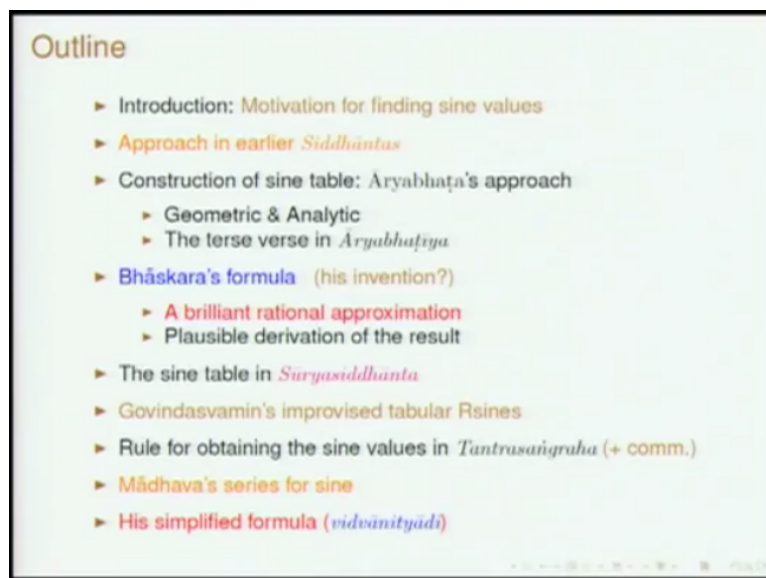


Mathematics in India: From Vedic Period to Modern Times
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Lecture - 32
Jyanayanam: Computation of Rsines

So in this lecture on computation of sines, which is referred to as Jyanayanam so I will be trying to cover as to how various approximations and various techniques have been evolved over a period of almost 2000 years. So as we see that this is a very, very important topic because almost all kinds of calculations are crucially dependent upon accurate computation of sines.

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First, we will try to give a brief introduction as to why this sine values are so important and then we will see the approach taken by the earlier Siddhantas. The earlier Siddhantas I mean the Siddhantas that have been compiled by Varahamihira in his text called Panchasiddhantika, so pancha is 5, so he has compiled so the 5 Siddhantas, which were prevalent so around his time so which is around 505 AD.

And whatever Varahamihira has noted down with reference to this sine computation, we will be just quickly tracing that. Then we will quickly recall so whatever we discussed when we discussed the Ganithapatha of Aryabhatiya so the 2 approaches, which we have taken and then we will spend quite a bit of time on an interesting rational approximation, which has been presented by Bhaskara.

So in his commentary on Aryabhata's bhashya as well as his work called Mahabhaskariyam so he has cited a couple of verses which give a very interesting rational approximation for finding sine. So we will also see how Bhaskara has approached this particular rational approximation. So there are 2 or 3 ways which have been attempted by various scholars.

So we will discuss one such method so which has been presented by K.S. Shukla in his translation of Mahabhaskariyam. Then we will quickly see the table which has been presented in Suryasiddhanta and then I will also touch up on this Govindasvamin's table of sine values. So he has also presented a table, which is a sort of improvised version of what Aryabhata has given.

And it is a very interesting table in the sense that it gives very accurate values so even around 9th century so and then we will discuss the verses which are available in Tantrasangraha, which present the technique for computation of sine values and then we will quickly go through the Madhava's verses as well as the infinite series and how he has handled the infinite series to compute sines.

So this is the idea, so as I was mentioning it is so crucial because the computation of time so in fact the reckoning of time whether it is masa, varsha, dina, tithi whatever it is so it is crucially dependent upon the planetary positions. So we say 1 year is over because sun has returned back to the same position, 1 month is over because the moon has come back once again in conjunction with sun and so on.

So all the yes crucially depends upon the computation of planetary positions accurately and as you will see even professor Sriram will be touching upon this so as to how the sine function appears in the computation of planetary calculations and therefore the sine function is extremely important if one has to even determine time so which is something indispensable.

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Introduction
Determination of time from shadow measurement

Figure: Zenith distance and the length of the shadow.

$$t = (R \sin)^{-1} \left[\frac{R \cos z}{\cos \phi \cos \delta} \pm R \sin \Delta \alpha \right] \mp \Delta \alpha.$$

If ϕ and δ are known ($\Delta \alpha = f(\phi, \delta)$), then t is known.

So I also showed this earlier so when we discussed this Sulvasutras that even the precise computation of time so can be had so using this kind of a formula which is presented so as you can easily see that sine and cosine functions appear so in this particular calculation as well as the inverse functions.

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Computing Rsines: Approach in earlier *siddhāntas*

► Varāhamihira has given the following Rsine values and relations in his *Pañcasiddhāntikā* (c. 505):¹

$$R \sin(30^\circ) = \frac{R}{2} \quad (1a)$$

$$R \sin(45^\circ) = \frac{R}{\sqrt{2}} \quad (1b)$$

$$R \sin(60^\circ) = \frac{\sqrt{3}}{2} R \quad (1c)$$

$$R \sin(90^\circ) = R \quad (1d)$$

$$R \sin(\theta) = R \cos(90 - \theta) \quad (2)$$

$$R \sin^2(\theta) + R \cos^2(\theta) = R^2 \quad (3)$$

$$R \sin\left(\frac{\theta}{2}\right) = \left(\frac{1}{2}\right) [R \sin^2(\theta) + R \text{vers}^2(\theta)]^{\frac{1}{2}} \quad (4)$$

The above Rsine values (1) and relations (2)–(4) can be derived using the *bhūja-koṭi-karṇa-nyāya* and *travastika* (rule of three for similar triangles). Equations (2)–(4) can be used to compute all 24 tabular Rsine values.

¹ *Pañcasiddhāntikā* of Varāhamihira, Ed. by T. S. Kuppanna Sastry and K. V. Sarma, Madras 1993, verses 4.1–5, pp. 76–80.

In Panchasiddhantika, we have these values of sine so very clearly stated. So sin 30, sin 45, sin 60 all that, so this we saw that easily by constructing a certain triangle so I will be able to see all these values can be easily (()) (05:02). Based on these values of sines and the sort of equations given below 2 to 4, I will be able to generate the 24 tabular Rsines, which we discuss while dealing with Aryabhatiyam.

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Construction of the sine-table

A quadrant is divided into **24 equal parts**, so that each arc bit
 $\alpha = \frac{90}{24} = 3^\circ 45' = 225'$.

- ▶ Āryabhata presents two different methods for finding $R \sin i\theta, (P_i/N_i) i = 1, 2, \dots, 24$.
- ▶ The Rsines of the intermediate angles are determined by interpolation (**I order or II order**).

Usually so a quadrant is divided into 24 parts, so p0, P1, P1, P2 and so on so each of them is taken to be 3 degree and 45 minutes 225 minutes so the idea is to obtain the sine values so at these intervals so this is what people have attempted. This is what Aryabhata has given in his table so which is referred to as (FL) so that is how it is. So this is the approach so which is generally taken.

