Mathematics in India: From Vedic Period to Modern Times
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Lecture - 31
Development of Calculus in India 2

So we will continue with the discussion of development of calculus in India. The earlier discussion was actually something like a pre-calculus the preparation for calculus or what Bhaskara called as (FL) for algebra.

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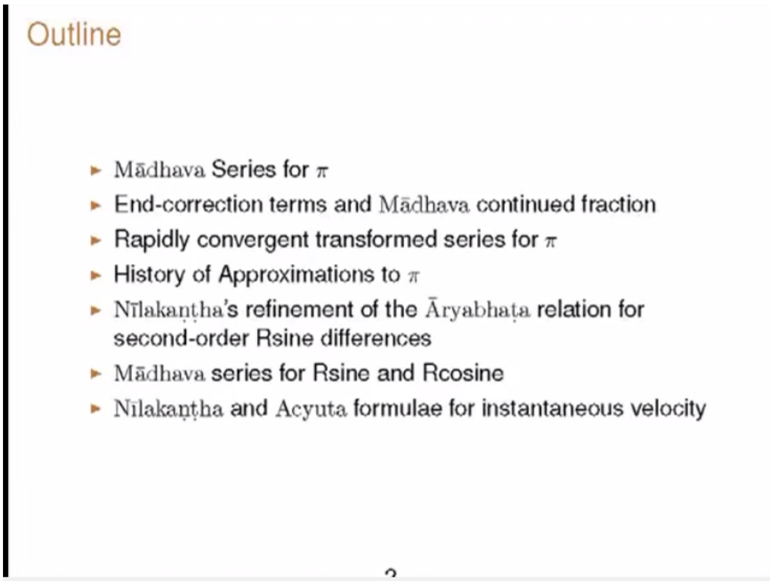

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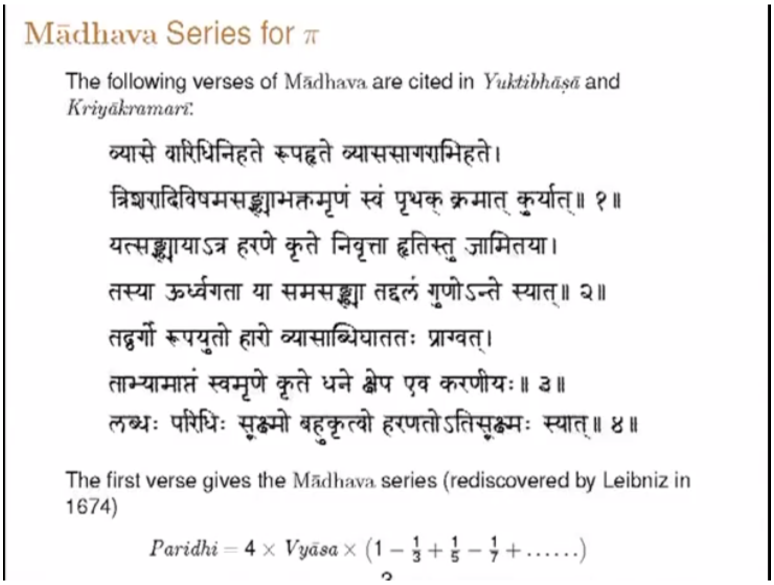

Similarly, now we will straight away go into the results derived by Madhava. So first is the Madhava series for pi. So the following verses are Madhava cited in both Yuktibhasa and

Kriyakramakari. (FL) so just these words will tell you this relation that the circumference is 4 times the diameter*1-1/3 3 (FL) is 5 that is 1/5 (FL) the odd numbers, (FL) divided, (FL) that is both negative and positive (FL) go on doing.

So that is the infinite series for pi/4 what later on was rediscovered by Leibniz in 1674. The other verses of Madhava have to do with the end correction term. We will come to it later.

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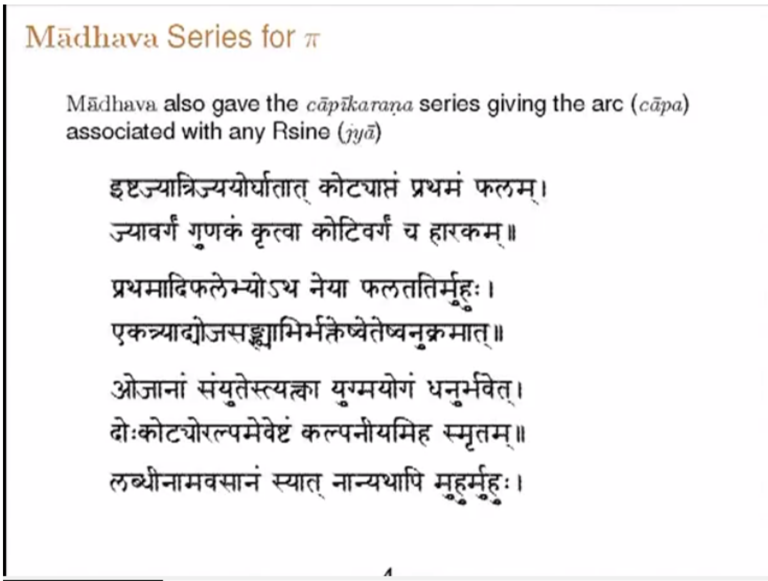

So this is the first result by Madhava. Then Madhava also gave the relation between the arc and the chord or the arc and the (FL) in the arc. So this is the relation, which is again cited in Kriyakramakari and in Yuktibhasa (FL) so let us first see the result.

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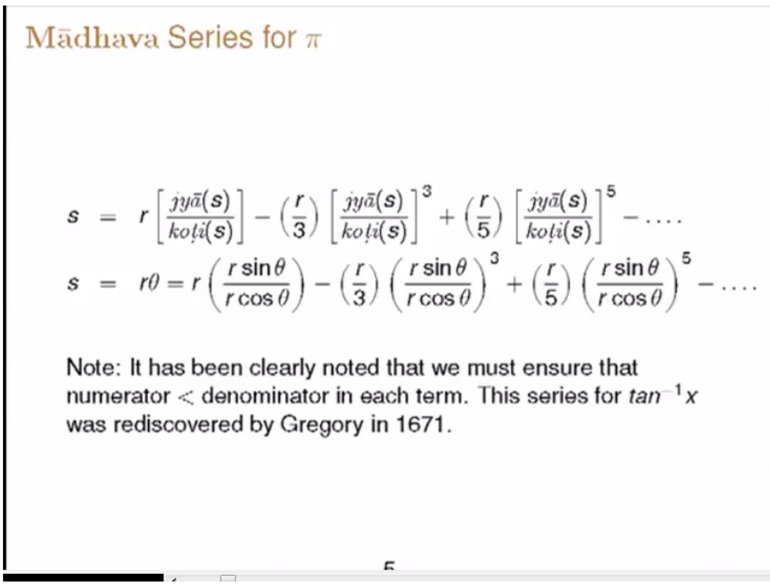

So jya * koti so jya is r sin, koti is r cos, so the ratio is the tangent of the arc. So this series essentially is what is called the tan inverse x series or called the Gregory series today.

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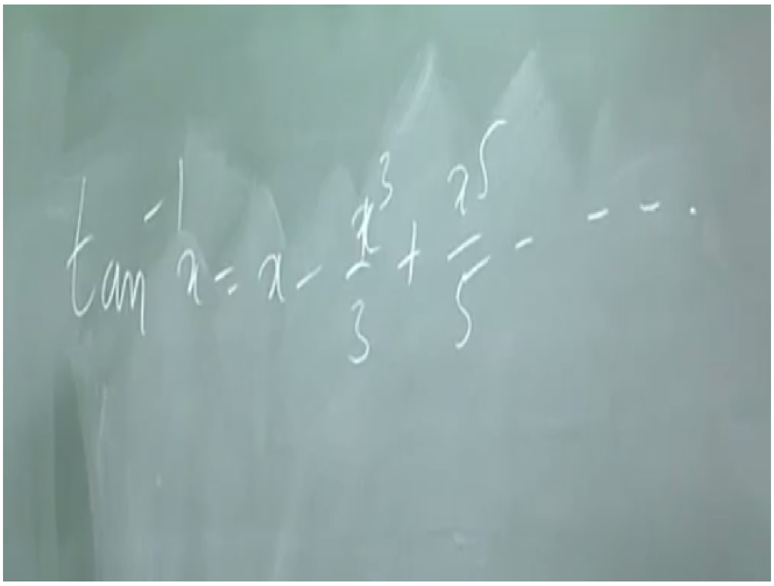

The versus of Madhava so (FL) the first term is the ratio of the jya/koti (FL) making sin squared/cos squared as the multiplier and the divisor obtain the successive terms (FL) that way obtain all the successive terms (FL) from the odd terms which are obtained this way (FL) so subtracting the odd from the even the rest will become the dhanu that is the arc. This is the capa so that is the chord.

(FL) so dhanu is actually the capa (FL) so between the jya and the koti you have to think of the smaller 1. So in the first quadrant it is sin/cos otherwise (FL) so unless you take the smaller 1 as the numerator and the larger 1 as the denominator you will never have the successive terms becoming smaller and smaller. So that is how this is called the capikarana verse.

So the jya has been converted to capa you know the jya, from the jya you know the koti so the ratio you take and this series will give you the capa or dhanu associated with the given jya. So this is the Gregory series. This is the other result of Madhava.

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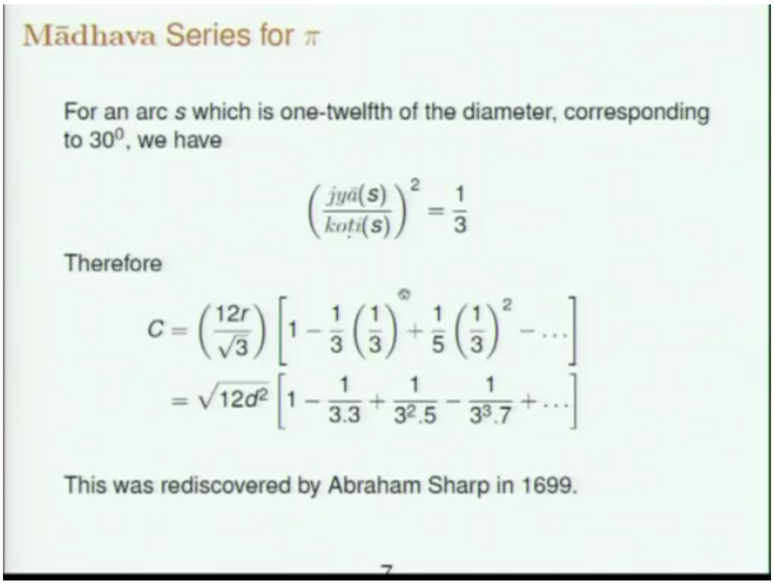

Next the third result is the result when you take the 1/12 as the diameter as the arc or the angle is 30 degrees then sin is 1/2, cosine is root 3/2 therefore the tan square is 1/3 of 30 degrees. So you obtain a series for the capa in terms of so you are considering tan inverse of pi/6.

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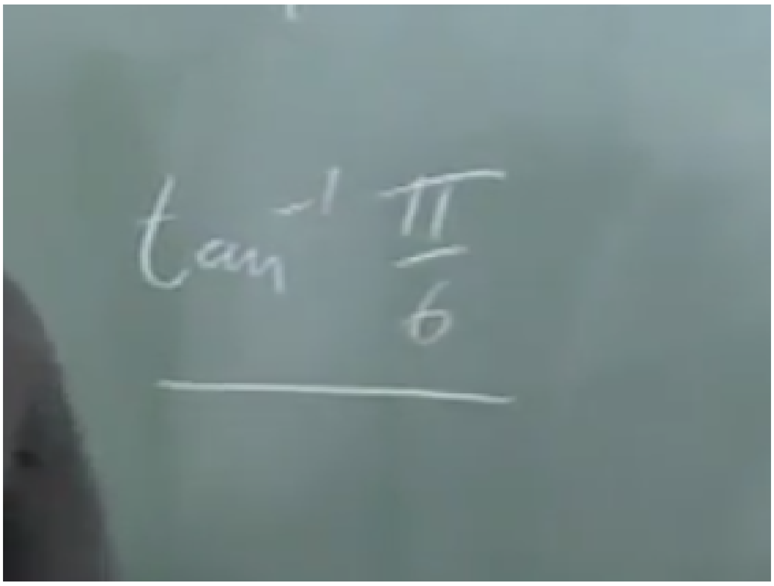

This series so it is since tan square it is 1/3, you have 1-1/3*1/3+1/3 whole squared etc, etc. So the relation between the circumference and the diameter is best expressed. This is the series which is somewhat faster converging than the Leibniz series. You can see 3 3 square, 3 cube appearing here. The Leibniz series was somewhat slow 1-1/3+1/5. So this series was later on rediscovered by Abraham Sharp.

