Mathematics in India: From Vedic Period to Modern Times

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Lecture – 26
Ganitakaumudi of Narayana Pandita 2

So, we continue with Narayana Pandita, so in this second lecture and Ganitakaumudi, I will be talking about the meeting of travellers, then progressions and the very important topic the **(FL)** some of sums, kth sum, so then a Cow problem, which is an application of this, so then some progress in cyclic quadrilateral Narayana Pandita has made, so that is the diagonals of a cyclic quadrilateral.

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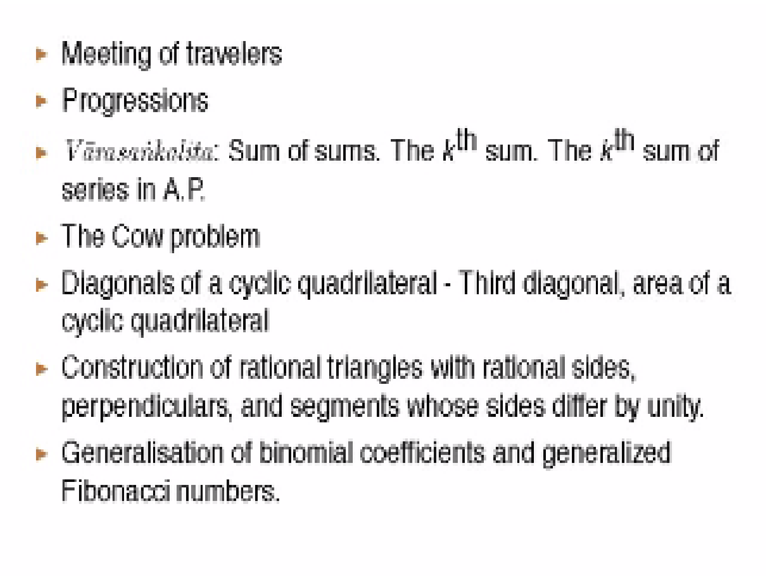

He introduces the concept of a third diagonal and so on, so then constructing of rational triangle with rational sides, perpendiculars and segments and then lastly, some generalization of binomial coefficients and generalize Fibonacci numbers.

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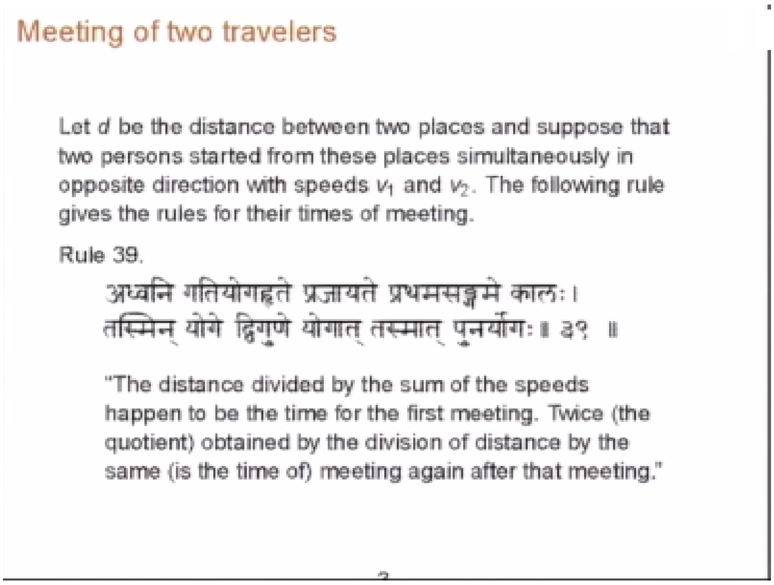

Suppose, 2 travellers are there; they start from 2 places; the distance between which is D, so then the 2 persons, they suppose they started from these places simultaneously in opposite directions with v1 and v2, so the following rule gives the rules for their times of meeting, so they will meet several times, so distance divided because some of the speeds happen to be the time for the first meeting twice the quotient obtained by the division of distance by the same is a time of meeting again after that meeting.

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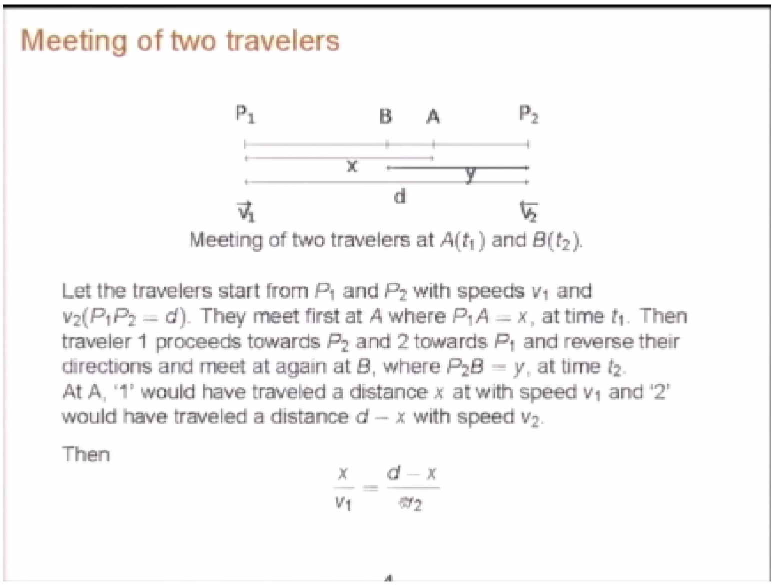

(FL) in the meeting again; second meeting, so it is illustrated in this figure, so one person is starting from here, other person is starting from here, so first they meet at; he is travelling with speed v1, this person with speed v2, so they meet let us say here first, so then clearly of this is x, so then this fellow is traveling distance x with speed v1, so x/v1, so that must be =, so this will be d – x; d – x/ 2.

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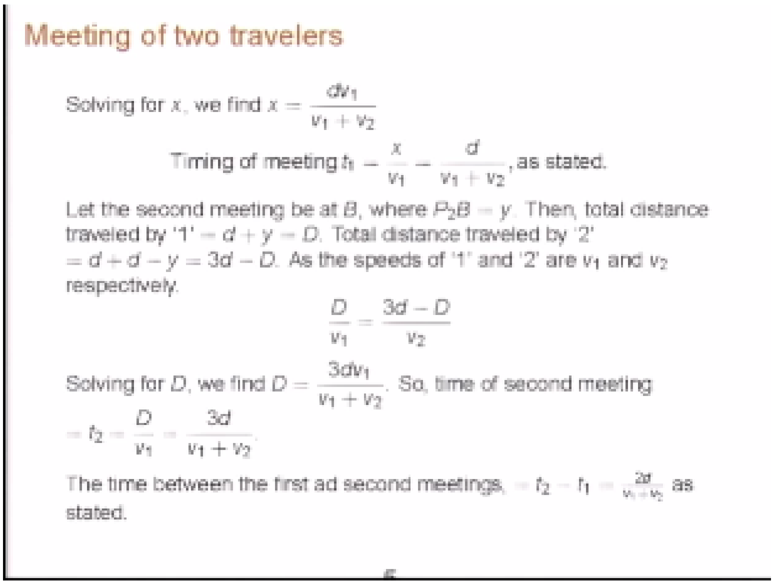

So, solving for x, we get dv1/ v1+v2, the time of meeting is x/v1 is v/ v1+v2, so that is this thing and the second meeting is at B, where you know again, so this fellow will proceed toward that and comes back. Similarly, he will also proceed and come back, so we can work out all these things, so the second meeting will be; it will be 3D v1/; 3D/v1+v2, that is from the beginning, so that will be the time of meeting.

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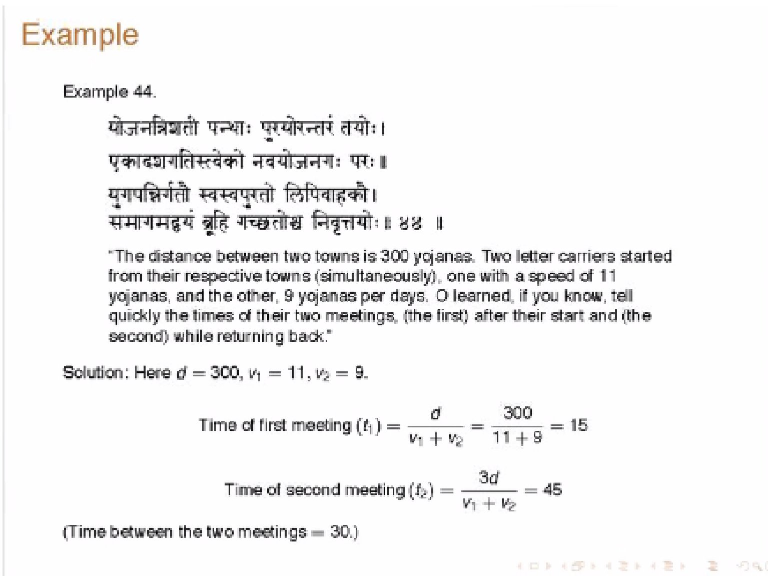

So, time between the first and second meetings will be 2D/ v1+v2, so that is a simple result. So, he has given some example for this, the distance between 2 towns is 300 yojanas (FL) so one is having the speed of 11 yojanas for some time per day, other 9 yojana (FL) so tell quickly the times of their meetings, so you can work it out; so time of first meeting is 15, then second is 45, after the; I mean from the beginning.

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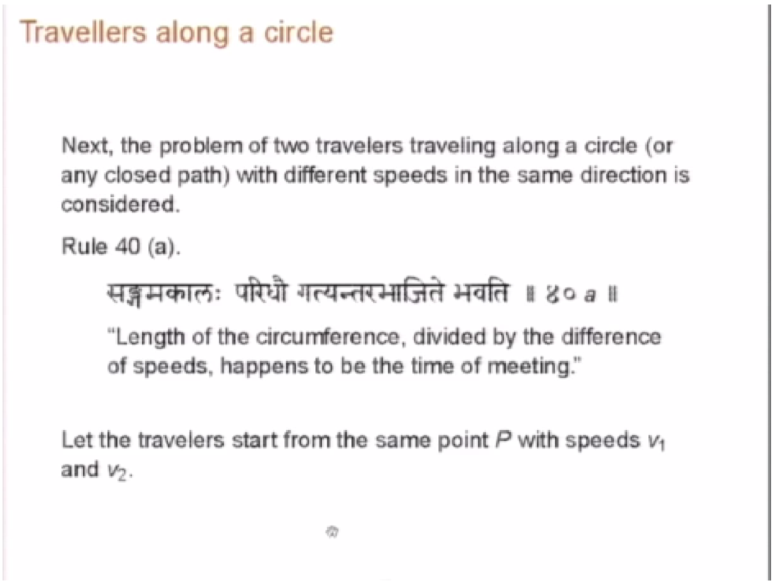

So, time between meetings is 30, okay. Then, traveling's along the circle.

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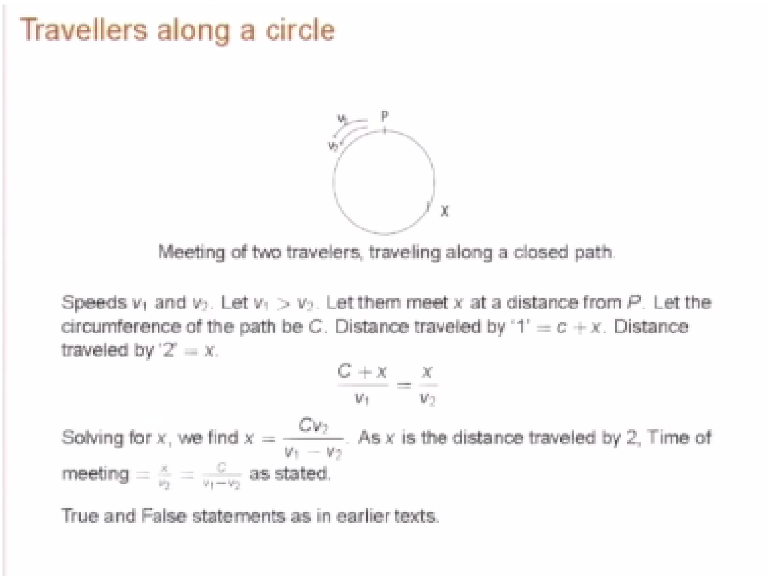

Suppose, 2 people are traveling in a circle like this, so then what is the; when do they meet okay, suppose they start with different speeds v1 and v2, so then see that v1 > v2, then suppose, let them meet at x at a distance from P, so that means if they are meeting that means the person was traveling fast, he would have completed one circle and then extra, you see and the second person will be meeting here.

So, the distance travelled by one, who is moving with higher speed is C+x, is this x and the distance travelled by 2 is x, so I have got C+x/ v1 is x/v2, so solving for x, we get x is = C v2/ v1 – v2 and time of meeting is C / v1 – v2, so this is you know, applied in; this is applied in astronomy, this called you know for to calculate what is the cyanotic period; see, suppose, if the planet you are seeing that it is moving around Earth as you observe it, so then they move with different speeds.

