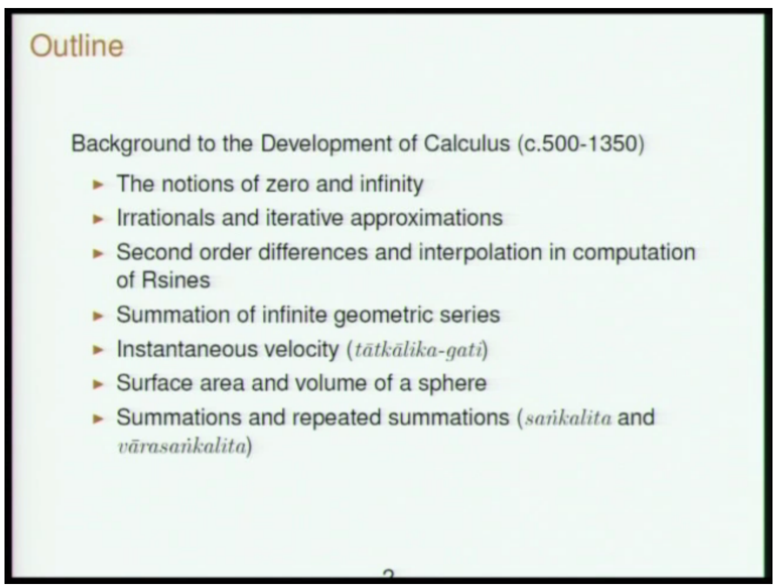
Mathematics in India: From Vedic Period to Modern Times
Prof. M.D. Srinivas
Department of Mathematics
Indian Institute of Technology – Bombay

Lecture - 30
Development of Calculus in India 1

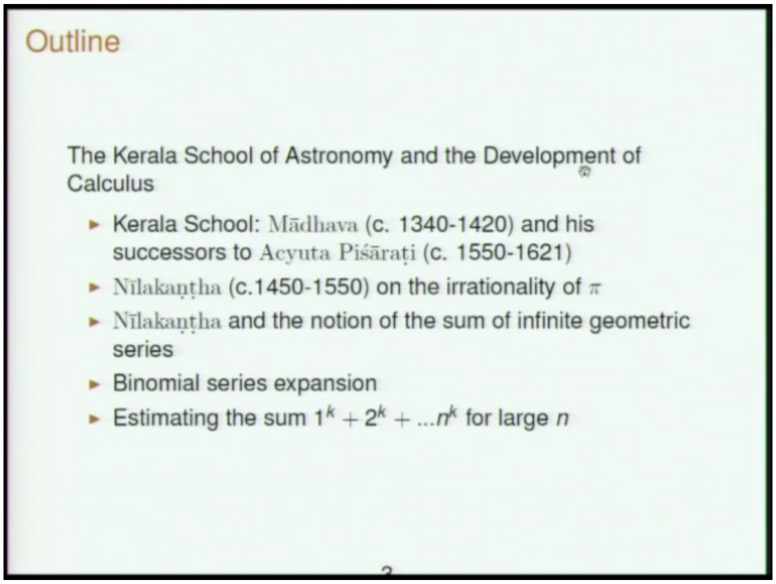
In the next talk we will discuss the development of calculus in India, mainly the work of the Kerala School. So, of course I will not define what calculus is. I will describe what is the work the Indian mathematicians did and then you can see what aspects of our understanding of calculus this what sort of contributes to, but this work of the Kerala School did not start in a vacuum, so there were various ideas which preceded it, so the first half of today's talk will be to trace these ideas in the earlier tradition of Indian mathematics.

(Refer Slide Time: 00:54)



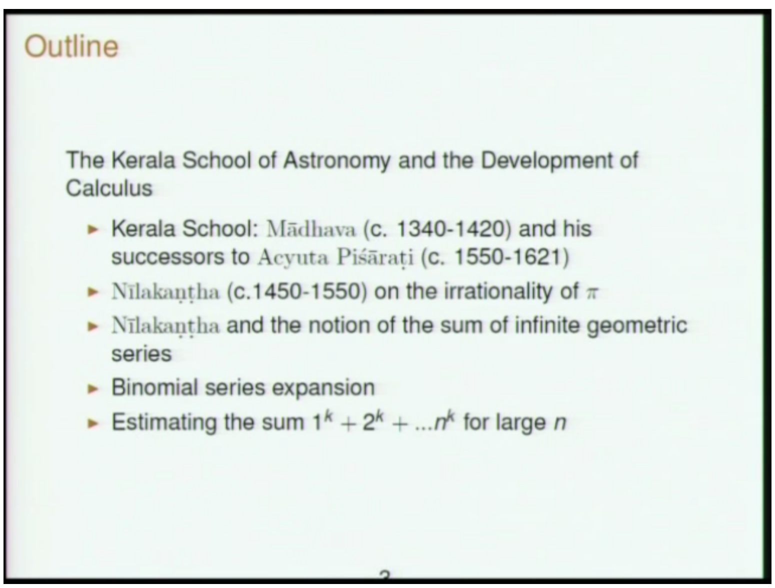
So how in the understanding their understanding of zero and infinity, how in the understanding of irrationals and iterative approximations to them particularly in their understanding of calculation of sine table, the use of second order differences, summation of infinite geometric series, the notion of instantaneous velocity we developed in the discussions in astronomy and of course in calculation of surface area and volume of a sphere in doing this summations of powers of integers. So, what does the work done from the time of Aryabhatta to Narayana?

(Refer Slide Time: 01:27)



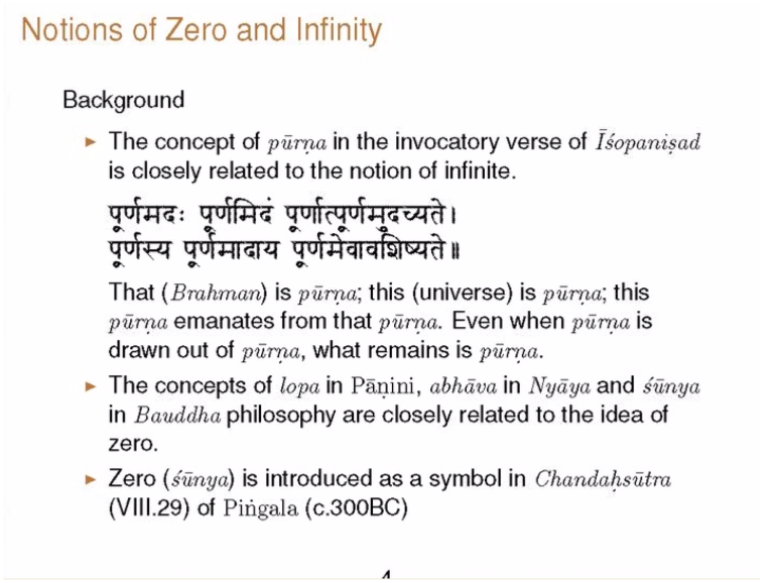
Which preceded the work of the Kerala School that I will cover in the first half of today's talk. Then I will introduce the Kerala School and discuss some aspects of their work. The main discussion of the work of the Kerala School will be done in the next lecture.

(Refer Slide Time: 01:44)



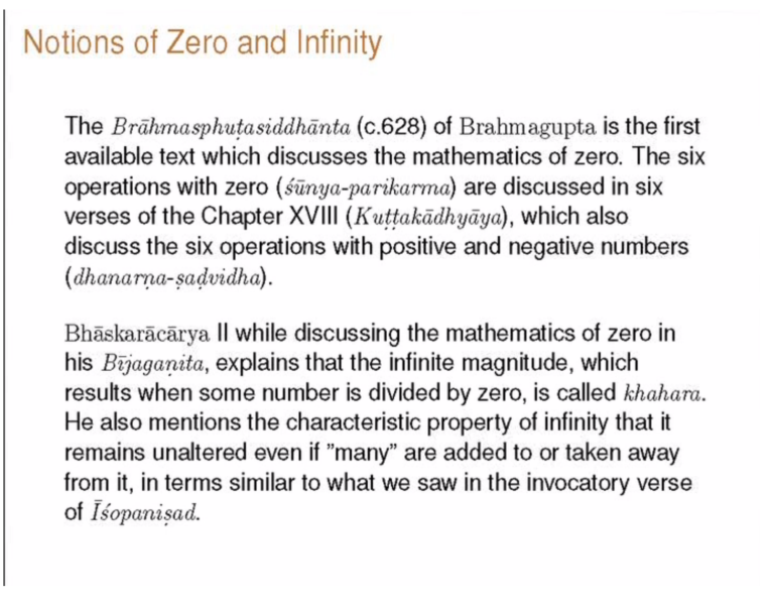
So some of these will be repetition, so the idea of zero and given the idea of infinities since you have been rarely well established from the ancient times.

(Refer Slide Time: 01:58)



This verse of (FL) has been quoted I think already which gives you an idea of the notion of infinity that the notion of zero can be trace to Panini and philosophical discussion in the Nyaya and the Bauddha School of philosophy.

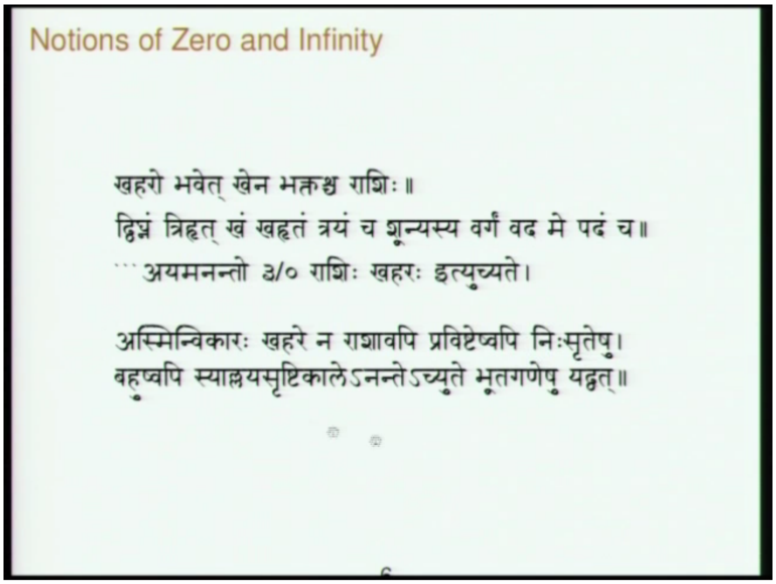
(Refer Slide Time: 02:13)



(FL) introduced the symbol zero in some other combinatorial calculation. The mathematics of zero of course was discussed in detail almost in the way that we do today in the brahmasphutasiddhanta. In this end set of six and eight verses Brahmagupta discusses the six operations with zero as well as the operations with negative quantities also. Now Bhaskaracarya tries to take this discussion further. Brahmagupta tried to define number divided by zero he called it (FL).

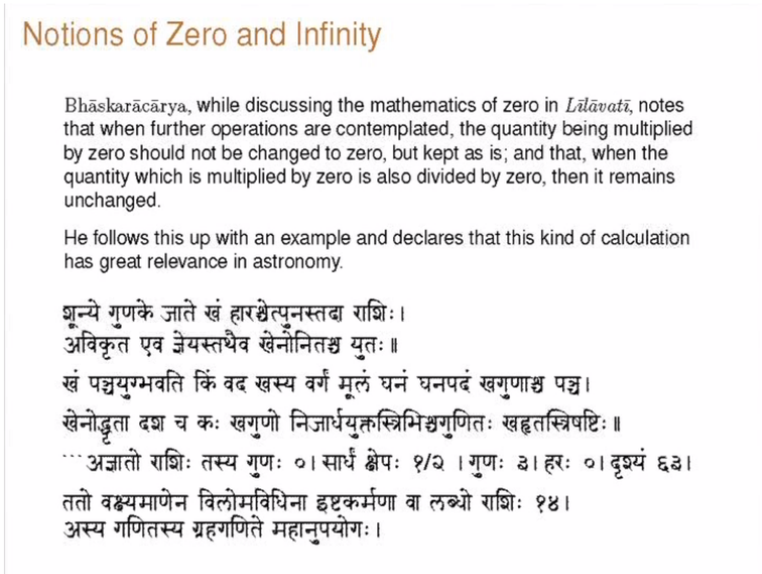
So Bhaskaracarya calls this as khahara and he mentions and of course say that this is (FL) this is infinite and he again describes this characteristic property of infinity that it remains unaltered even in many or added to it or taken away from it in terms which is similar to the way it was discussed in the (FL).