In fact, it is one of those series which relate to evaluation of pi to 100 decimal places I think towards the end of 17th century.

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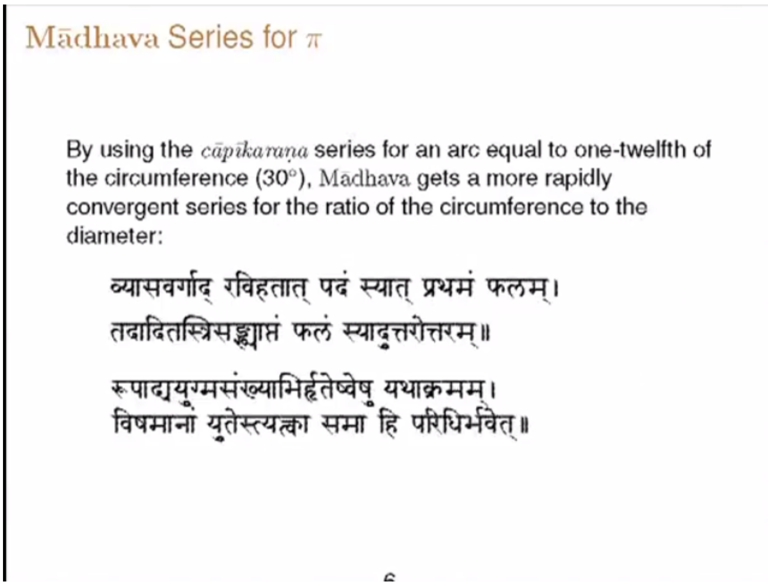

So the verse of Madhava (FL) 12 (FL) so the square of the diameter multiplied by 12. The square root of that is the first term, (FL) the further terms are successively divided by 1/3. The other odd terms etc also appear in the same way as before.

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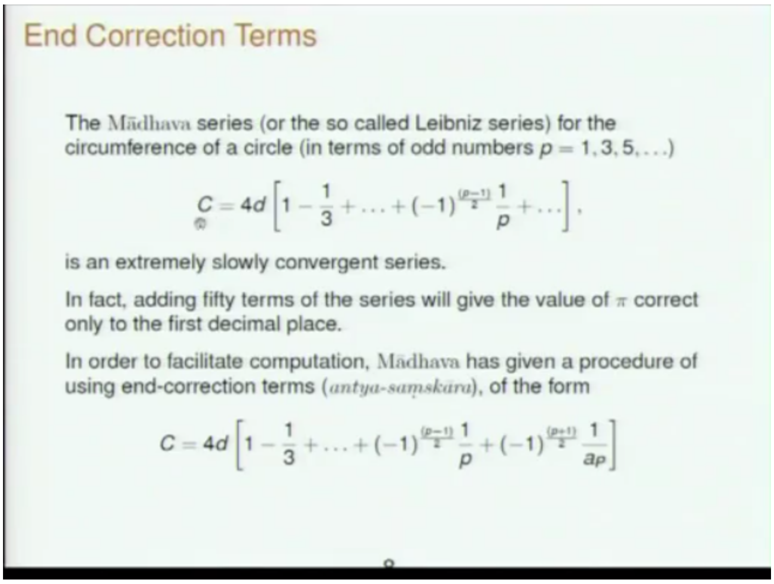

So these 3 series were enunciated by Madhava, but Madhava also enunciated a certain way of calculating using these series. The series themselves are not of much use called calculation. The Madhava series so if you write it this way where p are the successive odd terms. This is an extremely slow convergent series. Even if you take 50 terms in this series, it will give the value of pi correct only to the first decimal place that is 3.1 you can obtain.

So Madhava has given a procedure so what Madhava is saying is add a correction term like this after some odd term whatever odd term that you want you submit up to 1/273 or whatever then put a correction term. Obviously, the correction term depends upon the p value.

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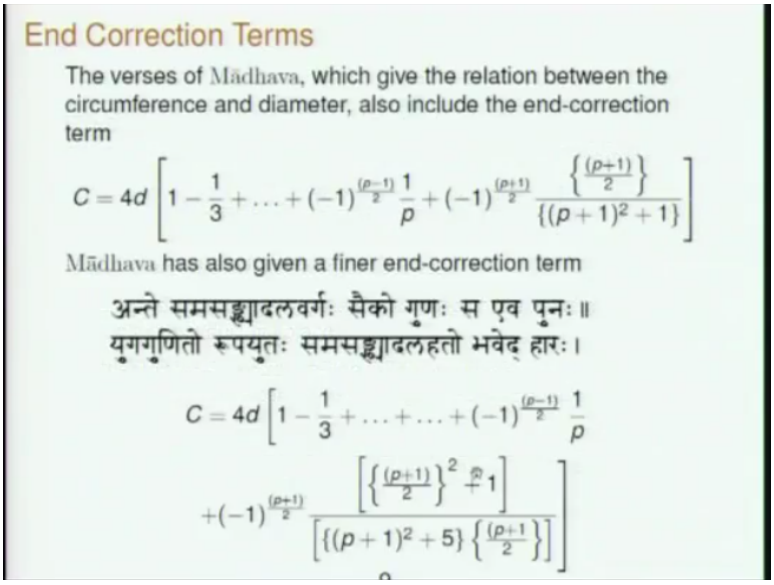

The versus of Madhava that we first quoted where the pi/4 series is enunciated that verse itself gives you the correction term that Madhava proposed so (FL) so by going on successively dividing by the odd numbers if you really get bored after some time so when you get tired or bored (FL) then take the following correction after that immediately. So once you get tired after dividing and summing these terms, which ever point you feel like you then do the following correction.

So what he is saying, the correction term proposed by Madhava is p+1/2/p+1 whole squared+1 let us see from the verse.

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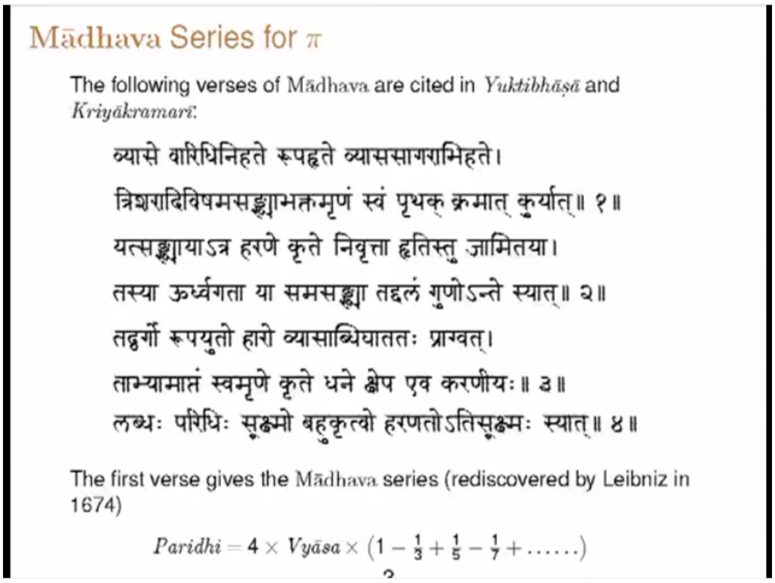

(FL) that is p+1 (FL) half of that (FL) so that will be the multiplier (FL) p+1 whole square (FL) added with 1 that will be the (FL) that will be the divisor (FL) the diameter is multiplier as before, (FL) so you added it or subtract it depending on the place in which you are there in the series, (FL) in fact the resultant value is quite accurate much more accurate than summing that series for many, many, many, many more times.

That is the Leibniz series or just the Madhava series if you go on summing it for 100 or 200 terms not much advantage can be obtained, using this correction term will give you much better results soon enough.

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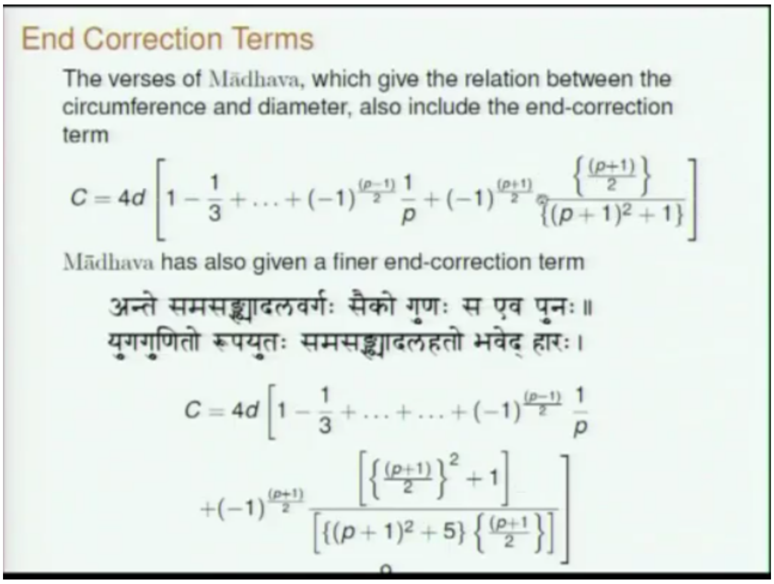

So the first correction term that Madhava proposed was -1 to the power p+1/2 p+1/2 p+1 whole squared+1. There is also 0th order correction, which was not stated by Madhava. We will see later on by analyzing this correction. “Professor - student conversation starts.” This is for any p.

You go on stop it whatever p when you feel tired put this correction you will get a result which is much better than going and summing many, many, many, many more terms in the series that is what he is saying “Professor - student conversation ends.” Then later on another versus of Madhava is quoted and this said to be a (FL) better correction, (FL) so (FL) is p+1 the end even number, p is the last odd number.

(FL) that is half of it (FL) square of it (FL) multiplying or adding 1 to it (FL) that is the numerator (FL) multiplying it by 4 times adding 1 to it (FL) p+1/2 dividing it by p+1/2 (FL) that will become the divisor. So simplification will give you p+1 whole squared+5 in the denominator. So this is called the more accurate correction of Madhava. The Shankara Variyar in Kriyakramakari says this is the (FL).

This is the much more accurate much more final correction term due to Madhava. So how does this help?

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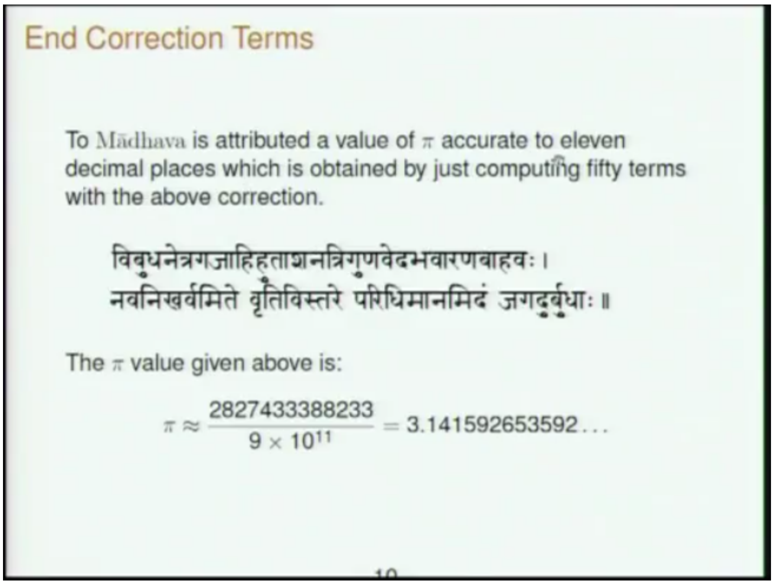

In fact, if you go and calculate 50 terms that is up to p=99 using the just this correction the correction of Madhava. Then you will obtain the following value which is attributed to Madhava. So this verse appears in the Aryabhatiya-bhashya Nilakantha Somayaji saying that the following verse was given by Madhava for an accurate value of pi. (FL) the wise say that the following is the circumference of a circle of diameter (FL) 9* (FL) 10 to the power 11.

So circle with diameter 9*10 to the power 11 the following is the circumference. What is it? (FL) is the number of devas 33, (FL) is number of eyes 2, (FL) 8 (FL) the elephants serpents that is 8 (FL) is fire, 3 is 3 itself (FL) again 3, veda is 4, (FL) is nakshatras 27, (FL) is again (FL) 8, (FL) is arms 2. So (FL) that is this number 282743388233 so this value of pi accurate up to 11 decimal places was given by Madhava.