When are they are in conjunction okay, so that is what is seen here. For instance, Sun and Moon when they are; apparently they are moving which respect to the earth, right in a circle; 2 different circles but of course, does not matter with even though, they are different circle, the angular speed will be what is; what comes okay. So, when they are in conjunction, so then that will be this Amavasya or new moon day, then again how much is that.

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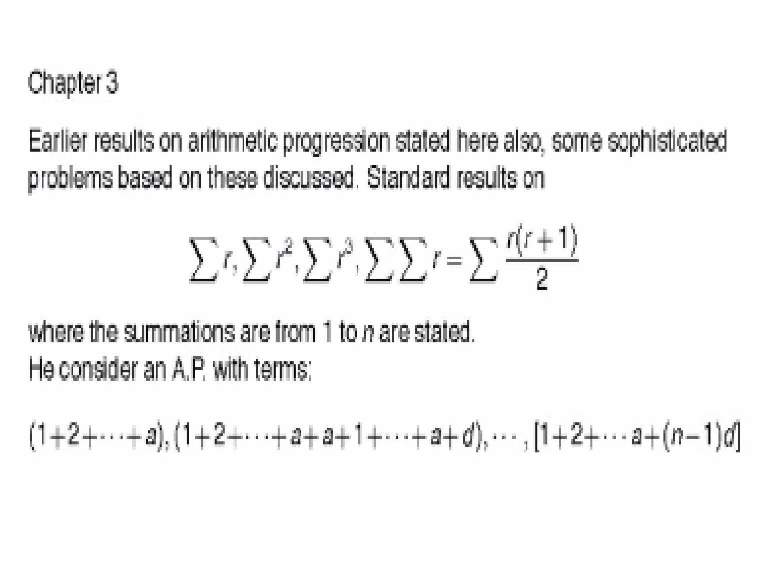

So, that they can calculate by circumference divided by difference in velocities, okay, so, next is an important result is doing, suppose you have got this, you know this first sum of the integer, second sum; sum of squared, sum of cube, so they are old results by the time Narayana Pandita wrote his work, it had been; they had been discussed threadbare in the earlier works and second sum is also; this second sum is also stated in Aryabhata onwards.

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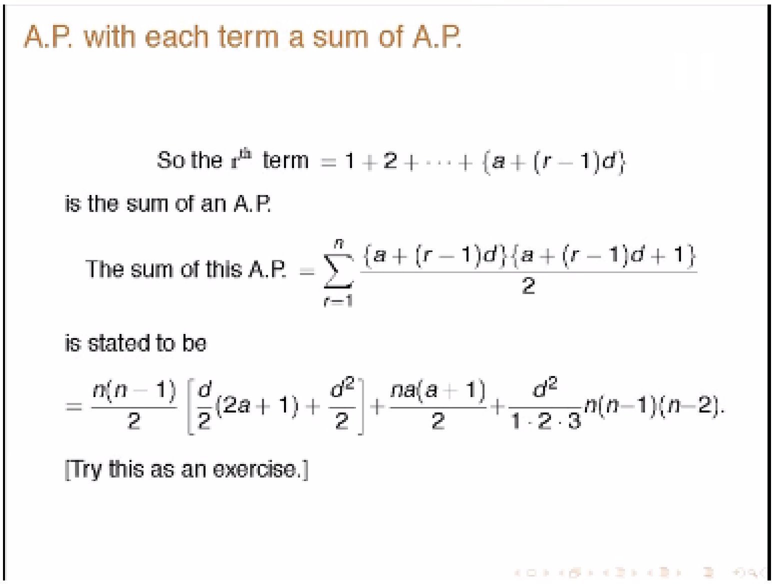

So, now we consider an arithmetic progression with these terms that is each term is a sum of an arithmetic series, okay, so the rth term will be this, 1+2 etc., a+r – 1*d, so sum of this arithmetic progression you see; you have to sum over this, r = 1 to n, so then you will get like this, so when he gives the; so slightly more general kind of a thing, you see, so from the sum of and sum of sums of integers, you are going to the arthritic progression kind of the thing.

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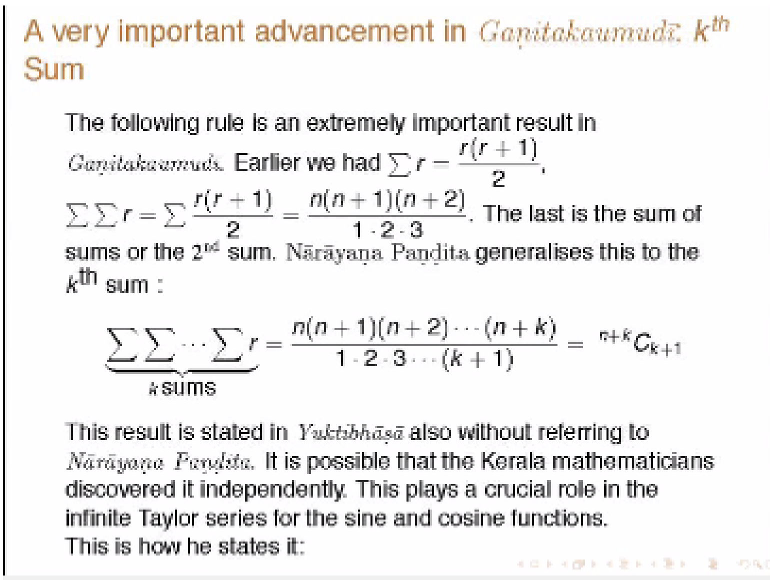

So, this will be there, so there stated is; okay I am not given this first, does not matter. So, now a very, very important advancement in Ganitakaumudi, I would like to talk about it. See, earlier we had the Sigma of R, you see sorry; this must be n*n+1/2, so that is when you sum from 1 to n, the sum of first n integers is n*n+1/2, then sum of sums, so that is n into; that is sum of this you know, so if you sum this once, so then it will be n*n+1/2.

Now, sum this from R is = 1 to n, so we write it like this, 1 to n, so that is n*n+1*n+2/ 1*2*3, so this is given by you know earlier mathematicians, the last is the sum of these; sums are the second sum; so, Narayana Pandita generalizes these to the kth sum, suppose you do it k times, so then that is = n*n+1*n+2 etc up to n+K/ 1*2*3/ K+1, so this is = n+k C k+1, so these are very important result.

These had not been stated in earlier in Indian mathematics literature, this result is stated in Yuktibhasa also without referring to Narayana Pandita, so probably they would have discovered it themselves also because I mean, that is at fairly advanced level that they are doing, so they might have done it because other things with they are in a borrowed from the always quoted, you know, **(FL)** they always quoted, so this is not quoted.

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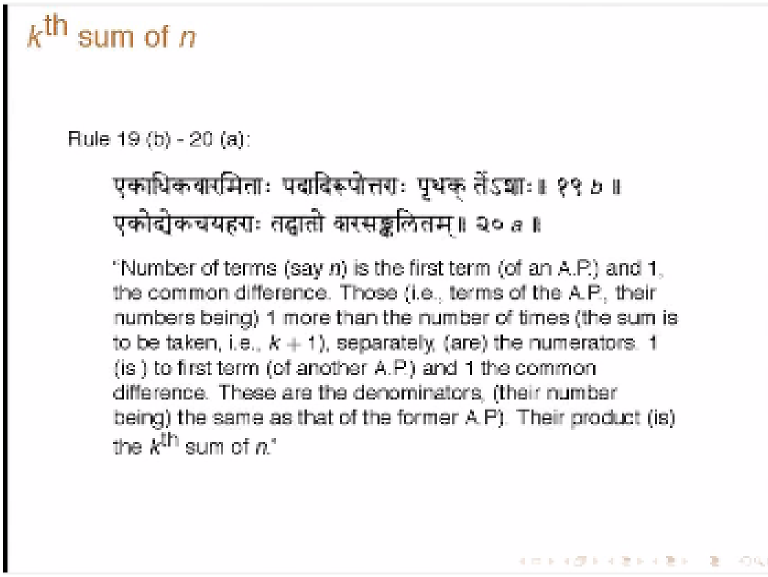

So, probably they do, anyway and this plays a very crucial role in the infinite Taylor series for sine and cosine functions and this is how Narayana Pandita states it; (FL) so number of terms say n, is a first term in arithmetic progression and one the common difference and those that is terms of the AP, their numbers being 1 more than the number of times, the sum is to be taken or the numerator.

Then, one needs to be the first term of another AP and one the common difference, these are the denominators and their product is the kth sum of n, so he is clearly given this result yeah, n*n+1*n+k up to because he is saying those one more than the number of terms, so this will go numb; first term is 1, then it goes up to one more number, sum of the number that is taken is; so, the n+k, to come in the last term, okay.

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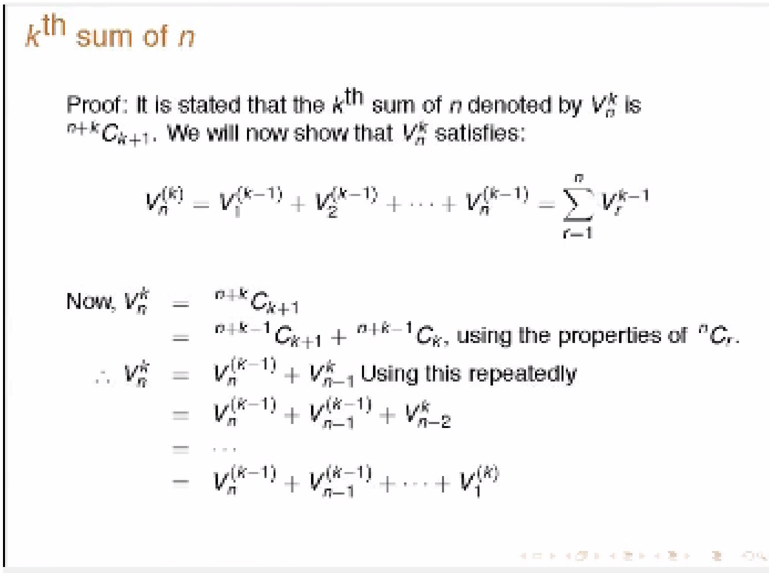

So, now suppose, I will use this notation for this V and k; kth sum of n, so we will; he has given this, so we should so; but in how he is treated as the sum of sum, you see, so he should be; if it is a sum like that, then it should be a sum of these k - 1 kind of a thing, right. So, first sum, then when you sum this, you should get the second sum, then you get the third sum etc., so he is giving the result for the kth sum.

So, we should show that it is actually k - 1 sum, when it is summed over things, you should get this okay. So, we have to show that we will show that this actually satisfies this thing that you know, that is k - 1 sum, when you sum from 1 to n, you get this result. So, now Vnk is n+k Ck+1, so using the properties of this nCr, you get these n+k - 1 Ck+1+n+k – 1 Ck, so V and K is = V n k – 1+Vn – 1 k.

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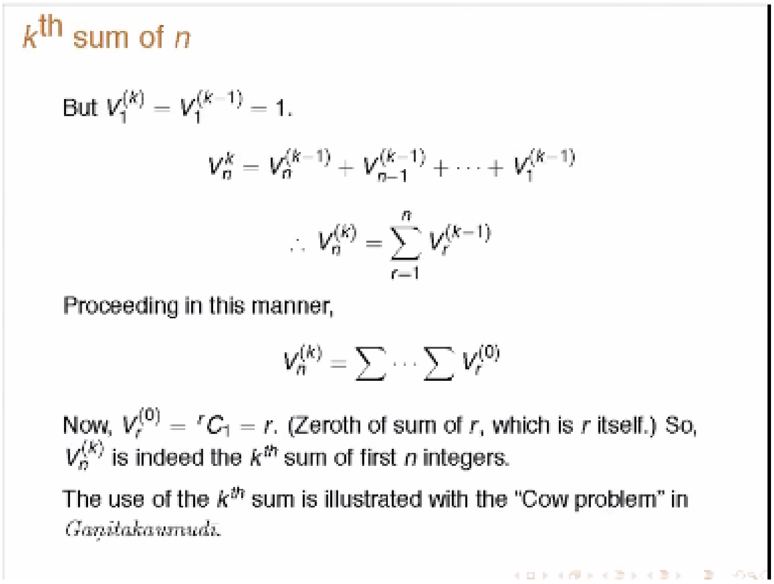

So, using this repeatedly, see this again we write as Vn – 1 k - 1+Vn - 2 k etc., finally you get it is, Vn k up to Vn k - 1 etc. but here, that also be V1 k, so that is a sum with one term, you see, so kth R k -1, it does not matter, so V1 k is = V1 k -1 that is always = 1, so V and K is this; so this is a sum and result is already given see, so proceeding in this manner and this again can be expressed as the sum of the lower order sum.