(Refer Slide Time: 03:17)



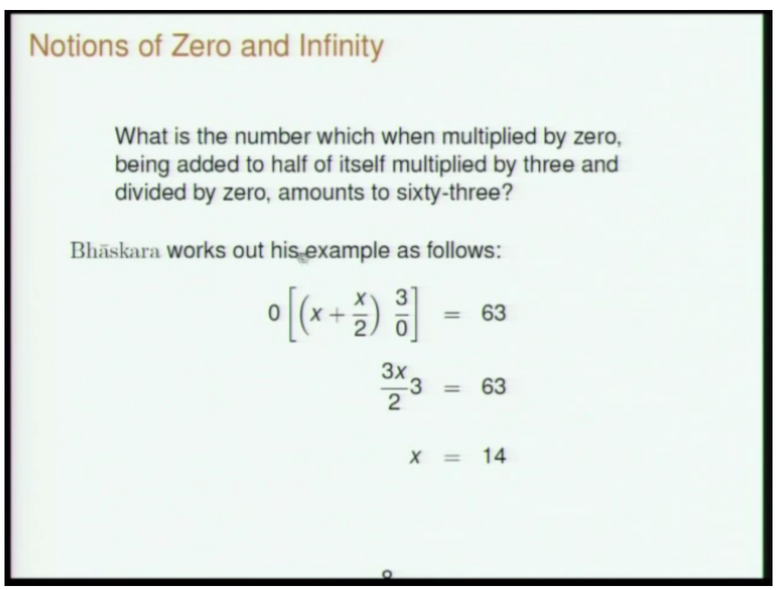
So (FL). So khahara is a number which is obtained then some number is divided by zero. So for incidence (FL) So this number is called in khahara and then he gives this words (FL) so which is very similar that there is no change in this (FL). In the same way as in the (FL) all the (FL) then they are emanated from him and when they enter back him during the (FL) there is no change in the (FL) in the same way there is no change in the (FL). This is very similar to the way the idea of infinity was sort of implicit in the discussion of (FL).

(Refer Slide Time: 04:21)



Now Bhaskara does a little bit more. This was discussed in the lecture of Lilavati, Bhaskara says (FL) that is when you are doing a multiplication with zero do not be in a hurry to cancel out and make it zero. Later on you may be dividing by zero so for that purpose it is better to keep showing it as it is and then he gives one example. So the example is (FL). So there is a problem.

(Refer Slide Time: 05:01)

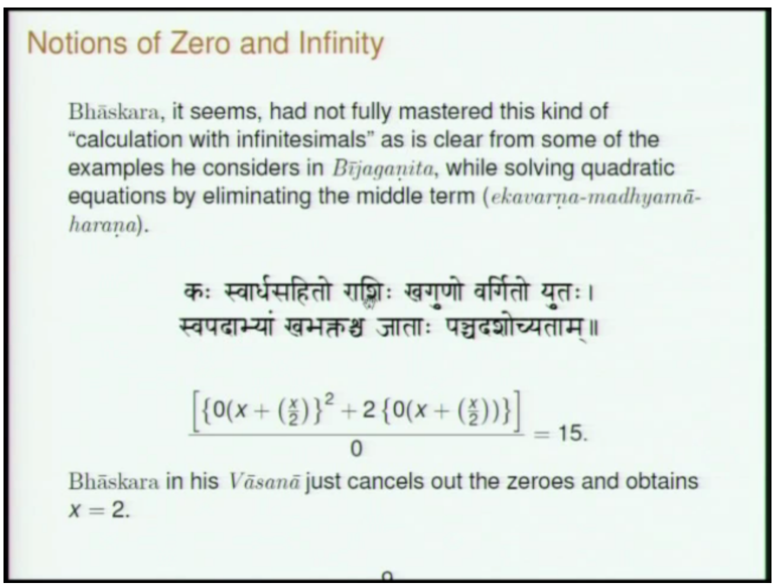


What is the number which when multiplied by zero being added to half of itself multiplied by 3 and divided by 0 amounts to 63 and of in his vasana Bhaskara tries to sought out to put down the working and then he says doing this (FL) you finally get the answer as 14 and finally he comments (FL) this kind of mathematics has much use in astronomy. So the problem he is doing is 0*x + x/2*3/0 = 63. So here Bhaskara is cancelling out the two zeros and obtaining x = 14.

So he is not treating 0/0 as indeterminate. He is canceling them out and as you can see Bhaskara though he is being quite clever he is saying that when zero comes in you keep it as it is and he then realizes that this kind of mathematics has much to do in astronomy that various situations you get a limit.

You take the value of a shadow so that tan 0, tan theta will come in that and tan 0 and tan 90 degrees both have a problem so in all these kinds of situations the kind of limits that you need to do to have to be carefully done in what Bhaskara is saying, but he himself is not fully able to handle this. So in (FL) he is giving an example and we see how Bhaskara is finding it difficult to handle this.

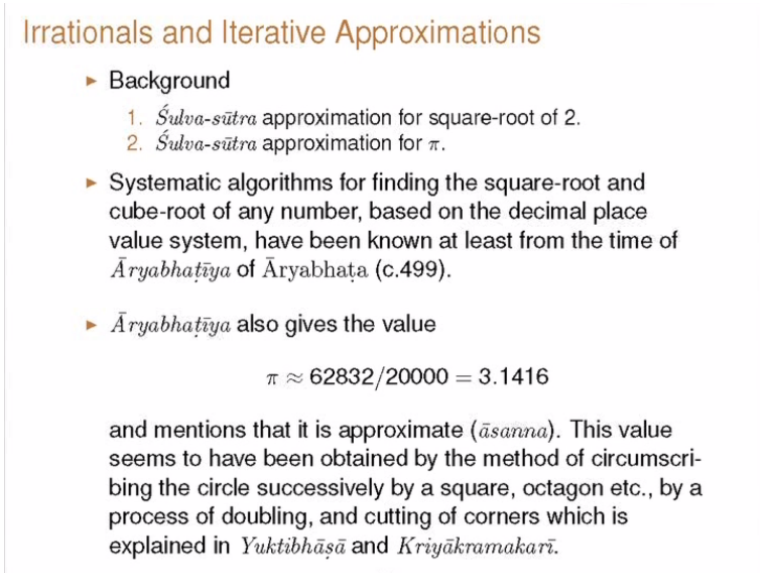
(Refer Slide Time: 06:36)



(FL). So 0*x + x/2 whole square + 2*0*x + x/2 / 0 = 15 and so Bhaskara not being fully adapt actually working out different orders of zero which are appearing here. He just cancels them out and that is of course not something which we can sort of justify today. he would have had to see different orders of different ways in which the quantities are becoming zero, zero square and zero (()) (07:17) are equal 14.

So he still not fully conversant with this mathematics of infinitism as you can see, but the idea is already there in his mind, but he is not able to fully put it into operation.

(Refer Slide Time: 07:29)

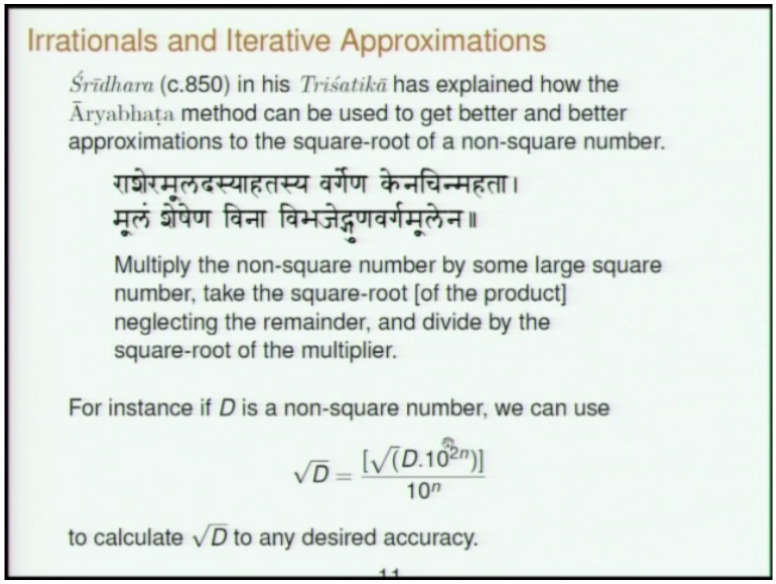


You later on see how the Kerala mathematicians are capable of keeping all different orders of magnitude and then take the limit right at the very end so they keep on keeping all the sums and everything and then go to the limit right at the very end so there will never been an error in what they will be doing. Now we come to the understanding of irrationals and iterative approximations for them.

This goes back to Sulva-sutra which has given an approximation per root two and also for pi. Now a systematic algorithm for square root and cube root for the first time was given by Aryabhatiya which was made possible because of the decimal place value system. Then Aryabhata is also given this value for pi and specifically mentions that it is asanna this is approximate and perhaps the way Aryabhata arrived at this might be using circumscribing this circle by square.

And then successively by octagon and a polygon of 16 sides etc. A method which is explained in Yuktibhasa and Kriyakramakari that is of course the other method which was discussed in the context of (FL) commentary on Lilavati where is talking of a hexagon inscribed in a circle and then 12-sided polygon and then 24-sided polygon and so on.

(Refer Slide Time: 08:54)



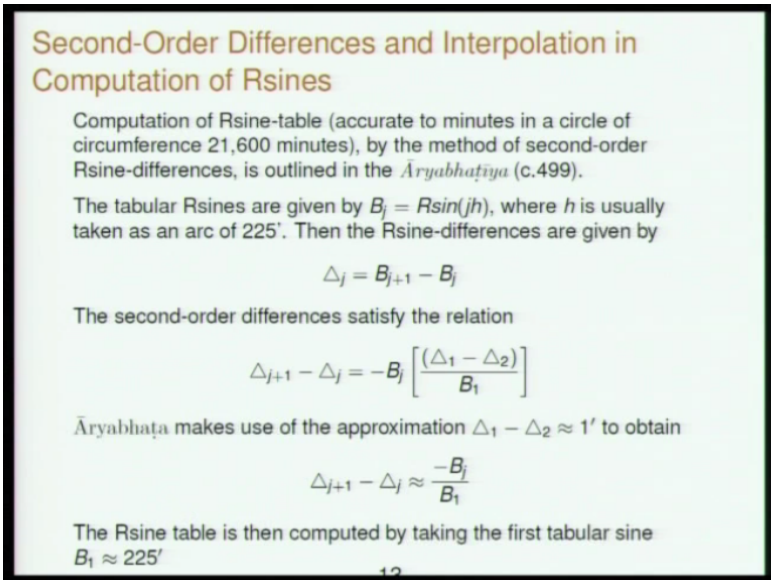
So he estimates the sides of the polygon and approximate the circumference of the circle by that also, but the Kriyakramakari that is the Aryabhata tradition seems to be inscribing the circle in a square and then in octagon and then in a so this we will see towards the end of the talk and proof sometime. Now the Aryabhata algorithm of course they were not dealing with they had a place where you notation, but for numbers below one.

What we today understand as the decimal notation was not adopted in India, it was discovered in the West Asian region. Now what is done for a quantity which does not have an exact square root. The Aryabhata algorithm is of course applicable so what is to be done so this was hindered in (FL) at some level by a specific statement of the way to proceed is Sridhara and Trisatika he says you take the number D which is not an exact square multiplied by a suitable power of 10.

And as suitable square power of 10 take the square rate and divide it by the square root of the power of 10 so you get better and better values of the square root. This is like a decimal expansion, but still treating the quantity as ratio of two integers. So (FL) this is the words of Sridhara as to how to approximate nonsquare quantities how to obtain better and better.

We of course discuss the other method of obtaining better approximations to square roots the more sophisticated one solving the (FL) equation and that this (FL) equation and (FL) can be specifically used for getting better approximation is explicitly mentioned in.

(Refer Slide Time: 10:40)



Narayana's Ganitakaumudi I gave this example yesterday also in the talk on Narayana. So this is about irrational and approximations to them now comes the calculation of sine and how second order differences have been made use of in the calculation of sine. As you have already seen Indians are using the radius times the sine function as the (FL) that is the sine function. Now you take some unit so you take your circle say that it is divided into 21,600 minutes.

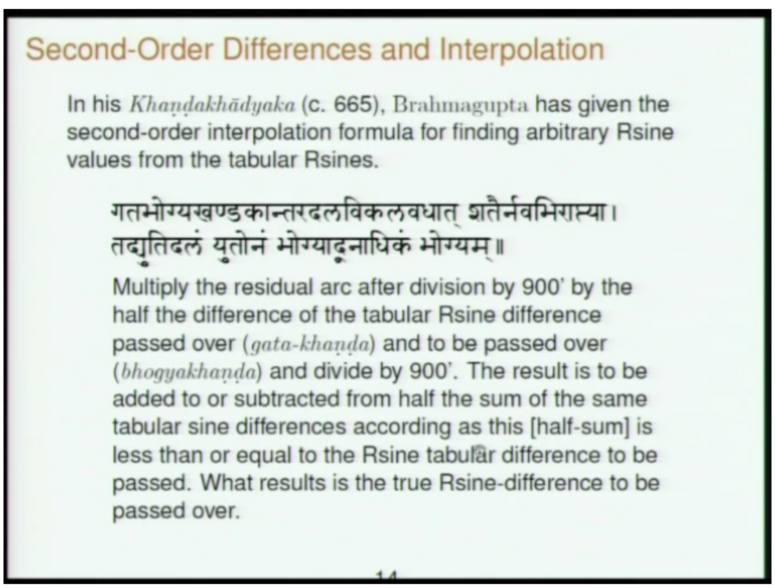
So 60 degrees and 60 minutes' product is 21,600 minutes so the circumference of the circle is measured as 21,600 minutes and you take the signs of arc which are of lengths in steps of 225 minutes each. So the quadrant will be covered by 24 R sines. So this is all be done later in more discussions of trigonometry so Bj is Rsin jH where j runs from 1 to 24 and h = 225 minutes so Rsin of 225, Rsin of 450, Rsin of 675 etc.