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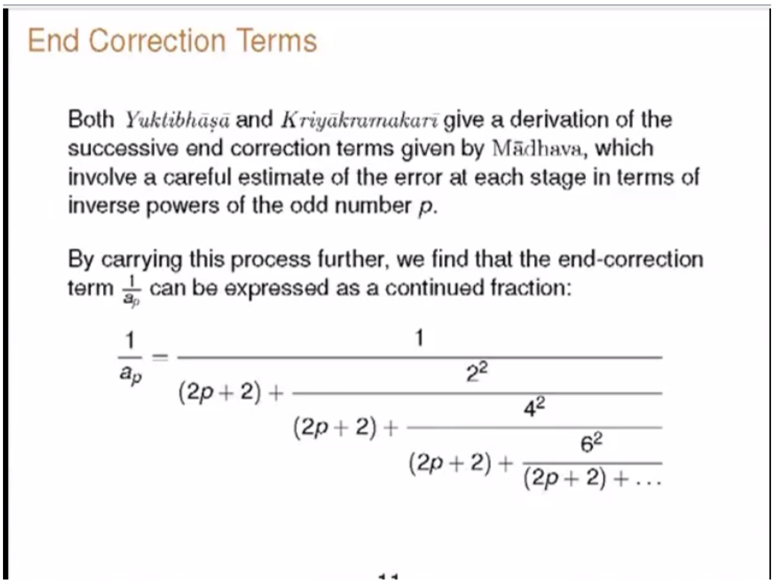

Now both Yuktibhasa and Kriyakramakari not only give the end correction term they also tell you the derivation of that they will give you an argument by which this p+1/2/p+1 square+1 are the more accurate correction of Madhava were obtained so if you carry out that procedure further and further you will see that actually that end correction term can be expressed in terms of what we saw in one of the other lectures the old familiar object a continued fraction.

This story of Mahalanobis going to Ramanujan and giving him a problem something he read in the newspaper that day and Ramanujan gave him answer. So then Mahalanobis is asked how did you find out, obviously this problem involved continued action. So that is how I think Madhava is also one of those persons I think who thinks in terms of continued fractions naturally.

So it is 1/2p+2+2 squared/2p+2+4 squared/2p+2 so 1/2p+2 is just the lowest order correction which we have not mentioned it all. 1/2p+2+4/2p+2 is the first correction of Madhava. This correction is given by 1/2p+2+4/2p+2. This is the 3rd order correction, 4 squared/2p+2 is this correction. So the next order correction will be 6 squared/2p+2 which is not mentioned in Kriyakramakari but if you follow the same method this what you will get.

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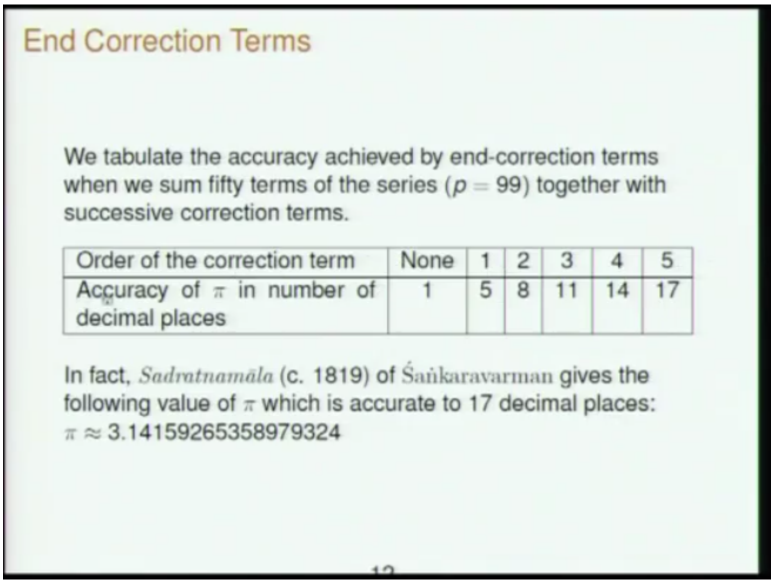

So today we can do we can do a tabulation so we can write down the Madhava series 1-1/3, put the correction term and find out what will be the accuracy of pi in number of decimal places. Say if you go 50 terms in the series, so if you put no correction terms as I said only 1 decimal place will be accurate, you have 1 correction term 5 decimal places, the second order correction term it gives you 8, first order is 1/2p+2.

That is not stated in the book, second order is the first correction given by Madhava. This is the correction which Madhava said is fairly accurate that gives you pi to 11 decimal places if you go 50 terms in the series. So if you use the next in the higher order correction, you will get 17 decimal places interestingly Sadratnamala of Sankaravarman gives the following value of pi correctly to 17 decimal places.

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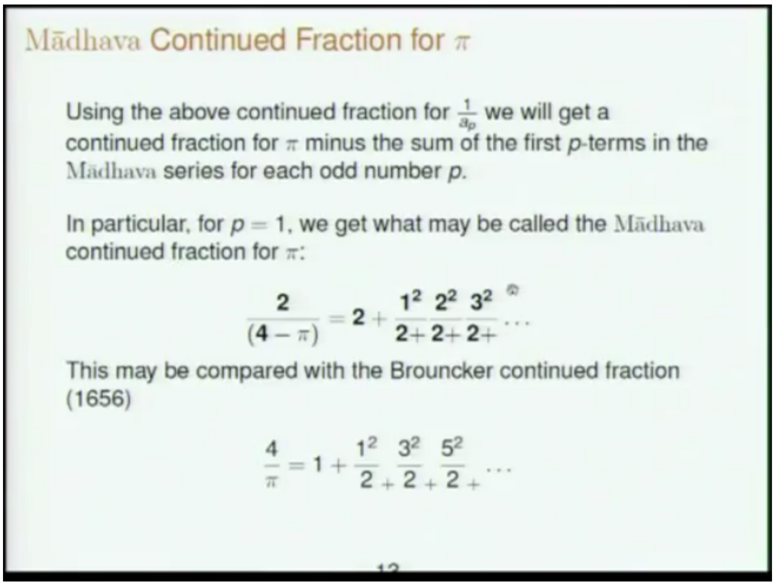

So now that we have an expression for ap right, we plug it in here now that we have a continued fraction for 1/ap we just plug it in here, then C-a sum of finite terms in the Madhava series=continued fraction. Now instead of going sum of finite series you just take the first term only and put in the continued fraction then you will get what we can call as the Madhava continued fraction.

So from that series you can immediately go once you have the correction terms written as the continued fraction we have a continued fraction and just for comparison I think in 1656 again (()) (15:42) Viscount Brouncker whose name is already familiar to us in the context of the (FL) the Pell's equation, he gave this continued fraction while he has had an infinite product which Viscount Brouncker converted into a continued fraction okay.

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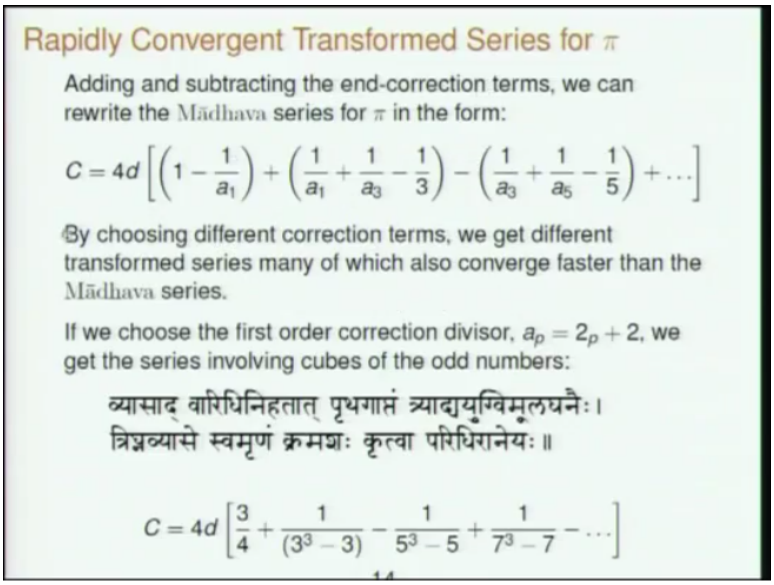

Now having given the correction terms Madhava found a simple way in which the series itself can be transformed to appear like a more faster convergent series once you have this correction touch you can use it and convert the series itself so that is the next thing. So I have been done the correction term so basically this is a way of rewriting the series itself.

This is how Yuktibhasa starts explaining 4d*1-a1, a1 is the correction term+a1 they are canceling+a3 will cancel with –a3, -1/3 will remain+a5 it will cancel with –a5 in the next term -*- +1/5 will remain. So this is only the original series, only you have interpolated the correction terms in-between and you are going to transform the terms of the series by doing this.

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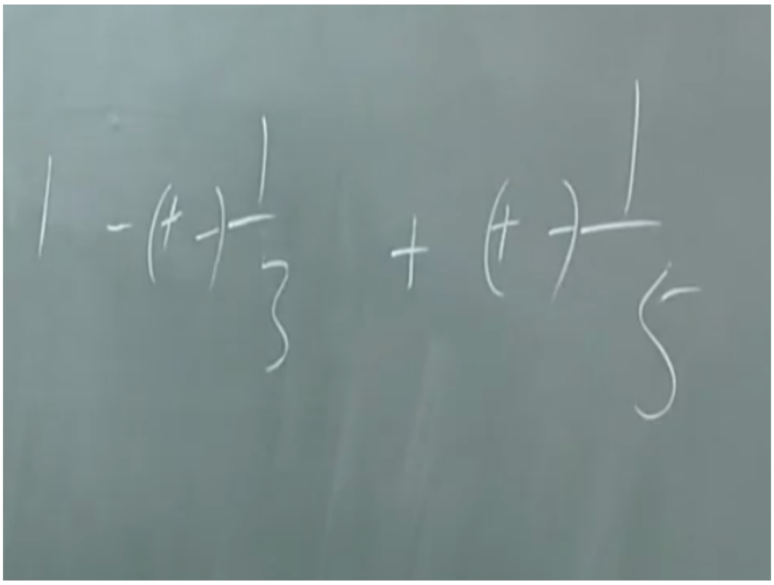

So it is like writing an infinite series so you have 1-1/3+1/5 so all that you do is put something here which you will all cancel out and regroup the terms. Put few things here, which will all cancel out and regroup the terms you will get a different series. Now the thing that you put in that is where the cleverness comes. Those correction terms are the best things to be put in.

Because that it what will transform the series into a faster convergent series. So this correction divisor 2p+2 which we did not use at all, that is the lowest order correction term that is to this series after 1/p you just say -1 to the power p+2 1/2p+2 that is the simplest correction term. Madhava's correction term was more complicated. Madhava's second correction term is even more.

If you use just this 2p+2 and plug it in here for 1/a1 2*1+2, for 1/a3 2*3+2 so put it in and go on then what you get you get a series like this and this series was expressed by Madhava this way. (FL) so (FL) 3 etc. (FL) odd numbers (FL) 3 cube-3 (FL) of which the root is subtracted. So those are the successive terms.

So here as you can see while the Madhava series goes like odd numbers in the denominator, here this series is going by cubes of odd terms in the denominator obviously it is faster convergent and this has taken into account the lowest order correction term 2p+2. Before that there is another transformation that you can do.

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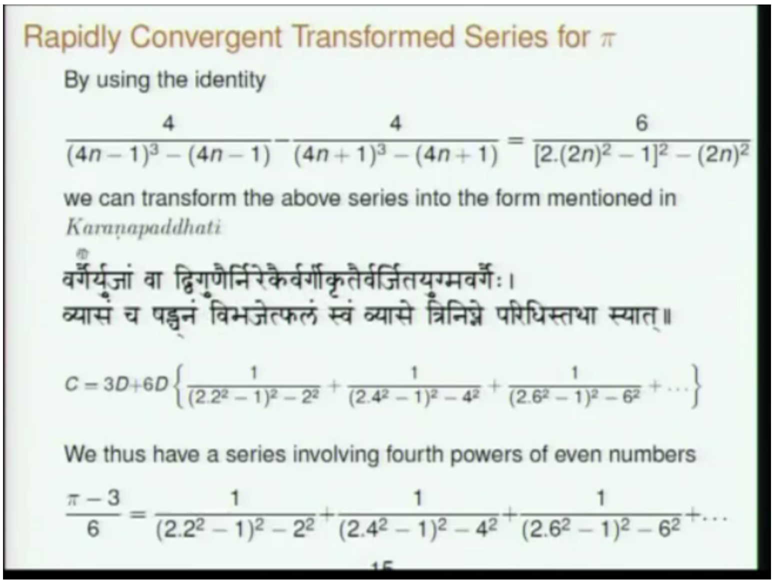

And that transformation is done in this Karanapaddhati one of the later works on astronomy so there in Karanapaddhati it is observed that these 2 terms can be grouped in this way. The successive odd cubed-the root so 4n-1 cube and 4n+1 cube-4n+1 this can be simplified this way. Then what happens you are going to get a series, which involves the fourth power of successive even numbers.