And you will also see in other lectures that people have attempted to give the sine values for 1 degree also, so they have reduced the interval to see that the accuracy can be improved upon. See if you know the value at p1 and if you know the value at p2, then if you want to find out the sine at some value in between p1 and p2, then generally we use first order approximation.

Second order approximation has also been given to see that we can have much better values but if you can reduce the interval itself and the table is available then it will be far more convenient and attempt in that direction has also been done by astronomers and mathematicians. So Aryabhata so has given this (FL). So in this verse which we explained in our Aryabhatiya lecture.

So we clearly showed that if you know the value of sine precisely say R 30 degrees and 1 more value.

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Finding tabular sines: Geometrical approach (contd.)

- ▶ Most of the Indian astronomers have presented their sine tables by dividing the quadrant (90°) into 24 parts.
- ▶ By the principle outlined above, it can be easily shown that all the 24 Rsines can be obtained provided the 24th, 12th and 8th Rsines are known.

- ▶ The circumference of the circle was taken by Āryabhata to be 21600 units.
- ▶ From that using the approximation for π given by him, we get $R = 24\text{th Rsine} \approx 3438$.
- ▶ Once this is known, it is noteworthy that in the proposed scheme of constructing the table, all that is required is extraction of square root, for which Āryabhata had clearly evolved an efficient algorithm.

See for instance if you know 24th, 12th and 8th so then you will be able to compute the entire sine table and this was explained. So I will not spend much time here. So once you know this value so 8th sine is basically 30 degrees and 12th sine is basically 45 degrees. So both of them can be obtained by simply looking at the geometrical construction, so one will be basically $R/2$. So once you know that so you will be able to compute the entire sine table.

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Finding tabular sines: Analytic approach

प्रथमाद्यापज्यार्थात् यैस्त्वं खण्डितं द्वितीयार्धम् ।
तत्प्रथमज्याधौत्रैस्तेस्तेरुनानि त्रेषाणि ॥

Let $B_k = R \sin(k \times 225')$, ($k = 1, 2, \dots, 24$), be the twenty-four Rsines, and let $\Delta_k = B_k - B_{k-1}$, ($k = 1, 2, \dots, 24$), be the Rsine-differences. Then, the above rule may be expressed as²

$$\Delta_2 = B_1 - \frac{B_1}{B_2} \quad (5)$$

$$\Delta_{k+1} = B_1 - \frac{(B_1 + B_2 + \dots + B_k)}{B_1} \quad (k = 1, 2, \dots, 23). \quad (6)$$

This second relation is also sometimes expressed in the equivalent form

$$\Delta_{k+1} = \Delta_k - \frac{(\Delta_1 + \Delta_2 + \dots + \Delta_k)}{B_1} \quad (k = 1, 2, \dots, 23). \quad (7)$$

From the above the discrete version of the harmonic equation follows

$$\Delta_{k+1} - \Delta_k = -\frac{B_k}{B_1} \quad (k = 1, 2, \dots, 23). \quad (8)$$

²Āryabhata is using the approximation $\Delta_2 - \Delta_1 \approx 1'$.

The other approach which has been taken by Aryabhata so this one can call as geometrical approach in the sense that you start with this geometrical figure, you make certain observations and then you will be able to compute the entire sine table. The other approach which has been taken by Aryabhata is the analytic approach. See here B_k represents the k th bhujā.

So the terminology for Rsine is bhuja and cosine is koti. So here so our idea is to obtain so all the values for B1 to B24. So this verse which you find in Aryabhatiya essentially translates to this kind of an equation. See equation 8 so wherein delta represent the first order sine difference, $\Delta_{k+1} = \Delta_k$ represents the second order sine difference and what you note here is so it is proportional to the sine value at that point.

The second order sine difference is proportional to the sine value so which essentially is the discrete version of the harmonic equation. So here when you say it is proportional and what you find as a proportionality constant is $1/B_1$ okay. So B_1 represents the jya of the first sine so generally it is taken to be 225 I mean so the value has been improved upon as we will see during the course of the lecture.

So how this constant has been improved upon over the period of almost 1000 years so people have sort of done and we will see that the value which has been presented by Kerala astronomers is astonishingly accurate okay. So that has to do with the kind of invention which has been done so in finding out infinite series for sine so that we will see as we progress during this lecture.

So this is about the Aryabhata approach to finding sine values so 1 is geometric the other is analytic.

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Āryabhaṭa's table for computing Rsines

- ▶ Using either/both the approaches, Āryabhaṭa having obtained Rsine values has presented a table in *Gīṭikā-pāda of Āryabhaṭīya* (verse 12).
- ▶ This verse³ lists the 24 first order Rsine-differences (in arc-minutes):

मसि मसि फसि धसि षसि ऋसि
 ङसि हस्य स्ककि किण्ण स्यकि किच्च।
 घलकि किच्च हक्क धकि किच्च
 स्म ष्ठ ङ्क ऋ ऋ ऋ ऋ कलार्धज्याः ॥

225, 224, 222, 219, 215, 210, 205, 199, 191, 183, 174, 164, 154, 143, 131, 119, 106, 93, 79, 65, 51, 37, 22, and 7—these are the Rsine-differences [at intervals of 225' of arc] in terms of the minutes of arc.

- ▶ In Āryabhaṭa's notation: म → 25; & सि → 200;

³This verse is one of the **most terse verse** in the entire Sanskrit literature that I have ever come across. Only after **several trials** would it be ever possible to read the verse properly, let also deciphering its content.

So both ways you basically construct the sine table. The sine table which has been presented by Aryabhata is in the form of a single verse, which essentially gives the Rsine differences,

the first order sine differences and as you can easily see the first order sine differences will keep decreasing. It starts with (FL) and then it ends with (FL) so this is how he has presented the entire sine table in one single Arya.

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Bhāskara's approximation to sine function

In his *Mahābhāskariya*,⁴ while presenting planetary parameters—the dimension of epicycles, their application, etc. —Bhāskara (c. 628) presents a brilliant rational approximation to sine function.

मन्व्यादिरहितं कर्म वक्ष्यते तत् समासतः । ॐ
चक्रोर्ध्वकसमुहात् विशेष्या ये भुजांशकाः ॥
तच्छेषगुणिता द्विष्टाः शोध्याः सांशेषाश्चितः ।
चतुर्धांशेन शेषस्य द्विष्टं अन्यफलं हतम् ॥
बाहकोटपोः फलं कृत्स्नं क्रमोत्क्रमगुणस्य वा ।
लभ्यते चन्द्रतीक्ष्णांशोः ताराणां वापि तत्त्वतः ॥

This in modern notation translates to

$$\sin x = \frac{x(180 - x)}{4[40500 - x(180 - x)]} \quad (0 \leq x \leq 180)$$

⁴Mahābhāskariya, VII.17-19.

