. So, now with respect to that, you measure this angle okay, incidentally only you observe the angle only, you all; distance measurements came out later, you see.

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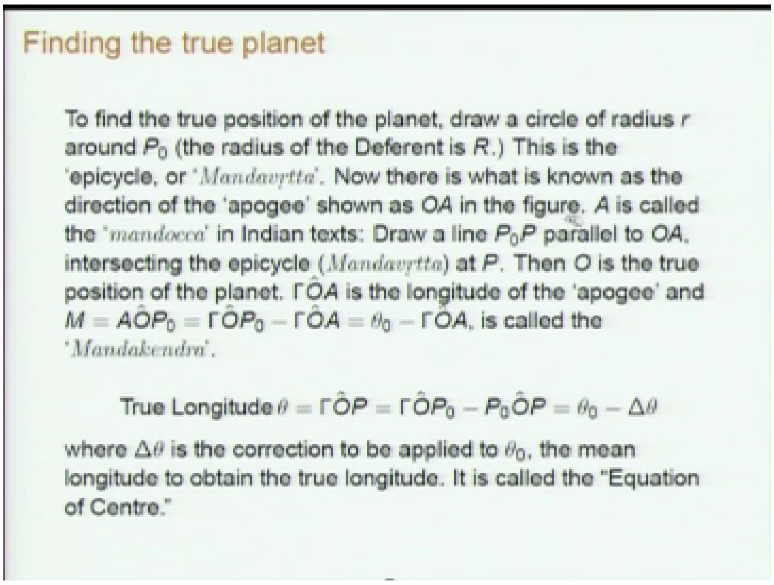

When you see some object you see essentially the angle with respect to certain directions, so this theta 0 that is called a mean planet okay, which is moving uniformly okay. So, now to take into account the non-uniformity, you know; you draw a circle of radius small r around P0, okay. So, around this mean planet, you draw a small circle, so this is called Epicycle or Mandavrtta, so these epicycle, okay and these are the radius r and it is what is known as an Apogee or we have called (FL) in India terminology.

So, this the direction of apogee and how we get it that a different matter, you see that we you have to discuss, which I assume you know that there is some kind of Apogee and that is a direction associated with it. Now, draw a line parallel to this direction of Apogee and let it hit the circle at capital P, so this is a true position of the planet, okay. So, got the point, this is P0, is the mean planet called (FL).

And P is the true planet called us (FL), okay. So, now you have to find this, so difference between them is you know P0 O P, that is the difference between the (FL) are the correction you have to apply to the mean planet to get the true planet, okay so P is; and that is where one can; 2 longitude is this; theta 0 - delta theta, I am calling this as delta theta, this small you know correction okay.

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Appearence of Sine function, Enter Trigonometry

Let $K = OP$. This is called the *manda-karṇa*. Extend P_0 to Q such that PQ is perpendicular to P_0Q. As P_0P is parallel to OA (by construction), $P\hat{P}_0Q = M$ and $PM = r \sin M$.

In triangle POQ, $P\hat{O}Q = P\hat{O}P_0 = \Delta\theta$, and so,

$$OP \sin \Delta\theta = PQ = r \sin M$$

$$\therefore K \sin \Delta\theta = r \sin M$$

$$\therefore \sin \Delta\theta = \frac{r}{K} \sin M = \frac{r}{K} \sin(\theta_0 - A)$$

$$\therefore \Delta\theta = \sin^{-1}\left(\frac{r}{K} \sin M\right) = \sin^{-1}\left(\frac{r}{K} \sin(\theta_0 - A)\right)$$

where $K = OP = [(R + r\cos M)^2 + r^2 \sin^2 M]^{1/2}$.

To know the correction $\Delta\theta$, one needs the sine function. One should also know how to find the inverse sine function, that is to find the arc from the sine.

This is how the trigonometric functions enter astronomy.

To find $\Delta\theta$ for any θ_0 and A, we should know $\sin(\theta_0 - A) = \sin M$, either by explicit construction or tabulated values.

And from the geometry book, you can refer to it later, see you draw a perpendicular some P to the line extended from OP0 onwards, this P, Q, O is 90 degrees, I mean it may not look like this in the figure but it is that and then simple geometry will tell you that you know one can show that this delta; suppose this is K, OP is K, this is delta theta, so then one can show that this delta theta is sin inverse or sin delta theta or K sin delta theta = r sin M.

Let me, so called Manda kendra, so which is this angle; angle between the (FL) mean planet and the Apogee, so your delta theta the correction will be sin inverse r/K sin theta 0 – A, where

A is the angle corresponding to the Apogee, okay. I mean this of course, you can look at it at leisure sometime what I am trying to say that this; so to know the correction delta theta, one needs the sin function, that is the important thing you see, details you do not bother now.

So, to know the correction you know, I have to know the sin function, you should know how to find an inverse sin function also, sin inverse is there, so that is to find the arc from the sin. So, this is how the trigonometric functions enter astronomy, so thus to find delta theta for any theta 0 we do not know a; we should know sin theta 0 - A sin M, either by explicit construction or tabulated value, okay.

So, these how it comes there, so I mean, many other developments in astronomy like this are intimately; I mean mathematics in the earlier days are intimately associated with developments in astronomy, so non uniform motion you know; you have to know science and all that, you have to develop a trigonometry. I mean even to describe the circle and all that you say, why did you say 360 degrees because earlier it was thought that the average year is about 60 days.

So, that is how you know, so now, it is not surprising that you know all the important works; important ideas in mathematics in the earlier days in India or from Aryabatta onwards, so they were always in astronomy text, you see and mathematics was a part of the astronomy text and later only Ganitasarasangara onwards, you will have independent treatises on mathematics.

I mean that is so even in the other Western countries also, I mean the Greek, European tradition also, so they were intimately related and many important developments in mathematics always occurred in relation to astronomy problems like instantaneous velocity of moon (FL) which was mentioned earlier, so that is you know what is needed in this thing and the instantaneous velocity intimately related to the development of calculus okay.

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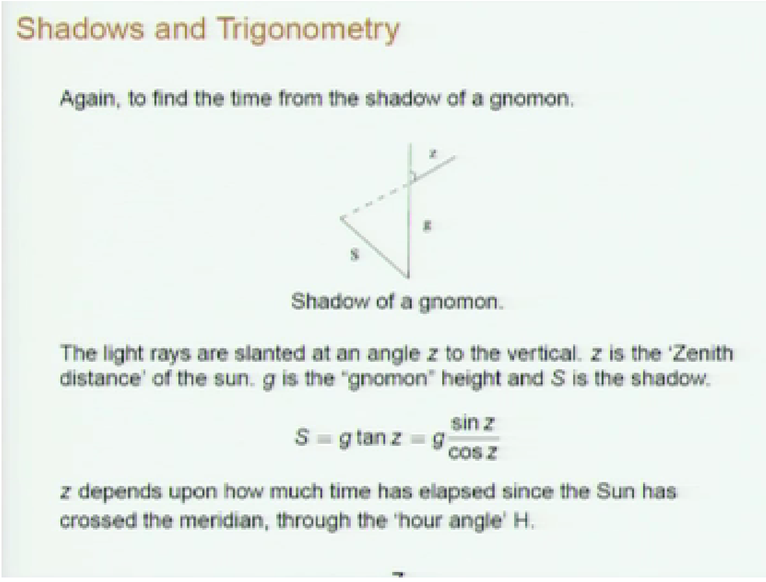

But we are you know bothering only about trigonometry here, so similarly to find that time from the shadow of gnomon, okay. Suppose, this is some (FL) you know pillar whose shadow we are observing and this is the; these are vertical direction and suppose the sun's rays are coming at an angle z, so then you can easily see that this shadow s is g tan z sin z/ cos z, so clearly there is a trigonometric functions are coming.