So it is even more faster converging so that is the next transformation (FL) so 2 4 etc squared multiplied by 2 and -1 squared once again and subtract the square of the number again (FL) 6D (FL) this is the versus (FL) which is giving you the series where the successive 4the power of the successive terms appear.

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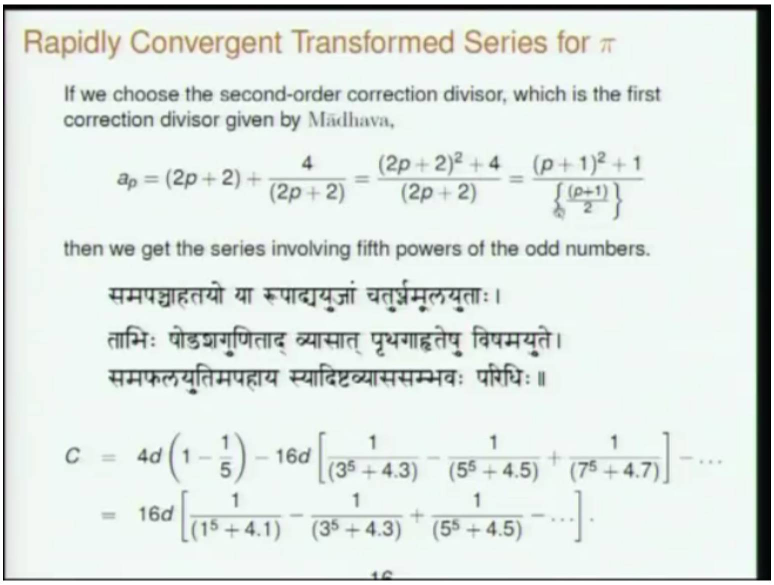

Now let us put the first correction stated by Madhava as ap. If you do that you are going to get 5th powers of successive odd terms in this series. So that is even more faster convergent so that is what is happening. So if you take 1/ap to be this and then plug it in this formula for the transformed series what you are going to get is this C is 4*1-1/5-16d 1/3 to the power 5+4*3-1/5 to the power 5+4*5 (FL) so same thing.

Fifth power-4 times the (FL) and these terms (FL) are given diameter this will be the circumference. So this goes like 5th power of odd numbers so much better convergent than the previous one.

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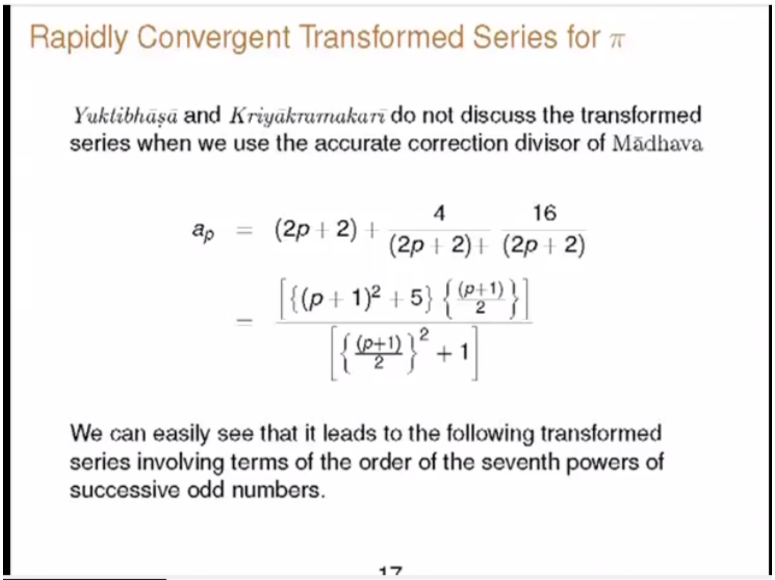

We go to the second correction divisor of Madhava what he called as the **(FL)** correction Kriyakramakari and Yuktibhasa do not mention what is the series that we get so but now we can write down what it is?

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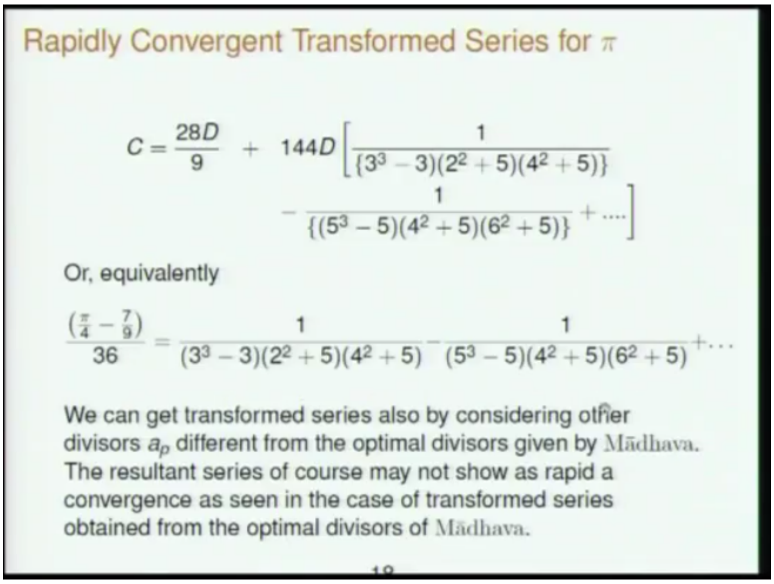

So we just put that 1/ap then we see we get a series which essentially goes like the 7th power of the odd numbers, which is equivalent to 7th power of the odd numbers. So if you put them each term will go like this 3 cube-3 2 squared+5 4 squared+5 5 cube-5 4 squared+5 6 squared+5 etc, etc. So this how the series goes.

Now the beauty of this Madhava transform "Professor - student conversation starts." This ap is Madhava's ap, two corrections will go back through the whole thing once again. You see Madhava has given 2 corrections, these are the 2 corrections given by Madhava. This will give you the 5th power, this will give you the 7th power, this is the 7th power, this one this correction term this transformation series is not given in Yuktibhasa Kriyakramakari.

This one is given, this is giving you that 3 to the power 5+4*3, the 1/2p+2 which was not given by Madhava that the 0th order correction that will give you the 3 cube-3, 5 cube-5. This is the 7th power. The Madhava correction actually corresponds to the 7th power. "Professor - student conversation ends." So the beauty of this kind of transformation so we are using this divisor the third order division of Madhava.

And that will give you the 7th powers, but the beauty of that series is you can put even divisor which are not this end correction terms even then it will give you a good transform series so some of these examples are mentioned in Kriyakramakari.

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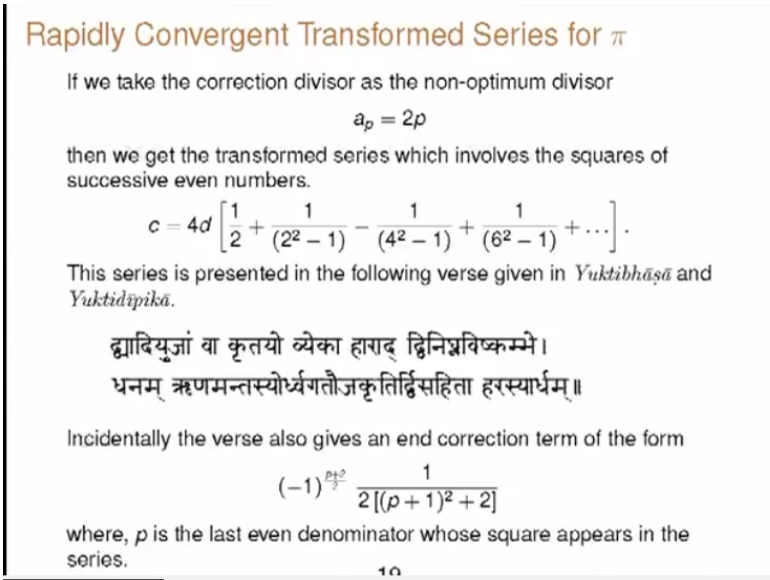

So you just take ap=2p not 2p+2 which was the optimal one, so ap=2p will give you the series. This is the series mentioned in Yuktibhasa and Yukthidipika. Yukthidipika is Shankara Variyar's commentary on Tantrasangraha. Kriyakramakari is Shankara Variyar's commentary on Leelavathi. Both of them contain Sanskrit versus attributed to Madhava, not only does it give the series it also gives you a correction term in the end.

So this is another example where an end correction term is also mentioned. So these are the transform series of Madhava. So with this we come to an end of Madhava's work on (FL) the relation between circumference and the diameter.

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A History of Approximations to π

	Approximation to π	Accuracy (Decimal places)	Method Adopted
Rhind Papyrus - Egypt (Prior to 2000 BCE)	$\frac{256}{81} = 3.1604$	1	Geometrical
Babylon (2000 BCE)	$\frac{25}{8} = 3.125$	1	Geometrical
Śulvasūtras (Prior to 800 BCE)	3.0883	1	Geometrical
Jaina Texts (500 BCE)	$\sqrt{(10)} = 3.1623$	1	Geometrical
Archimedes (250 BCE)	$3\frac{10}{71} < \pi < 3\frac{1}{7}$	2	Polygon doubling ($6.2^4 = 96$ sides)
Ptolemy (150 CE)	$3\frac{17}{120} = 3.141666$	3	Polygon doubling ($6.2^6 = 384$ sides)
Lui Hui (263)	3.14159	5	Polygon doubling ($6.2^9 = 3072$ sides)
Tsu Chhung-Chih (480?)	$\frac{355}{113} = 3.1415929$ 3.1415927	6 7	Polygon doubling ($6.2^9 = 12288$ sides)
Āryabhaṭa (499)	$\frac{62832}{20000} = 3.1416$	4	Polygon doubling ($4.2^8 = 1024$ sides)

So let us review where does all this work stand in the history mathematics? So we should take a quick look at history of pi. So the Rhind Papyrus and Babylon, they have values of this order 3.16, Sulvasutras we saw had 3.08, there is a slightly better value in Madhava **(FL)** Jaina Texts are opted square root of 10, Archimedes gave this unique validity, Ptolemy had this recurring decimal 3.14666.

He worked in sexagesimal place value system. The Chinese are supposed to have discovered 355/113 the next sort of fractional approximation after 22/7. This 355/113 is really very good, it is good to 6 decimal places. Now the method adopted is this what I mean by polygon doubling is essentially either you take a hexagon inscribed and hexagon circumscribing a circle.

And go on converting into a 12-sided figure 24-sided figure or you take a square inscribed and a square circumscribed go on doubling it to octagon etc. So as you can see for even for Ptolemy's value it is estimated that you need to have 6 steps and it is 384 sides polygon you have to consider. To get the Aryabhata value you need to take 4 steps from an octagon. It is 1024 sides are needed to get the Aryabhata value.

Now till the time of Aryabhata, there were no good algorithm to you and make this calculation. You should remember that this polygon doubling method essentially involves using right angle triangle arguments, which involves calculating square roots of quantities until Aryabhata's algorithm for the square root was available square root had to be calculated with much greater difficulty.

So once Aryabhata algorithm plus square root has available, it was only a matter of the amount of effort that you put in to go on calculating polygons of higher and higher orders to approximate the square the circle. So as you can see between Aryabhata, between this Chinese value and around the same time Aryabhata, there is really no great improvement till you come to this greater mathematician called Al Kasi who was in the West Asia Samarkand he did 6*2 to the power 27 sides.

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A History of Approximations to π

	Approximation to π	Accuracy (Decimal places)	Method Adopted
Mādhava (1375)	$\frac{2827433388233}{9.10^{11}} = 3.141592653592\ldots$	11	Infinite series with end corrections
Al Kasi (1430)	3.1415926535897932	16	Polygon doubling (6.2^{27} sides)
Francois Viete (1579)	3.1415926536	9	Polygon doubling (6.2^{16} sides)
Romanus (1593)	3.1415926535...	15	Polygon doubling
Ludolph Van Ceulen (1615)	3.1415926535...	32	Polygon doubling (2^{62} sides)
Wildebrod Snell (1621)	3.1415926535...	34	Modified Polygon doubling (2^{30} sides)
Grienberger (1630)	3.1415926535...	39	Modified Polygon doubling
Isaac Newton (1665)	3.1415926535...	15	Infinite series

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He was a very greater calculator Al Kasi and he obtained pi to 16 decimal places and Madhava obtained this 11 decimal places of course he had an infinite series with end corrections for doing so. Viete polygon doubling 2 to the power 16 sides, Romanus again and Van Ceulen I think supposed to have filled a whole book with his calculation 32 decimal places.

And Isaac Newton came up with an infinite series. This was I think during those 4 or 5 years when he was away from Cambridge due to plaque or whatever so he came up with this different infinite series we will see it in a minute. So these are the results for pi.