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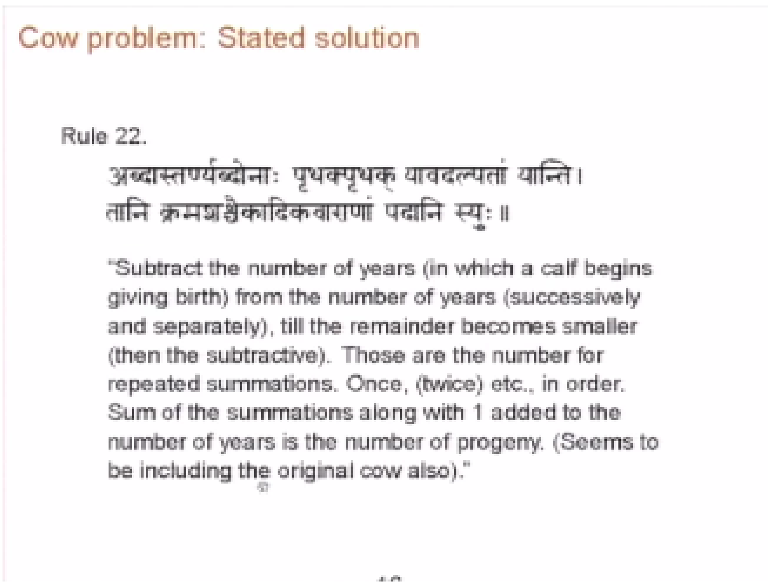

So, V and k is kth sum, we use to written on V r0, so now V r0 is r C1 is = r, right so that is 0 is sum of r, which is r itself, so V and k is indeed the kth sum of first n integers. So, now this is illustrated with a very famous Cow problem in Ganitakaumadi, so these are very interesting problems. (FL) he gives a solution itself first, subtract when it is as if when you supposed know the problem okay.

Subtract the number of years in which a calf begins giving birth from the number of years successively and separately till the remainder becomes smaller than the subtractive, these are the number for repeated summation (FL), this is first term, second number, etc. okay, some other summation along with 1 added to the number of years is the number of progeny, so this is the solution, he is giving.

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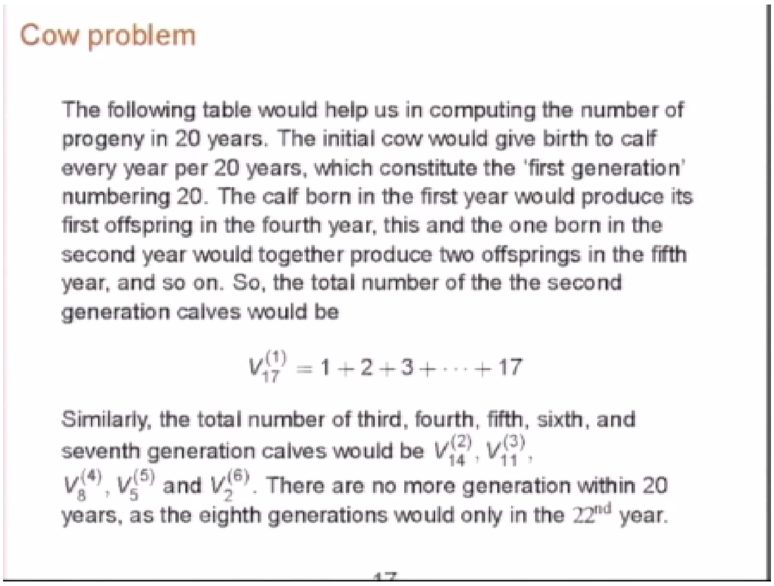

So, let us understand what is; I will just explain this slide before giving the table, see one cow is there, so initial cow okay or some primordial cow, let us say okay, so it is there. So, then every year, it is giving birth to 1 calf, okay you will assume that it is a female calf only.

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Every year it is giving 1 calf, so for 20 years okay, we are discussing that okay, now the calf born in the first year, would produce its first offspring in the fourth year. See, after 3 years only, it is there, okay, so this calf, he is giving; see the first generation of course; you see first generation of the cows, so then this is the second generation, you see, this is giving rise to this here and here you see, in the next year; fifth year, this will of course give one.

And here this also will give rise to 1 calf, and similarly, so this is 1+1+1 etc., so the next one is 1+2, so then in the sixth year, this also will start okay. So, all the you know; calf will you who are born in successive years, they will start producing their own calf as successive years, so they will be; that is the third column will be this, second column will be that. So, then again these you know, so these are fourth year.

From the seventh year onwards, there will be the third generation of course, so that is what is happening. The calf born in the first year will produce its first offspring in the fourth year, this and the one born in second etc., so total number of second generation calf will be 1+2 etc., 17 okay, first generation is 1+1, second generation is 1+2+etc up to 17 only; no 17, I will see that here.

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Year	1st generation	2nd generation	3rd generation	4th generation	5th generation	6th generation	7th generation
1	1						
2	1						
3	1						
4	1	$V_1^{(0)} = 1$					
5	1	$V_2^{(0)} = 2$					
6	1	$V_3^{(0)} = 3$					
7	1	$V_4^{(0)} = 4$	$V_1^{(0)} = V_1^{(1)}$				
8	1	$V_5^{(0)} = 5$	$V_1^{(0)} + V_2^{(0)} = V_2^{(1)}$				
9	1	$V_6^{(0)} = 6$	$V_1^{(0)} + V_2^{(0)} + V_3^{(0)} = V_3^{(1)}$				
10	1	$V_7^{(0)} = 7$	$V_4^{(1)}$	$V_1^{(1)} = V_1^{(2)}$			
11	1	$V_8^{(0)} = 8$	$V_5^{(1)}$	$V_1^{(1)} + V_2^{(1)} = V_2^{(2)}$			

Similarly, the total number of 3rd, 4th, 5th, 6th etc. will be V14 to V11, 3 see 3rd sum of 11, 2nd sum of 14, 3rd sum of 11, 4th sum of 8, 5th sum of 5 and sixth sum of 2, so no more generations within 20 years as the eighth generation would be only in the 22nd year. So, let me; the table it will be clear, you see the first generation is 1, 1, 1 etc., 1st year you see.

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Year	1^{st} generation	2^{nd} generation	3^{rd} generation	4^{th} generation	5^{th} generation	6^{th} generation	7^{th} generation
12	1	$V_9^{(0)} = 9$	$V_6^{(1)}$	$V_1^{(1)} + V_2^{(1)} + V_3^{(1)} = V_3^{(2)}$			
13	1	$V_{10}^{(0)} = 10$	$V_7^{(1)}$	$V_4^{(2)}$	$V_1^{(2)} = V_1^{(3)}$		
14	1	$V_{11}^{(0)} = 11$	$V_8^{(1)}$	$V_5^{(2)}$	$V_1^{(2)} + V_2^{(2)} = V_2^{(3)}$		
15	1	$V_{12}^{(0)} = 12$	$V_9^{(1)}$	$V_6^{(2)}$			
16	1	$V_{13}^{(0)} = 13$	$V_{10}^{(1)}$	$V_7^{(2)}$	$V_4^{(3)}$	$V_1^{(3)} = V_1^{(4)}$	
17	1	$V_{14}^{(0)} = 14$	$V_{11}^{(1)}$	$V_8^{(2)}$	$V_5^{(3)}$	$V_1^{(3)} + V_2^{(3)} = V_2^{(4)}$	
18	1	$V_{15}^{(0)} = 15$	$V_{12}^{(1)}$	$V_9^{(2)}$	$V_6^{(3)}$	$V_3^{(4)}$	
19	1	$V_{16}^{(0)} = 16$	$V_{13}^{(1)}$	$V_{10}^{(2)}$	$V_7^{(3)}$	$V_4^{(4)}$	$V_1^{(5)}$
20	1	$V_{17}^{(0)} = 17$	$V_{14}^{(1)}$	$V_{11}^{(2)}$	$V_8^{(3)}$	$V_5^{(4)}$	$V_2^{(5)}$
Sum	$V_{20}^{(0)}$	$V_{17}^{(1)}$	$V_{14}^{(2)}$	$V_{11}^{(3)}$	$V_8^{(4)}$	$V_5^{(5)}$	$V_2^{(6)}$

Table. Seven generations of the offsprings of the cow, born in 20 years.

It will go on in the next slide also, it goes on. Then the second generation you see, V1 0 is = 1, V2 0 is = 2 because the one born in the second year also is giving, so 3 etc. For second generation, if you sum, you will get 1+2 etc. up to 17 because it goes from 4 to 20, so it is 17; 1+drop to 17. The third generation will be; so this one you see, V10 that will give 1, then next in the 8th year; 1+2, you see that will become okay.

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Total progency

So the total progeny in 20 years is:

$$V_{20}^{(0)} + V_{17}^{(1)} + V_{14}^{(2)} + V_{11}^{(3)} + V_8^{(4)} + V_5^{(5)} + V_2^{(6)} = 20 + \frac{17 \cdot 18}{1 \cdot 2} + \frac{14 \cdot 15 \cdot 16}{1 \cdot 2 \cdot 3} + \frac{11 \cdot 12 \cdot 13 \cdot 14}{1 \cdot 2 \cdot 3 \cdot 4} + \frac{8 \cdot 9 \cdot 10 \cdot 11 \cdot 12}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} + \frac{5 \cdot 6 \cdot 7 \cdot 8 \cdot 9 \cdot 10}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6} + \frac{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7} = 2744$$

We have to add 1 if want to include the original (initial) cow. (Note $V_n^{(0)} = n$, $V_1^{(k)} = 1$ for all k).

So, next will be like that you know, so the second sum will come and in the 4th generation it will be; so like that V1 2 etc., so 5th generation, so V20, 0 that is only just 20, then V 17 1, so that is 17*17+1/2 and this will be 14, this is a 14*14+1*14+2/ 1*2*3, so like that, so this is the 17th generation of their offspring, so total will be total progeny produced by the gomata, he is you know 20, 17, 1, 14, 2 etc.

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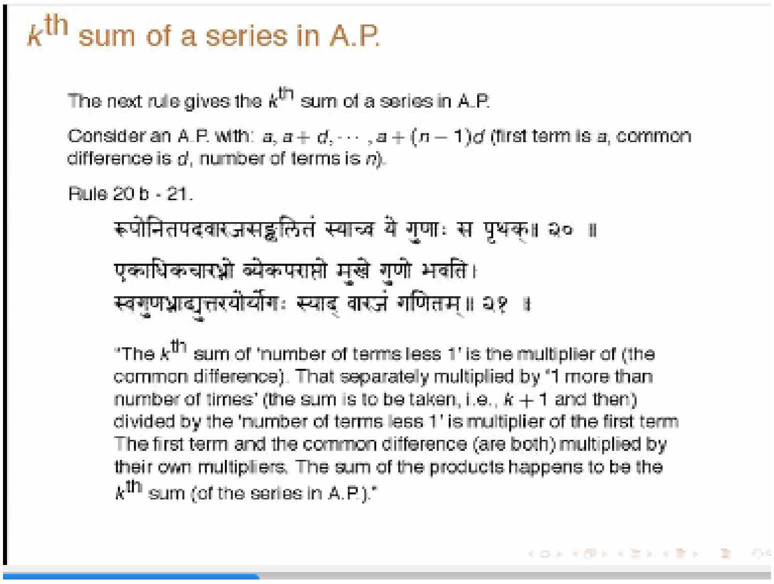

So, 20+17*18/ 1*2+14*15*16*1*2*3 etc., etc., okay so finally it will be 2744, if we had to add 1, if you want to going to the original initial cow also. Now, that will be the total number, so this is the direct application of the; so then of course, we talked about the kth sum of a series in arithmetic progression. So, consider an arithmetic progression a, a+d etc., to a+n – 1 d, first term is a; common difference is d, number of terms is n.

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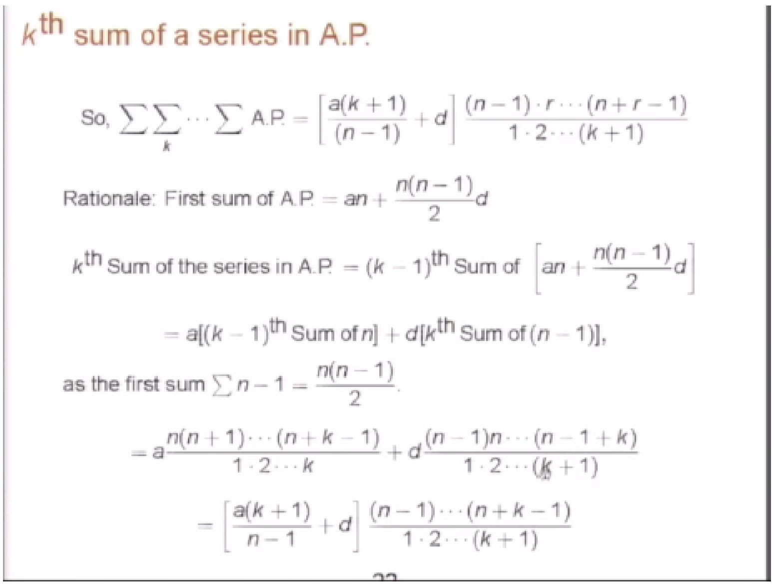

Then he has given this result (FL) the kth sum of n number of terms < 1 etc., I mean I might as well give the result hinder this thing, so what is doing is; there is an arithmetic progression, so kth sum of that, so that is given by this result. So, how do we understand is the first term is this you see, when you sum of arithmetic progression, the first term will be this, an+n*n – 1/2*d, right that is in a sum of an arithmetic progression.

