Mathematics in India: From Vedic Period to Modern Times
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Lecture-24
Bijaganita of Bhaskaracarya 2

We are continuing the discussion of Bijaganita of Bhaskaracarya, so this talk will be continue.

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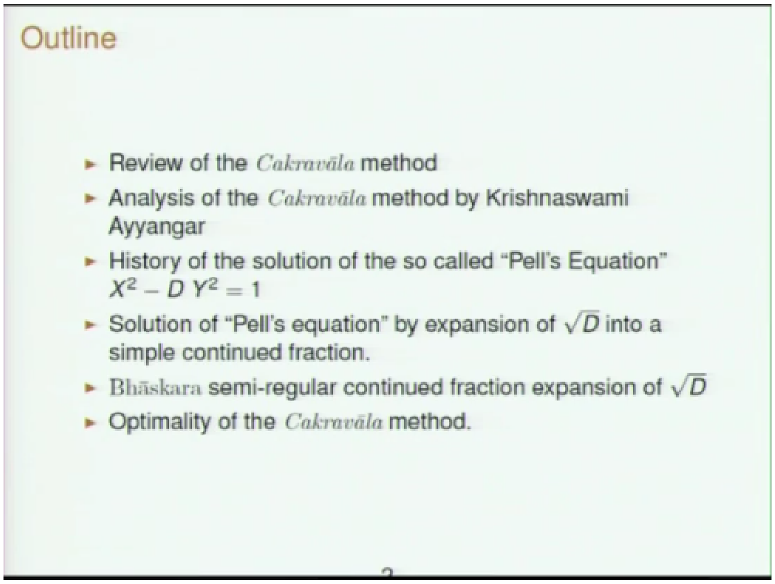

This is continuing with the analysis of the (FL) process of the (FL) method first I will remind you what we discussed yesterday the basic algorithm of the (FL) method then I will summarise the analysis mad by Krishaswamy Ayyangar then we will go to the currently known or what is thought in the textbooks are the solution of this equation, equation x2-D y2 =1 is known as the pell's equation how it was solved in European 6, 17^{th}, 18^{th} centuries.

Then I will compare the solution of Bhaskara or the older (FL) method of India with a (FL) method of solution. Finally he says something about the optimality of the (FL) method.

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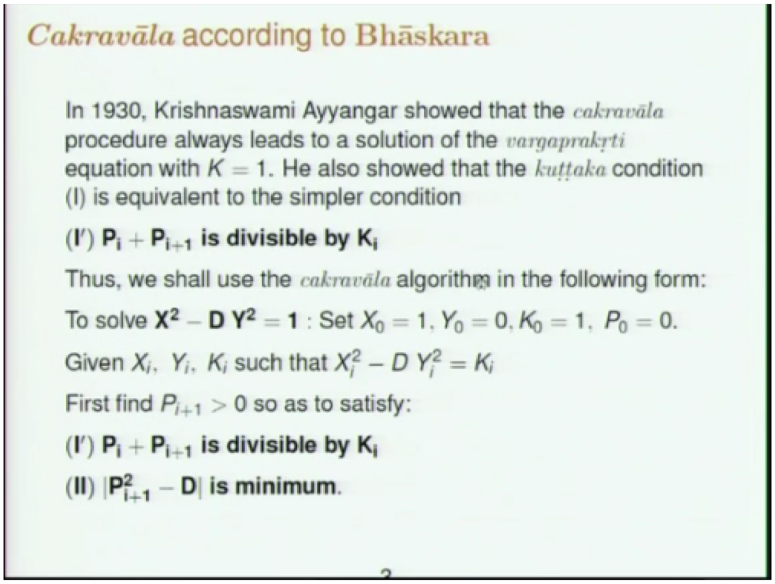

So, as I said the (FL) condition which was there in the (FL) method was replaced by the simpler condition that Pi+Pi+1 is divisible by Ki by (FL) and so the algorithm (FL) algorithm that we are using will be in this form use all x square-Dy square =1 remember D is a non-square integer and you want to find out x and y in integer. So, you start with initial values X0 =1, y0 =0, K0 =1, p0 =0. In any step Xi, Yi, Ki such as Xi square-D yi square =Ki find the Pi+1 such that these 2 conditions are satisfy Pi+Pi+1 is divisible by Ki, Pi+1 square+1 is minimum.