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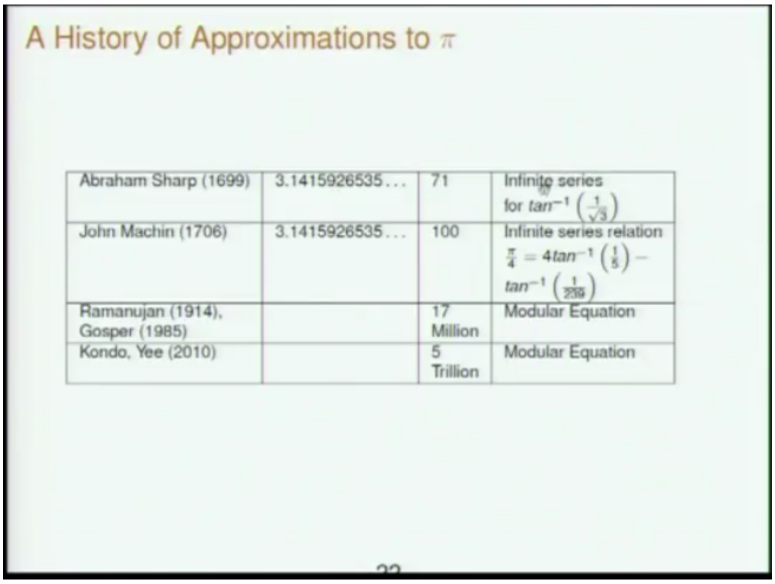

Now later on Abraham Sharp as we saw he had this tan inverse 1/root 3 series so he calculated it to 71 and later on a very different kind of formula came due to John Machin. This was calculated to 100 places. Now from here I am going to 17 million there are so many in between steps many, many great people come in between, but this is what is of interest to us.

There is a result of Ramanujan in 1914, which was used by Gosper in 1985 to calculate pi to 17 million which was a record at that time using what is called a modular equation and around I mean around our times in the last couple of years I think we are around 5 trillion decimal places of pi.

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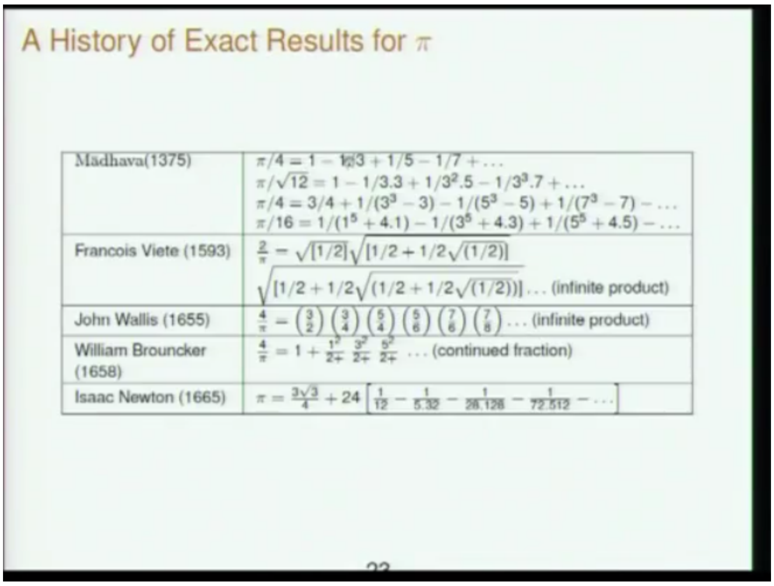

So exact results for pi, the first one is of course Madhava's and he had so many of them, many, many, many results. Vieta had this infinite product, Wallis also had another infinite product, Brouncker had this continued fraction, Newton had this infinite series. He only estimated first few terms. He invented some other series and obtained this. Even the sin series and cos series Newton evaluated a few terms.

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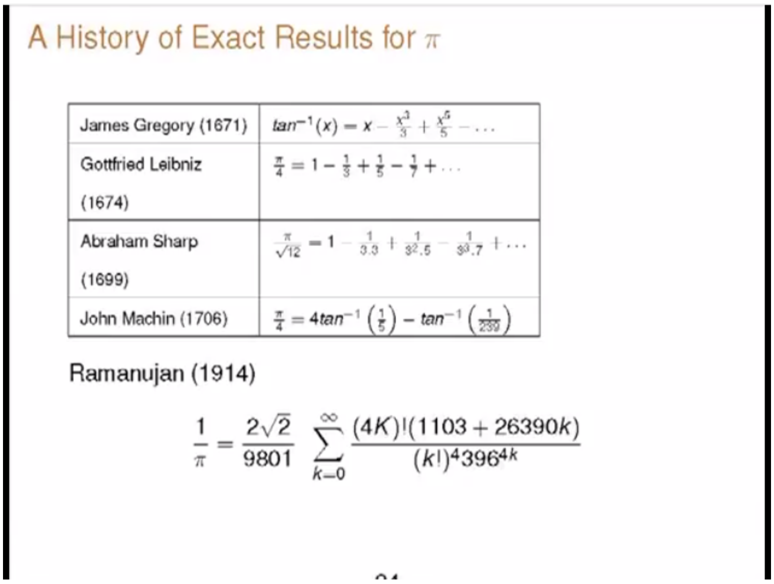

And then said the other terms have also going to be similar. Gregory's tan inverse x, Leibniz pi/4, Abraham Sharp, John Machin and this is the modular equation given by Ramanujan in a paper in 1914.

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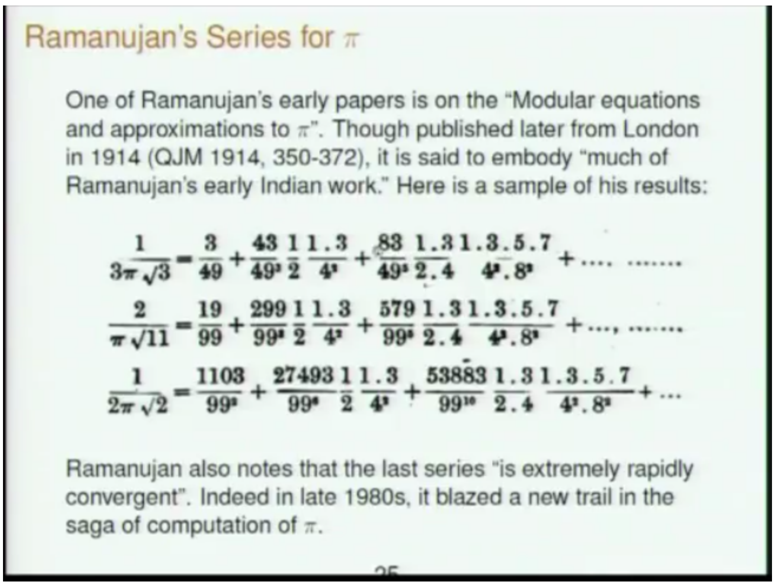

So here I am displaying an extract from the collected words of Ramanujan, the way the equations appear. So this was the paper in which appeared in quarterly journal of mathematics in 1914. So this was published after Ramanujan made to England but it is said to contain much of his older work only and he gave about 50 or 60 series in that paper and he says that this is one of the good series.

Each term in the series gives 8 decimal places of pi accurately and in fact in 1980s this series was used for a determining entirely new kind of algorithm obtaining pi.

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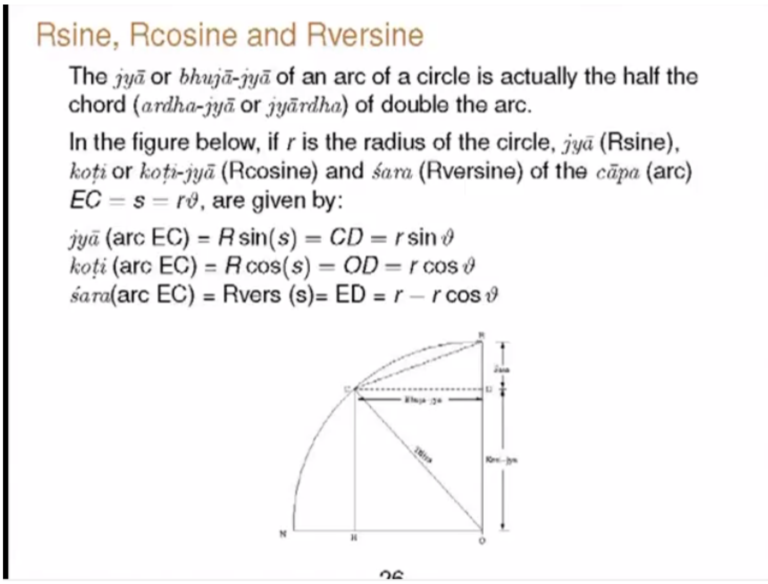

So with this the **(FL)** discussion we can finish. Now we come to the other major work of Madhava, which is establishing the sine series, cosine series and calculation of the sine table. So when you have an arc if BC is an arc, CD is its bhuja-jya, DO is its koti-jya and DE is its sara. These are the 3 quantities, so the jya, koti and sara are functions of the arc. Now the arc is measured in terms of minutes or in radian measure whichever may you want.

So the sara is called versine it is 1-r*1-cos, koti is r*cos, jya is r*sin.

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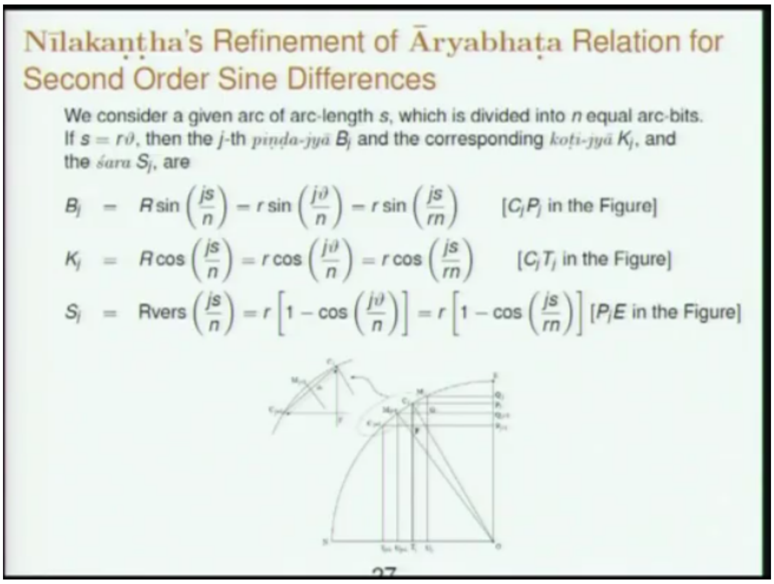

Now we had this Aryabhata relation to which I am again coming back and explaining to you the kind of refinement Nilakantha made of the Aryabhata relation to enable the calculation of sin values accurately. So the idea is the way Nilakantha arrived at that refinement I am just trying to explain to you. You take the arc, so the arc is something ES or wherever, divide it into n equal arc bits.

So the same thing will be later on useful in considering the Madhava's proof of the sine series also as given in Yuktibhasa. So whatever arc you have divide it into n bits and then the Jth and the bhuja, the Jth koti and the Jth sara can be defined. So CiCjCj+1 is the Jth arc bit when you are dividing whatever given arc which may be here into n arc bits, this is the Jth arc bit each of them are of equal length.

You think of a midpoint for that you think of the midpoint of the previous one, so Bj is the bhuja here, this is the koti Kj and this will be the sara. So Cj Pj is the bhuja, Cj Tj is the koti, this Pj E is the sara okay. Now these Bjs, Kjs are also called pinda-jya's, pinda-koti's, pinda-sara's. Pinda means gross or for a lump of. That in between points are the desired points, these are the tabular jya's are whatever the lump.

Now the jya's and koti's at the midpoint Mj+1 and Mi we can denote them by Bj+r, Sj+r, Bj-r, Sj-r so same standard notation.