First term a, common difference is d and number of terms n, so now we have to sum over this, so make it into r and an r is going from 1 to n, you see that is a; so kth sum will be k – 1 sum of this; k - 1 sum of an+n*n – 1/2*d, so k - 1 sum of n+d *; you see this is; this is; these are actually first sum of n – 1, so k – 1, so kth sum of n - 1 at the first sum n - 1 is n*n – 1/ 2, so finally you get this result.

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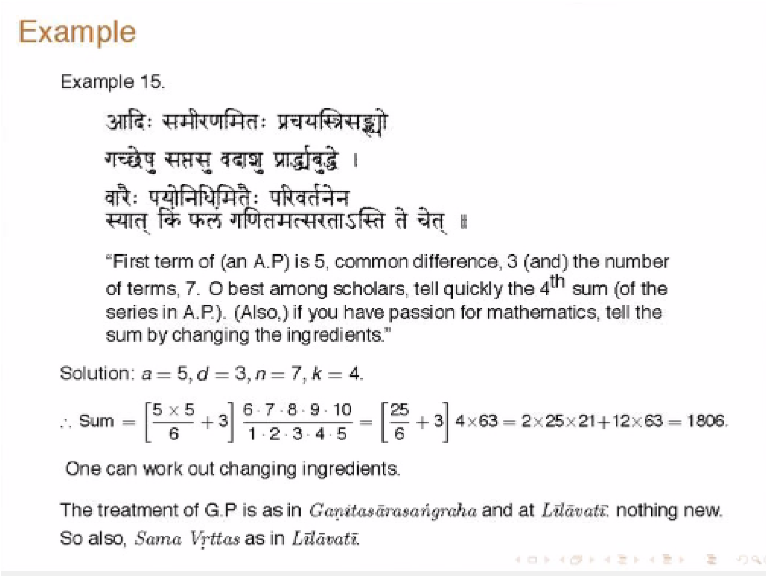

Because k - 1 sum of n is this, right n*n+1 you should go up to n+k - 1 and so k terms are there and below again, k terms 1*2*etc., k because d*kth sum of n – 1, so which is n - 1 you should start n – 1+k, so like this, so these are final result. So, for instance he gives an example (FL) the first term of arithmetic progression is 5, common difference is 3; 3 Sankhya, right and the number of terms is 7, number of (FL) oh, best among learners okay he is doing wooing his audience.

Tell quickly the 4th sum of this thing, you see (FL) you say passion for mathematics tells the sum in changing the ingredients also. So, a is 5, these three common difference n is 7, k is 4, so if you plug in these numbers here, you get the result, these 1806. One can work out changing the ingredients, what he is saying is make it 6 and other kind of a thing, so then geometric progression nothing new as far as I could see, apart you know from what is given in Ganitasarasangraha and Lilavati.

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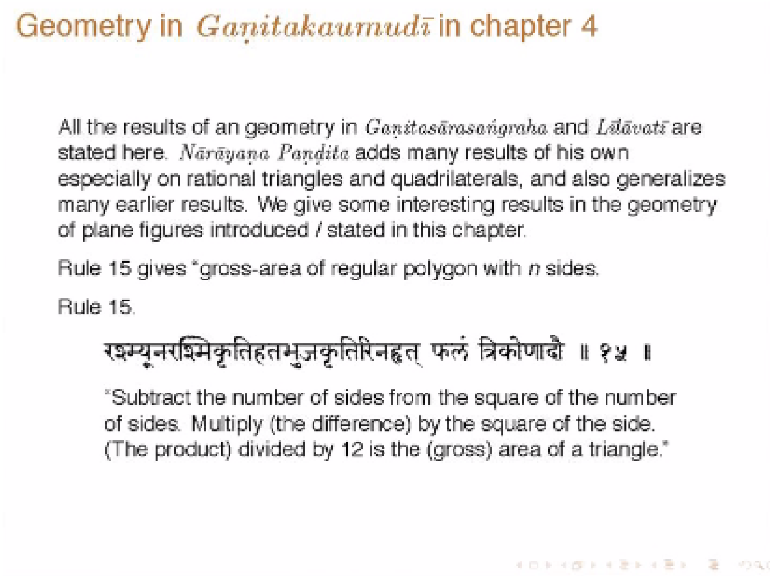

Then, Sama Vrttas and all those these things that is also same as in Lilavati. So, geometry in Ganitakaumudi in chapter 4, so all the results of geometry in Ganitasarasangraha and Lilavati are stated here and adds many results of his own especially, rational triangles and quadrilaterals and also generalizes many results, so you might have noticed this you know, I always try to generalize the results given in earlier works.

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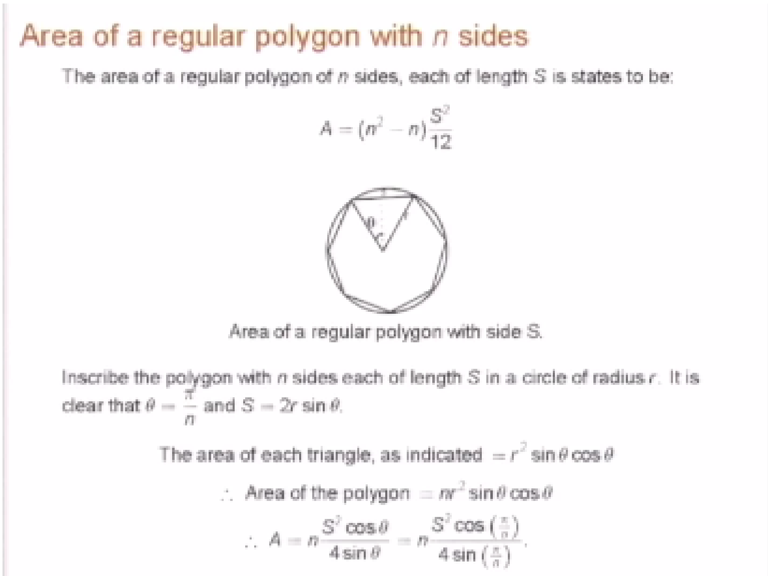

So, we give some interesting results in geometry of plane figures. See for instance gross area of a regular polygon with n sides, so he gives the n side, so suppose a side is s, so in the gross area, is say as n squared - n*s squared/ 12 where just see the side and it does not come out all right

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Area for large n

When n is large,

$$\sin\left(\frac{\pi}{n}\right) \approx \frac{\pi}{n} - \frac{1}{6}\left(\frac{\pi}{n}\right)^3 \approx \frac{\pi}{n}\left[1 - \frac{1}{6}\frac{\pi^2}{n^2}\right]$$

$$\text{and } \cos\left(\frac{\pi}{n}\right) \approx 1 - \frac{\pi^2}{2n^2}$$

$$\therefore A \approx n\frac{S^2\left(1 - \frac{\pi^2}{2n^2}\right)}{4\frac{\pi}{n}\left(1 - \frac{1}{6}\frac{\pi^2}{n^2}\right)}$$

$$\approx \frac{n^2 S^2}{4\pi}\left(1 - \frac{1}{3}\frac{\pi^2}{n^2}\right)$$

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Further approximation

If we put $\pi \approx 3$, (this crude approximation to π has been stated by Narayana), we obtain

$$A \approx \frac{n^2 S^2}{12}\left(1 - \frac{3}{n^2}\right) \approx \frac{(n^2 - 3)}{12}S^2$$

It is not clear what is the approximation which led the author to his result.

After many other 'gross' results Nārāyaṇa states;"The earlier gross rules have been stated for novice calculations. Due to occasional disagreement between (gross and exact) results, I have not much respect (for them)."

And even in approximation of large and; I find that it will be n squared – 3/ 12*a square, apart term of course, we are already made an approximation of pi being = 3, okay, so it is not clear what the approximation is; anyway it is small result here, so after many other gross results, Narayana states that you know, the earlier gross results have been stated for novice calculations, due to occasional disagreement between gross and exact results, I have not much respect for them, he himself is stated, okay.

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Diagonals of a cyclic quadrilateral

Diagonals of a cyclic quadrilateral.

Here *Nārāyaṇa* gives the standard expression for the diagonals of a cyclic quadrilateral. He also introduces the concept of third diagonal, which is very useful in deriving many results (including the expression for the area of a cyclic quadrilateral, $\sqrt{(s-a)(s-b)(s-c)(s-d)}$) which is proved in *Yuktibhaṣa*.

Rule 47-52 includes:

उभयश्रवणाश्रितभुजवधयोगौ तौ परस्परं विहृतौ।
प्रतिभुजभुजवधयोगा हतौ तु मूले चतुर्भुजे कर्णौ॥
सर्वचतुर्बाहूनां मुखस्य परिवर्तने यदा विहिते।
कर्णस्तदा तृतीयः पर इति कर्णत्रयं भवति॥ ४८ ॥

"Divide the sum of the products of the sides about both the diagonals by each other. Multiply the quotients by the sum of the products of opposite sides. Square roots of the, products are the diagonals in a quadrilateral.

In all (cyclic) quadrilaterals, the (new) diagonal obtained by the interchange of its face and flank side is the third diagonal."

So, it is only maybe to you know test the alertness of the students, whatever it is anyway but many of them in approximations do make sense. Then diagonal of a cyclic quadrilateral, so will not going into detail of this earlier result okay, so area for you know; that area of cyclic quadrilateral is this, then the diagonal of this thing is 2 diagonals, you know how to get it, see that is the result; famous result due to Brahmagupta.

Divide some of the products of the sides about both the diagonals to each other, multiply the quotient with some other products of opposite sides, square roots of the products or the diagonals in a quadrilateral, now comes the new thing, so new thing he is saying; he introduces the concept of a third diagonal, so which will be very useful in many results on cyclic quadrilateral.

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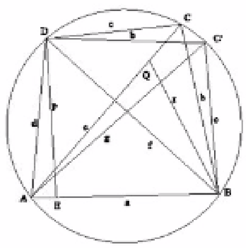
Cyclic quadrilateral: Third diagonal



A cyclic quadrilateral ABCD. AC, BD are the diagonals. Another cyclic quadrilateral ABC'D got by interchanging the sides BC and CD. AC' is the third 'diagonal'.

In the figure, $ABCD$ is a cyclic quadrilateral with sides, $AB = a$, $BC = b$, $CD = c$, and $DA = d$. $AC = e$ and $BD = f$ are its diagonals. Now on the arc BD, choose a point C' such that $arc\, DC' = arc\, BC$. Then naturally, $arc\, C'B = arc\, CD$. The corresponding chords are also. Therefore, $DC' = C'B$. So the quadrilateral $ABC'D$ is generated by interchanging the sides b and c in the original quadrilateral. Then $AC' = g$ is called the third diagonal.

In all cyclic quadrilaterals, the new diagonal; of course, he does not say the diagonal obtained by the interchange of its face and flank side is the third diagonal, so this is; this is; so this is your cyclic quadrilateral ABCD, so these are the diagonals, right BD and AC, they are the diagonals, so call them E and F. So, the expressions for that is you know, fairly straightforward, we have already done that.

Of course, the result may look slightly different because my order of the sides maybe you know you please check you know, so here you know it is ABCD, so AB and CD at the opposite face, so when you are comparing with other results, please mark this. Anyway, so now he is saying you know, see now you get another quadrilateral you know, by making DC prime is = BC, so they are essentially interchanging the side.

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Expressions for the three diagonals

We had already derived the expressions for the diagonals $AC = e$ or $BD = f$ in the material on *Brāhmasphuṭasiddhānta*. These are stated by Nārāyaṇa:

$$AC = (e) = \left[\frac{(ac+bd)(ad+bc)}{(ab+cd)}\right]^{1/2}$$

$$BD = (f) = \left[\frac{(ac+bd)(ab+cd)}{(ad+bc)}\right]^{1/2}$$

(The expressions may look slightly different, as the symbols for the sides is different here.)

The third diagonal $AC' = g$ is got by interchanging b and c in the original quadrilateral. Hence, by interchanging b and c in AC

$$AC' = (g) = \left[\frac{(ab+cd)(ad+bc)}{(ac+bd)}\right]^{1/2}$$

So, DC prime is = BC and BC prime is = CD so, essentially these 2 sides are and these diagonal are kept saying that the upper flanks you know you can say they are interchanged, so that is this thing and then so, AB C prime D that is the new cyclic quadrilateral and this is AC prime is the third diagonal, so that is the thing and a third diagonal and very easily, you can find out the third diagonal.