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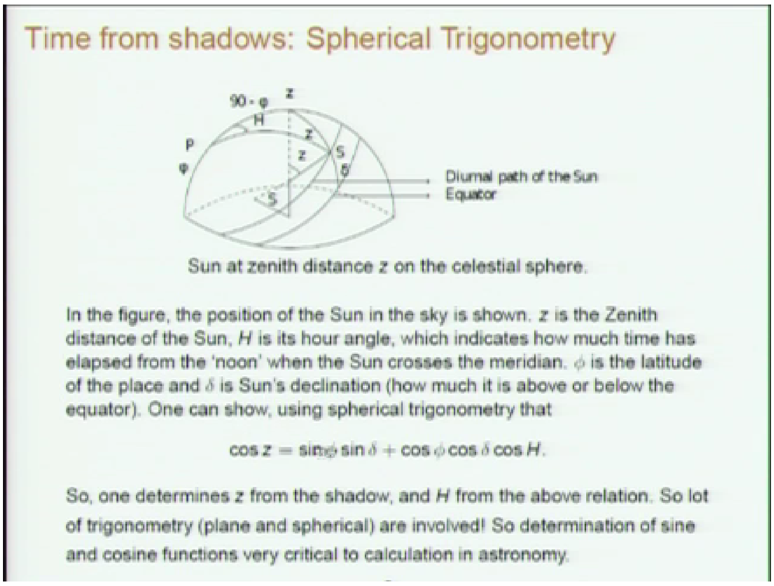

So, these are shadow and then to find the shadow time from shadows that also were mentioned one has to use so called spherical trigonometry, which I will discuss later. So, actually this is the formula which is relating the shadow; I mean this angle z and the time H, essentially it is related to the time okay it is called hour angle, so you can see that they are all related. So, you can find out H from z, if you know phi and delta.

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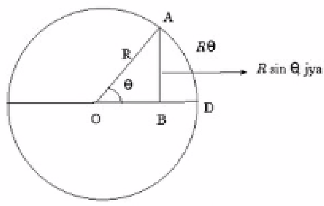
Indian *jyā*



The Indian *Jyā*.

In Indian astronomical and mathematical works, the circumference of a circle is taken to be $360° = 21600'$. The radius $R = (21600'/2\pi) \approx 3438'$. This is the '*Trijyā*'. Then for an angle θ, or an arc $R\theta$, the *jyā* or *jīvā* is $AB = R\sin\theta$ as shown in the figure. $OB = R\cos\theta$ is the *koṭijyā* or *kojyā* and $BD = R(1 - \cos\theta)$ is called *Utkramajyā* or Versed R Sine, or '*Śara*'.

Phi is a latitude of the place; delta is the so called declination. So, determination of sine and cosine functions, very critical to you know for time related things also daily motion, okay. So as important a thing as time, so hence the trigonometry you know, the criticality of trigonometry for astronomical calculations. So, now are going to discuss the Indian jya, for some of them have been done earlier but let me repeat it for completeness.

So, these here you have a circle okay with the radius capital R okay and then this is your angle theta and AB is the perpendicular like this okay, so then AB is called a jya okay. The radius R is taken to be 21,600 minutes divided by 2 pi that is nearly 3438, you have already come across this. So, essentially you are taking the length of the circumference to be 21, 600 which is the number of minutes in a circle; 360 * 60.

The radius of that circle is jya you know Trijya, which is a nearly 3438, this is Trijya, okay. So, then this R sin theta that is a jya; Indian jya is r sine theta, so it is a length, that at the length okay and this OB is called a Kotijya or Kojya and this BD is r * 1- cos theta is called Utkramjya or Sara and this also has been discussed okay, so these how it is. Of course, earlier it was called this; I will come to that.

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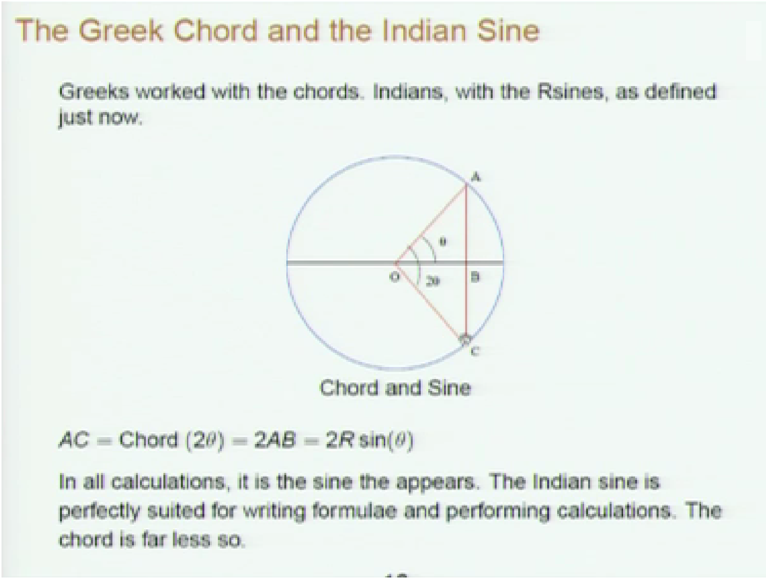

So, now the Greeks worked with chords and Indians with Rsines, okay. Suppose, we see the this is a relation; suppose you have a circle like this okay, so these angle theta, so AB is a Indian sine or sine theta, whereas chord, so this is does not touch the circle, it is only half of this chord is AB and chord is AC, right. So, AC is chord of angle 2 theta, which is 2AB, which is 2R sine theta, okay.

So, Greeks work mainly with the chords and Indians with these sines, you know all in fact, one exclusively with signs after Aryabhata okay and of course, earlier India also, this used to be called jya and then AB was called (FL) okay but later that was given up and this itself called was called jya, okay. Now, the important thing is the Indian sine is perfectly suited for writing formula and performing calculations.

If you actually want to go through that I wrote that figure you know, you have to calculate the correction and all that, it is always the sine function which comes okay. So, the sine function, so then if you know the formula and you can write it easily in terms of the sines not with chords, so in all ways in Indian text, there will be some formula you know, for calculating their positions of planets. Sun, Moon and planets okay.

It will all can be summarized you know with some parameters and all that you know, with some 2 pages you can summarize all the calculations okay and of course, I will tell how to compute sine and all that there is a different matter whereas, in the Greek this thing, it was go to some work like Ptolemy, it is difficult to find out how to calculate, so you know, you take this chord and then you go to the table and then find out this and so on and so forth.

I mean is as accurate as the Indian thing, I am not denying that but it is more difficult for computation and Indians always you know, enthusiastic about competition, it should have a quick picture of this thing, okay. So, the full theory, the philosophical and all that, that can wait a little so because for these computational purposes, this is most eminently suited and ideal actually, so this is the Indian sine.