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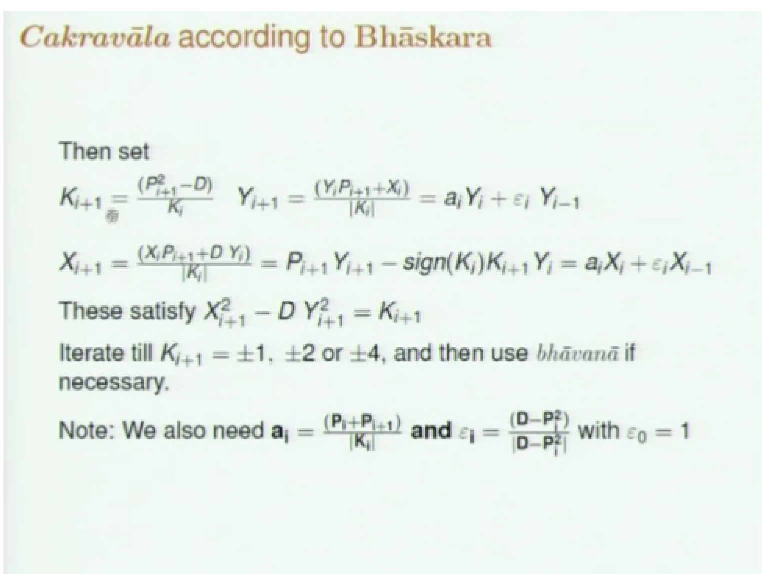

Then once Pi+1is found you can find out Yi+1, Ki+1 and Xi+1 and for that use the auxiliary quantity ai who significance we will see in the minute, ai is defined as Pi+Pi+1by Ki in epsilon 1 is +1 is D is greater than Pi square and xi square and epsilon 9 is -1, if D is less than Pi square, so this Xi+1 and pie i+1 that you are calculated satisfy Xi+1 square-Dyi+1 square is Ki+1. Now the Ki+1 value could anything, so you have to keep iterating the algorithm till you get +1 in which case the problem is solved.

But you can always do (FL) if you find -1 or +/-2 or +/-4 you can always do (FL) and go to the solution, so this is the (FL) algorithm that we discuss in the last class.

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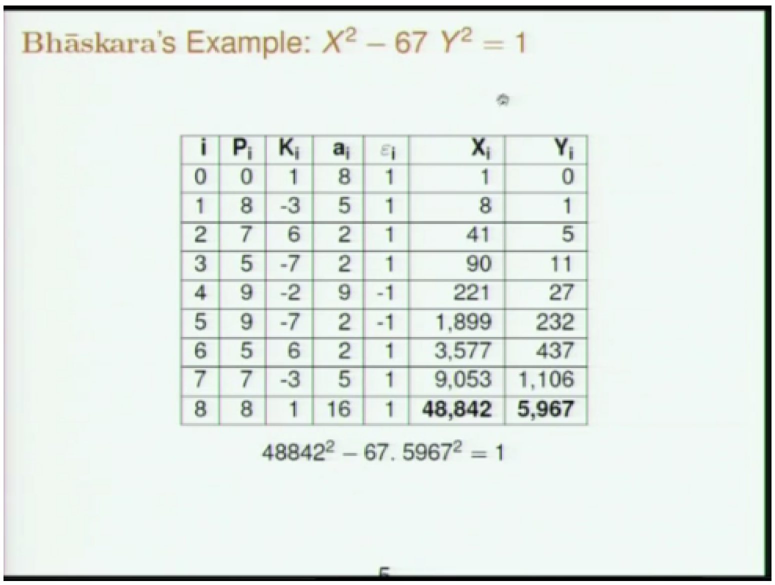