Because essentially, what have you done in getting this new diagonal, essentially these sides B and C have interchanged, right see this upper; this thing, so you should have seen this is become B and this is instead of B, it is become C, so that is the thing, only we are interchange B and C, so diagonal expression also is straight forward, only you have to; this thing you know, so instead interchange B and C.

So, ab+cd, so this will be sd+bc, then ac = bd, so these are third diagonal, so it plays the very important role in the proof for the area of a cyclic quadrilateral and also for the area of the circumdiameter, so that (FL) and the other speakers one of them probably professor Ramsubrahmanyam will prove that result, which is given in (FL).

So, for that this concept is very crucial; the third diagonal plays an important role. **”Professor – student conversation starts”** So, yeah that is what he has given know, interchange of sides; upper side, yeah, yeah, yeah now, this will come in various things, we will see that actually you know, the new diagonal has come has come but what is the use of that? Yes, several results as I said using this, it is very important for proving the expression for this area of a cyclic quadrilateral.

He had to use this concept of third diagonals yeah, that is the thing, no, no that is how they are using you see, so it as important as the first 2 diagonals of the original quadrilateral perhaps that is what they mean, yeah, yeah it is not the third diagonal the same quadrilateral, anymore this thing yeah, because if you interchange this, the order will be different but you will get the same these things you know, so maximum you can generate 3, probably yes, yes that is the thing. **“Professor – student conversation ends.”**

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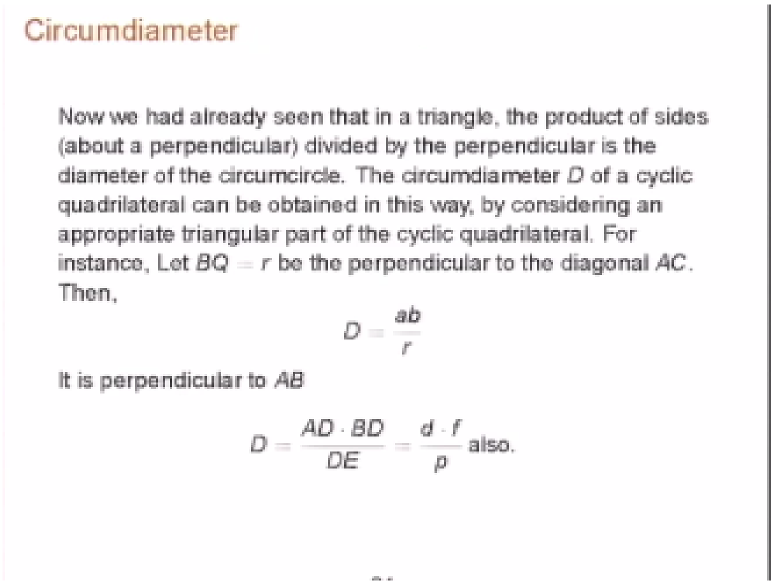

So, then you will get a circumdiameter have already given this result earlier, you see, see suppose your sides A and B suppose is a triangle, ABC, so then this BQ is perpendicular to AC, it may not appear in the figure like that but it is take it like that, so suppose this is at perpendicular R, so then he states that D is = ab/r, these are Brahmagupta result, which I do not have to say again.

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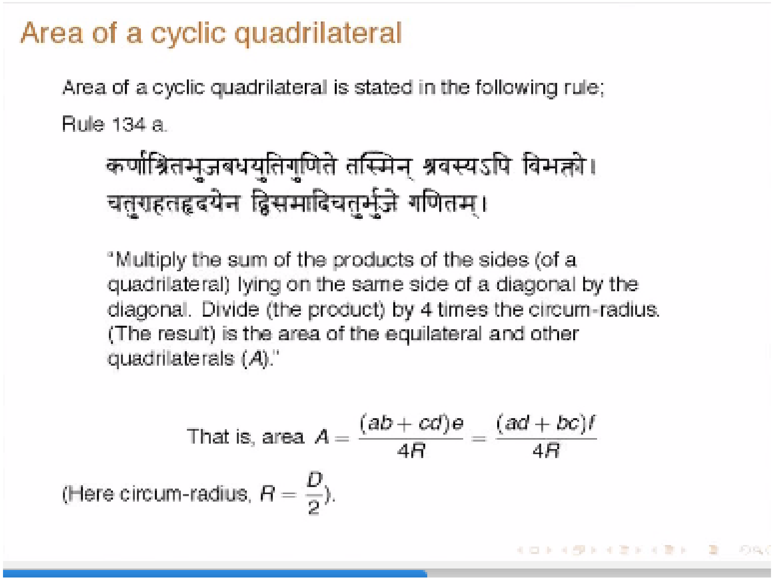

And it is perpendicular; you can also write it as d*f/ p, see suppose, you take instead of that you know this triangle, you see, so any triangle, you see the circumdiameter will be the product of the sides and divide be the perpendicular, you see so that is the thing, so this is the result here also. Then the area of a cyclic quadrilateral is stated in the following, multiply the sum of the product of the sides lying on the same side and divide the product before time the circumradius.

That is the area of the equilateral and other quadrilateral he says, so this is the expression for the area, so there would have; he is very clear because you know you can consider it as 2 triangles, you know say AB, these two triangles and you know, those perpendiculars are there, so from that you can get the result, okay because the half base into altitude and altitude itself is given in terms of the two sides and circum diameter that is the thing you see.

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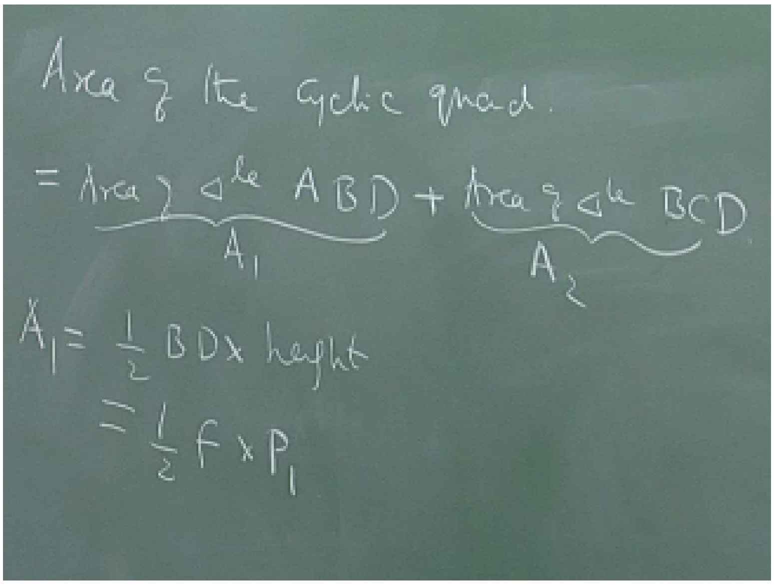

That is what you are doing, did you get the point. So, the ABCD is cyclic quadrilateral, the area of this cyclic quadrilateral can be thought to be the sum of the areas of the triangle ABD and BCD, so area of triangle ABD + area of triangle BCD, call this as A1 and call this as A2. Now, the area of this angle ABD is half base into height, you can consider BD as the base, half BD into height, so which is essentially to drop a perpendicular from a to this base BD, call it as p1.

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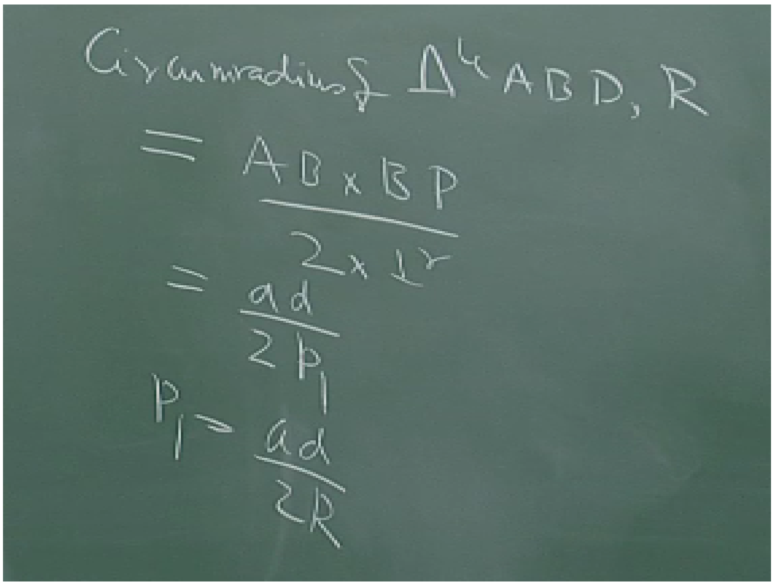

So, BD is f, so 1/2 f*p1, now the circumradius of the triangle; circumradius, if you call it as r, triangle ABD R, so this is the product of the 2 sides AB*BD/; pi is the perpendicular so this is essentially AD/ 2 *; now the perpendicular is p1, so you p1 is = ad/2R.

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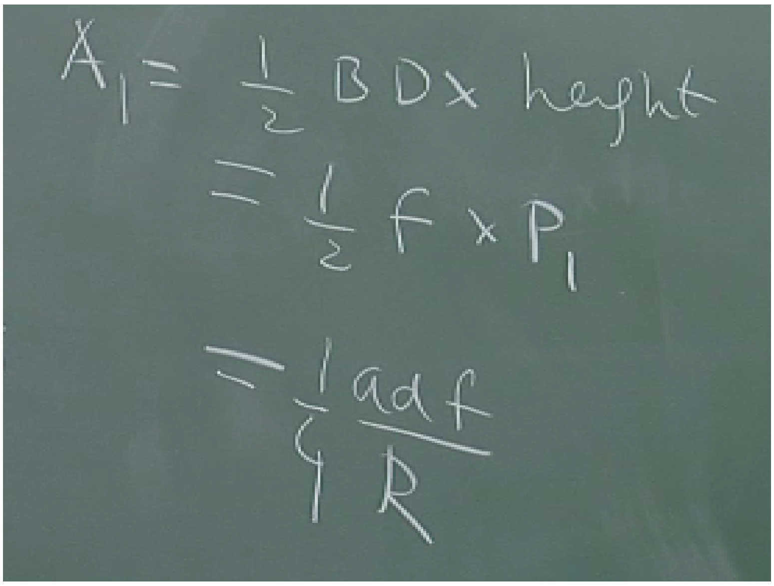

So, substituting that here, one can see that this is equal to 1/4*ad*F/ 4 R so the area of the triangle abd.

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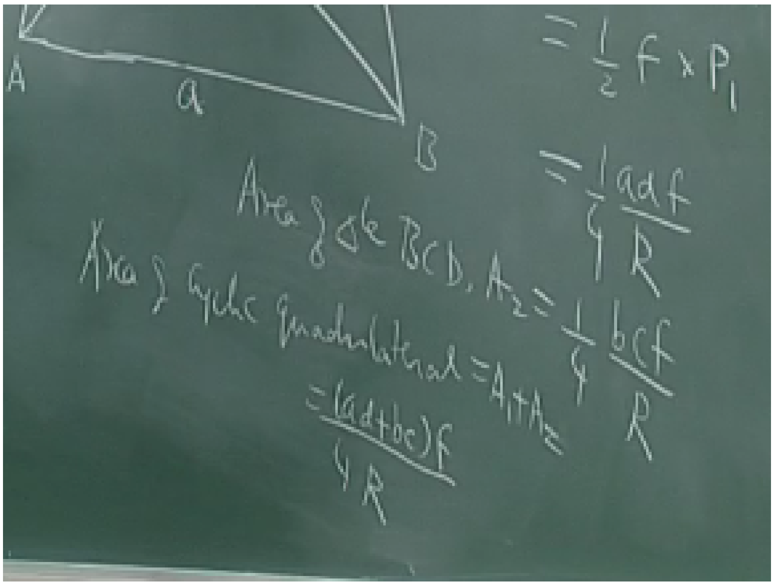

So, the similarly the area of the triangle BCD, so which is A2, so same logic will apply, instead of ad, you will get BC here, so BC into the base is the same F/ R, so the area of the quadrilateral; cyclic quadrilateral is equal to A1+A2 is = 1/4*ad+bc*f / R, so these what has been stated okay.