Now these are called the (FL) or the tabulated jh 24 sines. Now sine differences are Bj + 1 - Bj and second order sine differences are the difference of these delta j + 1 - delta j. So already at the time of (FL) he had a verse which essentially gave this relation that the second order sine differences are actually proportional to the sine themselves so this is a difference variant of the differential equation for the sine function you know that sine satisfies D square sin x/D square it is - sinx.

So essentially this is a difference variant of the. Now this relation is implicit in the words of Aryabhata I think it was discussed at some length. Later on (FL) makes this relation absolutely explicit in his commentary on Aryabhatia. Aryabhata used the approximation that the first order sine difference he said was 224, second order sine difference was 223 and so this delta1 is just 225 minutes, delta2 is 224 minutes so he made this approximation that delta1 - delta2 is 1 minute.

He also assumed that B1 is 225 minutes and he went induced this difference to calculate B1, B2, B3. B1 is assumed to be 225 and then you remove this delta1 - delta2 by saying that that is 1 minute so apparently this equation is dimensionally not correct that delta1 - delta 2 = 1 is sitting there. So using this difference equation the various sines these are much more sophisticated and simple calculating sines where other than using all those complicated trigonometric formula for half angle, then going from say the 30 degrees then to 15 degrees then to 7.5 degrees this is much better direct way of. So using this Aryabhata tabulated the sines.

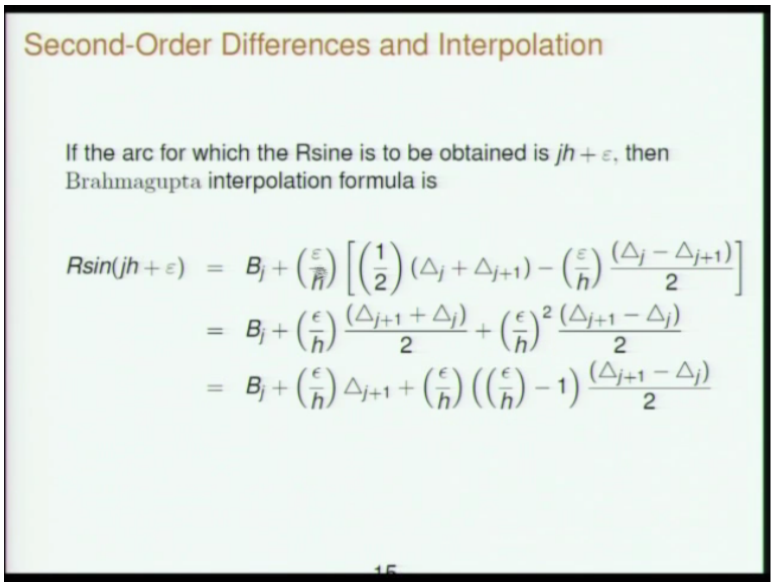
(Refer Slide Time: 14:03)



Now what do you do about the so this is for sine of 220/sin of 450 that is 3 degrees 45 minutes, 7 degrees 50 minutes etc, but about sine of 5 degrees or what about sine of 3 degrees 46 minutes. So you have to do interpolation. The first thing was to do linear interpolation, but Brahmagupta soon came up with the second order interpolation formula for finding sine values for in between value.

Brahmagupta gave a sine table in basis of 900 minutes that is sine of 15 degrees, 30 degrees, 45, 60 and 90 only 5 sine values he gave and then said calculate all other sine values by interpolation, but he gave such a beautiful interpolation formula that the value comes out to be fairly accurate this is in his Khandakhadyaka not in his(FL) this is in later (FL) work or the manual work (FL). So Brahmagupta is giving these verse.

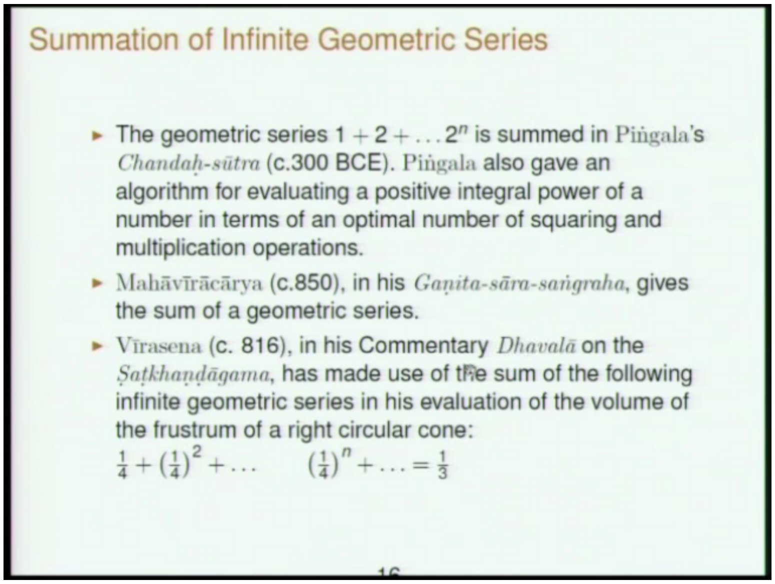
(Refer Slide Time: 14:59)



Just see the relation that he is giving. So Rsine jh is one of these values 220/450 675 minutes etc. For any value beyond this in between this so he says you start with approximated first by bj itself then usually you would have just said epsilon x Rsine cosine jh that will give you the first kind of term. He has given two terms. This is today known as the Newton Sterling interpolation formula to the second order.

So this is a prescription using this you have a sine table of 15 degrees only the sine of most of the angles came out to be good enough for the purposes of astronomy.

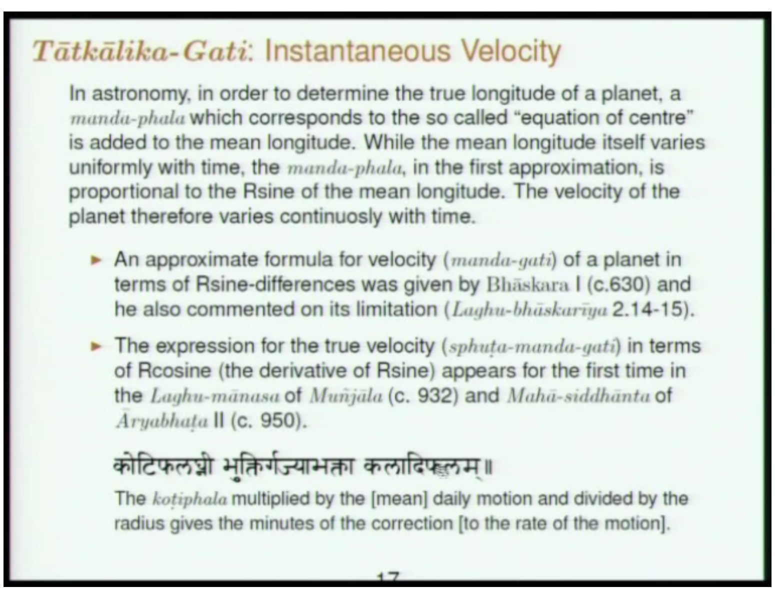
(Refer Slide Time: 15:43)



Now we come to infinite geometric theories. The finite geometric theories go back to Pingala's Chandah sutra itself these are how he is summing this series. Mahaviracarya gives the sum of a general geometric finite series like that. It is in the (FL) that the sum of an infinite geometric series is mentioned of course this result is well known. It forms the basis of one of the calculation in Archimedes work on the cylinder and cone.

So in the history of mathematics infinite geometric series were summed. They were summed in some peculiar way in Archimedes. So Virasena in his commentary Dhavala on Satkhandagama has given this explicit. He uses this in evaluating the volume of the frustrum of a cone.

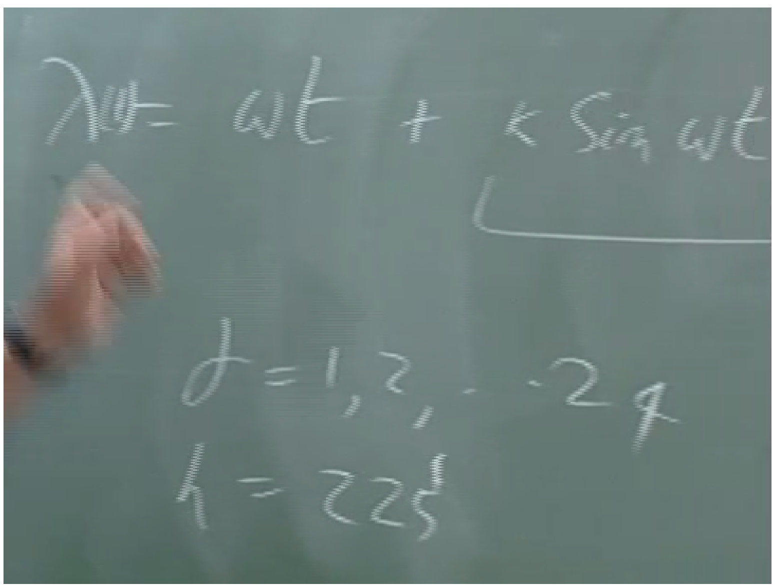
(Refer Slide Time: 16:35)



We will later on see how the Kerala mathematicians interpret a relation like this how do they understand this. Then comes the idea instantaneous velocity. Instantaneous velocity occurs in astronomy very naturally especially in considering the motion of a fast moving and whose place is changing constantly object such as the moon whose motion is fairly irregular.

So what is done in astronomy is first you always calculate the mean position. so assume a constant angular velocity calculate the mean longitude to that you add what is called the equation of center in today's astronomy, it was called manda-phala by Indian astronomers. to the first order of approximation this manda-phala is always proportional to the Rsine of the mean longitude. So it is something like so if.

(Refer Slide Time: 17:30)

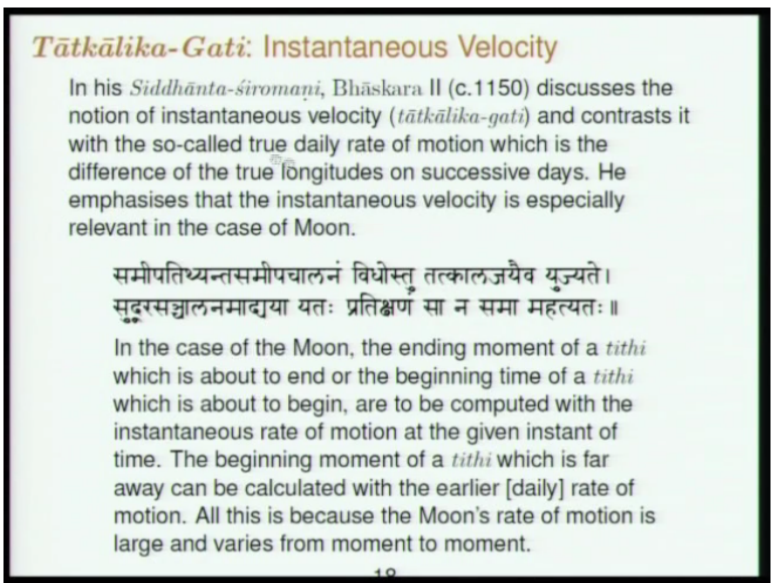


Lambda is the longitude it is something like. This just carried (FL) way of saying. So when the longitude is changing with this time dependence if you want to calculate the speed or (FL) or velocity obviously you will lead to know the derivative of the sine function at any instant if you want to go to the idea of instantaneous velocity. So to start within Bhaskara 1 in his (FL) gave an approximate formula.

The sines were as you saw the sines were tabulated in units of this, so you just subtract the tabular sines around the longitude of the lambda that you are considering so use the sine difference. So this was the first approximation to the velocity and Bhaskara later on immediately saw the limitations of this because the tabular signs change only when you cross 225 or 450 so it appears as if there is a jump occurring in the velocity of the object.

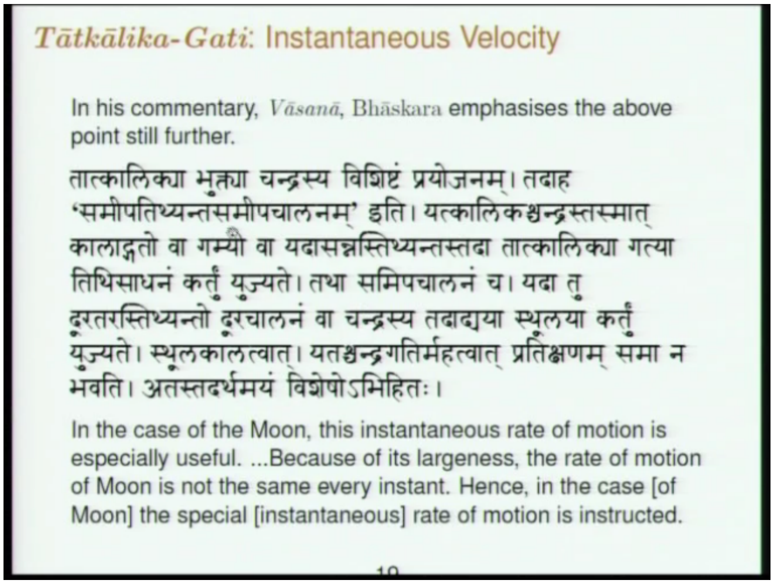
So obviously it is not something very physical. Now the expression for true velocity is sphuta-manda-gati in terms of Rcosine which is called the (FL) appears first in the (FL) written around 932 and there is another Aryabhata who has written a Maha Siddhanta in 950 so (FL) call is for the degrees, minutes of an angle so basically the mean daily motion kotiphala multiplied by the mean daily motion so omega*cos wt so this is roughly the correction that you have to put for the velocity of the value. Now, so this is discussed much systematically in Bhaskara who explains the importance of instantaneous velocity in astronomy he calls it (FL).