i	P_i	K_i	a_i	ε_i	X_i	Y_i
0	0	1	8	1	1	0
1	8	-3	5	1	8	1
2	7	6	2	1	41	5
3	5	-7	2	1	90	11
4	9	-2	9	-1	221	27
5	9	-7	2	-1	1,899	232
6	5	6	2	1	3,577	437
7	7	-3	5	1	9,053	1,106
8	8	1	16	1	**48,842**	**5,967**

$48842^2 - 67.\ 5967^2 = 1$

So, to the collect let X square-67 y square =1, so here the initial step is x is 1, y is 0, p is 0, a is 1. So next step ei+Pi+1 is divisible by Ki and Pi square is closest to 67 then we have to choose this to be 8, then 8 square-67/1 maybe-3, so the Pi and Ki you will be 8 and -3 and if you use the **the** formula gave you Xi and Yi will turn out to be 8 and 1. This of course it will be 8 square -67 into 1 square it is –p.

Next step, so 8 you have to add Di+1 such that it is divisible by 3, so possibilities are 4, 7 and 10 of them 7 square 49 is closest to 67 choose 7, so 7 square-67 divided by –p is 6, so in this case you get p and k to be 7 and 6. You can fill up this ai column also a0 is 0+8/1 that is 8, a a1 is 8+7 15 divided by 3 that is 5.

So, in the next step from 7 you to find out Pi+1, so it should be divisible the sum should be divisible by 6, 5 and 11 are the possibilities, 5 square is closer to 67 that 11 square choose 5 then you will get 5 and -7, so at this point you have 90 square-67 into 11 square is -7, next you will go you will immediately get the K of -2, so to 5 you have to add a pi+1 such that the sum is divisible by 7, possibilities are 2, 9 and 16 amongst them 9 has the square closest to 67.

So, you put 9 then 9 square-67 divided by -7 is -2, so you have which reach the (FL) of -2 you can do (FL) at this point 221 square-67*27 square is -2 you do the (FL) by 221 and 27 you obtain the final solution 48842 square- 67 into 5967 square =1.

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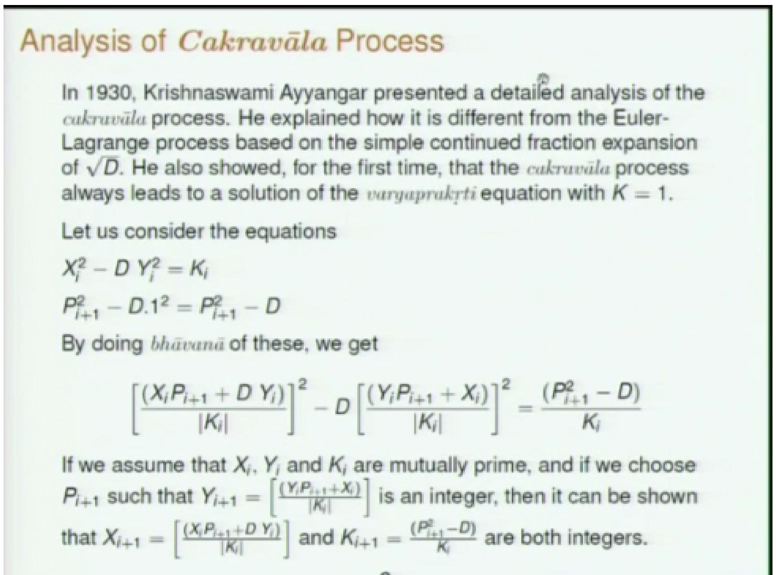

So, let us summarise how (FL) analyse this algorithm he proved that this process always leads to K =1, he also showed that the process is different from what is known as the Euler-Lagrange method which is based upon the simple continued fraction expansion of root D I will explain all these. So, now first step let us start with Xi square-D yi square =Ki, think of an auxiliary equation like this Pi+1 square-D into 1 square the Pi+1 square-D do the (FL) of these two.