So now I come to the interesting approximation which we find in Bhaskara's work. So before I proceed to explain the verse given by Bhaskara, I just wanted to say couple of things so Bhaskara starts his approximation to sine by saying (FL) so here the word (FL) refers to the verse of Aryabhata so which starts with (FL).

So (FL) so means so a certain process by which you can avoid the (FL) means the table the table of Rsines can be avoided. So the idea is if you want to get a certain value of sine so you look at the table and then choose that particular value and then find out the difference and then do that. So either it is first order, second order and then you get the actual value of sine. So he is saying (FL) means you can completely avoid the table.

So which means he is presenting a certain other procedure by which you will be not requiring the sine table at all. So what he presents is a certain approximate formula so this is not necessarily the invention of Bhaskara. So this is what one comes to understand from 2 factors, 1 is so this verse if you look at it is present in 2 of his works, 1 is this Aryabhatiya-bhashya.

So wherein he actually explains a certain verse in Gitika-pada and he says (FL) so if you want to do computation of this so without employing the table so he just goes on (FL) that is

what he says that is a certain hint wherein he says Aryabhata himself might have known. So this is what one can guess from that.

The other thing is see this Bhaskara in his own work Mahabhaskariya, so when he presents this verse so this is presented in such a way as if it is quite well known. So it is not presented as if it is something new which he is presenting. So it looks as if this approximation might have been available even during the period of Aryabhata. So one is not very sure whether this is the invention of Bhaskara at all, but anyway so this is the very interesting approximation.

He says (FL) so this is where it is over. So what does he is saying? (FL) chakra basically refers to 360 degrees, chakra is a circle and it is used to refer to 360 degrees. Chakra ardha is half of it so (FL) is degrees so (FL) means to be subtracted so (FL) so the degrees corresponding to the sine value, which you want to compute. In fact, this is applicable for not only sine, cosine also as well as (FL).

In fact, that is what Bhaskara towards the end he says (FL) so all that he says. This is a very interesting approximation. So you can easily follow that this particular line basically refers to $180-x$, then he says (FL) so (FL) means remainder. So remainder that is obtained by subtracting x from 180 (FL) multiplied so (FL) means it has to be placed in 2 places.

So whenever you have the same quantity appearing in our computation they will always say so keep it in 2 places, so do not get lost. So like we store in memory in computer, so that is what they say keep in 2 places. (FL) means they have to be subtracted, subtracted from what so (FL) so this (FL) refers to 0 (FL) is also 0 (FL) refers to 5 (FL) again 0 and (FL) is 4 so this has to be subtracted from this number.

And this has to be divided (FL) so (FL) is one fourth see that is why we divided this by 4 okay, $1/4$ th of this quantity so this (FL) here basically whenever you find in most. They sometimes explicitly state that it has to be divided so when you say divided you always use by so by is (FL) itself in many places will indicate the process of division. So here (FL) that is how it has to be understood.

Then so (FL) means wherever you have placed this so that has to be divided. So (FL) so then he says (FL) so it is actually referring to a certain difference process. (FL) actually refers to

the radius of the epicycle which perhaps will be made clear in the other lectures so (FL) so he actually describes a certain process of computing the planetary position and incidentally he gives this kind of approximation which can be employed to compute the sine function okay.

So (FL) see so whether it is sun whether it is moon (FL) means sun so it is really true see (FL) the one who has scorching rays okay. So (FL) okay. So this is something which is applicable over the range 0 to 180 okay. So usually the table is presented only up to 0, but this formula as you can easily see also captures the symmetry of the sine function.

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Bhāskara's approximation to sine function

मख्यादिरहितं कर्म वक्ष्यते तत् समासतः ।
 चक्रोपौञ्जकसमूहात् विशोभ्या ये भुजांशकाः ।
 तच्छेषगुणिता द्विष्टाः शोभ्याः साभेषसाभितः ।
 यत्पूर्वांशिन शेषस्य द्विष्टं...

मख्यादिरहितं कर्म	– operations without मख्यादि
वक्ष्यते	– is being stated
समासतः	– completely
चक्रोपौञ्जकसमूहात्	– from 180°
भुजांशकाः	– arg. of sine in degrees (x)
ये विशोभ्याः	– that which is to be subtracted
तच्छेषगुणिता (भुजांशकाः)	– $x(180 - x)$
द्विष्टाः	– kept in two places
शोभ्याः (एकत्र)	– have to be subtracted (in one place)
साभेषसाभितः	– from 40500
यत्पूर्वांशिन (विभजेत्)	– with $\frac{1}{4}^{th}$ of the result (divide)
द्विष्टं	– the value placed in the other place

So this I have almost explained so all that so (FL) okay.

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How good is Bhāskara's approximation?

x	Bhāskara's value	Modern value
0	0.000	0.000
15	0.260	0.259
30	0.500	0.500
45	0.706	0.707
60	0.865	0.866
75	0.966	0.966
90	1.000	1.000

- ▶ The formula gives **exact value** for certain arguments.
- ▶ However, it is noted that for the entire range $[0 \rightarrow 90^\circ]$ it is almost **99% accurate**
- ▶ This clearly speaks of:
 - ▶ The brilliance of **the discoverer** in arriving at an **excellent** approximation.
 - ▶ an **novel attempt** to obtain a **rational approximation** to sine function.
 - ▶ the **beauty and sophistication** of ancient Indian mathematics

So how good is the rational approximation, which has been presented by Bhaskara? This formula which has been given by Bhaskara so if you put x is 0 so you will get 0. So when you put x is 180 also, it is going to become so if you see that closely so for x=90 you will get 1. So even in the intermediate values you can see that it is exact for certain values but in the intermediate values also it is almost 99% accurate.

So it is a very, very good approximation and so whether Bhaskara founded or it has been available in the tradition whatever it is whoever be the discoverer so it is indeed an excellent approximation for sine and there has been some attempt which has made so I remember to see that this formula can be taken so to compute the sine value in a much more efficient way so even in the modern devices.

Anyway and it also actually speaks off a certain attempt, a normal attempt to capture this function. So the sine function is something which is ubiquitous and so one needs to capture its value in some form or other form so which will be as easy as possible so whenever we want to compute this values.

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Properties mirrored by Bhāskara's approximation

- ▶ We know that sine function
 1. is *symmetric* about 90° point
 2. is *concave* over the range 0° → 180°
- ▶ The formula given by Bhāskara clearly satisfies these properties

Bhāskara's approximation

$$\sin x = \frac{4x(180 - x)}{40500 - x(180 - x)}$$

- ▶ Isn't a mathematician's delight to arrive at an expression for sine function that *at once captures the properties* as well as serves as a *very good approximation* (≈ 99%) for the entire range (0 – 180°)?
- ▶ Here I may quote the statement made by Hardy⁵

The Greeks were the first mathematicians who are still *'real'* to us today. Oriental mathematics *may be an interesting curiosity*, but Greek mathematics is *the real thing*. . . .