(Refer Slide Time: 19:58)



So in Siddhanta Sironamni, Bhaskara discusses this that he emphasizes that you have to do this instantaneous velocity especially in the case of moon. So it is not good enough to take the velocity of moon to be the differences of the longitude of today and the longitude of tomorrow. That will be the average daily velocity of moon you have to know its velocity which is changing at every instant of time (FL) is the moon. For him the velocity you have to take especially when you want to calculate the time of onsite of tithi or the time to be elapsed for going to the next tithi.

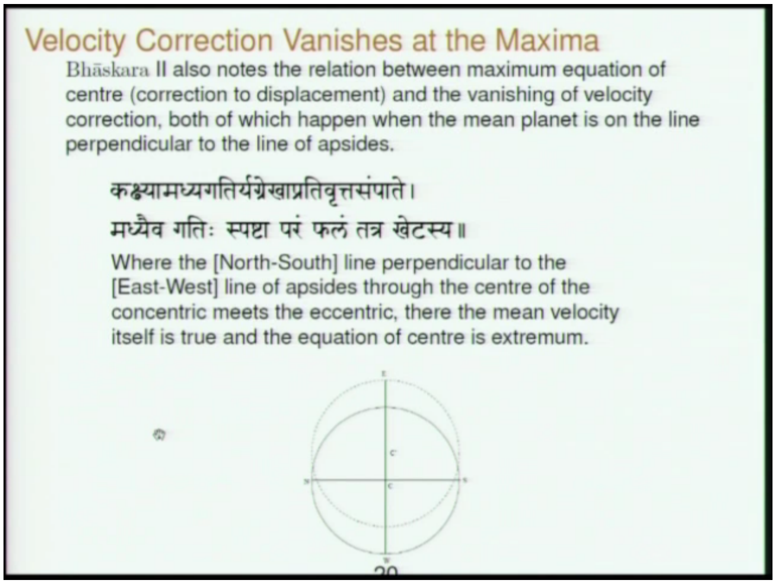
(Refer Slide Time: 20:47)



So the argument given Bhaskara as you know has written as Vasana commentary on his Siddhanta Sironamni so where he extensively explains as one of the extremely beautiful documents which explains all the principals of astronomy I mean prior to (FL) this is in fact one of the best text to understand how Indians or you viewed out various principals in astronomy.

The vasana commentary of Bhaskara on his Siddhanta Sironamni. (FL) That is why it has been said that you should do that. (FL). So in order to find out as I said when the beginning of the next tithi arises or how much time as elapses the previous tithi it is best to do the calculation with the velocity of the moon at that point. (FL) When the start of the tithi is far away that is when the amount is large enough you can use a more approximate calculation. (FL) The velocity of moon is not the same at every instant. It keeps changing every instant.

(Refer Slide Time: 22:10)

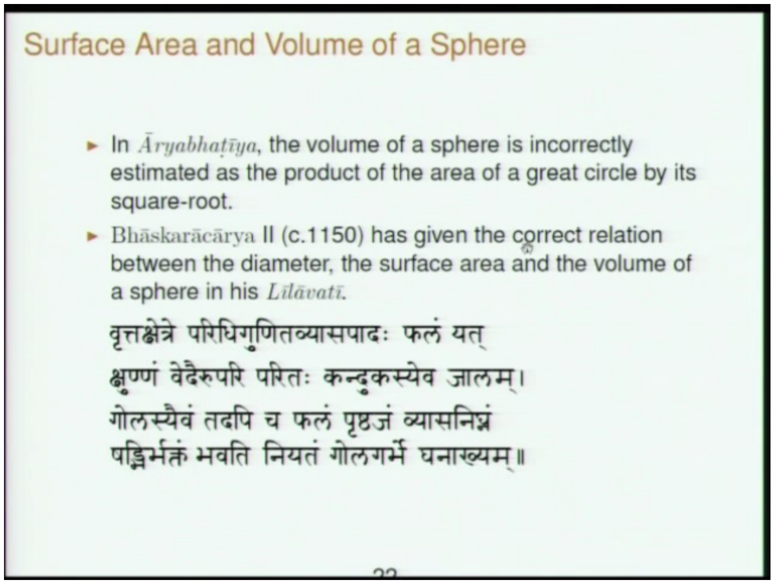


So there is a very clear physical understanding of the notion of the instantaneous velocity and for the sine function they have also arrived at the formula for the instantaneous velocity in terms of the cosine punch. Again another principal of calculus that when some quantities are maximum its derivative vanishes. This is implicit in the kind of discussions that Bhaskara does.

So here he is talking of the velocity correction. It vanishes when the equation of center is maximum. So this is again in the Siddhanta Sironamni and that the derivative has to change sign. When something becomes a maximum the rate of change of that quantity changes from positive to negative and it becomes zero.

At some point all this is explicit in Bhaskara's commentary of this principal. (FL) that the velocity correction moves from positive to negative then the equation of center when the correction is maximum so that is the kind of principal that Bhaskara talk. So when you take your astronomy seriously then in no way in which the principals or instantaneous velocity and related notions they will have to appear in your calculations and discussions.

(Refer Slide Time: 23:33)

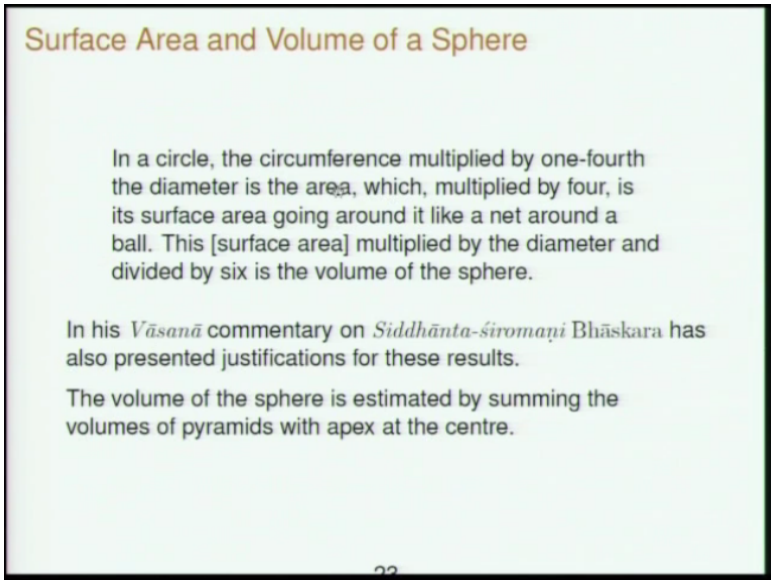


Finally, the area and volume of a sphere. So here we should all know that Aryabhata gave a formula which was incorrect. He said that the product of the area of the great circle by its square root is the volume of a sphere. The r cubed factor is right, but the coefficient is not correct. It is understood that in the way Sridhara and others between the time of Aryabhata and Bhaskara use this formula.

They were perhaps aware of 4 pi 3 as the kind of factor. their pi was used are infinite by assuming that they are giving the volume of a sphere as 4 pi/3 r cubed, but it is Bhaskara II in 1150 in Indian tradition he is giving and explicit correct statement for the surface area in the volume of a sphere.

So all these arguments that Indian tradition was influenced by Greek and we borrowed this and that we will become very suspicious when given a basic quantity like the volume and surface of the sphere which was well know by the time of Archimedes they were not borrowed by the Indians.

(Refer Slide Time: 24:48)

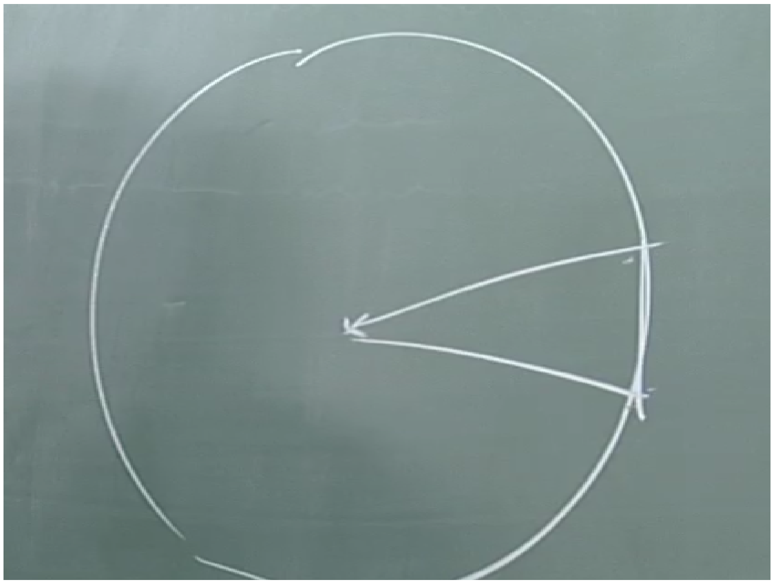


So in a circle, the circumference multiplied by one fourth the diameter is the area which multiplied by four its surface area going around like a net around a ball. This surface area multiplied by the diameter and divided by 6 is the volume of the sphere. So this is the verse in Lilavati which was discussed in the class (FL). So this is the four pi r square. (FL) this is for the volume.

Now in the Vasana commentary on Siddhanta Sironamni, Bhaskara placed to give an argument for this both the formula a kind of a proof. I think the proof for the volume was both proves were summarized in the lecture we will just recall a little bit in the lecture on the Lilavati. The proof for the volume of the sphere involved thinking of the surface of the sphere being covered it.

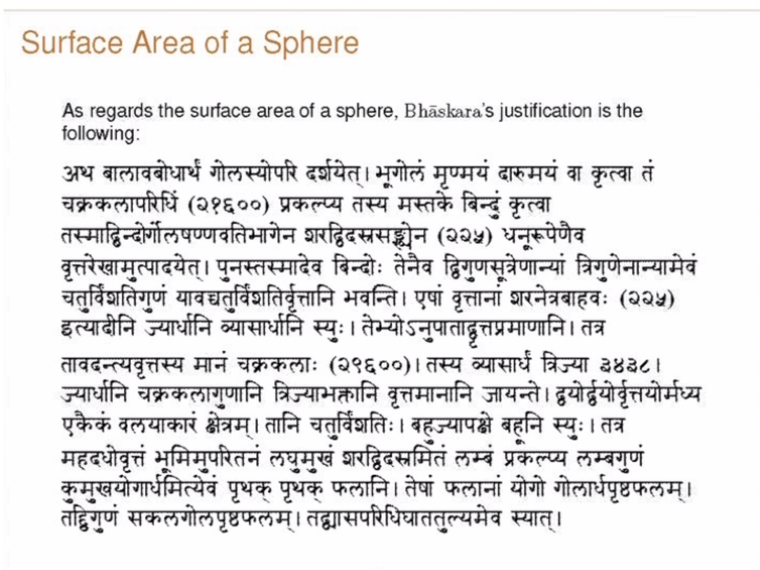
By various pyramids which have their vertex at the center of the sphere and assume that the volume of the sphere is approximated by the sum of the volumes of the pyramids and so in the limit you will just get it to one third times radius times the surface are of the sphere so that it is how the volume of the sphere is obtained.

(Refer Slide Time: 26:20)



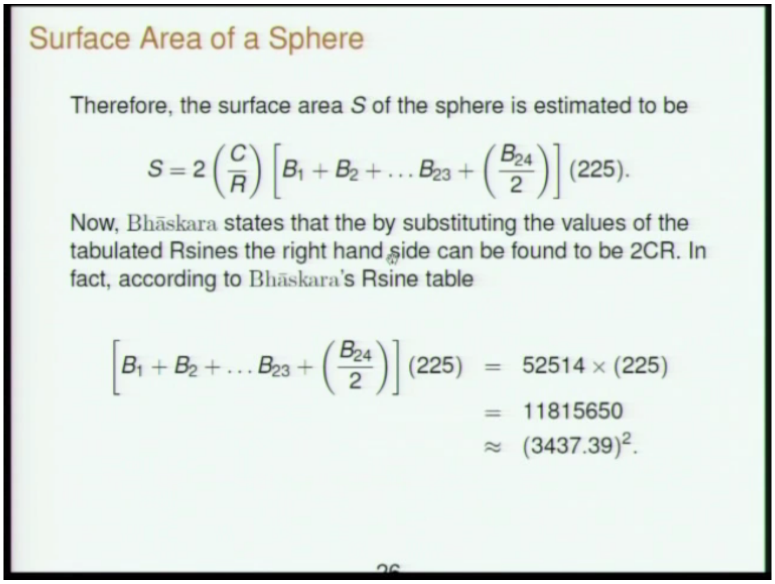
This is very similar to the so for both the area of a circle you think of it as begin covered by triangles like this and for the volume of the sphere you think of it being cover it by pyramids like this you will get the correct value of the volume of the sphere of area of the circle. In fact, this kind of an argument is even reproduced in the (FL) so it is considered valid argument for estimating the volume of a sphere.