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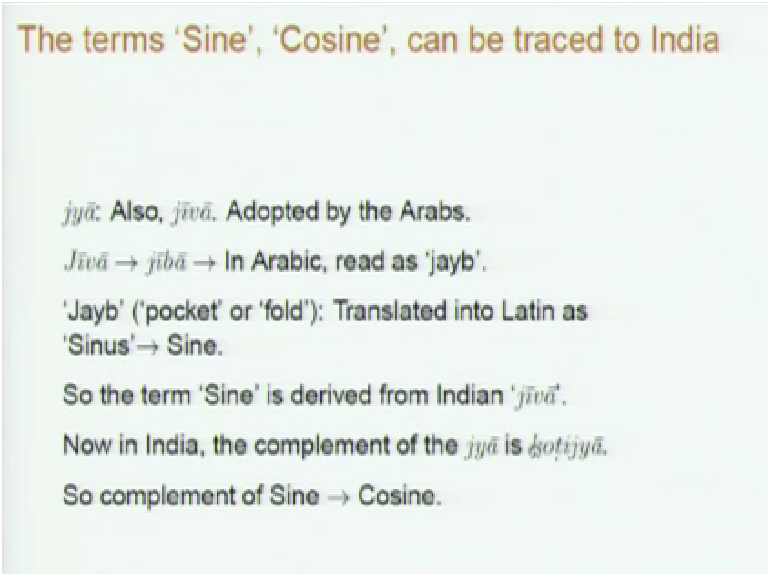

And actually, the very term sine cosine can be traced to India, see jya was also called jiva okay and this was adapted by the Arabs and when it went to Arabs around 6, 7 century, jiva; jiba also it became, mean Arabic is read as jayb, okay. So, jiva did not mean anything, jayb means some pocket or fold in Arabic, so they started doing jayb okay, which means pocket are fold and it translated into Latin, it is sinus okay.

This meaning, you know this got translated into Latin, then it became sinus and from sinus, it became sine okay. So, jiva to sine, though they sound different, so the term sine is derived from India jiva, so then Indian; in India, the complement of the jya is Kotijya okay, so that is we have already seen that you know, the complement of that you see; so this is AB, AB is sine, OB is cosine, right, so they are actually sine of 90 - theta it could be viewed like that also.

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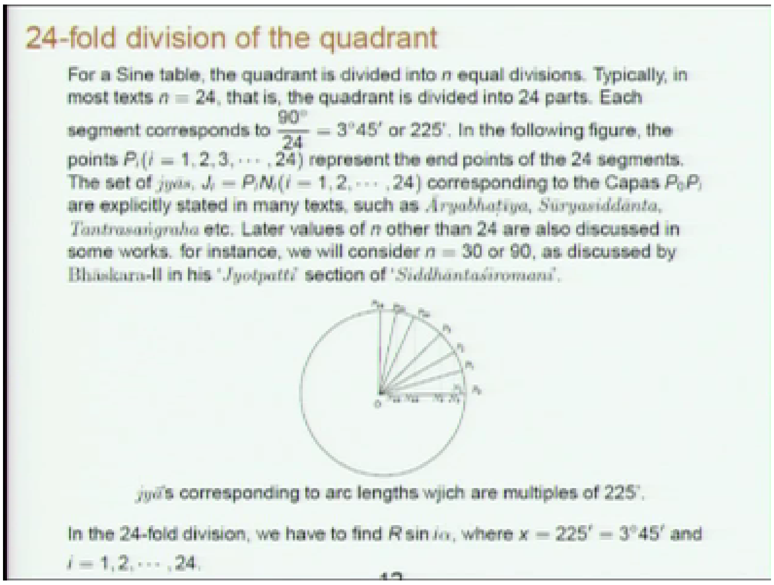

So, they always use kotijya or kojya in chart, so some complement of sine is cosine. So, the sine and cosine are very much you know to do with India like just like algorithm, we see it has mentioned earlier, it had to do with al-khwarizmi, who was interested in Indian ways of calculation, okay. Now, 24-fold division of the quadrant to find out the signs okay that I will not; you have come across this earlier also maybe. I should not be going to details.

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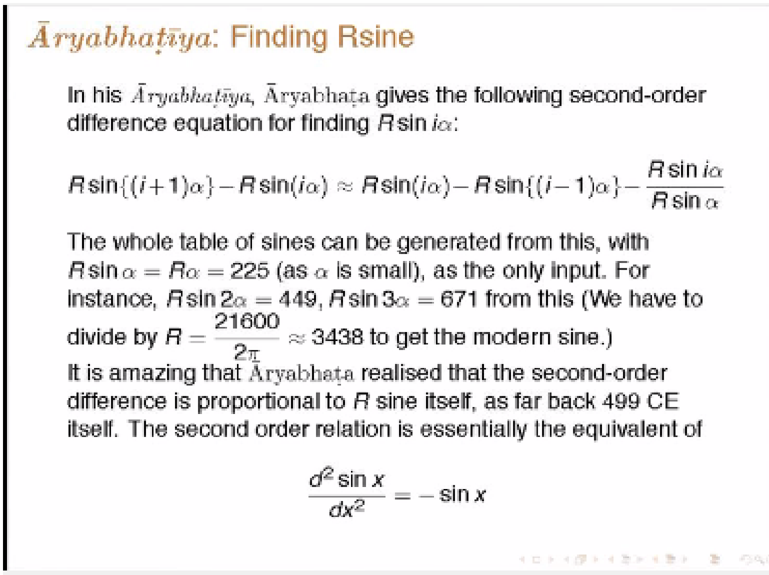

So, you find out the sine at any particular for any angle you first find out the sines corresponding to multiples of 3 degree 45 minutes at 3 quarter degree which is 90/24 okay, so find that out and then sine tables and all that much has been said about it but I just briefly summarized for completeness. So D is the difference equation; second order difference equation for finding Rsin alpha, I alpha okay.

The second order difference equation, so this is what was used by Aryabhata, right even in the previous lecture, you had this, so you are not going to detail. So, first Rsin alpha is 225 it is given in Aryabhatiya, next is Rsin 2 alpha is 449, see all those things I need not explain but a very crucial thing is that you know that D the second order difference equation and it is essentially equivalent of the second order differential equation d square b/ dx square sine x is = - sin x okay.

So, it is a very and for that time you know; at the time of you know this is, in 499, is a very important thing you know and remarkable you see, there is one professor Mumford who is the fields medallists, okay very important equal at or Nobel Prize in mathematics, he had come to Chennai about 5 years back or so, so he was really; if there are some lectures and history of mathematics and he was also talking about it, he was marvelling at this you know.

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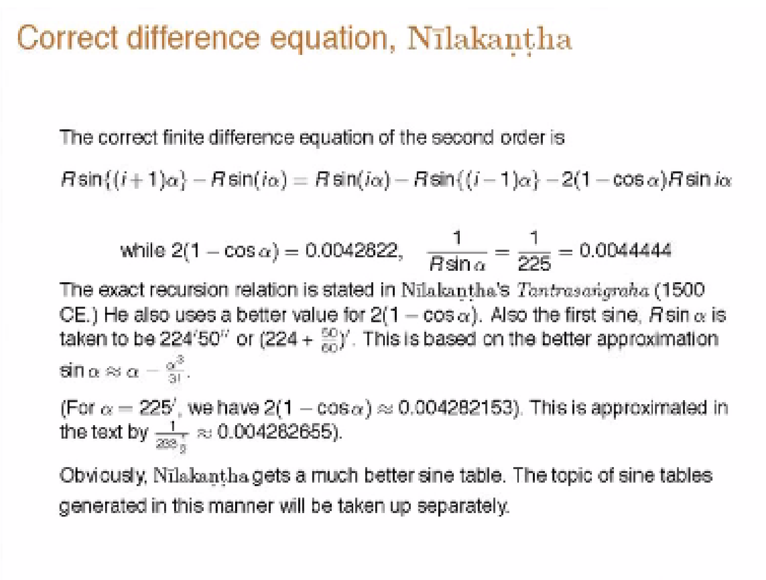

Fact that Aryabhata had something too close to that kind of you know, advanced and this is most optimal way of you know, generating a sine table. So, that is what emphasized me many people you see, from this you know the second order difference equation you can generate the whole thing and the cleverest way of doing things and that is where it is important, so you are not going to details of that, the exact second order finite difference equation is this.