⁵G. H. Hardy, *A Mathematician's Apology* Cambridge, 2nd ed. (1967) p. 80.

So this formula is also a very beautiful formula so in the sense that it actually captures this symmetry so which is there in the sine function and it also so since it is applicable for the range 0 to 180 so it also captures the concavity of this sine function and at this stage I would like to quote Hardy, he says the Greeks were first mathematicians so who are still real to us today.

Oriental mathematics maybe an interesting curiosity but Greek mathematics is the real thing. So this is the kind of statement which Hardy makes so in the early part of 20th century so in spite of knowing these kinds of brilliant discoveries. By brilliant discoveries we mean the infinite series for pi, its various fast convergent approximations as well as the series called trigonometric function such as sine, cosine and tan inverse functions.

In fact, it is not that by the beginning of 20th century, Hardy was not aware of some of these significant contributions made by Indians but maybe it has to do with what people call as the Eurocentric view.

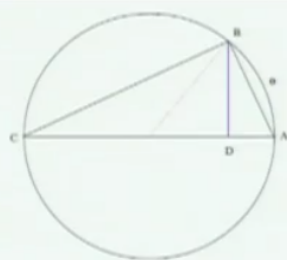
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Proof of Bhāskara's formula

In the figure, the area of the triangle ABC can be expressed in two ways:

$$A = \frac{1}{2}AB \cdot BC = \frac{1}{2}AC \cdot BD$$

or $\frac{1}{BD} = \frac{AC}{AB \cdot BC}$ (9)



Since the length of the chord < that of the arc, (9) may be expressed as an inequality

$$\frac{1}{BD} > \frac{AC}{AB \cdot BC}$$

or $\frac{1}{BD} = \frac{x \cdot AC}{AB \cdot BC} + y$

$$= \frac{2xR}{\theta(180 - \theta)} + y$$

or $R \sin \theta = \frac{\theta(180 - \theta)}{2xR + \theta(180 - \theta)y}$ (10)

So now I give you the approach to arrive at this interesting rational approximation, which has been given by Bhaskara so this you can easily follow so there are 2 or 3 approaches so which has been presented by scholars. So here I am presenting an approach which is based on the geometry and the kind of analysis, which has been done around that period by most of which is familiar to you by now.

So here just consider this triangle, so A, B, C so the area of this triangle can be represented in 2 ways, so one is half times AB*BC fine, so AB*BC base times the other is AC*BD fine. So from this 1/BD is AC/AB*BC. So you know that the length of the chord is always < that of the arc fine. So the sine function is basically finding the relation between the chord length and the arc length. So this is all the sine function is all about.

So now you can write this equation 9 as in any quality in this form so what has been done is so in the denominator so AB and BC have been replaced with arc lengths. So since the arc length is much bigger than the chord length, so this becomes the any quality so we wrote this way this. This by introducing x and y, I write it as an equation $1/BD=x$ times AC+y okay. AB we just take it as theta and therefore the other part BC becomes $180-\theta$.

So that forms the denominator so we have this equation. So you can easily see that so BD is nothing but $R \sin \theta$ fine. So BD is $R \sin \theta$ and therefore we have equation 10.

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Proof of Bhāskara's formula

Substituting $\theta = 30$ and $\theta = 90^\circ$ in (10) we have,

$$2xR + 4500y = \frac{9000}{R} \quad (11)$$

$$2xR + 8100y = \frac{8100}{R} \quad (12)$$

Solving the above equations for x and y we have,

$$y = -\frac{1}{4R} \quad \text{and} \quad 2xR = \frac{40500}{4R} \quad (13)$$

Using the above values in (10), we have

$$R \sin \theta = \frac{4\theta(180 - \theta)R}{40500 - \theta(180 - \theta)} \quad (14)$$

which is the same expression given by Bhāskara.⁶

⁶The above proof has been given by K. S. Shukla in his edition of the text *Mahābhāskariya* with translation and annotation.

If you substitute $\theta=30$ and $\theta=90$ so in the above equation so this is the equation so by simple substitution, you will be getting these 2 equations $2xR+4500$ is this and this. So by solving this for x and y, you will get y is $1/-4R$ and once you plug in this value into any of this equation, you will get x =this now you have the Bhaskara expression.

It is very simple. So this is something which might have been used by whoever who discovered this interesting approximation for sine. So this is basically the Bhaskara formula. So this derivation is presented by Shukla in his Mahabhaskariya.

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The sine table given in *Sūryasiddhānta* (II. 17-22)
The values presented are in *Bhūtasankya* system and the same as that of *Āryabhaṭīya*

तत्त्वाश्विनः अङ्गाभिकृता रूपभूमिधरर्तवः ।
खाङ्गाष्टौ पञ्चमून्येजा वाणरूपगुणेन्द्रवः ॥
मून्यलोचनपक्षैकाः छिद्ररूपमून्येन्द्रवः ।
वियद्यन्द्रातिधृतयो गुणरन्ध्राम्बराश्विनः ॥
मुनिपद्ममनेत्राणि चन्द्राङ्गिकृतदम्बकाः ।
पञ्चाष्टविषयाङ्गीणि कुञ्जराश्विनगाश्विनः ॥
रन्ध्रपञ्चाष्टकयम्माः वस्वद्वाङ्कयमास्तथा ।
कृताष्टमून्यज्वलनाः नागाङ्गिज्जिवह्वयः ॥
पट्टपञ्चलोचनगुणाः चन्द्रनेत्राङ्गिवह्वयः ।
यमाङ्गिवह्विज्वलनाः रन्ध्रमून्यार्णवाङ्गयः ॥
रूपाङ्गिमावरगुणाः वस्वङ्गिकृतवह्वयः ।

225, 449, 671, 890, 1105, 1315, 1520, 1710 ... 3178, 3256, 3321, 3372, 3409, 3431,
3438.

He is also giving some other algebraic derivation. So now I move on to the sine table which has been presented in *Sūryasiddhānta*. So one is not very sure as to when this *Sūryasiddhānta* was really composed so because we have the (FL) so in *Varahamihira's Panchasiddhantika* and *Sūryasiddhānta* seems to be an improvised version of that. So anyway *Sūryasiddhānta* presents this sine table in these verses.

And what has been used is *Bhūtasankya* system so if you recall this *Bhūtasankya* system some of these words see (FL) so (FL) is the number of (FL) so 25, (FL) refers to 2 it is 225, see (FL) is 9 so the number leaving out 0 (FL) is 9 (FL) is 4 (FL) refers to the number of (FL) and therefore this represents 449 and see (FL) the forum is unique for every entity and therefore (FL) refers to 1 (FL) is mountain so that refers to number 7.

So (FL) that is 7 and (FL) seasons number 6 so it is 671. This table which has been presented in *Sūryasiddhānta* is identical with the table which has been given by *Āryabhaṭa*, which is (FL) so the only difference is *Āryabhaṭa* has given the differences the first order sine differences so here it is presenting the value of sines themselves. For instance, the last value so if you look at see (FL) refers to 8, agni refers to 3, (FL) refers to 4 and (FL) refers to 3 okay.