Then you will get this square up on the right hand side right but take this product to the take this Ki to the denominator then you get an equation like this. If you divide the product of these 2 by Ki square then you will get an equation like this, so by doing just the (FL) of these 2 equation you have obtain this you can recognise these are the quantities appearing in the steps of the (FL) equation.

So, the first thing is to show that the quantity that appear in this algorithm are all very defined then non-negative, so that of the first thing that we have to do when we analyse the algorithm like this. So, what can be shown is it is very simple if you assume that Xi, Yi and Ki are mutually prime that means they have no common devisers and if we choose Pi+1 such that Yi+1 =YiPi+1+Xi by mot a is an integer that was the (FL) version of the (FL) algorithm by the first condition was put in the (FL) there.

This is chosen to be an integer by the algorithm then you can show assuming that Xi, Yi, ki of prime then you can show by this algebra equation that Xi+1 and Ki+1 are also both integers, so that is the first thing.

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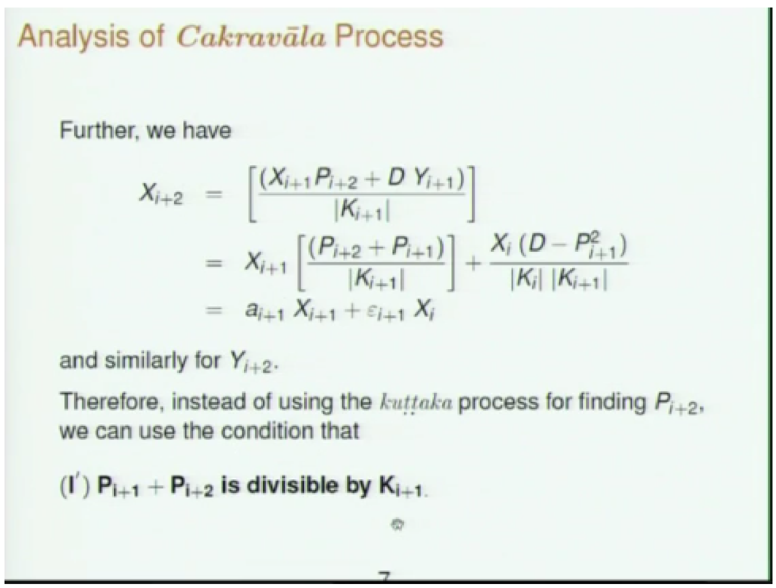

And if you write the algorithm for the next step you will get an equation like this and that equation can be simplified into this forum which will show you that is ai+1 or ai+2+Pi+1 by Ki+1 that is these have to be integers. So, (FL) version it is just obtain by second iteration of the algorithm therefore this (FL) 2 point in Krishnaswamy Ayyangar’s proof he showed first that the algorithm is well defined that these quantities are all integers.