So, cos 1 - minus alpha even the previous lecture, you saw that, so Aryabhatas values were slightly inaccurate, so improved by Nilakantha, okay. So, under first sine, you took as 224 minutes sorry, this might be 50 seconds; 2 * 24 minutes 50 seconds are 224 + 50/60 and these are the better approximation sine alpha is = alpha - alpha cube / factorial 3, okay and similarly this was you know is 2 * 1 - cos alpha.

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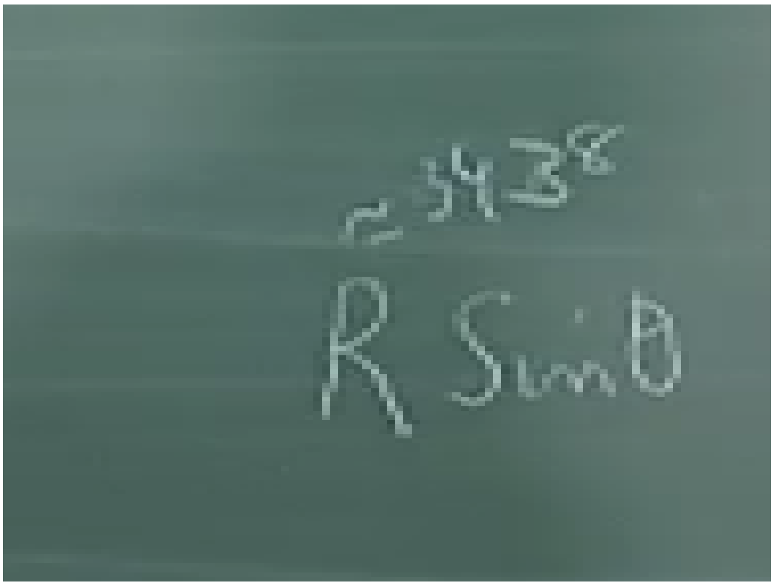

This was you know approximately, 1 over 233 and 1/2 which is this, so obviously Nilakantha gets a much better sine table. So, you already listened to this sine table know, so these how it is done and please remember again that you know the Indian sine, okay; so you have to divide essentially by 3438, to get the modern value of the sine, so that is all. So, when they say sine is Rsine theta right, so that is the jya and R is very close to H; sorry 3438 so that is what you have to remember, okay.

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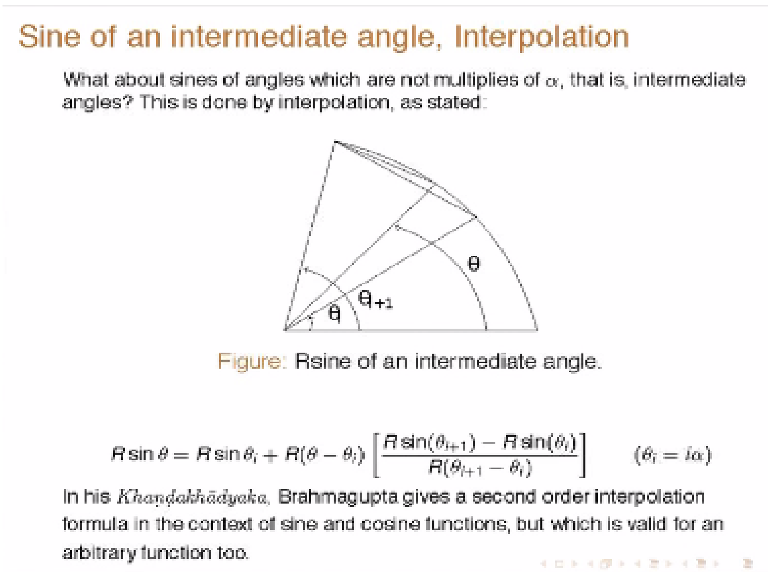

Now, how to get; for an intermediate angle, how do we handle the situation okay, so these 3 gives only the sines at regular intervals of 3 degree 45 minutes so yeah; so, for that you do the interpolation. So, Rsine theta; the first; is the first order interpolation okay, so suppose theta is there, it is close to theta I, let us say, so then Rs theta - theta *; so by the rule of proportionality, so what is then here is you know.

Suppose, that is, if theta I; you are finding the R sine theta for theta i in theta i + 1, so these are difference okay. So, for a difference corresponding to theta i + 1 and theta i, this is the actual difference between the sines and what is the value, what is the correction for an arbitrary difference, theta – theta i, so by the rule of proportionality, so if this is the change for this change in the angle, what is the change of sine for this change in angle, so that is this okay by proportionality you get this.

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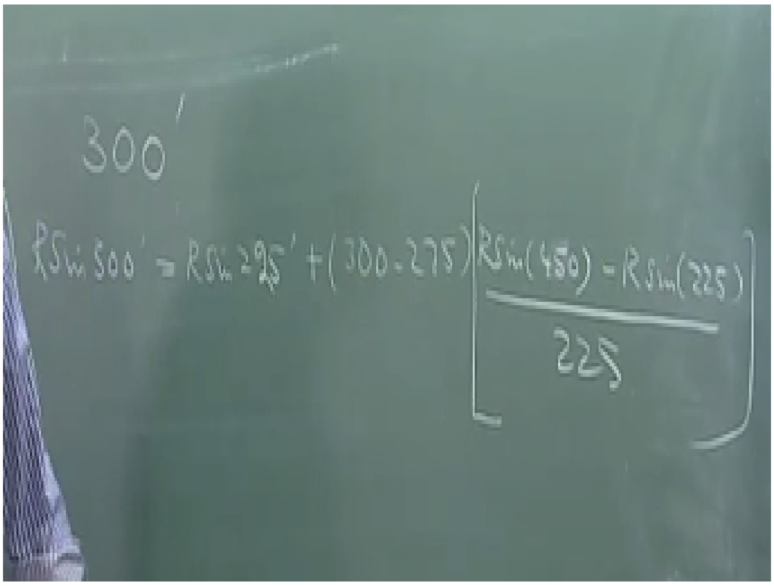

So, what I am trying to say is that you know, suppose you have; you want to find the; find this Rsine at 300 minutes. So, the first approximation is that R sin 300 minutes, so close to that is R sin 275 minutes, okay + 300 – 275, okay and R sin right 450; 450 or 250; sorry, I am sorry, 225 this must be 225; 450 - R sin 225 divided by 225, so this is the; and these are tabulated right, 225 you know, 450 you know, so these are tabulated values.