So this table is identical with the table that has been presented by *Āryabhaṭa* but it only presents in the form of *pinda-jya's* okay so whereas *Āryabhaṭa* gives the value of (FL) the differences in sines okay.

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How to find intermediate sine values?

Prescription of first order interpolation (based on *trairāsīka-nyāya*)

लिप्ताभ्यस्तत्त्वेनोत्राताः गता ज्याः श्रेयतः पुनः ।
 गतगम्यान्तरप्राच हतास्तत्त्वयमेः क्षिपेत् ॥ २ ॥
 दोःकोटिज्ये नयेदेवं ज्याभ्यहापं विपर्ययात् ।

- Suppose we want to find the *jyā* of θ where $i \times 225 < \theta < (i + 1) \times 225$.
- Let $\phi = i \times 225$, whose *jyā* is known (J_i).
- Now, *jyā* θ is given by

$$jyā \theta = J_i + \frac{\delta\theta \times (J_{i+1} - J_i)}{225}$$

How do we obtain the intermediate sine values? So this is the general prescription so which is based on this (FL) or the first order interpolation. So it is clearly states (FL) for instance suppose AOC so represents an integral multiple of 225 so they give the values for every 225 so if you know the angle AOC and if you want to find out see AOB the sine corresponding to AOB.

So then all that you do is so you take the value corresponding to AOC which we write as J_i this can be simply pulled out from the table and for this so this COB which I have marked as $\delta\theta$ which is an increment so you use this (FL) rule so if for 225 minutes so which is GC so this is going to be the difference. Then for $\delta\theta$ what is going to be the difference? So by rule of 3 you add this.

So this is what has been stated in this verses see (FL) see (FL) is already (()) (28:32) (FL) is that which is going to be the next value. So (FL) is difference multiplied and divided by 225 so this gives you the value for any intermediate sine argument.

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Improved values of tabular Rsines

- ▶ Govindasvamin (9th cent.) in his commentary on *Mahabhaskariya* presents a set of corrections to the values of the tabular Rsine presented by Aryabhata.
- ▶ He observes:

ॐ

मख्यादयो हि न्यूनाधिकावयवाः । तेन ज्याच्छेदविधानात्
अवयवावगतिः । तथा च अवगता एते तत्परादाः -

सप्तत्रिंशत्त्रिंशति वियदुणां नेत्राश्विनेत्रं मुनिपञ्चवेदाः ।
द्वाङ्गयष्टयः पण्णयनद्विरामा वेदाश्विभूतं रविष्टकृष्णान् ॥
रन्ध्राभ्रपङ्कं गुणापावकाष्टौ चक्षुर्वियत्सप्तस्रचन्द्रसूर्याः ।
रुद्राश्विचन्द्रा मनुमत्सोमा दम्भाभ्रनेत्रं नयनं द्विसूर्यम् ॥
अश्विपङ्कं वसुनेत्ररन्ध्रं चन्द्राश्विविदा वसुष्वाष्टचन्द्रम् ।
रन्ध्रेषु वेदं नवरूपमिध्मं श्वाभ्राश्वयस्मत्सप्तगुणध्मसंख्यम् ॥
- ▶ The values listed in the verses are: 9" 37"', 7" 30"', 2" 42"', 4" 57"' ...

Now I move on to the sine table so which has been presented in Govindasvamin's vyakhyana of Mahabhaskariya. This is a very elaborate commentary which has been written by Govindasvamin and here he presents a set of verses which primarily give the correction to the Aryabhata values. So in fact it starts like this (FL) see (FL) means something which is less something which can be greater.

So the values which have been presented in the (FL) table so this (FL) which has been given by Aryabhata. (FL) so sometimes it can be so it is basically the approximation so rounding off has been done. So it could be slightly less, it could be slightly greater. So by specifying the measure by which it is less or more, so we will be able to get the exact value so that is what he is trying to hint here.

And he gives (FL) the word (FL) is used to refer to thirds so we know degree, we know minutes, we know seconds so if you go one more step 160th of a second is referred to as (FL) so here he says so (FL) so the verses in which I am going to list the numbers are basically in (FL) and then it will be in (FL). So here he says (FL) see (FL) is 7, agni is 3 (FL) means hole so why does it represent 9? In the body, (FL) okay so in the body there are 9 holes, 7 in the face and 2 below okay so that is how it is.

So he gives basically the correction which has to be done to the Aryabhata values so to get the accurate values.

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Improvised values of tabular Rsines

θ in min.	R sin θ according to		
	Aryabhata's	Govindasvami	Madhava(also Modern)
225	225	224 50 23	224 50 22
450	449	448 42 53	448 42 58
675	671	670 40 11	670 40 16
900	890	889 45 08	889 45 15
1125	1105	1105 01 30	1105 01 39
1350	1315	1315 33 56	1315 34 07
1575	1520	1520 28 22	1520 28 35
1800	1719	1718 52 10	1718 52 24
2025	1910	1909 54 19	1909 54 35
2250	2093	2092 45 46	2092 46 03
2475	2267	2266 38 44	2266 39 50
2700	2431	2430 50 54	2430 51 15
2925	2585	2584 37 43	2584 38 06
3150	2728	2727 20 29	2727 20 52
3375	2859	2858 22 31	2858 22 55
3600	2978	2977 10 09	2977 10 34
3825	3084	3083 12 51	3083 13 17
4050	3177	3175 03 23	3175 03 50
4275	3256	3255 17 54	3255 18 22
4500	3321	3320 36 02	3320 36 30
4725	3372	3371 41 01	3371 41 29
4950	3409	3408 19 42	3408 20 11
5175	3431	3430 22 42	3430 23 11
5400	3438	3437 44 19	3437 44 48

So this is the table, so Aryabhata has taken for 225 as 225 and Govindasvamin actually says so 224, 50, 23 so how do you get this see you can see that see so in this you have to remove 9 37, 37 (FL) and 9 (FL) so you will get this value 224, 50, 23. So what is interesting to note is these values which have been given by Govindasvamin are very close to the values which are given by Madhava having obtained the infinite series.

And then giving the accurate values so once you have the infinite series in place so obviously you can get very, very accurate values depending upon the number of terms which you consider so it is a very fast converging series. So what Govindasvamin so even without that has so this also is sort of indicative as to how they have been trying to improvise the sine values so meticulously.

So this Govindasvamin is in 9th century and Madhava is 14th century and Aryabhata 5th century.