(Refer Slide Time: 26:48)



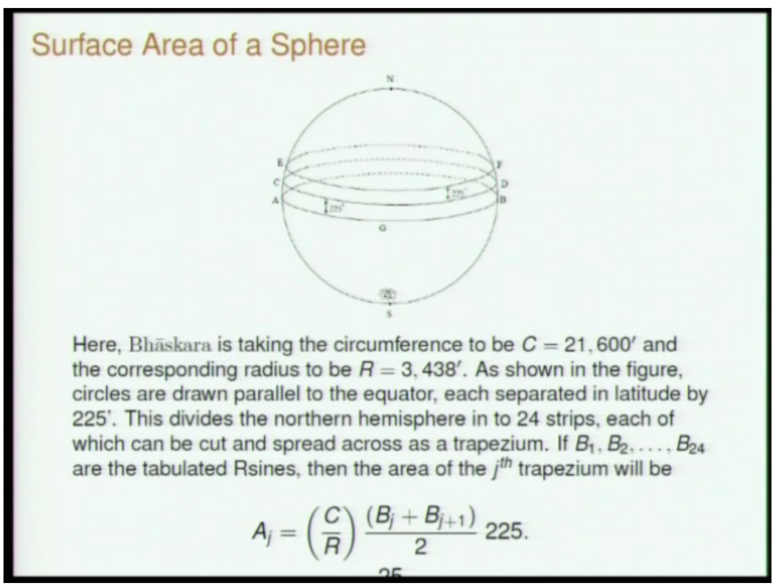
Now as regard the surface area of a sphere Bhaskara says that we have to give an argument especially for those who are unable to comprehend the result straight away and so he gives a argument which is essentially.

(Refer Slide Time: 27:05)



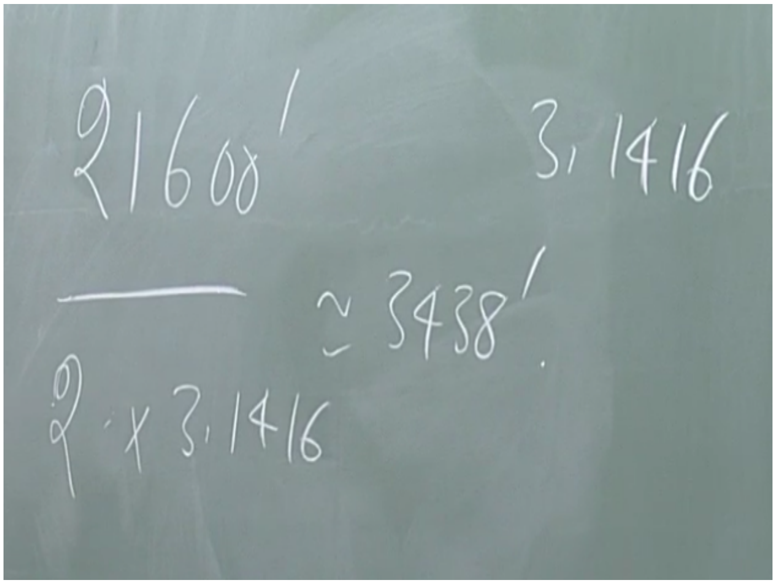
It was summarized in the lecture there, but just to recollect this because this point may be important so.

(Refer Slide Time: 27:09)



Bhaskara what he is doing is he is considering the northern hemisphere of the sphere and of course the circumference if 21,600 minutes then correspondingly with Aryabhata's value of pi the radius turns out to be 3,438 minutes. IF you take the circumference.

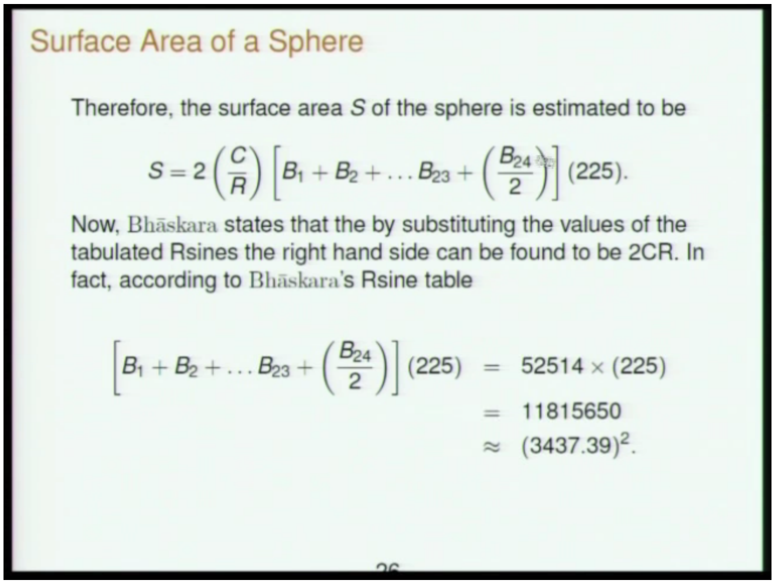
(Refer Slide Time: 27:27)



To be 21,6000 Aryabhata value is 3.1416. So you take twice pi here. So this will come out to the 3,438 minutes. So for the radius whenever you use 3,438 minutes it means you are using roughly Aryabhata's value of pi. Now so you draw circles parallel to the equator and so divide the sphere into various strips like this and cut the strip and stretch out its area.

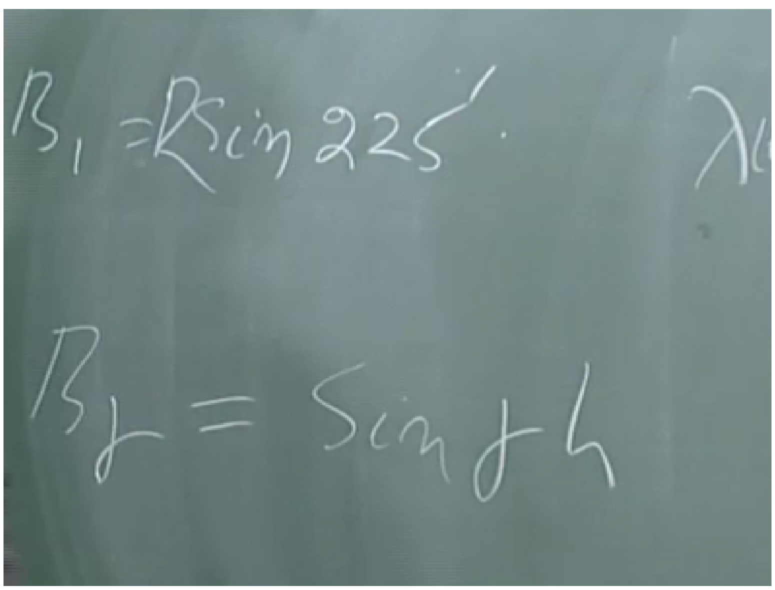
So each of them will look like a trapezium and the height of the trapezium the radius of each of those circles will be the Rsines at the corresponding angle, the height of the trapezium will be constant it is just 1/24th of the arc length from top to bottom so you have this Aj, the area of the jth trapezium written the top side and bottom side/2*the height 225 minutes*C/R, C/R is the ratio to get take the radius at the appropriate level.

(Refer Slide Time: 28:44)



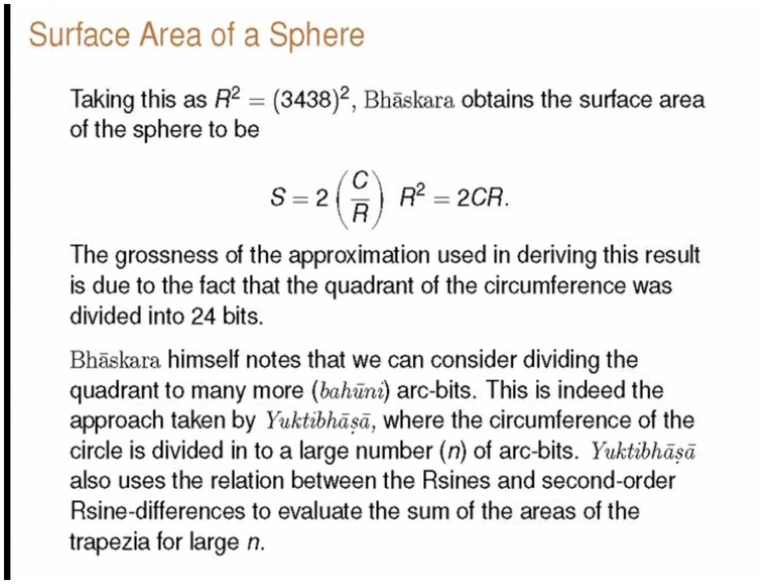
So the surface area of the sphere is estimated to be this. Now Bhaskara says he will do a numerical summation of these Rsines. So Bhaskara has a sine table. B1, B2 are all as I said this is Bj.

(Refer Slide Time: 28:58)



So B1 is sine of so Rsine of 225 minutes where the radius is 3438. So Bhaskara says let me just numerically sum the sines so once the numerically sums this signs he gets a value like this. using his own sign table so you will get a value like this it is roughly 3437.39 square.

(Refer Slide Time: 29:26)

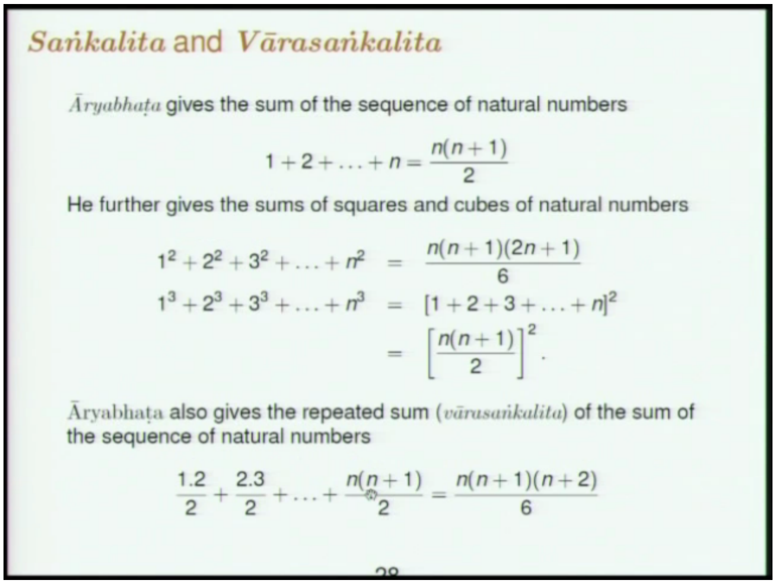


So Bhaskara says of course do not see where this is R square equal to 3438 square and therefore I have proved that the surface area is twice the circumference times the radius. Now of course the grossness is there in of approximation in deriving it. It is not that Bhaskara was not aware of the grossness of the thing if you go back to his statement (FL) so divide this into 24 strips. (FL) So if you want greater accuracy you can divide the upper hemisphere into as many bits as you want.

So it is not that he is not aware of the face that he is making an approximation, but making an approximation to convince you that this is the result he knows the result and this make an approximation to so he is doing a numerical integration to enable you obviously so if you want to be serious about this argument you need a more sophisticated argument.

So these indeed the kind of approach given in Yuktibhasa where the circumference of the circle is divided into a large number n of arch bits and then Yuktibhasa it is like doing the interval sine theta, d theta, cos theta, but Yuktibhasa uses the relation between R sines and the second order R sine differences to straight away evaluate that some in the limit.

(Refer Slide Time: 30:51)

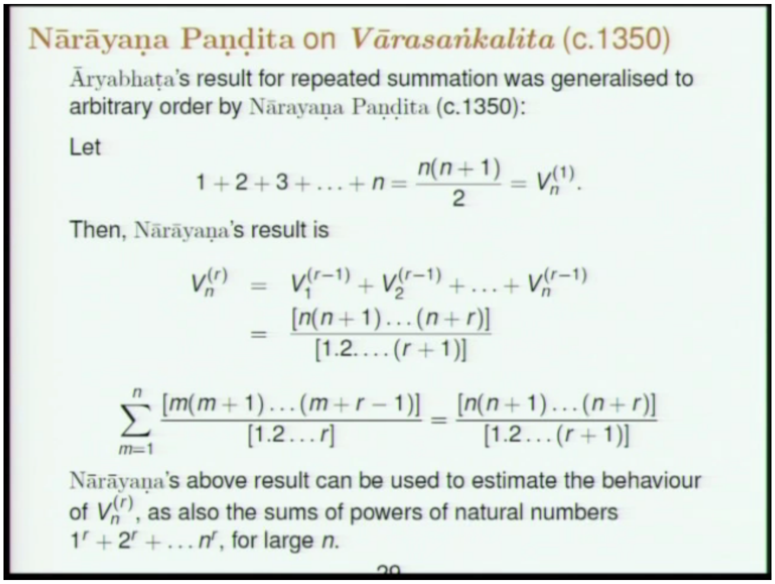


So lastly about the development of calculus in the pre Kerala era the topic we wanted. "Professor - student conversation starts" They provide volume because all their measures are variant in Rsine so it will be beneath the Rsine the linear units are now minutes so minute squared and minutes cube will be the corresponding one so they have separate area units and volume units which are given in the first chapter the (FL).