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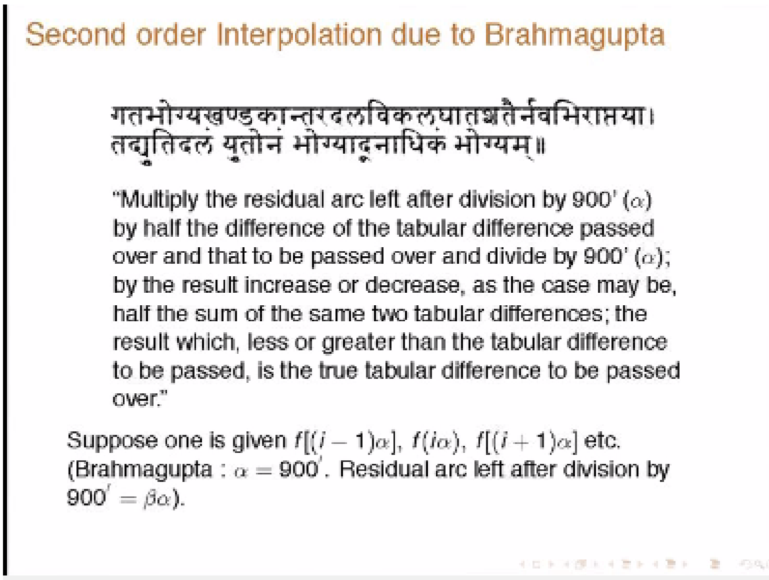

So, for a difference of 225, it is sin defined you know, this is what you have to add for a difference of 300 minutes; sorry 225, you have to add this by rule of proportionality, about this assumes that you know it is linearly varying in that interval which is not true, so one should have a more clear you know, better approximation to take into account the fact that sine is not; you know varying so uniformly, it is not a linear function okay, to take that into account.

So, the second order interpolation due to Brahmagupta, so he says in his other famous work under (FL) so multiply the residual arc left of the division by 900 by 1/2 the difference of the tabular difference passed over and that to be passed over and divided by 900. By the result, increase or decrease that the case may be half the sum of the same 2 tabular difference, the result which less or greater than the width it should be; difference to be passed either 2 tabular difference to be passed over, it is easy mechanical translation.

So, what he is trying to say the following, so we see first we will get what he is trying to say and actually Brahma Gupta in the (FL) is the Kannada work you see. So, for the fast calculations without going to too much of theory, so in that the sine tables are you know a very rough values are given, you know that it has interval of 15 degrees. So, there is the 900 minutes, so that is why 900 is coming.

So, the 90 degrees is divided into 6 portions, so he is giving the values for the 6; these things. So, intermediate values; how do you get? So, because these intervals are so large, so more; better approximation for the intermediate values is needed, so that is why he is saying this. So, what he is saying is following okay, suppose one is given in fact, he does not say sine or cosine and all that obviously, it is for sine and cosine from the context.

Actually, it is valid for any function and any function suppose, you are given some function at the interval of alpha, you see so, alpha, 2 alpha, 3 alpha like that. Suppose, you are given if the values is i - minus stage of this; f of i – alpha, then f of i alpha and F of i + alpha.

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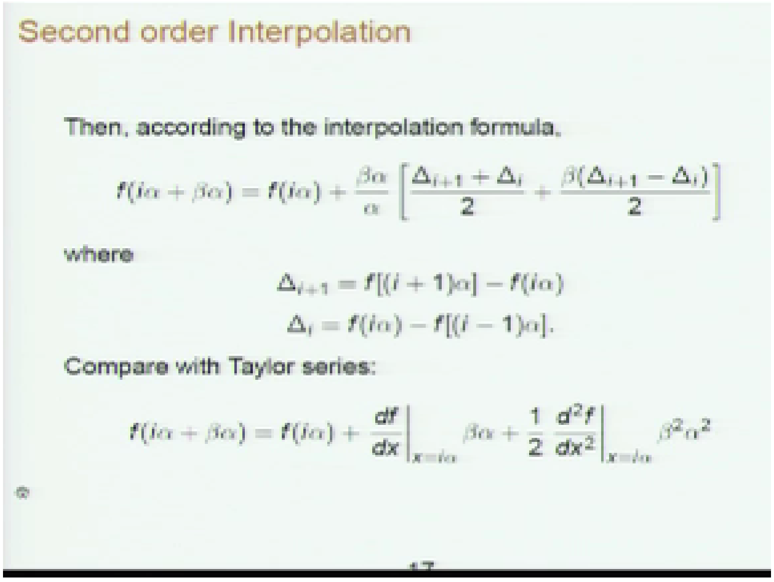

So, then what he is saying is that suppose, beta is a fraction, so f of i alpha + beta alpha is f of i alpha + beta alpha/ alpha * delta of i + 1 + delta i/2 + beta * delta of i + 1 - delta i/2, that is what he is saying and what is delta I, that the difference you see; i alpha and i - 1 alpha, so that is the difference and this is the next difference, you see (FL) we are talking about right, the difference between the sines that successive this thing; values of this multiple.

So, that is that, so we are taking these and then you find out that what he is saying. So, now compare with the Taylor series; the modern Taylor series of course, he does not talk about it, I am saying just to for a comparison, if you take this, this is how you do your Taylor series in modern times F of i alpha is you know. Suppose, you know the function at this value, then to find this you have to know the derivatives of df at various orders.

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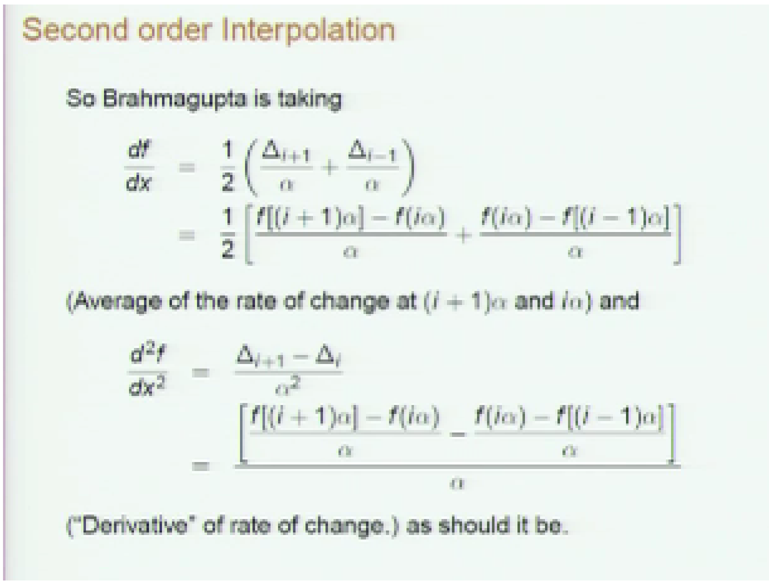

And first order this is this and second order, you have to have this kind of a term; the second order and also it is called Newton Stirling formula; second order this thing and so Brahma Gupta is taking this essentially, he is taking df/dx as this so this is a change and this is amount of change in the angle, so these are rate of change. So, he is taking the average of the rate of change at 2 points; i alpha and i + 1.

Average rate are change at i + 1 alpha and i alpha and d squared f / dx and he is saying the difference; difference between these you see, these are rate of change at i + 1 alpha, these are rate of change in i alpha. Essentially, it is related to the second derivative as we call it. So, it is very amazing that he is discussing this in 650 AD or whatever you know, that is what it is right. So, only you have to plug in properly.

As I told you, I mean what he says is this but we can very clearly see you know, that alpha is 900 minutes and all the other things is the half the difference of the tabular difference, he is saying; all the things he is saying and passed over and divided by 900, all lesser or greater than the tabular difference, so all the ingredients are there and essentially he is giving this formula which we can understand like this.