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Computation of tabular Rsines (*Tantrasaṅgraha*)

विलिप्तादशकोना ज्या राश्यष्टांशधनुःकलाः ॥
 आद्यज्यार्धात् ततो भक्ते सार्धदिव्यभिस्सतः ।
 त्यक्ते द्वितीयखण्डज्या द्वितीया ज्या च तद्युतिः ॥
 ततस्तेनैव हारेण लब्धं शोध्यं द्वितीयतः ।
 खण्डात् तृतीयखण्डज्या द्वितीयस्तद्युतो गुणः ॥
 तृतीयः स्यात् ततश्चैवं चतुर्धाद्याः क्रमाद् गुणाः ।

- The first *piṇḍajya* is often taken to be 225' based on the approximation,

$$R \sin \alpha \approx R\alpha = 225'. \quad (15)$$

- In contrast to the above, it is stated to be $225' - 10'' = 224' 50''$.

- This is based on the approximation $\sin \alpha \approx \alpha - \frac{\alpha^3}{3!}$. Thus we have,

$$R \sin \alpha \approx \frac{21600}{2\pi} \left(\alpha - \frac{\alpha^3}{6} \right) = 224.8389' \approx 224' 50''. \quad (16)$$

So that has been a certain effort continuously going on to improve upon the values. So coming to this Tantrasaṅgraha so which is composed in 1500 so Nilakantha presents the following verses. So he says (FL) in the first line is of importance the more or less the procedure is same as that of Aryabhata so I will skip to the rest okay. So (FL) means reduced by less so *jya* here refers to the first *jya*.

So *jya* of (FL) is 30 degree (FL) is 1/8th of that so 1/8th of that corresponds to so which is 225 minutes so he says so 225-10 seconds so gives you the first *jya*. So (FL) so one can easily see that so this is based upon this approximation so *R*sine alpha is simply taken as *R* alpha, sin alpha is alpha so by Aryabhata and therefore 225 is taken as 225 so if you use this next level of approximation, so alpha-alpha²/3 factorial one can easily see that so you get this so 224.83 and this is what it is.

So this is what has been given. It is quite clear that he starts with this approximation instead of taking sine alpha as alpha.

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Computation of tabular Rsines (*Tantrasaṅgraha*)

विलिप्तादशकोना ज्या राश्यष्टांशधनुःकलाः ॥
 आद्यज्यार्धात् ततो भक्ते सार्धदेवाश्विभिस्ततः ।
 त्यक्ते द्वितीयस्रष्टज्या द्वितीया ज्या च तद्युतिः ॥¹⁰
 ततस्तेनेव हारेण लब्धं श्रोत्र्यं द्वितीयतः ।
 स्रष्टात् तृतीयस्रष्टज्या द्वितीयस्तद्युतो गुणः ॥
 तृतीयः स्यात् ततश्चैवं चतुर्धाद्याः क्रमाद् गुणाः ।

► The above verses essentially present the following equations for generating the successive *jyā* values.

$$j_{i+1} = j_i + \Delta_{i+1} \quad (0 \leq i \leq 23) \quad (17)$$

$$\Delta_{i+1} = \Delta_i - \frac{j_i}{233\frac{1}{2}} \quad (1 \leq i \leq 23). \quad (18)$$

► Since $\Delta_1 = j_1$, is known, all the *jyā*s can be generated using the above equations recursively.

So then what you said in the later lines of these set of verses is basically this relation so which is same as the relation which has been given by recursion relation given by Aryabhata but for this constant factor which appears here. So this relation can be employed to get the entire sine table once you know the first value once J_1 is in place the rest of the things can be easily obtained from this recursion relation.

So in fact (FL) in fact in this verses (FL) see (FL) is one half, deva refers to number 33 (FL) refers to 2 so 233 and a half see so this is the divisor. So this is what has been given.

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Evolution in the recursion relation

► From the time of Āryabhaṭa (499 CE), the astronomers have been using the recursion relation, to obtain the values of the 24 *R*sine – differences.

► Either these values themselves, or the *jyā* values have been listed explicitly in the form verses.

► The **exact recursion relation** for the *R*sine differences is:

$$\Delta_{i+1} = \Delta_i - j_i \cdot 2(1 - \cos \alpha) \quad (\alpha = 225) \quad (19)$$

► While, $2(1 - \cos \alpha) = 0.0042822$, the different. approximations that have been employed are:

$$2(1 - \cos \alpha) \approx \frac{1}{225} = 0.0044444 \quad (\text{Āryabhaṭa})$$

$$2(1 - \cos \alpha) \approx \frac{1}{233.5} = 0.0042827 \quad (\text{Nilakantha})$$

$$2(1 - \cos \alpha) \approx \frac{1}{233\frac{32}{60}} = 0.00428204 \quad (\text{Śaṅkara Variyar}).$$

To quickly summarize we use the recursion relation to obtain the sine table and one can easily see that exact recursion relation will have this factor. The proportionality constant will be 2 times 1-cos alpha okay and this 2 times 1-cos alpha has been written as 1/225 by Aryabhata

and Nilakantha actually gives this factor to be 1/223 and a half see that is what we saw (FL) and in fact Shankara Variyar so further improves upon.

And then he says 223 and 32 (FL) so that is what he says so you can easily see that the modern value is so 42822 and so this is a much better approximation than 233 and a half. So this is how they have attempted to improve upon.

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Rsine table due to Mādhava
 The values presented are in *Katāpayādi* system and are correct to thirds

श्रेष्ठं नाम वरिष्ठानां हिमाद्रिर्वेदभावनः ।
 तपनी भानुसूक्तज्ञो मध्यमं विद्धि दोहनम् ॥ १ ॥
 धियाज्यो नाशनं कष्टं छन्नभोगाश्रयाम्बिका ।
 मृगाहारो नरेन्द्रोऽयं वीरो रणजयोत्सुकः ॥ २ ॥
 मूलं विशुद्धं नालस्य गानेषु विरला नराः ।
 अश्रुद्धिगुप्ता धोरश्रीः शङ्कुकर्णो नगेश्वरः ॥ ३ ॥.....
 धीरो युवा कथालोलः पुज्यो नारीजनैर्भगः ।
 कन्यागारे नागवह्नी देवो विश्वस्थली भृगुः ॥ ६ ॥
 तत्परादिकलान्तास्मू महाज्या माधवोदिताः ।
 स्वस्वपूर्वविशुद्धे तु श्रिष्टास्तत्खण्डमोर्विकाः ॥ ७ ॥ इति ॥

► The first and the last values are: 224' 50" 22''' and 3437' 44 48''' .
 ► These values have been arrived at by considering terms up to θ^{11} in the series expansion of $\sin \theta$ – ascribed to Mādhava.

And Madhava actually gives a set of verses so which not only gives up to seconds but also up to thirds. In fact, the last line if you note here (FL) so (FL) is starting from thirds then you move on to (FL) then you move on to (FL) so (FL) see Aryabhata has taken so this to be 3438 so 3437 so 44 and 48 so that is what it commence to see so here this is given in Katapayadi notation.

So we saw that so in Katapayadi so (FL) see so both of them refer to 212 (FL) refers to 0 ma is 5 (FL) again 0 so this is what you get so similarly here (FL) the last word so (FL) see (FL) so that is how you construct the sine table which has been given by Madhava and one could see that Madhava would have obtained so this kind of a table by using the sine series.