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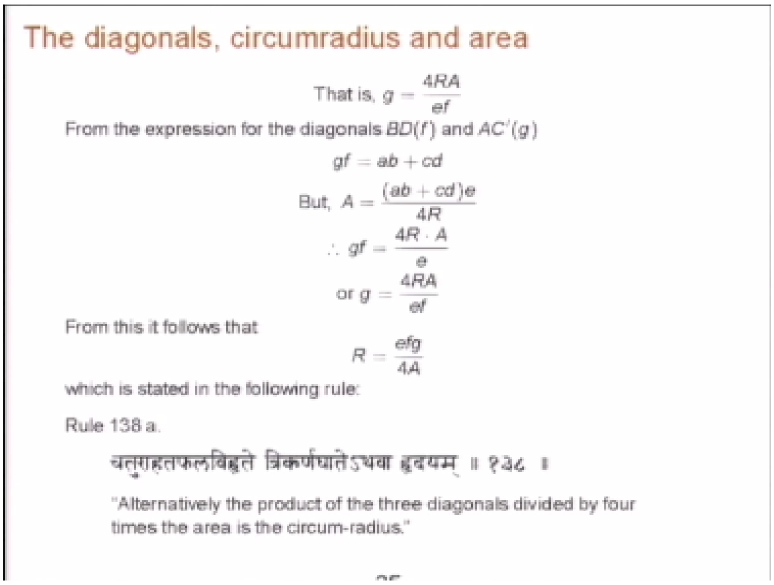

So, an alternate expression yeah, see for instance, he gives a result that the radius circumradius is = efg/ 4A, so this is where you know, said neatly it comes that the radius is = the product of all the three diagonals divided by 4 times the area. So, (FL) circum radius alternatively, the product of the 3 diagonals divided by 4 times the area, so that is circumradius so this will you know introduce some symmetry in this thing you know.

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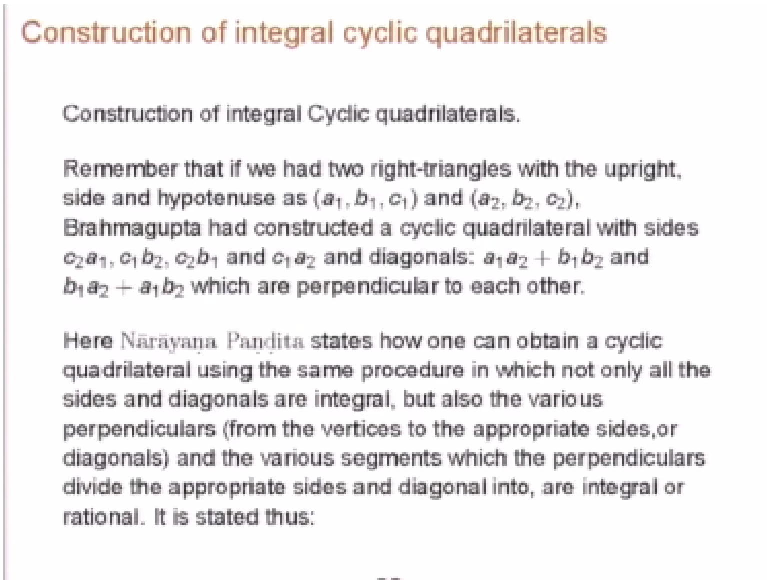

So, all these things are somewhat equivalent and radius comes out neatly as the product of all the 3 diagonals divided by the 4 times the area. Then, construction of integral cyclic quadrilaterals, I will not have too much time, I am not going to that in detail.

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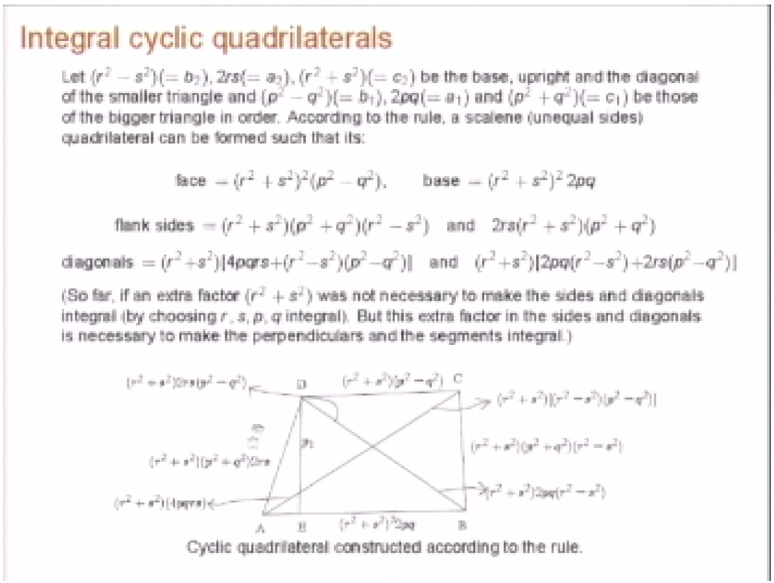

See for instance, he will start some 2 triangles; right triangles like this, r squared - s square 2rs r square + s square P squared – q squared 2pq P squared + q squared, see if our r,s , p, q etc. are rational number, this also will be rational okay, so then actually you can get, if you are integer they will be integers, so you can get an integral quadrilateral using these starting from this, where the sides and perpendicular etc. are given in this figure.

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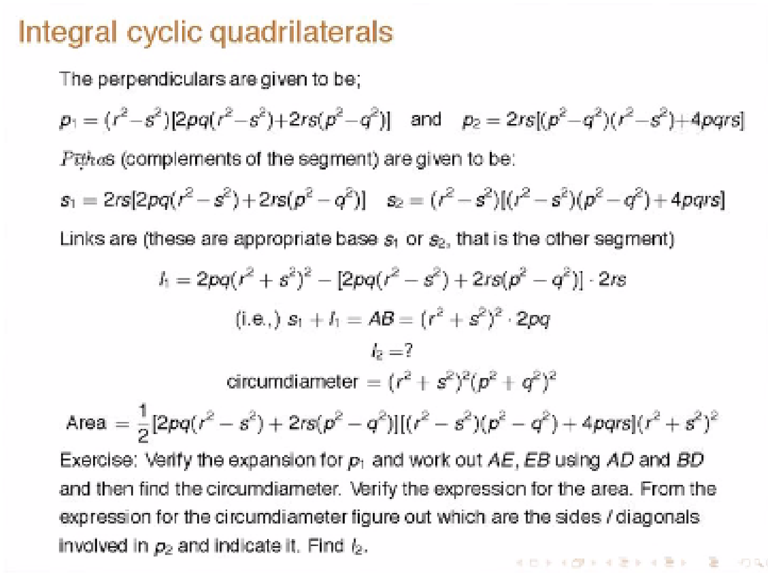

So, what is important is not only the sides but the various perpendicular, the various segments etc., they are all integral here you see, so it is highly quite a non-trivial result, so it will take some time to explain but he has constructed, that is why there is some extra factor you know, r square + s square everywhere otherwise, that need not have been there. So, that this is to make all the other things also, the perpendiculars and segments also integral, so use this result.

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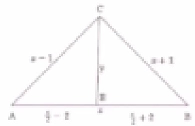
Construction of rational triangles, whose sides differ by unity

The following rule gives the procedure:

Rule 118.

द्विगुणेष्टमिष्टकृत्या त्रिहीनयाप्तं च तत्कृतिस्त्रिगुणा।
सैका मूलं द्विगुणं भूः सैकोनाऽधिका बाहू ॥ ११८ ॥

"Divide twice an optional number by the square of the optional number less 3. Add 1 to thrice the square (of the quotient). Twice the square root of the sum is the base. 1 added to and subtracted (from the base) are the flank sides."



Rational triangle whose sides differ by unity.

We can conceive of the triangle as above, with the sides $AB = x$, $AC = x - 1$ and $BC = x + 1$. The perpendicular $CE = y$ divides the base AB into segments $AE = \frac{x}{2} - 2$ and $EB = \frac{x}{2} + 2$. We want the perpendicular and to segments also to be the rational.

So, then he gives a very interesting result consistent of rational triangles, whose sides differ by unity. So, he says that; so this is, you know what we have want is a triangle besides x – 1, x+1 and here yeah x, x -1, x +1 okay, so where all the sides are integral, you can say what is so great you know because you can choose any x and you can keep you know a triangle is determined by 3 sides.

But what he wants is all the perpendicular segments, they should all be integral okay, so that is what he is doing, he is giving this thing. So, the construction is as follows; divide twice an optional number by the square of the optional number less 3, add 1 to thrice the square of the quotient, thrice square root of the sum is the base, 1 added to and subtracted from the base are the flank sides.

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Rational triangles whose sides differ by unity

We should have,

$$(x-1)^2 - \left(\frac{x}{2} - 2\right)^2 = (x+1)^2 - \left(\frac{x}{2} + 2\right)^2 = y^2$$

This leads to

$$\frac{3}{4}x^2 - 3 = y^2$$

$$\text{Choose: } y = \frac{3 \cdot 2n}{(n^2 - 3)}, \quad n \text{ an integer.}$$

$$\begin{aligned} \therefore x &= 2\left[3\left(\frac{2n}{n^2-3}\right)^2 + 1\right]^{1/2} \\ &= \left[\frac{12n^2 + (n^2-3)^2}{(n^2-3)^2}\right]^{1/2} \\ &= 2\left[\frac{n^4 + 6n^2 + 9}{(n^2-3)^2}\right]^{1/2} = 2\left(\frac{n^2+3}{n^2-3}\right). \end{aligned}$$

The solution for x is as stated in the rule. Clearly, x is rational.

So, what he is saying is that you know, (FL) So, essentially finally, he is giving, if you are having this kind of a situation, so then using the theorem of the right triangle, one can easily see that the solution for this is if n is this, I mean x is this in terms of n, then this will be a rational triangle with all the perpendiculars and segments also being this thing okay.

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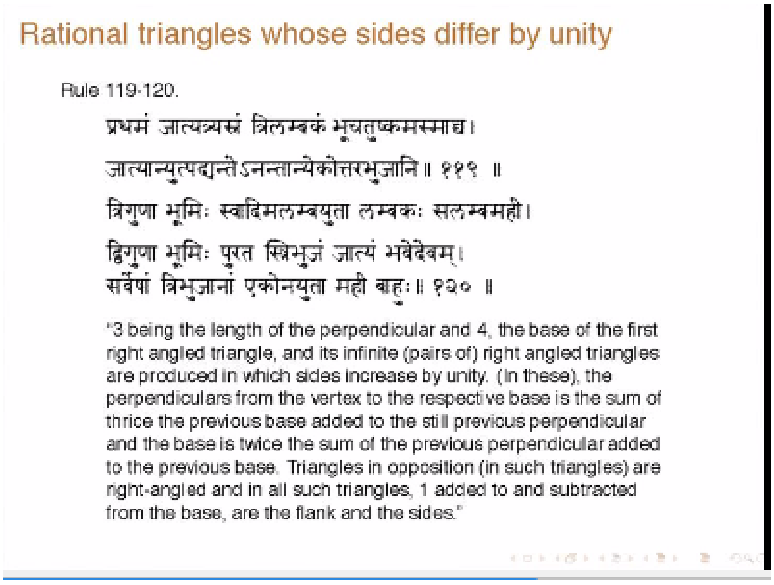

Now, very important thing is now, he will generate an infinite number of triangles with this kind of a procedure, so that is what is stated here. I will give the; I will tell you what he wants to do see, so it essentially sees, you are having this kind of a thing, you see this is the triangle right, these are triangle and considering the fact that these are right angled triangle, one can easily get this result, 3/4 x squared – 3 = y square, okay.

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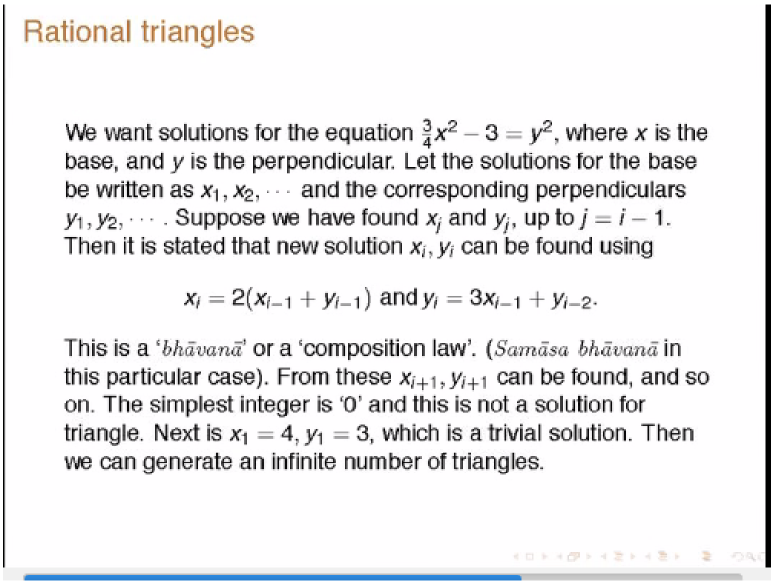

See, X and Y should satisfy this relation you see, where Y is the perpendicular okay. So, now he will say that you know where x is the base by the perpendicular, so let the solution for the base be written as x1, x2 etc. and a corresponding perpendicular y1, y2 etc. Suppose, we have found xj and yj up to j is = I – 1, so then it is stated that new solution xi, yi can be found using xi is = 2*xi – 1+yi -1 and yi is = 3*xi -1+yi -2, okay.