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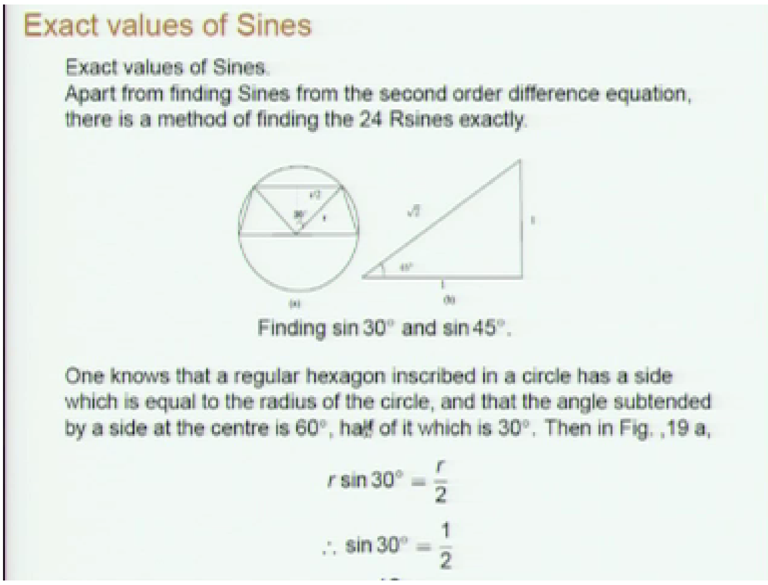

So, you can get more accurate values for the intermediate angle that is what I am trying to say, so that is the second order interpolation formula for which Brahmagupta is justly famous, then apart from tables one can do with; I know the exact values for the 24 hour sines without the professor Ram Subramanian was also mentioning that, so without doing the tables, one can find out all the 24 hours' sines is a good geometrical method okay, that is what he was saying.

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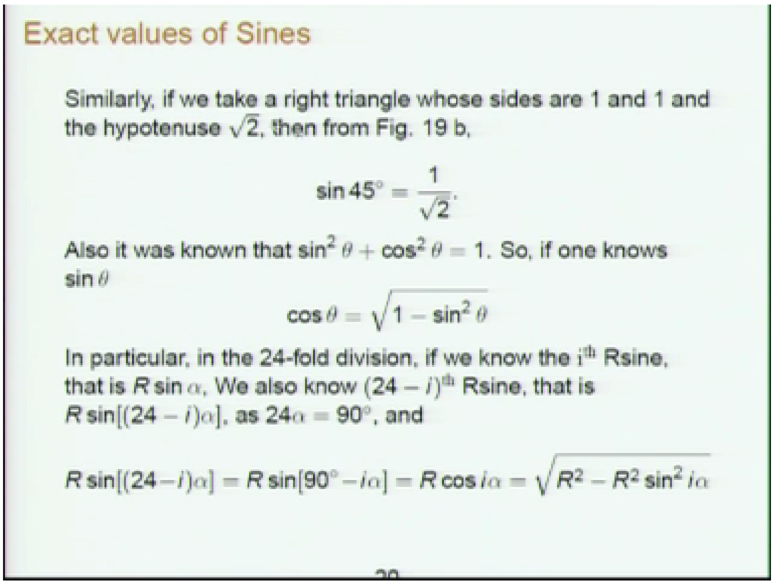

See for instance, if we inscribe a hexagon, you can see that Rsine 30 degrees is = R/2 okay, so these are hexagon this side, so this angle is 60, half angle is 30, so this is r/2, these r/2, this is r, so sine 30 is r/2 basically, sin 30 is r/2. Similar sine 45 degrees is you know; 1/root 2, right, so that is also given in; these are explicitly stated in the works, various textbooks, books in Indian mathematics and astronomy.

And so you know sin 30, you know sin 45 and if you know sin theta, you can find out cos theta from this. In particular, in the 24-fold division, if you know the i th R sine, that is R sin, which will be i alpha, we also know 24 – i th Rsine that is Rsine 24 - i alpha as 24 alpha = 90 degrees, so 24 – i or 24 – i alpha is Rsin 90 - i alpha, which is Rcos i alpha, which is this. So, what I am trying to say is; for instance, suppose, i is 2, okay then, Rsine 22 alpha.

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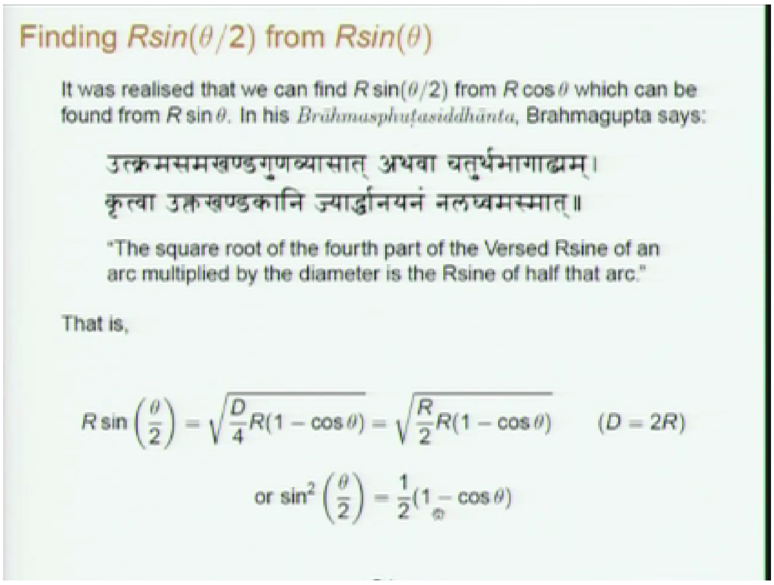

Essentially square root of R squared - this one; this one; right, so Rsine 22 alpha is square root of R squared - R squared sine squared 2 alpha. So, from 2, we can find out 22, so like that and it was realized that we can find Rsine theta/2 from Rcos theta, which can be found from Rsine theta. So, in his Brahmasphutasiddhanta, Brahmagupta says, (FL) The square root of the fourth part of the versed sine of an R, multiplied by the diameter is the Rsine of half that arc, that is what it means.

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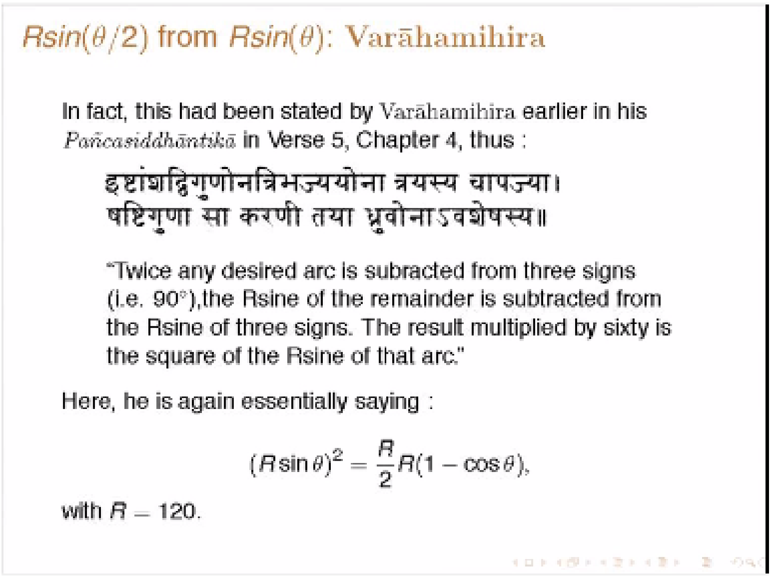

So, R sin theta/ 2 is essentially is a giving this D / 4 * this, so this is square root of R/2 * R * 1 - cos theta. So, essentially saying to say the sine squared theta/2 is = 1/2 of 1 - cos theta, this we know right, so this formula also was known. So, in fact this is; we shall be noticed earlier by Varahamihira himself. Varahamihira was just almost a contemporary of Aryabhata, slightly later Aryabhatia 499, and see, Pancasiddhantika around 520 or something like that.