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Comparing the *jyā* values of different texts

Āryabhaṭīya, Sūryasiddhānta, Tantrasaṅgraha, Laghuviśvaytti, Mādhava & modern

Dhams of Cāpa symbol used	Cāpa Length (mi)	Notation used	Value of the <i>jyā</i> (in minutes, seconds and thirds)				
			As in AR/SS	From TS	From LW	Given by Mādhava	Modern
J_1	325	J_1	325	324 50	324 50 22	324 50 22	324 50 21
J_2	450	J_2	449	448 42	448 42 58	448 42 58	448 42 57
J_3	675	J_3	671	670 39	670 40 16	670 40 16	670 40 16
J_4	900	J_4	890	889 44	889 45 17	889 45 15	889 45 15
J_5	1125	J_5	1105	1105 00	1105 01 41	1105 01 39	1105 01 38
J_6	1350	J_6	1315	1315 32	1315 34 11	1315 34 07	1315 34 07
J_7	1575	J_7	1520	1520 26	1520 28 41	1520 28 35	1520 28 35
J_8	1800	J_8	1719	1718 49	1718 52 32	1718 52 24	1718 52 24
J_9	2025	J_9	1910	1909 51	1909 54 46	1909 54 35	1909 54 35
J_{10}	2250	J_{10}	2093	2092 42	2092 46 19	2092 46 03	2092 46 03
J_{11}	2475	J_{11}	2287	2286 35	2286 40 10	2286 39 50	2286 39 50
J_{12}	2700	J_{12}	2431	2430 45	2430 51 40	2430 51 15	2430 51 14
J_{13}	2925	J_{13}	2585	2584 32	2585 38 37	2584 38 06	2584 38 05
J_{14}	3150	J_{14}	2728	2727 14	2727 21 31	2727 20 52	2727 20 52
J_{15}	3375	J_{15}	2859	2858 15	2858 23 42	2858 22 55	2858 22 55
J_{16}	3600	J_{16}	2978	2977 02	2977 11 30	2977 10 34	2977 10 33
J_{17}	3825	J_{17}	3084	3083 03	3083 14 23	3083 13 17	3083 13 16
J_{18}	4050	J_{18}	3177	3175 53	3176 05 07	3176 03 50	3176 03 49
J_{19}	4275	J_{19}	3256	3255 06	3255 19 50	3255 18 22	3255 18 21
J_{20}	4500	J_{20}	3321	3320 24	3320 38 11	3320 36 30	3320 36 30
J_{21}	4725	J_{21}	3372	3371 27	3371 43 24	3371 41 29	3371 41 29
J_{22}	4950	J_{22}	3409	3408 05	3408 22 20	3408 20 11	3408 20 10
J_{23}	5175	J_{23}	3431	3430 07	3430 25 35	3430 23 11	3430 23 10
J_{24}	5400	J_{24}	3438	3437 27	3437 47 39	3437 44 48	3437 44 48

So to have a comparison so here is where we present and what is noted is the modern value and the value which has been given by Madhava are almost identical so up to thirds so you see that so 3437 44 48 3437 44 48 okay so up to thirds so they almost coincide and the kind of accuracy to which it has been computed has to do with the accurate value which they have been able to find out using the series so that I will explain now shortly.

(Refer Slide Time: 39:06)

Infinite series for the sine function

► The verses giving the ∞ series for the sine function is⁷ –

निहत्य चापवर्गेण चापं तत्तत्फलानि च ।
 हरेत् समूलयुग्मेः त्रिज्यावर्गहतेः क्रमात् ॥
 चापं फलानि चाधोऽधो व्यस्योपर्युपरि त्यजेत् ।
 जीवात्यै, सङ्ग्रहोऽस्यैव विद्वानित्यादिना कृतः ॥

► $N_0 = R\theta$ $D_0 = 1$

► $N_1 = R\theta \times (R\theta)^2$ $N_{i+1} = N_i \times (R\theta)^2$

► $D_1 = R^2(2 + 2^2)$ $D_i = D_{i-1} \times R^2(2i + (2i)^2)$

► जीवा = $\frac{N_0}{D_0} - \left\{ \frac{N_1}{D_1} - \left(\frac{N_2}{D_2} - \left(\frac{N_3}{D_3} - \dots \right) \right) \right\}$

► जीवात्यै = For obtaining the *jyā* (Rsine)

⁷ Yuktidīpikā (16th cent) and attributed to Mādhava (14th cent. AD).

So Madhava so this has to do with the initial value of R in fact it is usually see even without this sine series so Govindasvamin has been able to obtain the table which is fairly close to the Madhava table which is what I showed you so it depends on the value of R that to take so if you take R to be 3438 and then do your computation then the accuracy that you will be able to achieve will be far less than the value of R that R is basically (FL).

So in all your computations so you will have this (FL) which is the radius and if you are able to compute the radius accurately then I think you will be able to improve upon significantly okay. So the verses which present the sine series go like this (FL) in fact there are 2 things which I have to explain. So this was cited earlier, 1 is the terms which actually represents the various terms in the series in the verse.

And finally this (FL) so (FL) *capa* refers to the arc, so which you have been repeatedly saying (FL) always refers to the arc, *jya* refers to the chord so (FL) refers to the square of this. So (FL) is having multiplied so (FL) so you have to multiply this by this (FL) means see once you multiply $R \theta^2$ by $R \theta$ so you get $R \theta^3$ so this also has to be multiplied by $R \theta^2$ that is why he is saying (FL) means the results that you obtain.

So (FL) we have to do this multiplication so (FL) so basically what you will get is so $R \theta^2$ so $R \theta^3$ $R \theta^4$ to the power 5 and so on. Then having obtained them he says (FL) means divide so what should be the divisor (FL) refers to even number okay so (FL) refers to square of the even number so (FL) see (FL) means the base has to be added, squaring is an operation so it can be done on any number.

So (FL) means the base so $2+2$ square $4+4$ square so this is what is refer to so basically this quantity $2i$ is $2i$ square so (FL) and so on. So basically it amounts to this kind of a series and he says finally (FL) is same as *jya* if you want to find out *jya* so then you do this computation so this amounts to the infinite series for sine function okay and finally he says (FL) so having given a certain infinite series for this *jya* so he says (FL).

So a shorter form, a convenient form okay so (FL) is a shorter version okay so (FL) see we saw (FL) so (FL) means the verse which has (FL) so similarly (FL) means the verse which has *vidvan* as the first term (FL).

(Refer Slide Time: 43:10)

How to use the series to evaluate Rsines?