But whenever you are dealing with circle the circumference etc are measured in minutes of circumference "Professor - student conversation ends." So again the standard results which we keep seeing in perhaps the next lecture and the lecture after let us. 1 to n is n*n + 1/2. 1 square plus two square plus etc n square is n*n + 1*2*n + 1/6 then the sum of the cubes these two results will well known in the Greek literature.

Also this cube perhaps is not known outside India and in Aryabhata's book 1 + 2 +3 + n squared then these repeated summations of this sum of 1 to 2 is n*n + 1/2 what is sum of n*n + 1/2 that is what is 1*2/2, 2*3/2 etc. n*n + 1/2 this is n*n +1*n + 2/6. So all these are there in Aryabhatia.

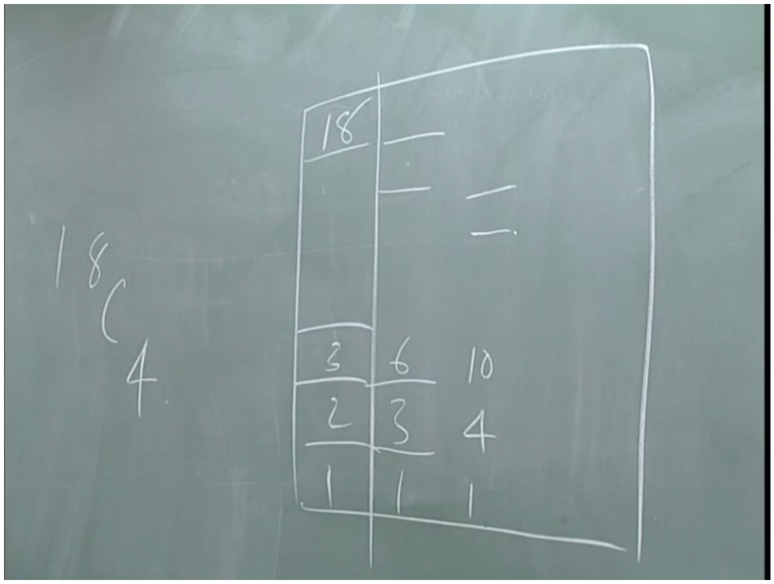
(Refer Slide Time: 32:19)



So now as was emphasizing the lecture on (FL) it has a nontrivial result which is an extension of the earlier result it is extension which is of an arbitrary order. So Narayana gives this expression for (FL) from which he will works out the famous cow problem. So what is Narayana Pandita's formula called this summation of 1 to n as (FL) of order one. The (FL) of order R is given by sum of 1 to n of (FL) of order R – 1.

And Narayana Pandita's says that this is given by n*n + 1 etc*n + r divided by r + 1 factorial or if you write it in extent so it will be like this. Now it is important to realize that Narayana Pandita's is not pulling this result out of a hat. This result I perhaps told you while discussing the way (FL) is expressing the number of combinations of n things or things selected from n. He has a (FL) and what was (FL).

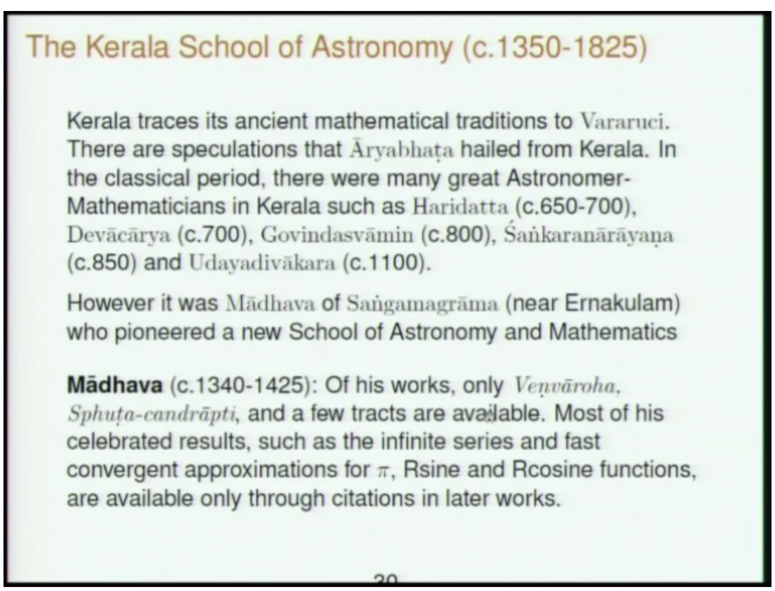
(Refer Slide Time: 33:41)



For 18C4 what did (FL) say he said write 1, 2, 3 etc up to 18 and in the next column you write sum of this 1 then 3, 6, etc and come up to this point and the next column will be 1, 4, 10 etc and come up to a column so sum of sum, sum of sum of sum and the fourth sum and he showed you that the fourth sum gave you 18C4 and that is essentially Narayana Pandita's relation.

So there is really nothing new in Narayana Pandita's relation he is writing it in of course in a as a formula of order r so Varasankalita of order r, but I think this idea is fairly implicit in Indian mathematics from earlier times. Of course this plays a crucial role in the development of calculus as we are going to see later.

(Refer Slide Time: 34:44)



Now we will say something about the Kerala School of astronomy. So it is not that astronomy started in Kerala with Mhadava or any such thing Kerala was sparely well known Vararuci, the author of the famous (FL) I said to be from Kerala many scholars speculate especially those from the Kerala region that Aryabhata headed from Kerala and then it is well known that these was very well known astronomer, mathematicians like Haridatta.

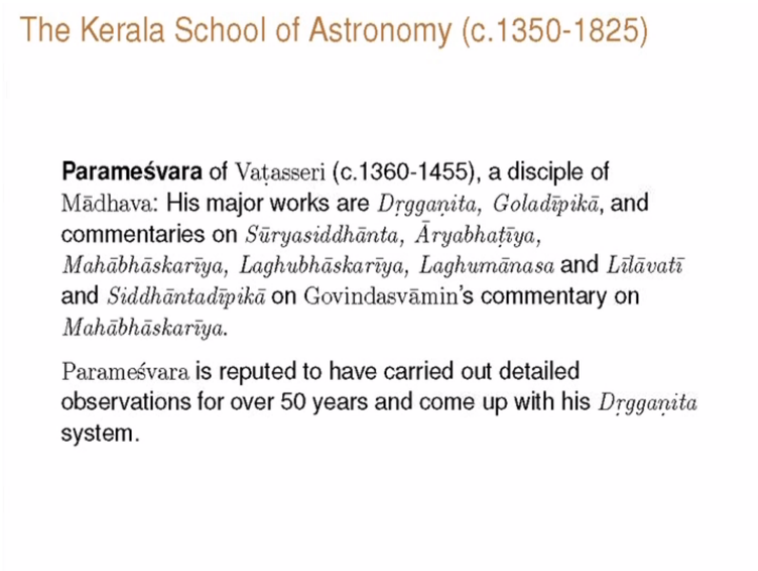
He started the (FL) system of astronomy which is famous in Kerala even now, Devacarya, Govindasvamin who wrote a commentary on the (FL) of Bhaskara1. Sankaranarayana, who wrote a commentary on the (FL). So as you can say the Aryabhata tradition he is popular in Kerala much prior to even the Madhava period and Udayadivakara whose name came up in the context of (FL) that his commentary(FL) is where the (FL) versus on (FL).

So there have been various mathematicians, astronomer. There is a talk of a observatory in I think Sankaranarayana's commentary on (FL) he talks of the observatory in (FL), but it was Madhava who started new school of astronomy he said to hail from Sangamagrama near Ernakulum. Now none of Madhava's work and you can see Madhava is roughly a contemporary of Narayana Pandita slightly junior contemporary is around till 1425.

Because of the dates available in (FL) you can see something like that. None of his works are available except two tracts both deal with calculation of the instantaneous position of moon a better method for calculating the instant. It improves the (FL) and also it gives you a way of calculating the longitude of the moon exactly at eight different points in a given day without doing any approximations.

It is some interesting methodology so you can see that this Madhava must have been a great mathematical sort of manipulator and genius even by looking at (FL). Now, but what he is more famous is for the various results on infinite series for pi fast convergent transformed series for pi, series for sine and cosine and many formulas for calculating sine and cosine many approximations and all these sites are attributed to him so (FL) are sited in later works. We will see the names of the later scholars so that is how he is much more famous Madhava of course is called (FL) by later scholars.

(Refer Slide Time: 37:39)



That he was supposed to be a specialist on spherics. (FL) is a direct disciple of Madhava he is in Vatasseri it is in the point where the river (FL) or what is called (FL) today meets the sea. His major works of Drgganita. Goladipika he supposed to have started new Drgg system of astronomy.

What we call as (FL) even today and he wrote large number of commentaries on various earlier works and he wrote a very detailed commentary on Govindasvamin's commentary on Mahabhaskariya a super commentary called Siddhantadipika. So Nilakanta says Paramesvara carried out detailed observations for over 50 years and came up with this Drgganita system. Then comes (FL).

(Refer Slide Time: 38:31)

The Kerala School of Astronomy (c.1350-1825)

Nīlakaṇṭha Somayājī of Tṛkkaṇṭiyūr (c.1444-1555), student of Dāmodara (son of Parameśvara): He is the most celebrated member of the Kerala School after Mādhava. His major works are *Tantrasaṅgraha* (c.1500), *Āryabhaṭīyabhāṣya, Golasāra, Candracchāyāgaṇita, Siddhāntadarpaṇa, Jyotirmīmāṃsā* and *Grahasphuṭānayane Vikṣepavāsanā.*

In *Tantrasaṅgraha*, Nīlakaṇṭha presents a major revision of the traditional planetary theory, which, for the first time in the history of astronomy, gives a correct formulation of the equation of centre and the motion in latitude of the interior planets. In his later works he discusses the geometrical picture corresponding to his modified planetary theory, according to which the five planets, Mercury, Venus, Mars, Jupiter and Saturn go around the mean Sun, which itself goes around the earth.

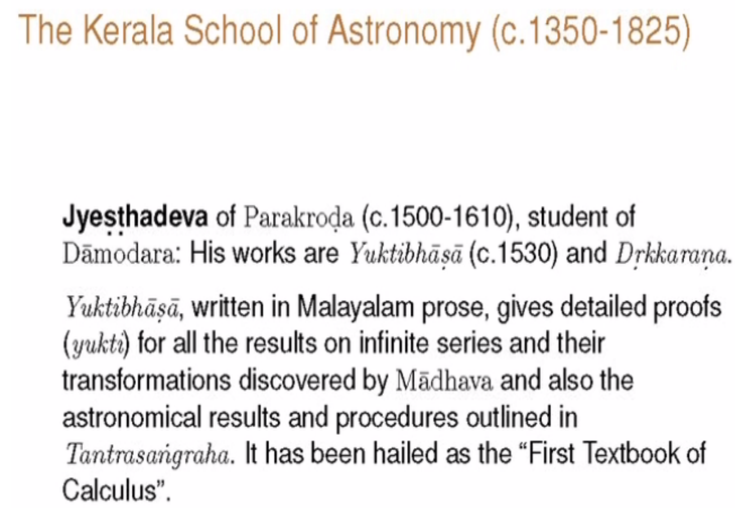
Who is perhaps amongst the most well known members of the Kerala School after Madhava. He is the student of Damodara who was son of Paramesvara. Nilakantha Somayaji of Trikkantiyur which is near (FL) in just border of Mallapurum district is the most celebrated member of the Kerala School after Madhava. His major words of Tantrusangraha which is written in 1500, the commentary on Aryabhatia.

Small works called Golasara, Candraccjayaganita Siddhantadarpana, Joytirmimamsa which is a discussion on the methodology of astronomy that what is a theory, what is observation how are they to be related what is the purpose for discussing a (FL) text and all that. A very small track called Grahasputanayane Viksepavasana. So apart from his mathematical equipment Nilakanta he presents major revision of the older planetary theory of the Indian astronomy.

In general, and in that he changes the formulation of the equation of center for the interior planets and so he comes up with a correct formulation of both the motion and latitude and the equation of center for interior planets something which has not been done either in Indian tradition or in European tradition or in Islamic tradition prior to his time.

So this was the major departure from the traditional models of astronomy so that it is given in Tantrusangraha written in 1500. In his later works he also discusses what is the model of planetary motion, geometrical model so where he speaks on the five planets Mercury, Venus, Mars, Jupiter, and Saturn go around in the mean Sun which itself goes around the earth.