So, there he describes 5 systems of astronomy and so on. Panca; Pancasiddhantika and there he describes some trigonometry, so that is what he says, (FL) so, twice any desired arc is subtracted from 3 signs that is 90 degrees. The Rsine of the remainder is subtracted from the Rsine of 3 signs. The result multiplied by 60 is the square of Rsine of that arc, okay, so that is the thing.

See, I should mention that sometimes you know the capital R is not always taken to be 3438, sometimes some other values also are taken for simplicity in calculation. So, Varahamihira is taking R to be 120, so that is why he is saying 60 is coming you know, in reverse (FL), so there is R/2 and even the Siddhanta siromani, baskara will take a shorter sine table you know, so he will take because for faster computations, he will take R is = 120 there also.

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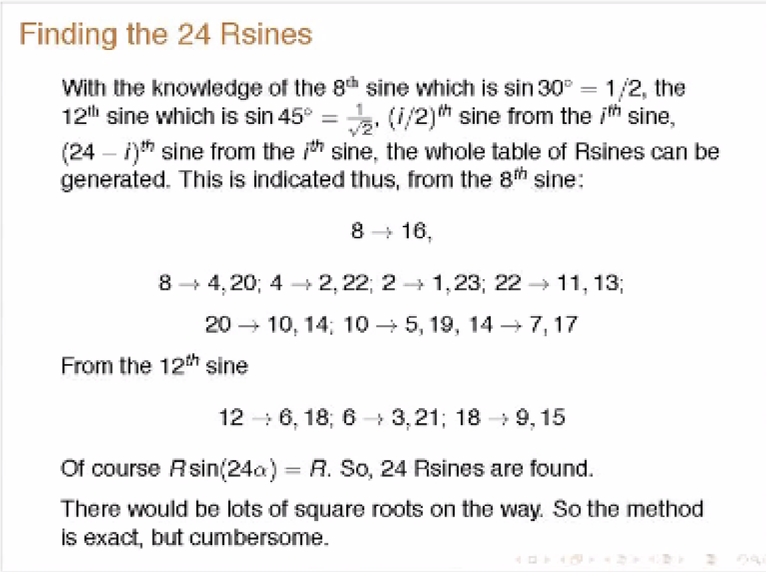

So, like that there is only a constant here, so this is essentially, so this formula that you can find out sorry; I am wrong here it must be 1 - cos 2 theta, right; twice any desired arc, 2 theta; this is not correct, it is cos 2 theta. So, essentially cos 2 theta is found from sin 2 theta. So, if you know sin 2 theta, you can find sin theta or if you know sin theta, you can find sin theta/2. So, essentially, so you know sin 30; you know sin 45, you know sin 90, sin theta/ 2 from sin theta.

And you know how to get sin 90 - theta from sin theta, so using this one can construct it; so with the knowledge of the 8th sine which is sin 30 degree is = 1/2, the 12^{th} sine which is sine 45 degrees is = 1/ root 2, i/2th sine from ith sine, 24 – ith sine from ith sine, one can find out the whole you know. So, from 8th, you can find out 16, 8th sine and from 8 you can get, you know from the 1/2 sine theta/2 formula, you can get 4 and 20, from 4, you can get 2, I mean that second sine, 20 second sign.

From 2, you can get first sine and 23rd sine, from 22, we can get 11 sine and 13 sine, you see, I mean this 13 means, 22 sorry; 11; 24 – 11, right, from 20, you can get 10 and 14, from 10, you can get 5 and 19 like that and from the 12th sine, you can get this, so you can see that everything is covered, so all the 24 sines can be found using this method and there will be lots of square roots on the way.

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Bhāskara's *jyotpatti*: Finding *sin*(18°)

Bhāskara's '*Jyotpatti*' (Generation of Rsines) is a part of '*Golādhyāya*' which ia a part of '*Siddhāntaśiromaṇi*'. It gives the value of sin 18° and sin 36°.

Verse 9.

त्रिज्याकृतीषुघातात् मूलं त्रिज्योनितं चतुर्थभक्तम्।
अष्टादशभागानां जीवा स्पष्टा भवत्येवम्॥

"Deduct the radius from the square root of the product of the square of radius and 5 and divide the remainder by 4; the quotient thus found will give the exact Rsine of 18°."

So, it states:

$$R \sin 18^\circ = R\frac{[\sqrt{5}-1]}{4}$$

So, the method is exact but cumbersome, so like you know finding the circumference inscribing polygons with larger and larger number of sides, one can keep on doing it, can get a larger; very high accuracy but it is very cumbersome after them, you have to do lot of, you know, square root and square root, square root of like that. Now, Bhaskasras Jyotpatti, it is called some verses or so or 28 I do not remember.

So that is; at the end of this so called Goladhyaya of Siddhantasiromani, so Siddhantasiromani and his spherial astronomy part is at the end is jyotpatti, he calls you know, so it interesting you know, so what it means a generation of science okay, so that is the Indian; you get the numbers and of course, with all accuracy and with all following all the logic okay but that is the important thing and it on the theory you can you know learn what we do it later also.

So, for instance deduce a method for finding sine 80 degrees in this; so sine 80 degrees does not come this sine 24 he says, it does not come, so he gives a method so he says (FL), so deduct the radius from the square root of the product of the square of radius and 5 and divide the remainder/ 4, the quotient thus found will give the exact Rsine of 18 degree.

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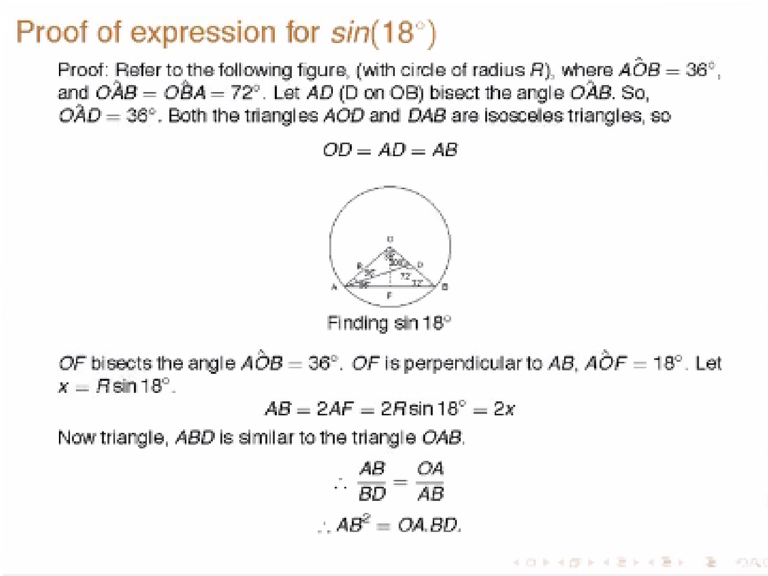

So, he is saying that Rsin 18 degrees is = R * root 5 – 1/4 because he does not use the; in the (FL) and I will explain this method. So, essentially what is; what one has to do is the following; so the circle of radius R, so here AOB is 36 degrees; AOB is 36 degrees of course, it looks much larger than that for clarity, assume it is 36, okay. Then, OAB is 72, this angle is 72, and let this AD bisects this you know OB.