- ▶ Though Mādhava came up with the infinite series, it would certainly be impossible to use the series expansion

$$R \sin \theta = R\theta - \frac{(R\theta)^3}{3!R^2} + \frac{(R\theta)^5}{5!R^4} - \frac{(R\theta)^7}{7!R^6} + \frac{(R\theta)^9}{9!R^8} - \frac{(R\theta)^{11}}{11!R^{10}} + \dots$$

for obtaining the values of Rsines. ∞

- ▶ However, since the series **converges pretty fast**—because of the factorial in the denominator—it would suffice to use a few terms in it.
- ▶ Noting this, Mādhava gave **explicit values** of the magnitudes of the terms starting in the reverse from sixth and up to the second in (the RHS of) the above equation when the arc $R\theta = 5400' = 90^\circ$.
- ▶ These values are mentioned in *Yuktidipika* using the *katapayadi* notation in the following verse.

विद्वांस्तुल्लवः कवीशनिचयः सर्वार्थशीलस्थिरः
निर्विद्धाङ्गनरेन्द्ररुद्रिगदितेषु क्रमात् पञ्चसु ।

So we will see that vidvan verse now okay. So the series which has been used by Madhava so though it is the very, very important discovery from the mathematical view point so think of so using some infinite series to obtain something, it is almost impossible. So similarly so this you will see even with reference to pi so you have some series.

See pi will be approximately equal to various other rational approximations but equality will be holding good only when you have infinite series so though the terms will be decreasing but then the equality is valid only if you consider all the terms but you cannot do that. It is almost impossible for anyone to compute or any device to compute what one has to do is one has to have a good approximation for that so which will be discussed later.

And here so this series is actually a very fast converging series so you need not consider so many terms so Madhava has considered up to 6 terms in the series and so having done that so he has composed another verse so which can be conveniently used so you have some mark value and plugging in every time and then using the series is going to be very, very cumbersome.

So we have R square appearing so R is that 34, 37, 38 so whatever the number of minutes and all that. So you cannot be doing every time and he has given a certain convenient way of doing and so basically the coefficient which are present here so have been handled and computed and some numbers have been given so those numbers can be easily plugged in along with the argument value and you will be able to find the sine values.

So (FL) so (FL) so 5 terms have been specified here. So each of them actually represent a certain number vidvan means normally a scholar.

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How to use the series to evaluate Rsines?

Term no. in RHS	Sanskrit equivalent in <i>katapayādi</i>	Mādhava's value according to <i>Yuktidīpikā</i>	Modern value
VI	विद्वान्	0'0''44'''	0'0''44.54'''
V	तुन्नबलः	0'33''6'''	0'33''5.6'''
IV	कवीशनिचयः	16'05''41'''	16'05''40.87'''
III	सर्वार्थशीलस्थिरः	273'57''47'''	273'57''47.11'''
II	निर्विद्धाङ्गनरेन्द्ररुक्	2220'39''40'''	2220'39''40.10'''

We find that the values given by Mādhava are indeed **very accurate**. Thus for finding **arbitrary value** of sine we use the equation

$$R \sin \theta = R\theta - \beta^3(2220'39''40''') + \beta^5(273'57''47''') - \beta^7(16'5''41''') + \beta^9(0'33''6''') - \beta^{11}(0'0''44''')$$

where $\beta = \frac{R\theta}{5400}$.

But vidvan in this context refers to number 44 say (FL) so in Katapayadi notation vidvan similarly (FL) refers to this number (FL) 0 cha is 6, ya is 1 see in Katapayadi so basically he has listed so 5 numbers (FL) so this basically correspond to see so (FL) 1, 2, 3, 4, 5 leaving out the first so he has given the values of the coefficients so in this particular forum.

So as you can easily see the last 6th will be much smaller so because you have a 11 factorial appearing here okay. So this is the order he has given in the reverse order so (FL) so one can now rewrite that equation see so wherein all this are taken out so 3 factorial R square etc so all this I mean pulled out and basically a set of numbers have been given and with this one will be able to compute the sine value.

Since it is a very fast converging series the 6 terms are more than sufficient to get very accurate values. So that is how he has composed this verse. In fact, the later half of the verse gives how it has to be done. As we saw it is quite interesting to note how? So various techniques have been evolved over a period of time in order improve the accuracy of the computation of sines.

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Concluding Remarks

- ▶ It was quite interesting to know the evolution of different techniques in India to evaluate the sine function – which is ubiquitous – as accurately as possible.
- ▶ Broadly speaking, the approach taken by Indian astronomers and mathematicians can be classified as:
 - ▶ to improve the accuracy of the sine table (which forms basis for interpolation as well as $\sin(A + B)$)
 - ▶ to obtain a good rational approximation
 - ▶ arrive at infinite series.
- ▶ The absolute logical rigor with which the results have been arrived at is indeed remarkable.
- ▶ Why were they worried about very accurate values of sines ?
- ▶ Accuracy of *Trijya R* → Accuracy in the computation of sines → Accuracy in planetary positions → Accuracy in the determination of tithis, and so on, → Avoid incompleteness.⁸

⁸नास्ते कालावयवकलना...श्रीतस्मार्तव्यवहृतिरपि छिद्यते तत्र धर्माः ।

So if one were to look at there are basically 3 approaches, 1 is to improve the accuracy of the table which has been presented for finding the sine values, which will be used for doing either second order approximation or even if you want to use so perhaps professor Sriram will be explaining in this lecture on (FL) so \sin of $A+B$ etc even there you need a very accurate sine table which forms the basis.

So that has been done, then another way is to use the rational approximation where you do not have to have the table at all, you can just plug in the value and you will be able to get the sine value so this is another way and third of course is infinite series and this has also been tamed to have a nice forum so by (FL) and we will see in the lectures on proof the logical rigor so with which we have approached this problem to arrive at is also quite remarkable.

And why were they interested in very accurate values? If you look at whether it is geometrical approach or it is other things so the value of (FL) plays a very crucial role. So (FL) basically refers to the radius so you need to determine the radius so radius once it is precisely known, then you find out root of R square-something whatever you want to do so that will also be quite accurate.

And that was required to compute the sines and which was required to get the accurate planetary positions, which was required to determine the tithis etc precisely so when this tithi ending so when is this eclipse occurring so all that and this in turn was required in order to know the precise occurrence of a certain event okay. So that is what we call as say for instance we say that this lagna that lagna from when does this lagna begin.

So all that is computation of time, in fact people were really concerned about the precise determination of ending moment of tithi so whether we should celebrate Deepavali today or we should celebrate Deepavali tomorrow, whether we have to ekadashi today or tomorrow. So that depends on precise computation of sunrise time and the precise ending moment of tithi.

So in fact one of the astronomers of recent times so (FL) at the beginning of his work, he quotes this verse see (FL) their concern is (FL) so all that whether it is (FL) or (FL) or our transactional thing so all of them will get shattered (FL) therefore he says (FL) so therefore I engage myself to compose this work, which will give you much better position of this planets so which in turn can be used to determine the precise moments of occurrence of various series.

So thus we see that the sine plays a very crucial role in various things. So with this we conclude our talk on sine function. Thank you.