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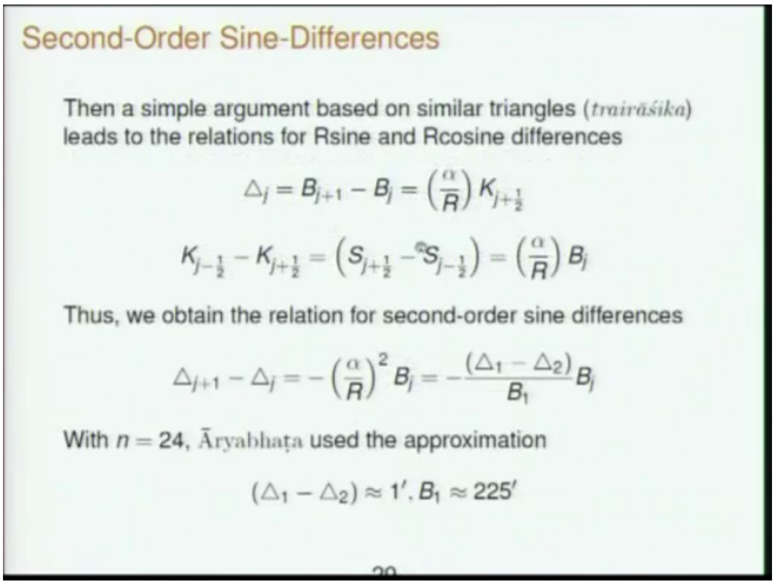

And all that you will need is there are so many perpendicular lines coming this way, coming that way and just these 2 radii so, so many similar triangles will be formed. So you choose the 2 appropriate similar triangles and you will get relations. So the Bj-Bj+1-Bj okay what is this alpha? Alpha is the called length of Cj Cj+1. Cj Cj+1 itself is s/n where s is the total arc, it is divided into n equal bits so Cj Cj+1 arc is s/n.

The alpha is the chord, so this one this chord associated that is the alpha Cj Cj+1 is s/n, alpha is the associated chord. So you will get Bj+1-Bj is alpha/r the koti at the midpoint and the different of the koti will be the bhuja at the midpoint and you couple both the equation you get this equation for the second order sine difference. So that is the essential derivation of Aryabhata relation given in the Aryabhatiya bhashya of Nilakantha Somayaji.

So the sign difference one minute Bj+1-j is related to the koti-jya at the midpoint. The difference of the koti-jya's or the difference of the sara's = the bhuja-jya at the midpoint combining the 2 the second order sine difference is alpha/r whole squared*Bj and if you put j=0 the same thing will be this therefore this can be related to (()) (34:41).

"Professor - student conversation starts." No, no, no, this is to give you angle on how to calculate sine at any point you are taking a given arc, dividing it into n small bits and each of them from the start to the end of each arc bit you call it up pinda-jya that is all. You are interested in sine of a large arc which were divided into n arc bits.

So later on when we discuss the proof of this I think I will be going into the proof of this then I will tell you why this is being done even for the (FL) you are dividing the arc into n bits, this is the same way **"Professor - student conversation ends."** In fact, we need not have gone through these n bits, it is just the fact that I have a transparency, which have the same diagram.

We could have thought of each of these units as 225 minutes not as s/n. The same relation will hold. The sine differences are proportional to the sine at Bj. The second order sine differences. First order sine differences related to the cosine and the first order cosine differences are related to the sine and that is about arc. Now what Aryabhata did was to use this equation with the following 2 approximations.

Delta 1-delta 2 as 1 minute and B1 as 225 minutes, so B1 is the first arc bit, which was of length 225, the corresponding sine was equated to the arc in the first approximation. Sin theta being taken nearly equal to theta for a small theta.

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Mādhava's Sine Table

θ in min.	R sin θ according to		
	Āryabhaṭīya	Govindasvāmi	Mādhava(also Modern)
225	225	224 50 23	224 50 22
450	449	448 42 53	448 42 58
675	671	670 40 11	670 40 16
900	890	889 45 08	889 45 15
1125	1105	1105 01 30	1105 01 39
1350	1315	1315 33 56	1315 34 7
1575	1520	1520 28 22	1520 28 35
1800	1719	1718 52 10	1718 52 24
2025	1910	1909 54 19	1909 54 35
2250	2093	2092 45 46	2092 46 03
2475	2267	2266 38 44	2266 39 50
2700	2431	2430 50 54	2430 51 15
2925	2585	2584 37 43	2584 38 06
3150	2728	2727 20 29	2727 20 52
3375	2859	2858 22 31	2858 22 55
3600	2978	2977 10 09	2977 10 34
3825	3084	3083 12 51	3083 13 17
4050	3177	3175 03 23	3176 03 50
4275	3256	3255 17 54	3255 18 22
4500	3321	3320 36 02	3320 36 30
4725	3372	3371 41 01	3371 41 29
4950	3409	3408 19 42	3408 20 11
5175	3431	3430 22 42	3430 23 11
5400	3438	3437 44 19	3437 44 48

24

So Aryabhata made this approximation so with that he obtained a sine table. We have seen that sine table earlier lecture. So Aryabhata's sine for 225 minutes is 225 itself. The first sine difference is 224 minutes, the second sine difference is 222 minutes, third sine difference is 219 minutes like that it goes. Now what does Nilakantha do? He derives this equation, which is exact.

Now you have to do calculations with it so as to give you approximations and he will give you approximations which are better than what Aryabhata gave so that you get better results.

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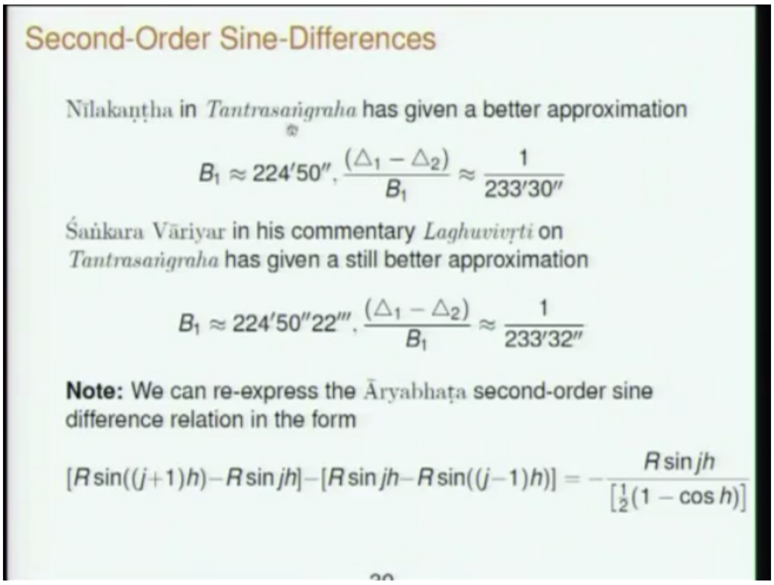

So Nilakantha's approximation is sine of 225 minutes take it as 224 minutes 50 seconds and this ratio the first sine difference by the first sine you take it as 233 minutes 30 seconds. What would this have been in Aryabhata? This 1/225 itself, delta 1-delta 2 is 1, B1 is 225 so this would have been taken as 1/225 by Aryabhata. Nilakantha is giving a slightly better value. Shankara Variyar in his commentary on Tantrasangraha has given a still better approximation.

B1 is 224 minutes 50 seconds and 22 thirds and this ratio is 1/233 minutes 32 seconds, so you use this in this equation then you already you know B1 and you know this ratio also, you can use this immediately to calculate the sine difference from that the next B, next sine difference, next B like that you can calculate B1, B2, B3 etc. This is the difference method that Aryabhata gave.

Only Nilakantha has given you better approximation, he has refined the Aryabhata's relation for second order sine difference. He derived it in Aryabhatiya-bhashya. He derived the exact form of it, but it is understood that Aryabhata itself had it somewhat along these lines only and he improved upon Aryabhata's. Essentially in today's notation, this is the J+1 sine difference.

This is the Jth sine difference that is proportional to the Jth sine/by certain quantity which depends on the arc bit h and it is this quantity which is being approximated here successively.

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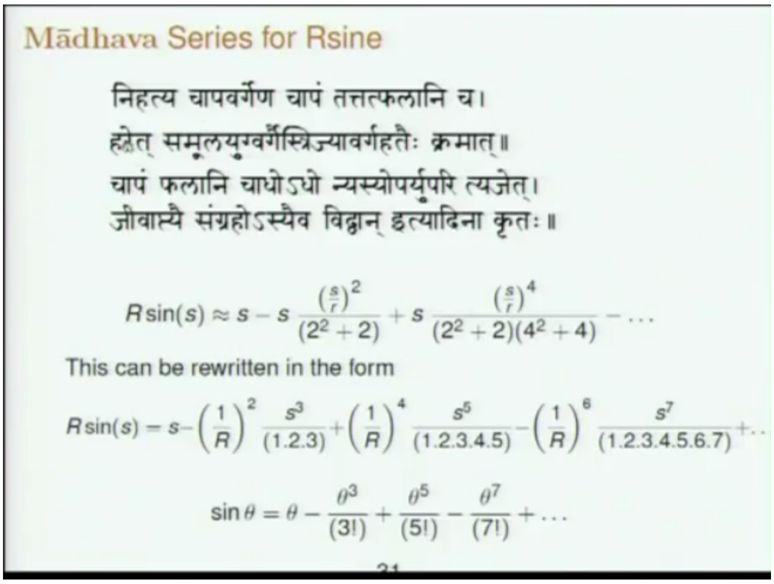

Okay now we will go to the sine series given by Madhava. So I had just described Nilakantha's refinement of Aryabhata's second order sine difference relation. Now we go to the infinite series for sine that Madhava gave. So this is the verse (FL) so this is your arc square of the arc/radius (FL) so 2 square+2*4 square+4 etc, etc you can convert that into 3 factorial, 5 factorial, 7 factorial etc.

So it is the well-known sine series and in the end he says that the same series is summarized by what is called vidvan etc. We will say what it is in a minute.

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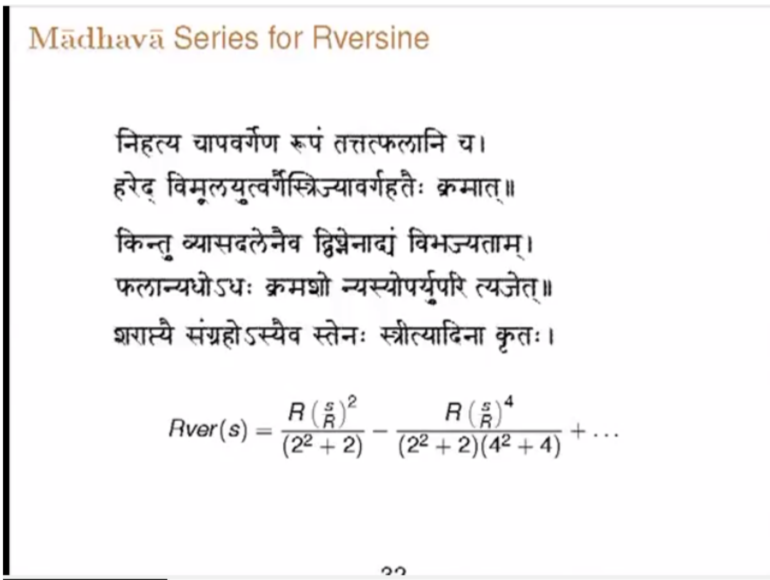

Then Madhava has given another set of verses for the versine of the sara that is 1-cos. So what is the versine series?

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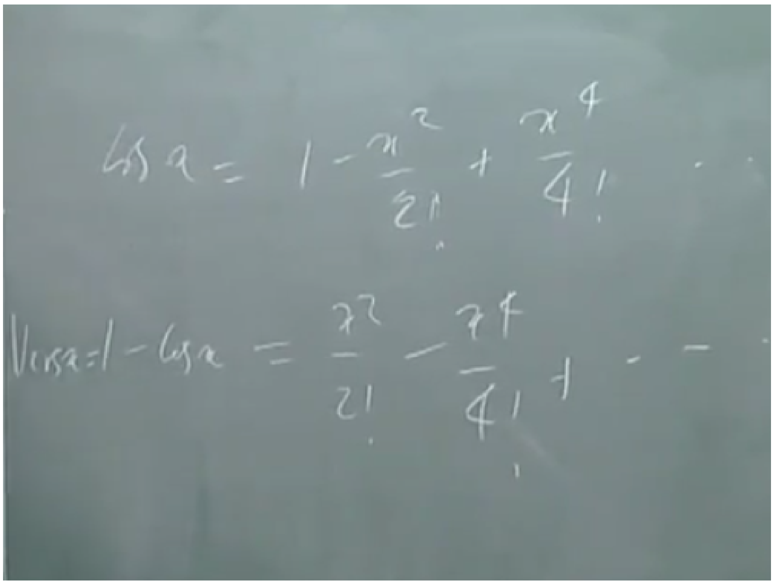

So cos x is 1-x square/2 factorial+x4/4 factorial etc so 1-cos x which is versine x so this equal to x squared/2 factorial-x4/4 factorial+x right. So this is the series Madhava is giving (FL) so the same kind of thing.