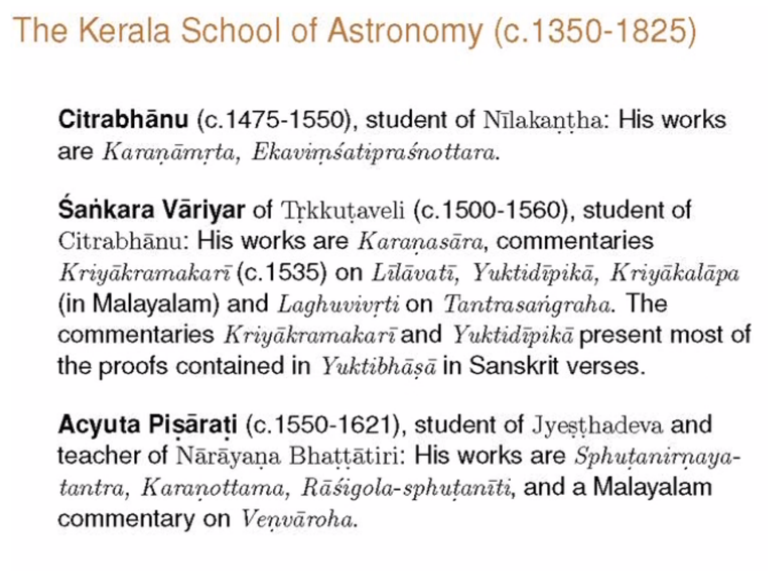
(Refer Slide Time: 40:17)



So he is famous for that, but the person whose work we are going to discuss in great detail is Jyesthadeva who is also a student of Damodara so is a junior contemporary of Nilakanta. His main work is Yuktibhasa or (FL) which is written in Malayalam. He has written a small astronomical tract called (FL). Now Yuktibhasa is of course it details with yukti and is in written in the (FL) or the language that is spoken.

And it gives the proof for all the results on infinite series and all the astronomical results and procedures which are discussed in Tantrasangraha so it is generally now today it has been hailed as the first text book of calculus.

(Refer Slide Time: 41:06)

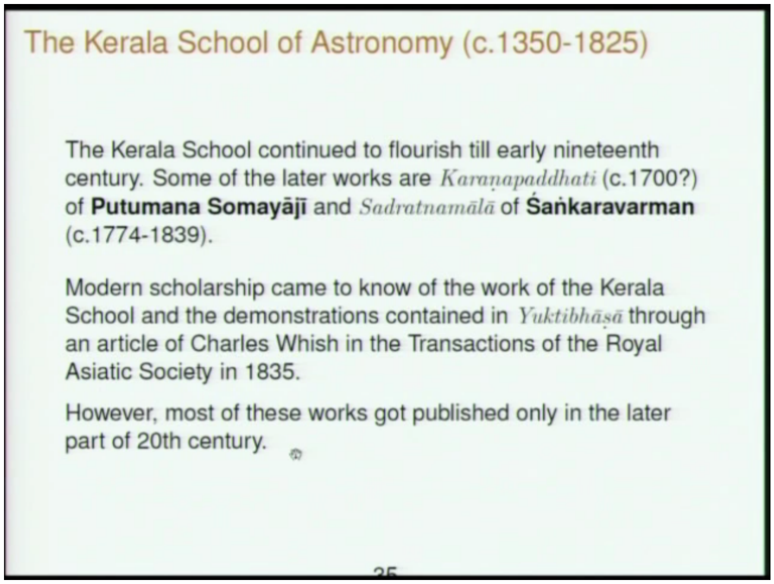


Citrabhanu is a student of Nilakantha. This (FL) is actually a generalization of the (FL) so given two sort of combinations of A and B how do you find A and B so this (FL) 21 such possibilities. You give A + B, A - B, A square + B square and A - B and like that (FL) is a work like that. (FL) is an astronomical manual. Sankara Variyar is very famous. His mantra Trkkutaveli because he rewrote what was given in (FL) in Sanskrit in the (FL) commentary has rewritten much affect in pros.

And in the commentary called Yuktidipika on (FL) he has re-written it in verse and so this something that is accessible to the (FL) tradition of mathematics and astronomy because (FL) text book was available in Malayalam only. Acyuta Pisarati the student of Jyesthadeva and he is more well known as the teacher of the famous author of Narayana Bhattatiri. Narayana Bhattatiri as you know was a great scholar of grammar also. So Acyuta Pisarati also written a book on grammar.

He is actually from Trkkutaveli. His verses are put in (FL) and a Malayalam commentary on Madhava Venuaroha.

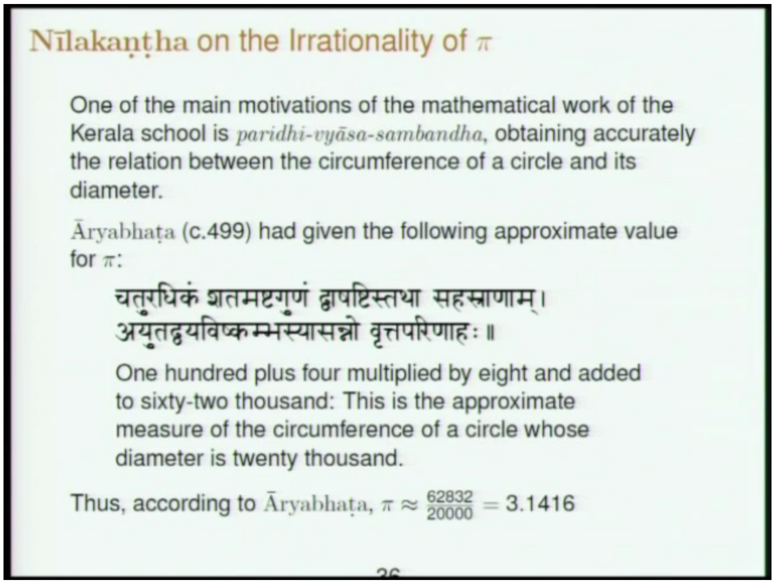
(Refer Slide Time: 42:33)



Now till (FL) the direct line from Madhava is stressable and the activity is continuing with great intensity. It is sort of (()) (42:42) mostly by the political situation of Kerala which started with lot of (()) (42:50) with the Dutch and the Portuguese playing a major role. In the Kerala School there is a work called Karanapaddhati by Putumana Somayaji both his place and time are not fully established still.

Of course, in 1820 raja Sankaravarman of (FL) wrote a work called Sadratnamala and when this Charles Whish who sort amongst the first modern scholars to write about the Kerala School in an article in 1835 raja Sankaravarman was alive at that time. so many of these four to five Kerala works were mentioned in the article of Whish, but most of this works got published only in the later part of 20th century.

(Refer Slide Time: 43:37)



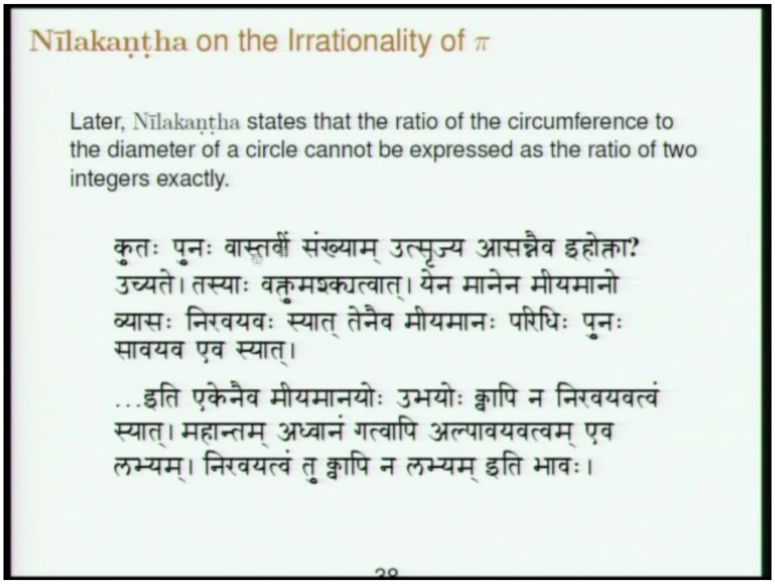
So few topics of the Kerala School we will discuss now the rest we will discuss in the next lecture on calculus. So the first is the issue of irrationality of pi and best discussion of this is found in the commentary of Nilakantha on Aryabhatia. So the verse of Aryabhata clearly says (FL) so its approximate value of the circumference when the diameter is 20,000. When the diameter is 20,000 62932 is an approximate value of the circumference.

So the value of pi is approximate. So Nilakantha in his detail commentary of Aryabhatia while discussing the rule of square root the algorithm for square root itself he explains that the square roots are not quantities which can be fixed that the procedure for square root of a nonsquare quantity is something that you can keep on carrying on for obtaining better and better approximations only. So while describing that he says (FL).

So even when you multiply a number by a large power of 10 calculate the square root and divide it by the square root of that power of 10 so you keep doing that (FL) So the true value of a (FL) a square root of a nonsquare quantity can never be determined and then it goes on later on to say in the same context that (FL). So Aryabhata is later on going to say this one Nilakantha is saying that he is going to give you an approximate value of the circumference for a given value of diameter. (FL).

So in order to calculate the circumference given the diameter there has to be series of inferences, a series of calculations you have to make. (FL) even while doing it this square root operation keeps coming repeatedly and therefore only the fact that the circumference can only be given approximately. (FL) So we will speak about it at that time.

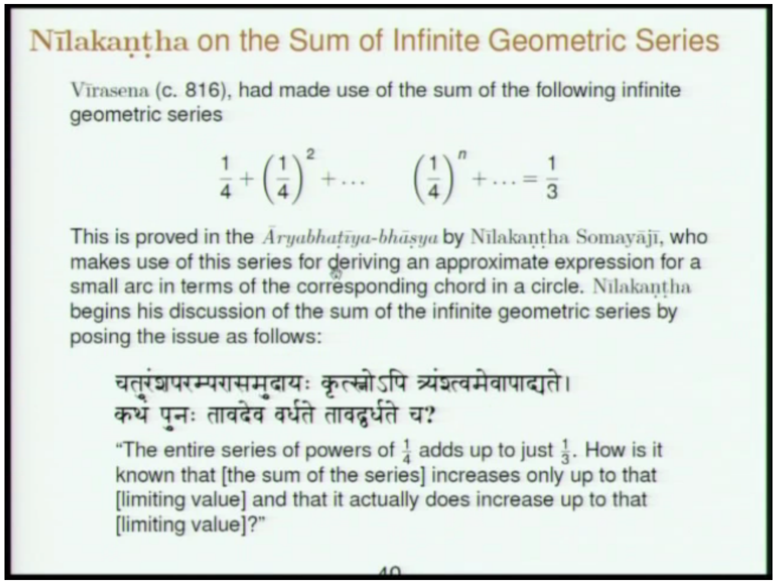
(Refer Slide Time: 46:02)



So when he comes to the Aryabhata verse Nilakantha makes the clearest statement in Indian tradition of the irrationality of pi. so basically he is saying why is Aryabhata giving only the approximate value because the exact value cannot be given by whatever unit you measure the diameter to be an integer in the same unit the circumference will not be an integer.

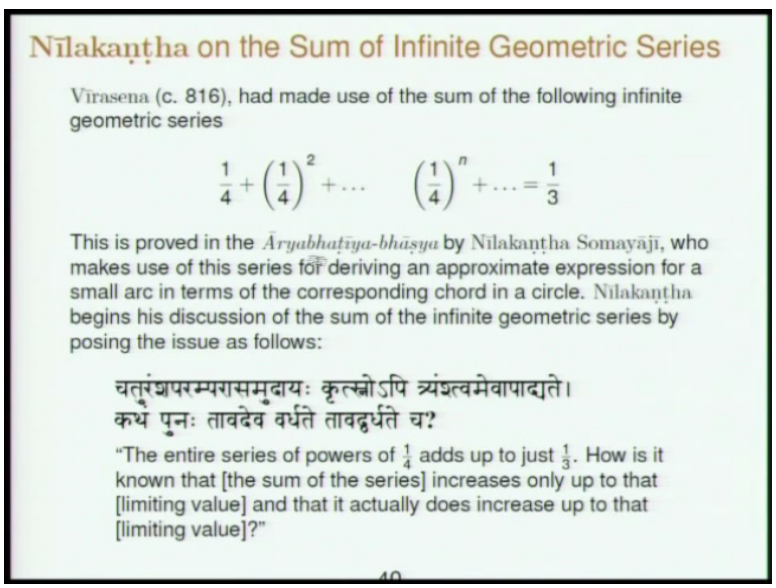
And this will be so however smaller unit you may go to or whatever unit you measure the circumference to be an integer in the same unit the diameter will not be an integer the same will be so whatever unit you may go to that is the meaning of this is the translation of that.

(Refer Slide Time: 46:48)



Now another important discussion by Nilakantha is on the infinite series. The geometric infinite series as I said the series itself is quite simple fairly well known in history.

(Refer Slide Time: 47:03)



Now Nilakantha is provided a detail discussion of the series. While doing something else he is trying to find the approximate value relation between the chord and the arc in a circle. so he is saying (FL) that is (FL) is 1/4, (FL) that plus that square that to the power cubed etc. (FL) the entire (FL). This entire family of all powers of 1/4 added together will become one third so then the question is (FL) how can you assert that it will go only up to one third and how can you assert that it will reach one third.

So this is the question that how do you understand an infinite series and he has posted it and then he tries to explain it the way that more or less the way we understand infinite series.

(Refer Slide Time: 48:03)

Nīlakaṇṭha on the Sum of Infinite Geometric Series

Nīlakaṇṭha obtains the sequence of results

$$\frac{1}{3} = \frac{1}{4} + \frac{1}{(4.3)}$$
$$\frac{1}{(4.3)} = \frac{1}{(4.4)} + \frac{1}{(4.4.3)}$$
$$\frac{1}{(4.4.3)} = \frac{1}{(4.4.4)} + \frac{1}{(4.4.4.3)}$$

and so on, from which he derives the general result

$$\frac{1}{3} - \left[\frac{1}{4} + \left(\frac{1}{4}\right)^2 + \ldots + \left(\frac{1}{4}\right)^n\right] = \left(\frac{1}{4}\right)^n \left(\frac{1}{3}\right)$$

So he gives a sequence of results one third is one fourth + 1/4*3. 1/4*3 is 1/4*4*1/4*4*3. Like that and so from this you can have a result like that. One third - sum of n terms in the series is 1/4 to power n*one third. So now this is the way we understand it today also the limit minus sum of n terms in the series will become smaller and smaller and n becomes larger and larger and larger and so that is what essentially Nilakantha is saying.