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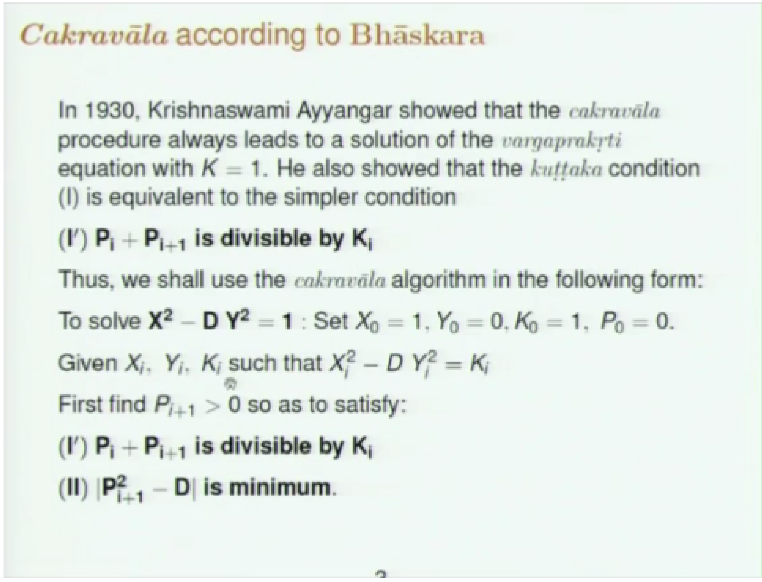

If you start with an initial situation then you showed that the (FL) condition is equivalent to a condition like this, so this algebraic relation shows you that ai is an integer if I know the Pi+Pi+1 by Ki+1 is an integer in that what gives you the equivalents between the 2.

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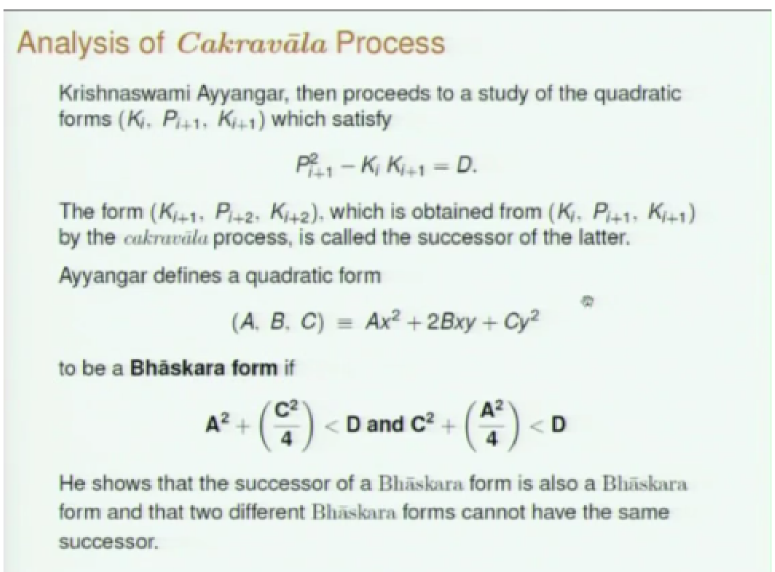

Then comes the proof that is always converges to k =1 that somewhat more complicated that is based upon what is known as the theory of quadratic forms I will not tell you detail a quadratic form A, B, C is something like this Ax square+2Bxy+Cy square (FL) is using this Ki is Pi+1 and Ki+1 to make a quadratic forum and that Pi+1 square-Ki Ki+1 =D, 2 quadratic forms are equivalent if B square-AC is the same for both of them.

So, you have a class of all quadratic forms of the forum Ki+pi+1Ki+1 such that they this disdetermine D then he defined quadratic form to be a Bhaskara form when this kind of a condition is satisfy and now in the (FL) what is happening you start with K0 and a P0 then you go to P1 first from that you will find K1 also, so in the (FL) you will successively get successive quadratic forms.

So, this successor of a Bhaskara form by doing (FL) process Krishnaswamy Ayyangar showed is also a Bhaskara form.