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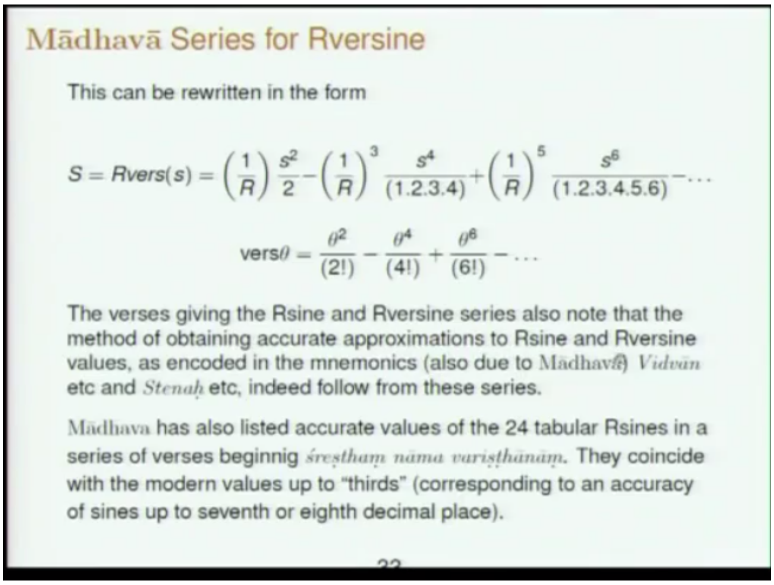

And ultimately that will lead to this series for the versine. So in the end Madhava says (FL) so this whole series is to be obtained to get the sara or the Rversine and the same thing has been summarized in (FL) etc. So what is this vidvan? What is this (FL) etc.? This will be explained in another lecture on sine table that will come later. So basically Madhava not only gave the series, he also gave a set of mnemonics by which you calculate the first few terms of the series.

So using these mnemonics for the first few terms of the series you can calculate any sine or any versine for any angle to a given level of accuracy. So this (FL) etc. these are the mnemonics for the first few coefficients of the Rsine series and the Rversine series. From that you can calculate the sine and versine for any angle that any arc that you want to a particular level of accuracy.

So Madhava always couples an exact result with an approximate way of calculation. He gave the pi series and then in the same set of versus he gave you the first correction term also. In the same way, he gave the sine and cosine series, at the same time he mentions that this can be quickly approximated by using vidvan and (FL) kind of mnemonics. So Madhava also listed the 24 tabular Rsines.

What are these 24 tabular Rsines for 225 minutes, 450 minutes, 675 minutes for which Aryabhata had given the sine table? So those are collected in a set of verses which start with srestham nama varisthanam etc verses, which are quoted in various commentaries (FL) of Shankara Variyar on Tantrasangraha. So they coincide with the modern values up to the third.

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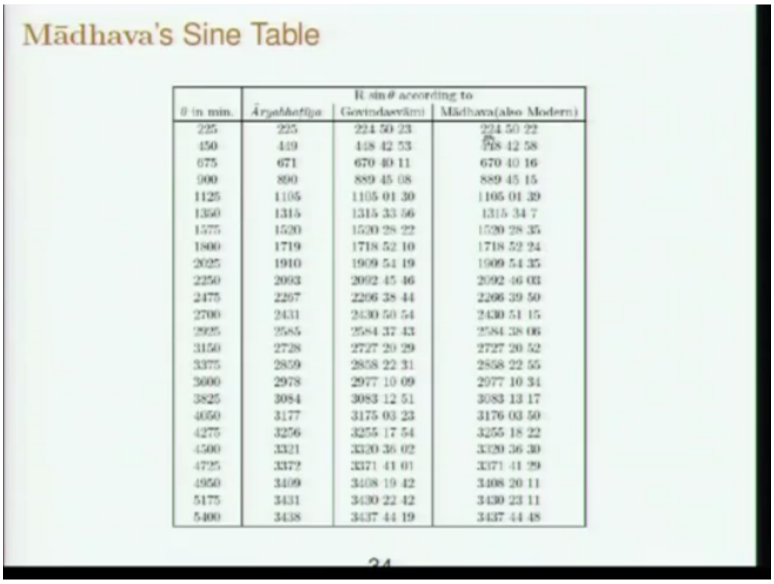

θ in min.	R sin θ according to Āryabhaṭīya	Govindasvāmi	Mādhava(also Modern)
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1125	1105	1105 01 30	1105 01 39
1350	1315	1315 33 56	1315 34 7
1575	1520	1520 28 22	1520 28 35
1800	1719	1718 52 10	1718 52 24
2025	1910	1909 54 19	1909 54 35
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3150	2728	2727 20 29	2727 20 52
3375	2859	2858 22 31	2858 22 55
3600	2978	2977 10 09	2977 10 34
3825	3084	3083 12 51	3083 13 17
4050	3177	3175 03 23	3176 03 50
4275	3256	3255 17 54	3255 18 22
4500	3321	3320 36 02	3320 36 30
4725	3372	3371 41 01	3371 41 29
4950	3409	3408 19 42	3408 20 11
5175	3431	3430 22 42	3430 23 11
5400	3438	3437 44 19	3437 44 48

So this is the sine table of Aryabhata, which was improved upon by Govindaswami, which is further improved upon by Madhava. So this is in minutes, this is in seconds, this is in thirds, so up to thirds it will coincide with the modern values that we know for sine. So if you think of it in decimal terms, it is accurate up to 7 or 8 decimal places. So this is the sine table for 24 Rsines given by Madhava.

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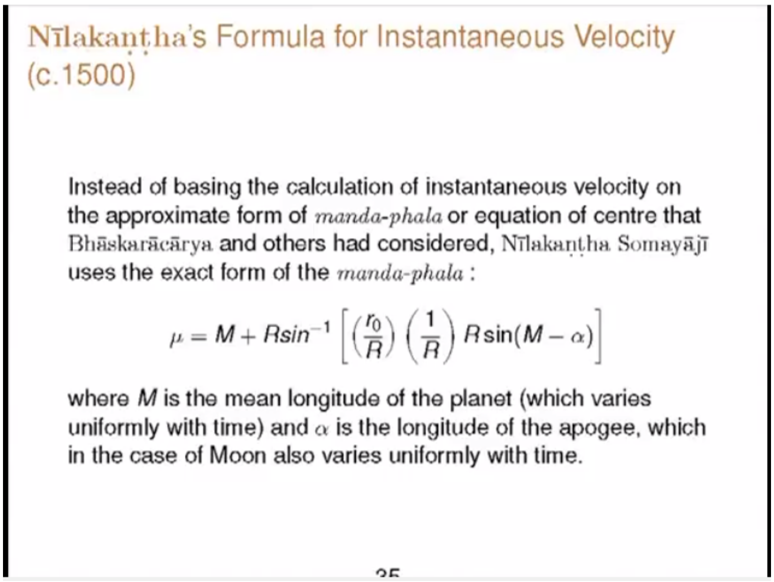

Now we will finally go into a topic which was the question of the instantaneous velocity. What was the kind of improvement that was done by the Kerala astronomers to what Bhaskara had done earlier? So Bhaskara had used the equation of center in the simplest form, but the actual equation of center is something like this. It is sine inverse Rsine of the anomaly.

This is the mean value, this is the apogee and these are the coefficients, this is the mean epicycle, this is the mean longitude, so this is the anomaly the difference between the mean longitude and the aphelion or the apogee depending on the object for which you are writing the equation of center. Now when this anomaly is small, you can rewrite the sine as angle itself.

So the whole thing becomes much simpler, but Nilakantha or you can rewrite the sine inverse as x itself so the equation of center looks much simple, it is just a sine term but the actual think is the sine inverse function of the sine of anomaly that you have to consider. This M is the time dependent quantity here. So when you calculate the velocity you should differentiate this with respect to this time dependent quantity.

So what is involved is the derivative of the sine inverse function is what will appear and Nilakantha in his formula for the instantaneous velocity obtained the derivative of the sine inverse function that is something what we all know.

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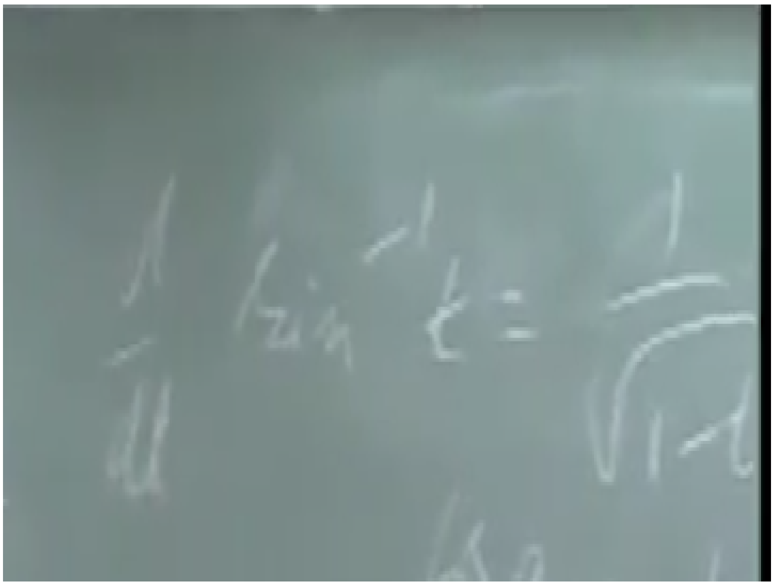

So this equivalent of this is what we obtain. So this occurs **"Professor - student conversation starts."** No, no, equation of center itself will take you from the uniform circular motion to non-uniform motion so it is equivalent to moving in an elliptical orbit which will also give you a non-uniform angular motion so to the first order in eccentricity the equation of center will mimic the elliptical orbit.

So in the elliptical orbit also you say that the area law of Kepler will tell you when you are close to the (()) (45:49) the thing will move faster, when you are far away it will move slower so at the same kind of thing that so for small eccentricity this equation of center is as good as working with a ellipse. The other aspect of the ellipse is the distance, which we were not considering at all.

One is only working about the angular motion, you are looking at the stellar object the planet in relation in the background of stars and the actual distance of the planet is the different kind of an issue. **"Professor - student conversation ends."**

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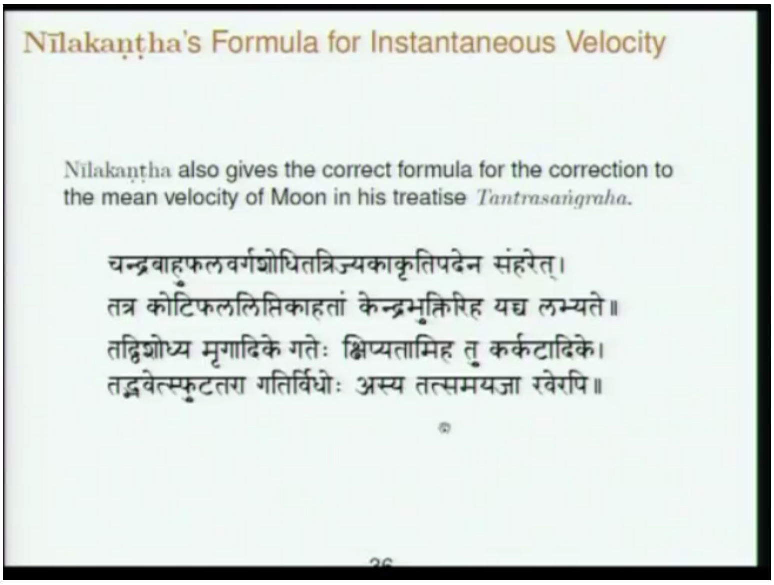

So in Tantrasangraha, Nilakantha has quoted this verse (FL) so this is basically the 1/square root of 1-s square that is characteristic of the sine inverse x.

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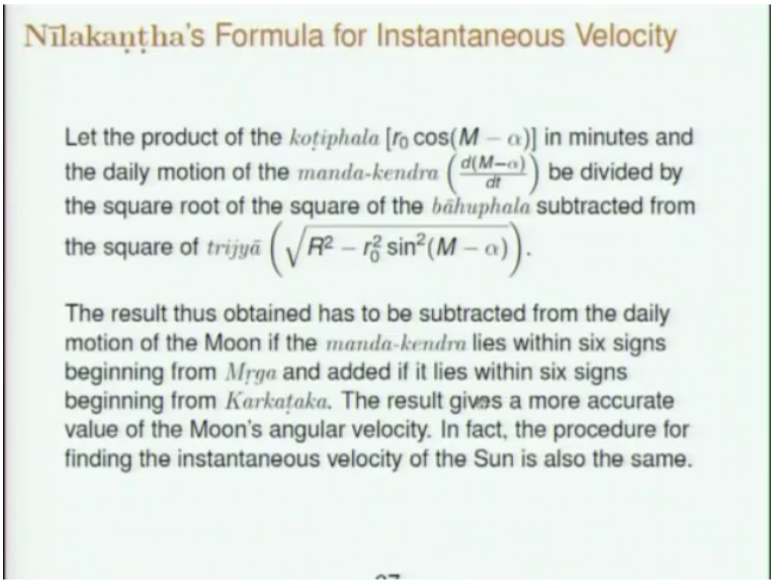

And he says specifically this should be done for the case of moon but one should do it for sun also if need be.