(Refer Slide Time: 48:39)

Nīlakaṇṭha on the Sum of Infinite Geometric Series

Nīlakaṇṭha then goes on to present the following crucial argument to derive the sum of the infinite geometric series: As we sum more terms, the difference between $\frac{1}{3}$ and sum of powers of $\frac{1}{4}$ (as given by the right hand side of the above equation), becomes extremely small, but never zero. Only when we take all the terms of the infinite series together do we obtain the equality

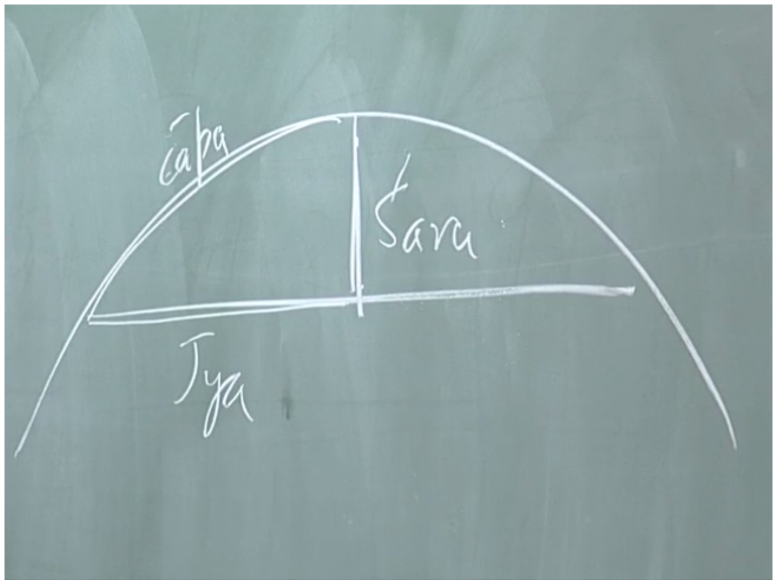
$$\frac{1}{4} + \left(\frac{1}{4}\right)^2 + \ldots + \left(\frac{1}{4}\right)^n + \ldots = \frac{1}{3}$$

Incidentally, Nīlakaṇṭha uses the above series to prove the following relation between the *cāpa* (arc), *jyā* (Rsine) and *śara* (Rversine) for small arc:

$$C\bar{a}pa \approx \left[\left(\frac{4}{3}\right) \acute{S}ara^2 + Jy\bar{a}^2\right]^{\frac{1}{2}}$$

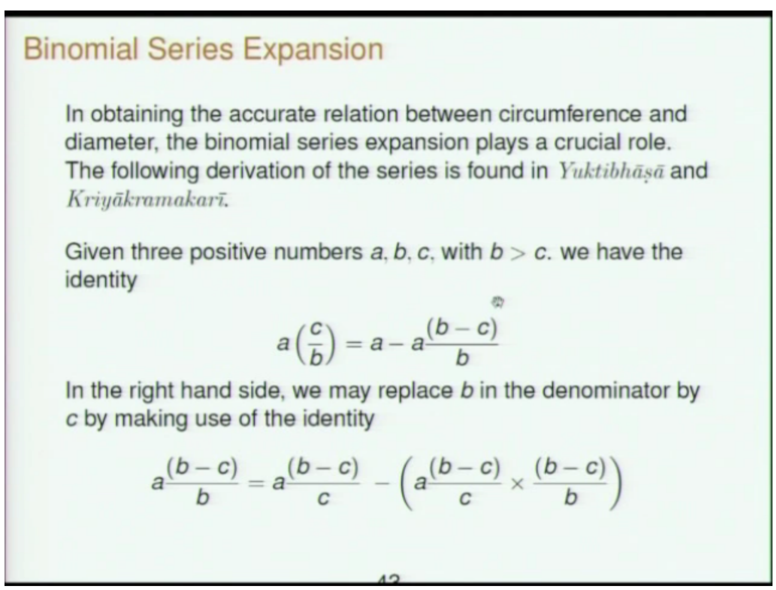
(FL) then goes on to present the following crucial argument as we sum more terms the difference between one third and the sum of powers of 1/4 becomes extremely small, but never zero only when take all the terms of the infinite series together do we obtain this equality. Incidentally the relation Nilakantha was trying to do was something that is likely better than (FL) is equal to (FL).

(Refer Slide Time: 49:05)



So this is Sara this is Jya so it is just trying to obtain an approximation for the. So this is Capa so this is Jya and this Sara. Next is the binomial series.

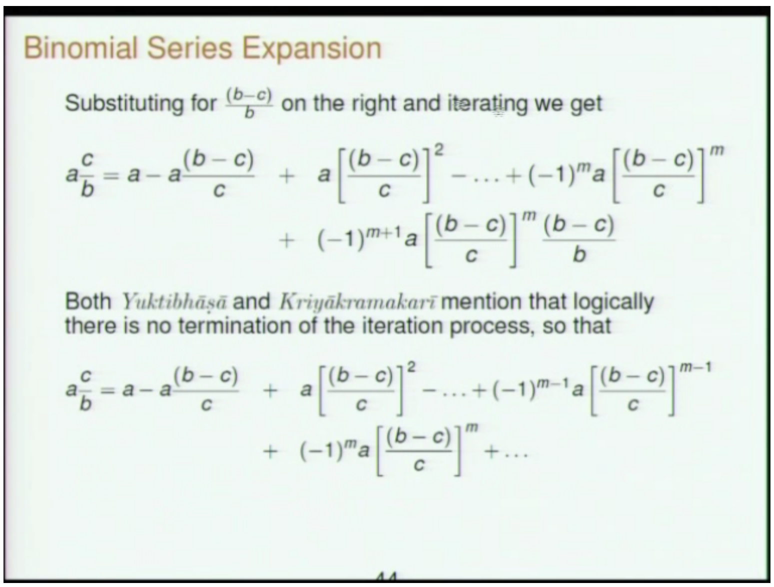
(Refer Slide Time: 49:23)



Again this is obtained by an iteration of an algebraic identity so it is obtained in a way because this is the context in which it appears in the astronomical context that is in the derivation of the Madhava series so let us try to see the binomial series in a complex way that is formulated. Let a, b, c be three positive number b > c. then you have an algebraic identity a*c/b is a - b -c/b. Now the question is this b - c/d can be replaced by b - c/c if you have something called (FL).

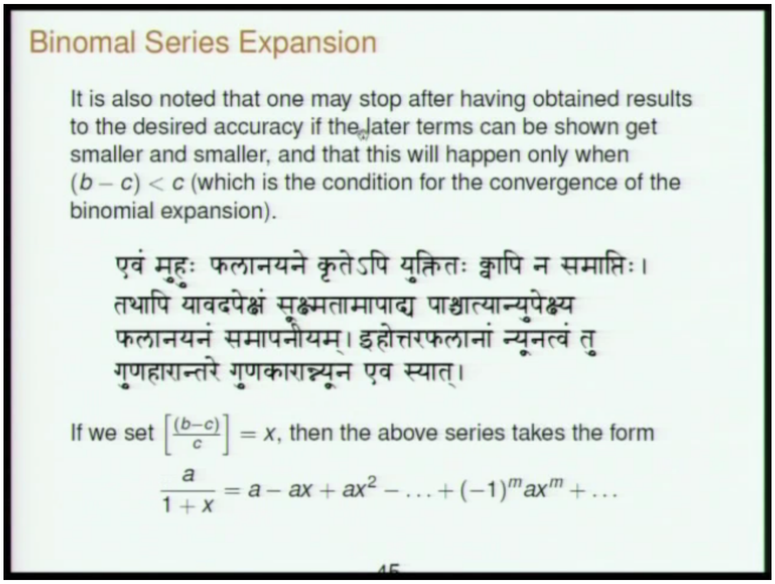
So this a*b - c/b can be written as a*b - c/c - (FL). Now on the right hand side again a b - c/b is appearing so for that again you can apply the (FL) replace it by b - c/c.

(Refer Slide Time: 50:20)



So you can iterate this b - c/b with this equation once again. You do it once you get like this in the end you still have b - c so this b - c/b up to end terms will come like this and now they say this iteration process can be we can go on indefinitely and so you can write an infinite series.

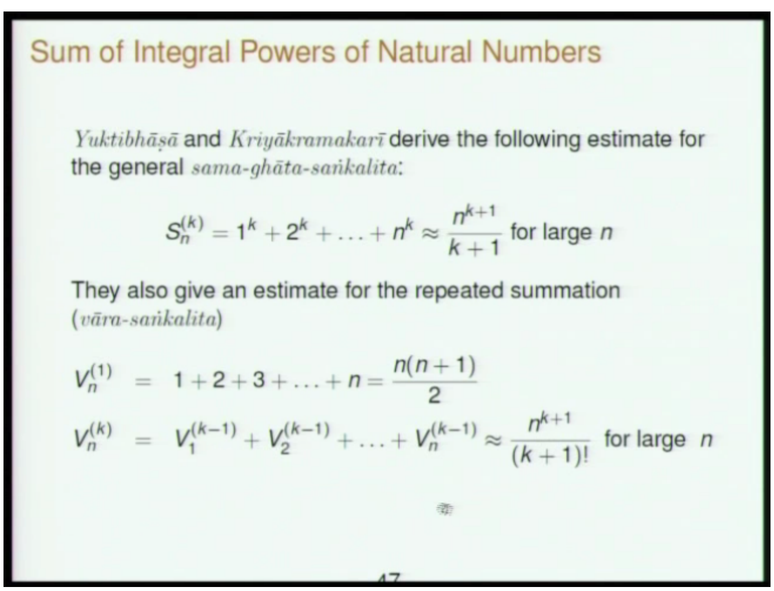
(Refer Slide Time: 50:41)



So the crucial thing is that the condition for the convergence of this series is also noted and the fact that the way the condition is understood. (FL). So (FL) is using the (FL) going from b - c/b to b - c/c. So (FL) means repeatedly if you do this iteration it will never stop and you can never say that it is going to stop. (FL) So what he obtained the kind of accuracy that you want. (FL) forget about the succeeding terms (FL) you can stop the calculation. (FL) so the succeeding terms will become smaller and smaller only if b - c < c.

So basically the kind of series that they are talking about this standard binomial theories that we know.

(Refer Slide Time: 51:49)



Lastly, the next thing that comes in (FL) is the estimation of the sum of powers of integers and estimation of the repeated sums of integers. both the crucial quantities this will appear in our derivation of the Madhava series for pi this will appear in the derivation of the sine series and both these are extremely interesting estimates and they play a crucial role in the history of calculus the world over.

So both (FL) and (FL) which are words of early 16th century, they are discussing this these topics were discussed in Europe in early 70 and mostly in later. In 17th century so this sum 1 to the power k + 2 to the power k plus n to the power k when n is large goes like n to the power k + 1/k +1 that is the dominant behaviour of it for large n is proved both in (FL) and (FL) and this repeated summation it goes like n to the power k + 1/k + 1 factorial for large.

Then is also proved there. Of course this second one can straight away be inferred from (FL) formula, but (FL) prepares to prove this again by mathematical induction which we will see in a later class. Again this one was well known for K = 1, 2, 3 but for a general k it had to be proved by mathematical induction only.

So these are the four kinds of results of the Kerala mathematicians that I have discussed in this part their understanding of pi, their understanding of convergence of infinite series, their understanding of binomial series and more than anything else their estimate of sums of powers of integers and estimate of repeated sums of integers. So all this will be used in the derivations.

(Refer Slide Time: 53:46)

References

1. *Gaṇitayuktibhāṣā* of Jyeṣṭhadeva (in Malayalam), *Gaṇitādhyāya*, Ed., with Notes in Malayalam, by Ramavarma Thampuran and A. R. Akhileswara Aiyer, Trichur 1948.
2. *Gaṇitayuktibhāṣā* of Jyeṣṭhadeva (in Malayalam), Ed. with Tr. by K. V. Sarma with Explanatory Notes by K. Ramasubramanian, M. D. Srinivas and M. S. Sriram, 2 Volumes, Hindustan Book Agency, Delhi 2008.
3. *Kriyākramakarī* of Śaṅkara Vāriyar on *Līlāvatī* of Bhāskarācārya II: Ed. by K. V. Sarma, Hoshiarpur 1975.
4. *Tantrasaṅgraha* of Nīlakaṇṭha with *Yuktidīpikā* of Śaṅkara Vāriyar, Ed. K. V. Sarma, Hoshiarpur 1977.
5. K. V. Sarma, *A History of the Kerala School of Hindu Astronomy,* Hoshiarpur 1972.

That (FL) does of Madhava results, but in the next class we will first discuss what are the results of Madhava which we understand as we think of him as a founder of calculus so let us see what are the kind of results that Madhava comes up with so that will be discussed in the next lecture of calculus.