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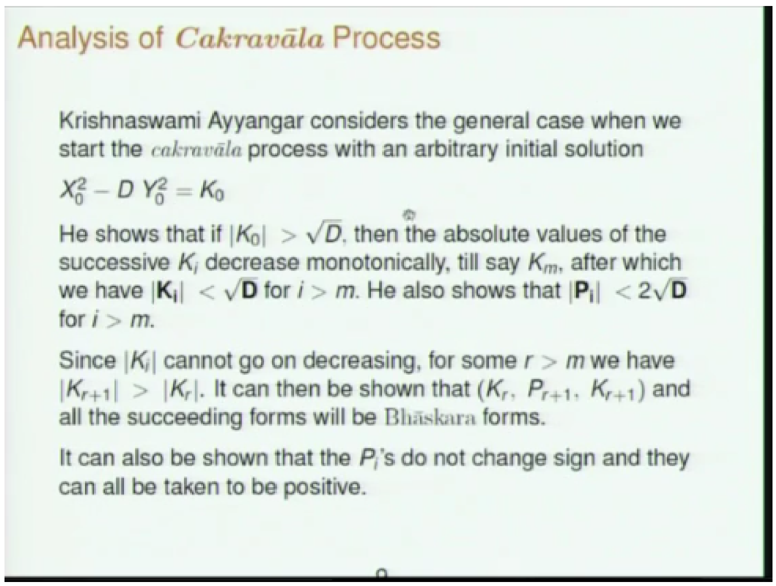

And then he even showed that you can start with any value of K and do (FL) you will eventually come to a value of K which is less than root D and later on you will move within the region K less than root D and P less than 2 root D. So, both case and P's are bounded by root D and 2 root D and K and P are integer and if they are bounded by root D and 2 root D after sometime they have to repeat in a cycle.

So, already the cyclic property of the algorithm is proved finally he showed that the amongst the set of all equivalent cycles of by doing the (FL) process.

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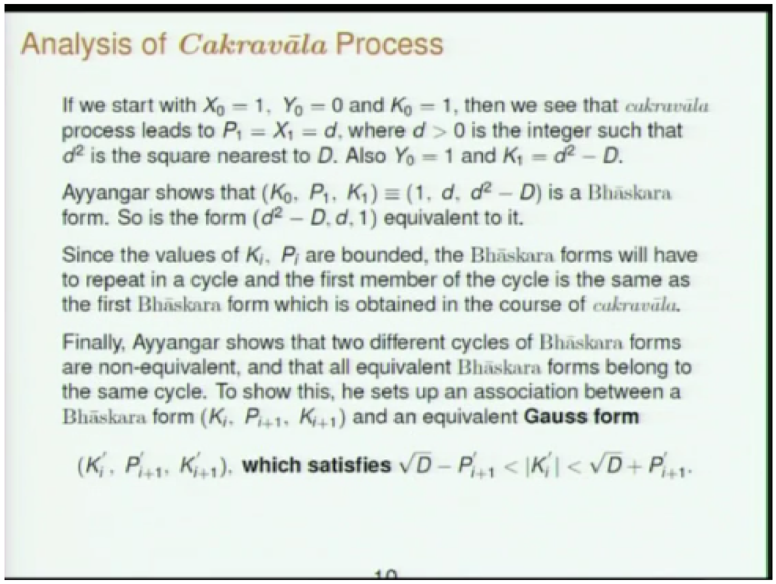

The Bhaskara forms go in a cycle he showed that any 2 cycles of Bhaskara forum are not equivalent and all equivalent Bhaskara forums coming the same cycle to do that proof he use the older proofs that were done by (FL) or quadratic forms are what was known as the (FL) forms and therefore he was able to show that you start with any K whatsoever you start with any xi any yi such that any k is there Xi square-D yi square =Ki go on doing (FL) you will always first come to a value of less than root D.

And eventually you come to the vale 1 and so this was the proof of (FL) now we quickly go to the Euler-largrane method, so what is the history of this. So, in1657 sharma the famous French mathematician he post a challenge to the British mathematicians first.