So, this is what he is given in the verse, 3 being the length of the perpendicular and 4 the base of the first right angle triangle and its infinite pairs of right angle triangles are produced in which sides increased by unity. In these the perpendicular from the vertex to the respective base is sum of the thrice the previous base added to the still previous perpendicular and the base is trying to sum of the previous perpendicular added to the previous base, so this is what he is stating.

So, this is in the spirit of yesterday and today the bhavana, the bhavana or a composition law (FL) in this case right, so we are getting from Burma Gupta right, xi, yi. If you have a pair of integers which is satisfying the (FL) equation, x squared - dy squared is = K, there we talk of two of them; x1 squared/d, yi squared is = k1, then x2 squared – g, y2 square is = K2) so then you can generate another pair, which will satisfy this with K1, K2 are the (FL).

So, that was discussed in detail, how to get that; how that x is in terms of x1, x2 and things like that, so these are another bhavana kind of a thing, so from these xi+1, yi +1 can be found and so on. Let us impress the equation of course, is you know this 0, so that is not a solution then take next is x1 is = 4, y1 is =3, so then here 3/4 x squared - 3 is = Y square, so that is satisfied. In this case, one segment of the base is 0 and other segment is the base itself, so that is 4.

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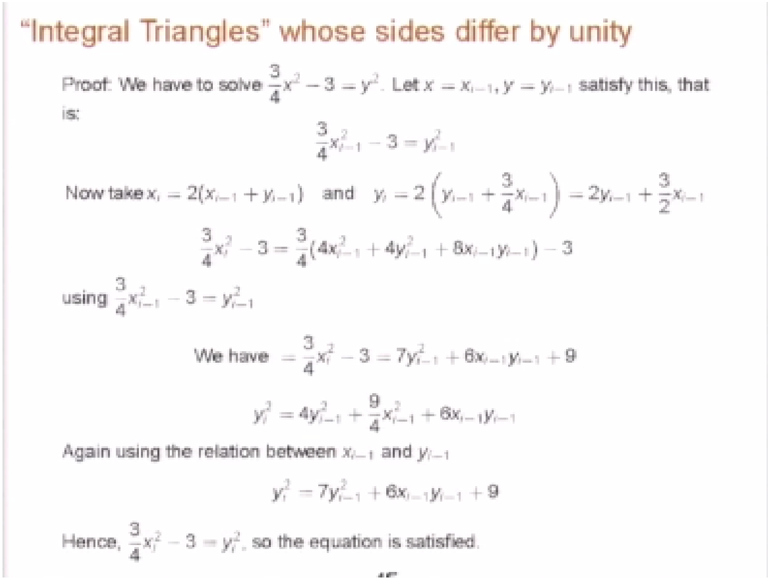

The upright is 3 and the hypotenuse is 5, so we can prove this, I will not be going to; prove in a simple proof okay, so induction I mean, so that is one if you are generated up to I, the next phase that is what is given here, you are generating the next set of values, so I will these quite some you know, just some manipulations with right triangles and so on.

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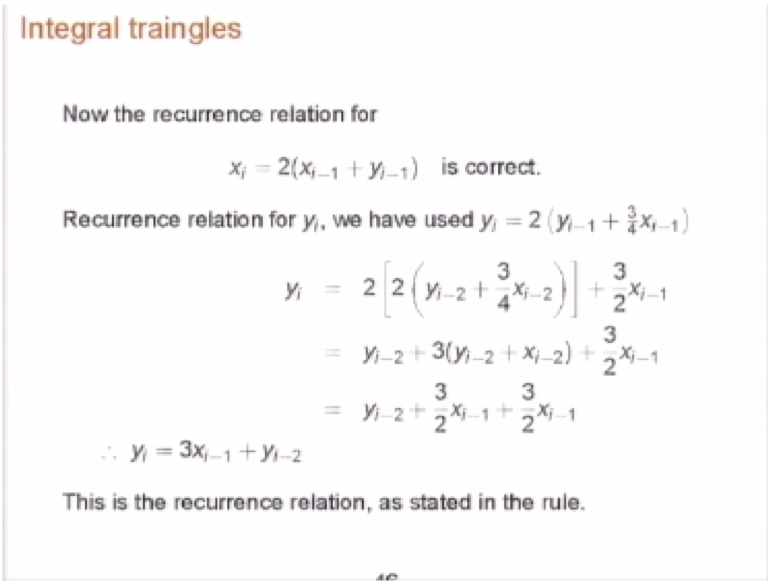

So, these are sequence, so this correct, one can show that this is you know, this will satisfy all the criteria.

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Generating the triangles

Generating the triangles.

Take $x_0 = 2, y_0 = 0$. These satisfy the equation.

$$\therefore x_1 = 2(x_0 + y_0) = 4 \quad y_1 = 2(y_0 + \frac{3}{4}x_0) = 3$$

So, $x = x_1 = 4, y = y_1 = 3$. The other sides are $x - 1 = 3$, $x + 1 = 5$ and the segments are $\frac{x}{2} - 2 = 0, \frac{x}{2} + 2 = 4$.

$$\text{Next } x_2 = 2(x_1 + y_1) = 14, y_2 = 3x_1 + y_0 = 12$$

So, $x = x_2 = 14, y = y_2 = 12$ (perpendicular). The other sides are $x - 1 = 13$, $x + 1 = 15$, and the segments are $\frac{x}{2} - 2 = 5$, $\frac{x}{2} + 2 = 9$

So, say for instance, if you take x0 =2, y0 is = 0, you start from that, so these will satisfy those equations, x1 you take this and y1 4, 3, okay, then you start to 4, 3, other sides are x -1 =3, x+1 =5 and the segments are 0 and 4 and as I told this will collapse but the next will be 14, 12; x2 will be 14, y2 will be 12, so x = x2 = 14 and 12, this is the perpendicular, the other sides are 13 and 15; 13, 14, 15 are the sides and the segments are 5 and 9.

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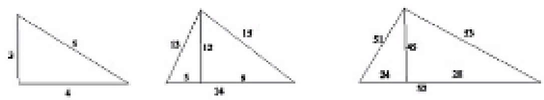
Generating the triangles

$$\text{Next } x_3 = 2(x_2 + y_2) = 52, y_3 = 3x_2 + y_1 = 45$$

So, $x = x_3 = 52$ (base), $y = y_3 = 45$ (perpendicular). The other sides are $x - 1 = 51, x + 1 = 53$, and the segments are $\frac{x}{2} - 2 = 24$, $\frac{x}{2} + 2 = 28$.

These are depicted in the figure below:



First three triangles with sides differing by unity, generated by the algorithm stated in Rule 119-120.

These are integral solutions (for sides differing by unity, as well as the perpendicular and the segments). Clearly there are an infinite number of solutions.

So, this is essentially the first kind of a; so the one of the segments is 0 here, so 3, 4, 5, it will not collapse, sorry it is just you know one segment is 0, it is a valid triangle, it is a right angle triangle that is all, okay. So, next is you know this triangle 13, 14 and 15 that you will get and from that you will get, 51, 53, 52 with these as the perpendiculars and all that, so these are integral solutions for sides with differing by unity as well as perpendicular and segments.

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Combinatorics

Chapter 13 on *aṅkapāśa* or 'combinatorics' is a very elaborate one, containing many new results on permutations and combinations. We first consider the generalised 'Fibonacci' sequence described here.

The Fibonacci sequence is 1, 1, 2, 3, 5, 8, 12, $\cdots$. If P_n denotes the n^{th} term in the sequence, where we start with $n = 0$, it satisfies the recursion relation:

$$P_n = P_{n-1} + P_{n-2}$$

They are related to the number of ordered partitions of a number into parts containing 1 and 2 only. $P_0 = 1$, by convention. We have

$$\begin{aligned}
1 &= 1, P_1 = 1\\
2 &= 1+1 = 2, P_2 = 2,\\
3 &= 1+1+1 = 1+2 = 2+1, P_3 = 3,\\
4 &= 1+1+1+1 = 1+1+2 = 1+2+1 = 2+1+1 = 2+2, P_4 = 5,\\
5 &= 1+1+1+1+1 = 1+1+1+2 = 1+1+2+1\\
&= 1+2+1+1 = 2+1+1+1\\
&= 1+2+1+1 = 1+2+2 = 2+1+2 = 2+2+1, P_5 = 8,
\end{aligned}$$

and so on.

So, (()) (42:13) infinite number of solutions, so these kind of nice tricks that he has. So, then of course, I will not be dealing too much with combinatorics but a little bit I will tell, with that there is in the thirteenth chapter called Ankapasa, it is very elaborate one containing many rules, new results on permutations and combinations. So, he considers a generalized Fibonacci sequence, he described here, so we already heard about it.

Fibonacci sequence is 1, 2, 3, 5, 8, 12 etc. and if Pn denotes the nth term in the sequence, it satisfies this relation. The same thing as what professor Srinivas talked about he was using the symbol S instead of P, so Sn they seem and he was also explaining how it occurs, you know naturally in this you know, suppose you assign the value 1 to a laghu and 2 to a guru okay, then with a given number of Matra, that is the number; total number, you know then, how many possibilities are there?

You see, suppose there is only 0 of it is; 1 only okay, 1 is 1, sorry 8, 13 okay, so suppose you have only 1, the total is 1, then it can only be 1, then suppose is 2, then it is 1+1; these are the partitions or it can be written as 2 that is 2 laghu or 1 guru, right and 3 is 1+1+1, so 3 laghu and also or 1 laghu and 1 guru returning to this thing say, so totally there will be 3 ways, so like that, so and this is what will lead to you know, this relation also.

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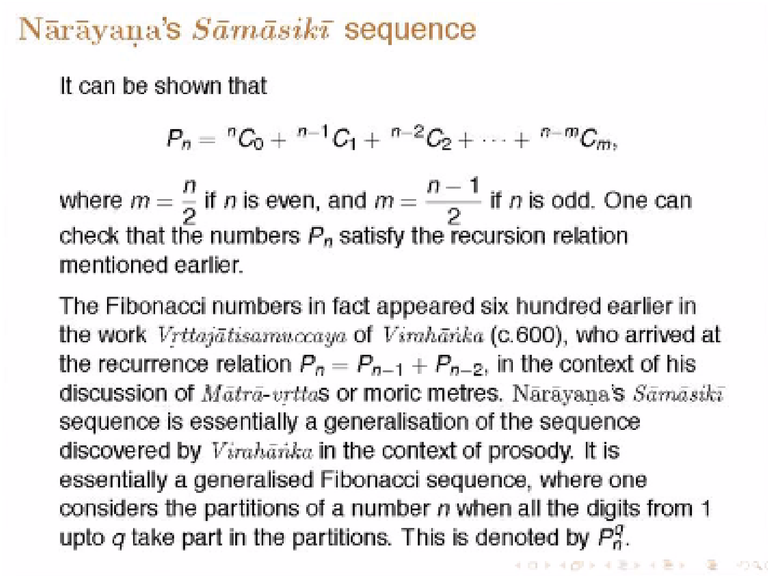

This nice relation and you will get the things, so he generalizes this and it can be shown that this Pn will be = nC0+n – 1C1 etc., up to n-m Cm, where this m is n/2 if n is even and m is = n – 1/2, if n is odd, I think this also was I think done in that lecture. One can check that this will satisfy the recursion relation like this okay. So, remember the origin of this kind of how would this go; this is understood okay.

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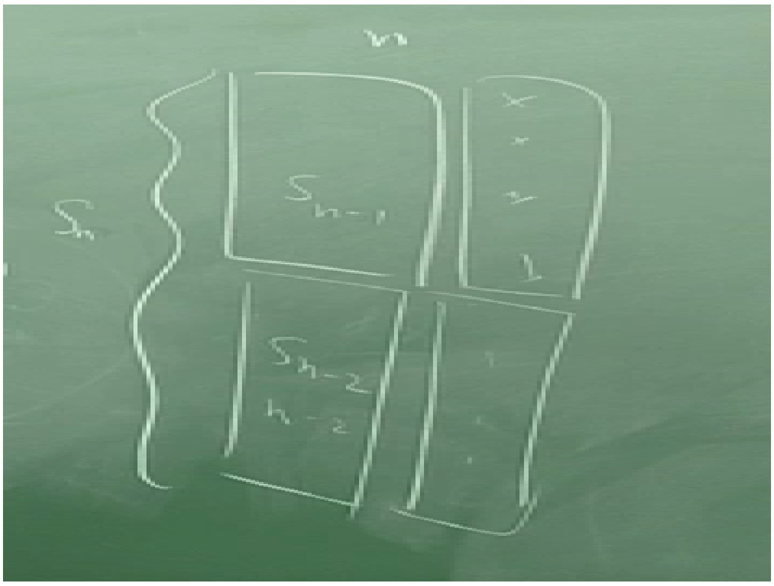

Suppose, you have got this total is n; n - 1 and n -2, okay so then what was explained earlier was that you know suppose, you have got this n - 1 Matra kind of a thing, so then you add a laghu to each of them okay, so that will become n, you see total will be n and similarly if you have n – 2, okay n -2, you add a guru to each of them okay at the end, so this number is S n – 1, so this number is S n – 2, the total will be a Sn.