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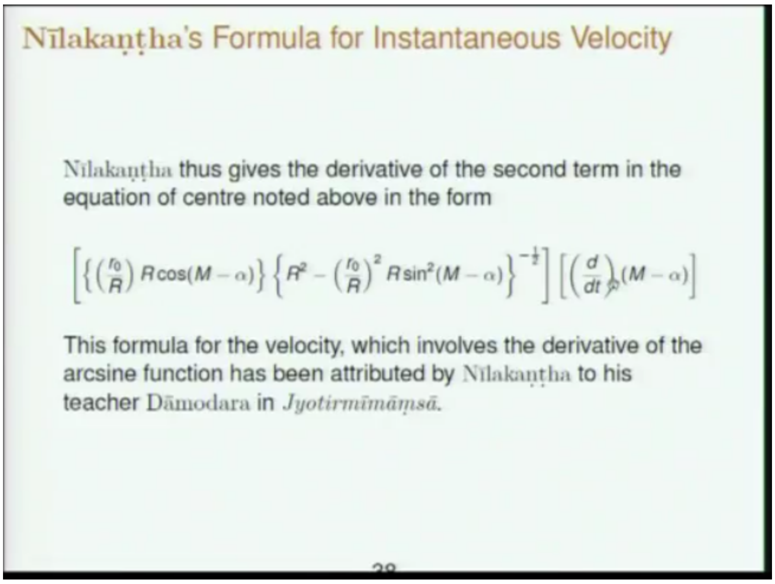

So in terms of the formula so if you take this is the equation of center this was the equation of center that Nilakantha was considering and the derivative that he has obtained is something like this. Nilakantha in his Aryabhatiya-bhashya has tried to give some explanation but this is not really the kind of explanation that we would like to know for the derivative of the sine inverse function today.

But he has tried to explain how this term 1/square root of 1-s squared comes? This verse Nilakantha actually in Jyotirmimamsa he explained that this was his actually due to Damodaran so it is not his discovery. This is due to his teacher Damodra who is the son of Parameshwara.

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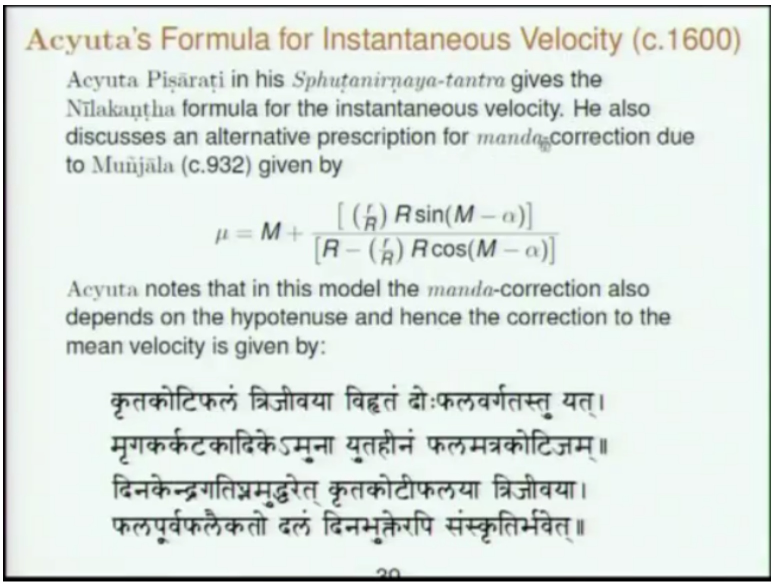

Now Acyuta Pisarati in Sphutanirnaya-tantra, he has given the derivative of the ratio of 2 functions. He constructs the equation of center. He of course gives Nilakantha's result of derivative of sine inverse of x. He also constructs the derivative for another model of planetary motion. This model of equation of center is due to Munjala given in (FL). It involves a variable denominator here which is depending on the anomaly.

So the M is the function of time here, so derivative of this will involve knowing the derivative of ratio of 2 functions.

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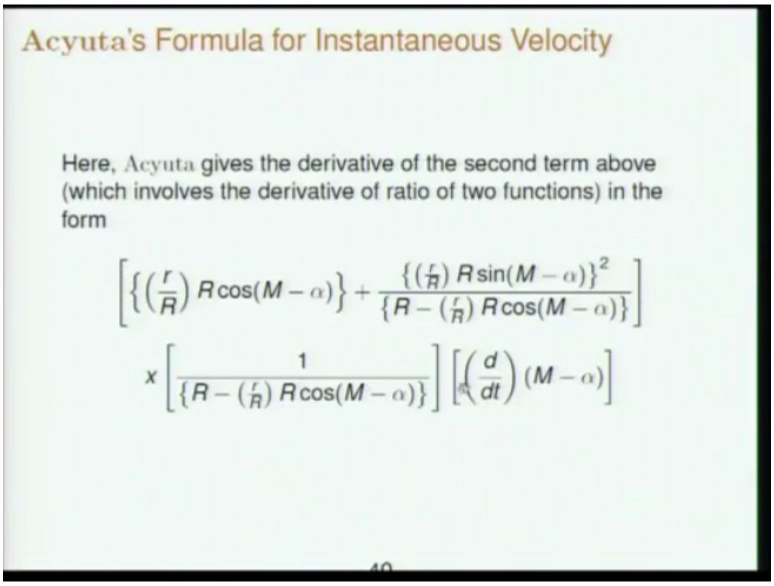

And again Acyuta Pisarati is able to construct the derivative in the same way that we used today. So in terms of the work of Kerala School, there are perhaps other things also. We have only analyzed 2 or 3 books. We have analyzed what the results as derived in Yuktibhasa. We have analyzed the results as given in (FL) and its commentaries (FL) Yukthidipika. We have analyzed Nilakantha's discussion in Aryabhatiya-bhashya.

But there are many other words of Kerala astronomy, we do not know what other kind of mathematical developments are encoded in them or worked out by them. Perhaps there may not be many more or perhaps there are a lot more. We cannot say right now, but what has been done is fairly substantial.

So they start with infinite series for pi and various approximations by end correction terms, transform series, something similar to continued fractions also. Then we have the infinite series for the sine and cosine functions, then approximations for calculating them, then the notion of instantaneous velocity and its implications in astronomy. So many, many aspects of infinite decimal calculus are there in the work of the Kerala astronomers.

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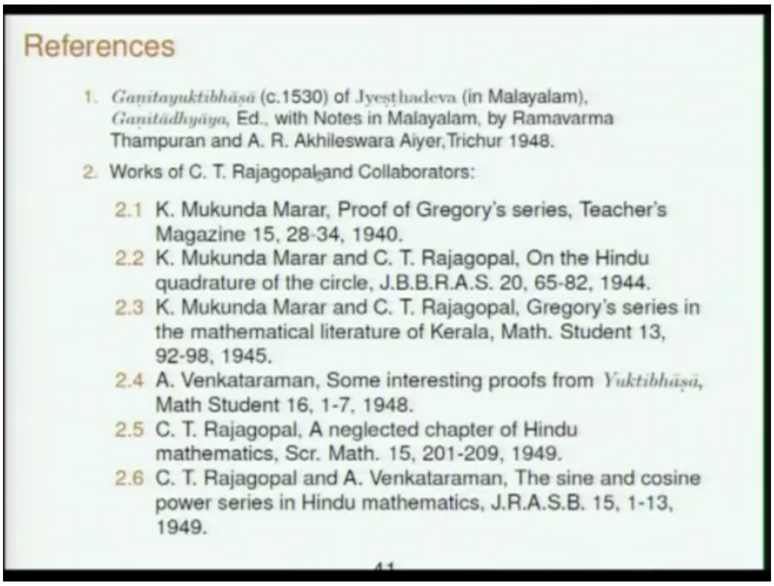

I have tried to summarize some of those that we have seen in this various books. The proofs of this we will discuss perhaps in the next and the last lecture and some of these proofs will be discussed as given in Yuktibhasa. So these results will be repeated once more again and again by discussing their proofs. So this Ganithayukthibhasya as I said was first published in 1948.

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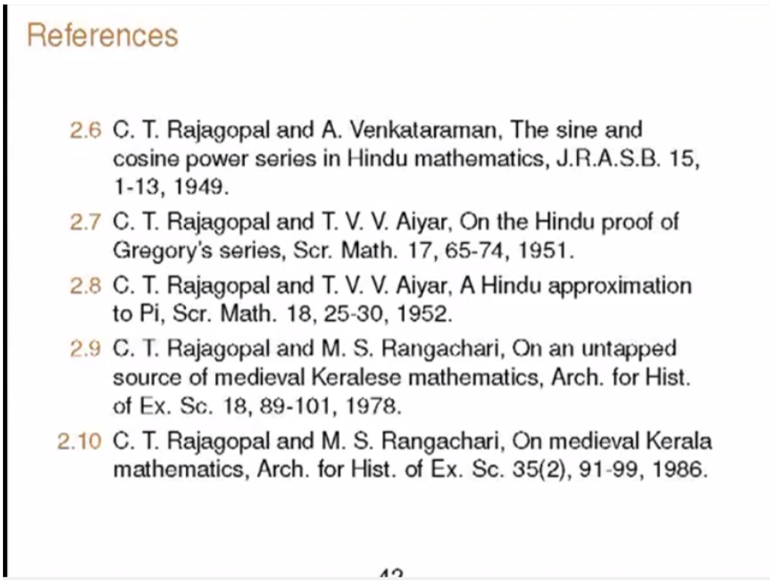

And around the same time a series of articles by C.T. Rajagopal and his collaborators appeared which discuss the proofs in Yuktibhasa. Then this English translation by K.V. Sarma and notes. This Kriyakramakari was edited by K.V. Sarma in 1975, Tantrasankara with

Yukthidipika edited by K.V. Sarma in 1977 and a new edition in English translation with detailed notes by professor Sriram and Ramasubramanian appeared in 2010.

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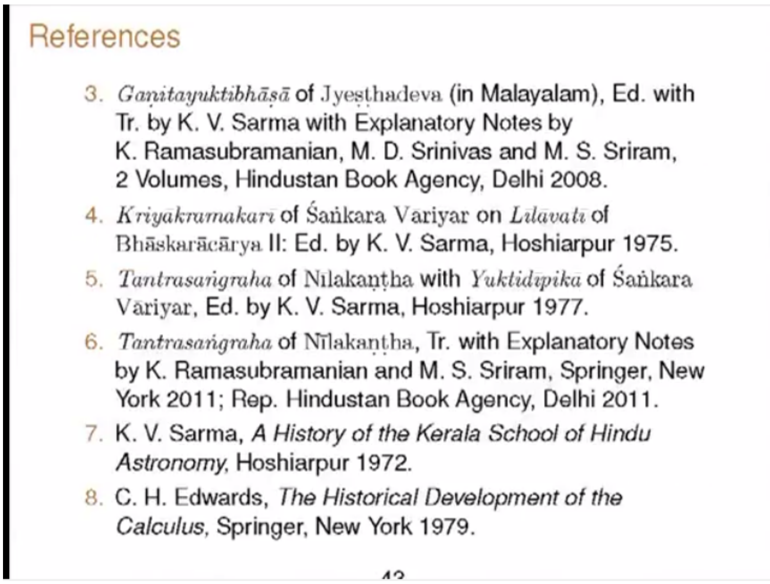

This is a good summary in English of Kerala mathematics so summary of the work and discussions in Yuktibhasa. Raju's book is a detailed discussion of the mathematical foundations of calculus in India and his own hypothesis that this was exported to Europe basically by the (()) (51:13) machineries sometime during 15 or 16 century. Since the argument for transmission normally has been made is one of priority.

So the Greek side discovered something, they had the epicycle model by the time of Ptolemy or by the time of Apollonius and Aryabhata is considering epicycle model in 1499 so this was transmitted to India. So if you want now you can revise this kind of argument and say Bhaskara had the Chakravala equation in 1911 and 1950 and (()) (51:41) is considering this anything in 1650 and Nilakantha in Yuktibhasa they have all these interesting results on calculus in around 1500.

And they are being rediscovered in 1650, 1670 so priorities in argument for cultural transmission and not really textual evidence for words being taken away translated as study. Then this is good enough argument. This is in summary what Raju is trying to say many, many, more things he is trying to say. George Joseph is another person who has popularized the study of Kerala mathematics through several books.

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Berggren and Borwein is a detailed history of pi in various cultures and civilization. Borwein and Borwein are one of those people who were specialized in fast computing algorithms for Ramanujan. There are several papers on Ramanujan by them also. So with this we stop our brief introduction to calculus in Kerala mathematics.