So, we are taking this angles a isosceles triangle, okay this is 36, this is 72, this is 72, this is radius okay and your angle OAD, so this is 72/ 2, which is, 36, so this is 36, these 108, and this is 72, these 72 and these also an isosceles triangle, AD is = AB and AD also = OD okay, it did not appear like that, I am sorry, the figure is not to scale. So, from these things one can find out the sine 18, so for instance OF bisects this angle AOB and OF is perpendicular to AB, AOF that is 18 degrees.

So, now let x is = Rsine 18, so here AB is = 2AF and AF is RSine 18, so you call it as 2x, so one can see that one can show that ABD; the triangle ABD and OAB, they are similar, one can show that because both of them 72, 72, so ABD, ABD, so this is 72, this is 72, this is 36. Similarly, OAB, this is 72, this is 72, this is 36, so they are similar, so you get a AB/ BD is = OA/AB, so AB squared is = OA * BD.

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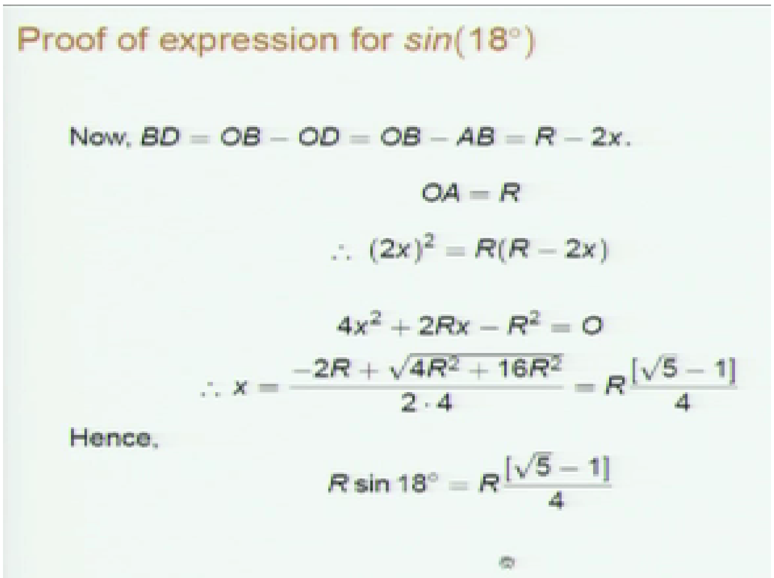

So, now BD is; BD is OB – OD, so it is R – 2x and OA is R, so essentially you get from this you know, AB squared is = OA BD that translates into 2x squared is = R * R - 2x, so finally you get x is equal; so this x; Rsine 18, so 4x squared + 2Rx – R squared is = 0, so these are quadratic equation to get the solution, so you will get root 5 – 1/4, so Rsine 18 is = R * root 5 – 1/ 4, so these are angle, which we have discussed in detail.

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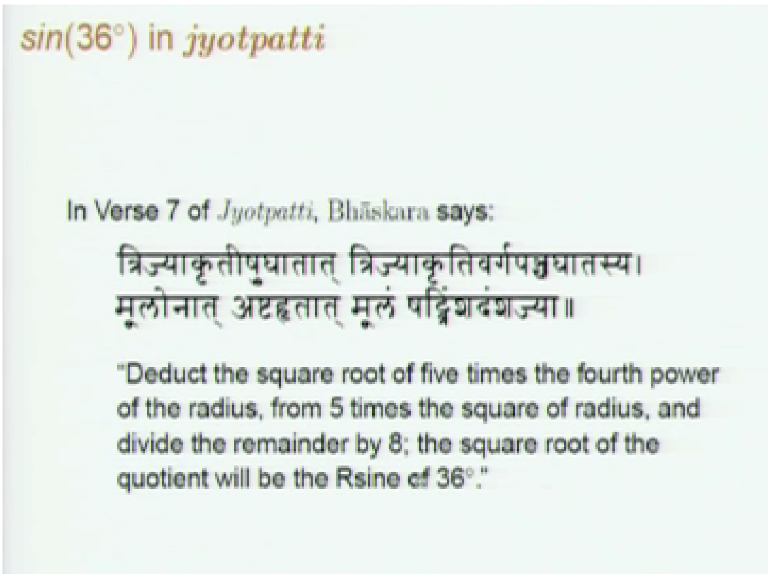

Similarly, in other words, deduce the Rsine 36, so (FL) deduct the square root of 5 times the fourth power of the radius from 5 times the square of radius and divide the remainder by 8, the square root of the quotient will be the R sine of 36 degrees.

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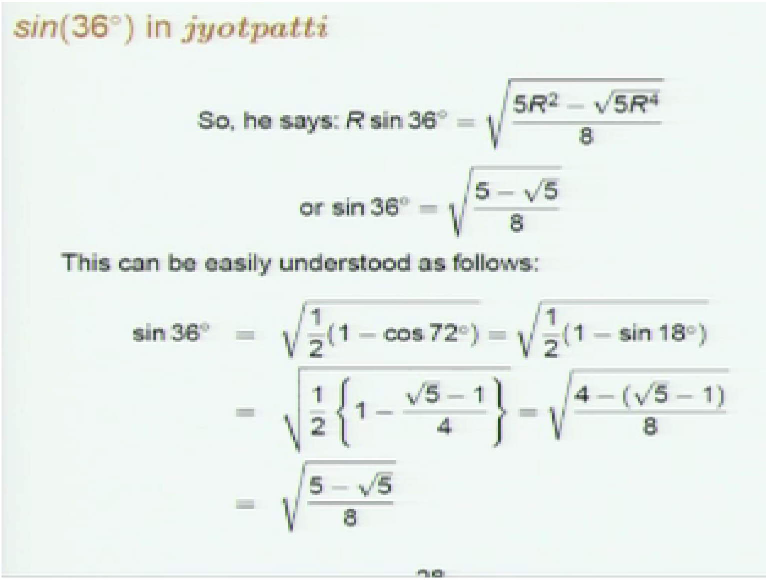

So, using similar methods, one can do in fact, this is R sine 36, he is is giving this; this is a thing 5 root of 5 and then inside the root, there is another root, so sine 36 is giving as root of 5 - root 5/ 8 and take the square root of that, so this can be understood as follows because sine 36 is = square root of 1/2 of 1 - cos 72, okay and cos 72 is sin 18 and sin 18, you have found, right; square root of 1/2 * 1 – root 5 and finally you get this.

So, this is how, he has of course, here discussed more things in jyotpatti, so we will come to that later for instance, he will tell how to find out; you know, you found 24 hour signs, then he will tell how to find out 30 Rsines, that is you know if you divide 90 * 30 divisions, then how to find that is; find a sin at the interval of 3 degrees or sin of 1.5 degrees and then he will go to sin of 1 degree, you know sin 1, sin 2, sin 3, etc, how to find, so those things also he will do in jyotpatti.

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So, which we will do in the next lecture in fact, you will say some very important things, important results of that related to sine and cosine in the jyotpatti, so the famous, now, you all of you know sin A + B, sin A cos B + cos A sin B, so those things we will discuss in that; so next lecture, we will start with this more things on jyotpatti, so the references are given here, thank you.