The total number of the arrangements said that the total becomes n, right so that is total is the same, so that is the logic here total is n, actually yeah, the number Sn. So, now you go to the Fibonacci numbers in fact, actually appeared 600 years earlier in the work with (FL) who arrived at the recurrence relation Pn is = Pn – 1+Pn - 2 in the context of a discussion of Matravrttas or moric meters, this was discussed in earlier lecture by professor MD, Srinivas.

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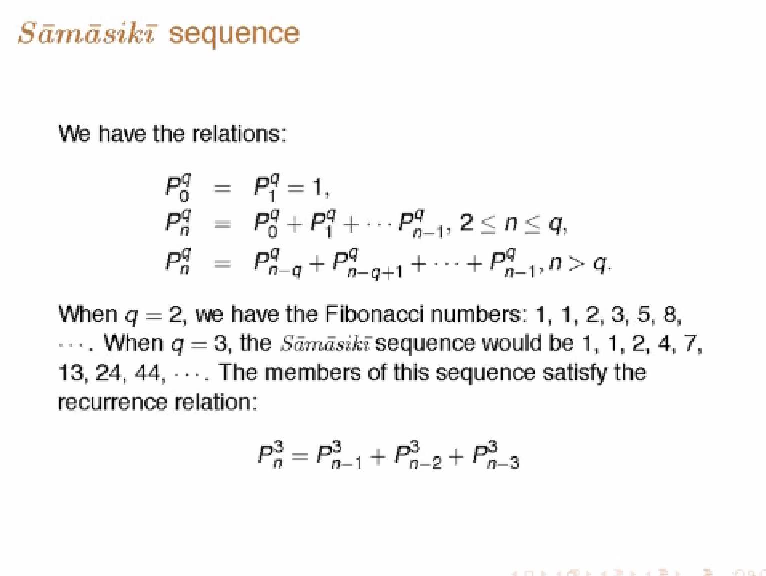

Narayana Samasiki sequence is essentially a generalization of the sequence discovered by Virahanka, in the context of prosody, it is essentially a generalized Fibonacci sequence, where one considers the partition of a number when all the digits from 1 to q, take part in the partitions, so this is denoted by P and Q n subscript and q superscript. So, then in that case, one can show that we have got these relations.

P0q is = P1q is = 1 and then Pnq is = this and similarly Pnq is = this up to you know summing you know this is n - 1 q terms are there, q terms are there in the sum, so when q is equal = 2, we have the Fibonacci numbers 1, I mean the 0 term also I putting, 1, 1, 2, 3, 5, 8 of course, next is 13, not 12 and q is = 3 is; if you take q is = 3, now the Samasiki sequence will be 1, 1, 2, 4, 7, 13, 24, 44, etc., so here Pn3 is sum of the previous 3 okay.

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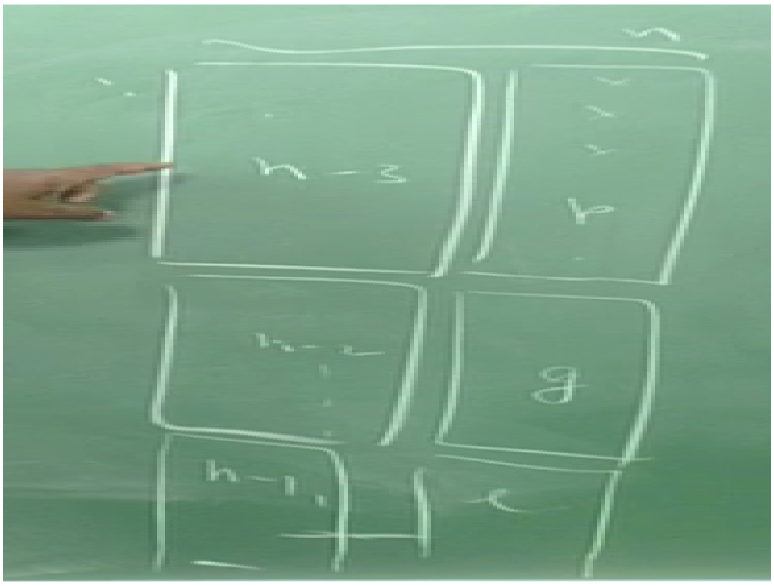

So, 4 is 2+1+1; 7 is 4+2+1; 13 is 7+4+2 like that you see, so this also could be understood in a similar manner, so instead of these 2, you know now laghu and guru suppose you have laghu, guru and Pluta okay, so then you see, what is the number of combinations with total matra to be n, and you see here; what you do is; suppose, you can start with the n - 3 okay, which is the total this thing n – 3.

So, note to it add this Pluta, which carries a value 3, okay, so then you will get total to be n. Similarly, n – 2, you take and add a guru, which will carry the number 2, right so this will be n – 2+this thing that will be become this, so and similarly n – 1, you add the laghu here, so this also will give rise to the total number n, so if you have to sum of all these arrangements that will be this Sn3, I mean Pn3 or Sn3, okay.

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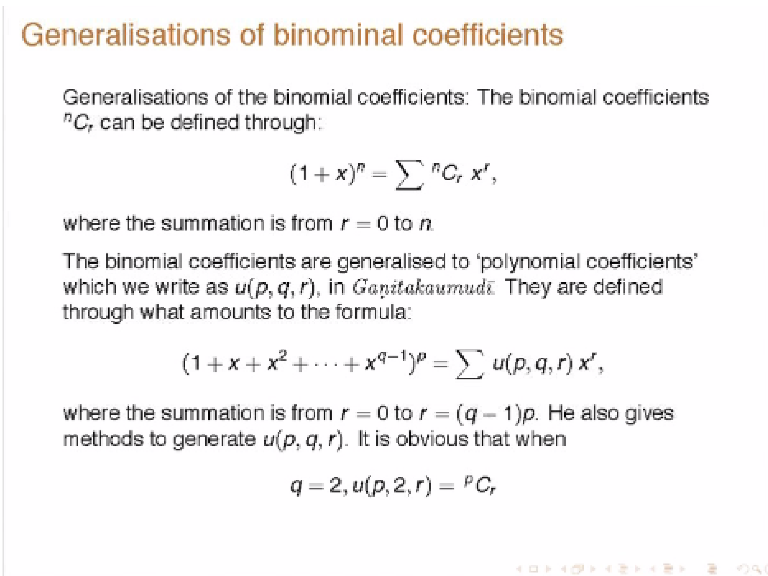

So, Pn3 is you know Pn3 n - 1+n – 2, n – 3, so that is how we can understand and you can generalize it to q, so as you can see he is always in the business of generalisation, okay. Then, he generalized the binomial coefficients also, the binomial coefficients are defined through this; 1+x, you know that you get from this, nCr*x to the power of r, right; sum is =; r is = 0 to n, so this is the binomial coefficients.

So, now he is generalising the polynomial coefficients in Ganitakaumudi, so what it; does is take 1+x+x squared + etc. + x to the power of q - 1 whole to the power of p, so here only the first power of x is coming, now you are going up to q - 1 power okay. So, then you can write it as upq because this; see here only n is there, one index, so here pq is there, they also fixed these things, r of course varies, so u pq r x to the power of r where the summation is from r is = 1 to;

Because the highest power is here is p*q - 1 and he gives the methods to generate qp, was he does not write it in this fashion, you see, i will talk about you know the number 1, 1, 1, 1 etcetera multiplied by several times and so on you see but it amount to that; at the amount to that. So, then if you take q is = 2, of course it will be reduced to this binomial theorem and you get pCr, p2r.

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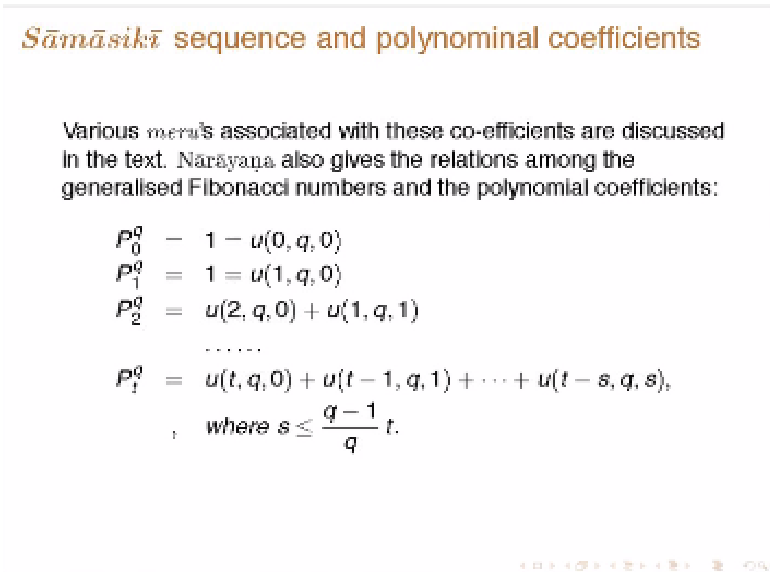

Otherwise, and then he will give the relations between these generalized Fibonacci numbers and these use, so there are very interesting. So, it is fairly quite advanced you know, it is just 14th century, remember you see all these things became you know very common currency much later in other countries, so it is quite seemed to be comfortable with this; fairly advanced topics.

So, I given a glimpse of Ganitakaumati, some of the important results, some more things will be discussed by professor Srinivas, so I will stop here. **”Professor – student conversation starts”** That same thing as you know whatever is given in that Ganita Sarasankara I told know, same thing is repeated yeah, see many other topics, they retain those things you know, so because they do not want to leave anything, all those things plus something addition.

So, it comes in Ganita Sarasankara, it comes in Lilavati, it comes in; nothing more is added as far as they know I do not think, yes added anything to that. Yeah, when we are talking about the third diagonal, when the sides b and c interchange you get, ac prime, suppose we interchange the sides a and d, can we get another diagonal; just out of curiosity? Ah, that also it can be done.
“Professor – student conversation ends”
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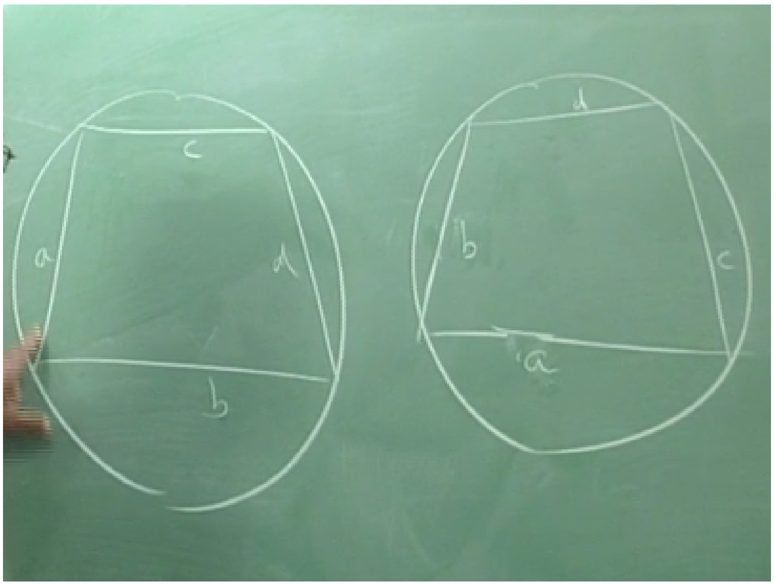

But relatively I think, finally you will be essentially only 3 will be there, see that will be you know rotating the whole figure, that is what I think, it would be I essentially rotating the whole figure, the order is what is important know. See here you what you are saying, so whatever is you see, a, b, c, d okay, so next we are getting a, b, c here and d, they are interchanged, right. So, now you are saying, suppose we interchange b and a okay, b and a let us say suppose, we do, so then this ac and db, so this will be ac, db.

Essentially it will be this triangle rotated, we are getting the same thing you see; ab; see these are the original thing, now you are saying interchange this, so ba, so essentially what I am saying is it is this only rotated you know, so ba, cd so ba, cd okay in the other way, I mean the relations between various things will remain the same yeah, so you will not get anything new. Essentially you will get 2 figures, you know, so that is what is this.

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Double sigma is the second summation; second summation, so first Sigma r is 1+etc up to n and second is for; you do the first sum; so n*n+1/2, replace n by r, then r*r+1/2 that to sum from 1 to n, so that is what it stands for second sum means that yeah, yeah you have just written like that, the references are given here, thank you.