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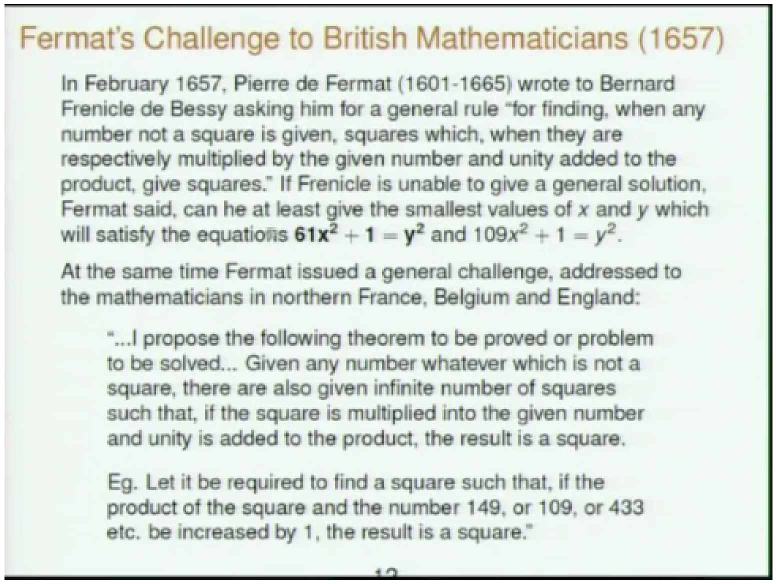

You wrote to his friend (FL) saying that can you solve these 2 equations 61x square+1= y square and 109x square+**y** 1=y square for x and y in integers already you see that the 61 x where it is coming there and within couple of months he wrote this as a problem and sent it as a challenge to mathematicians in Frnace, Belgium and England the question was the same thing can you solve the equation x square-D y square =1 in integers.

And he say people know the kind of an equation but the important thing is to put the restriction that you want solution in integers and this is the birth of the theory of numbers in modern mathematics as and the rail says that this was the birth of theory of numbers essentially the problem. Now it was sent to England William Brouncker was the president of royal society and his friend John Wallis was the professor in Cambridge prior to Newton.

So, both of them wrote down the solution first they wrote down the rational solution that Brahmagupta had written down in 628 and 72 (FL) said you are cheating and ask you for integral solution rational solution (()) (12:39) so, then of course they sat down and worked out the integral solutions (FL) had asked for 4 particular cases (()) (12:50) had asked for D =61, D =109, so he had post the problem for 149, 109 and 433.

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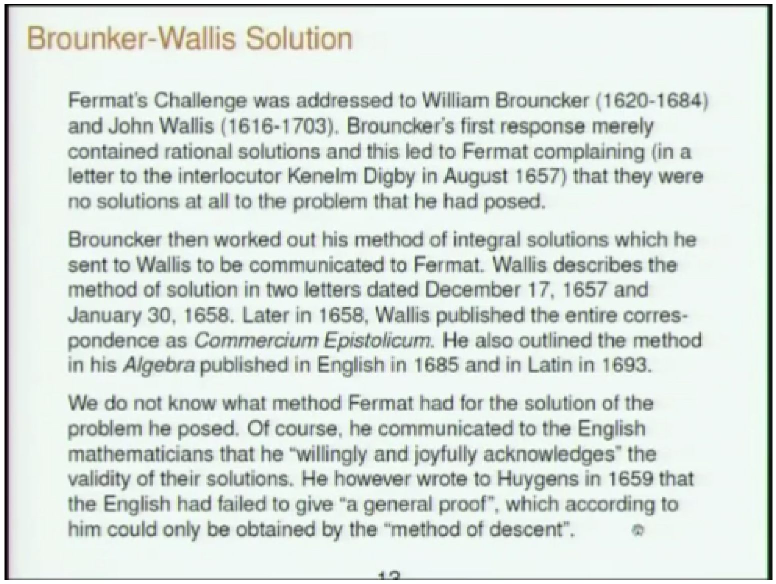

He knew that they are going to involve large number of steps, so to check that you are proficient in (FL) you solve the (FL) for all this (FL) equation for all this 5 examples 61, 4 examples right.

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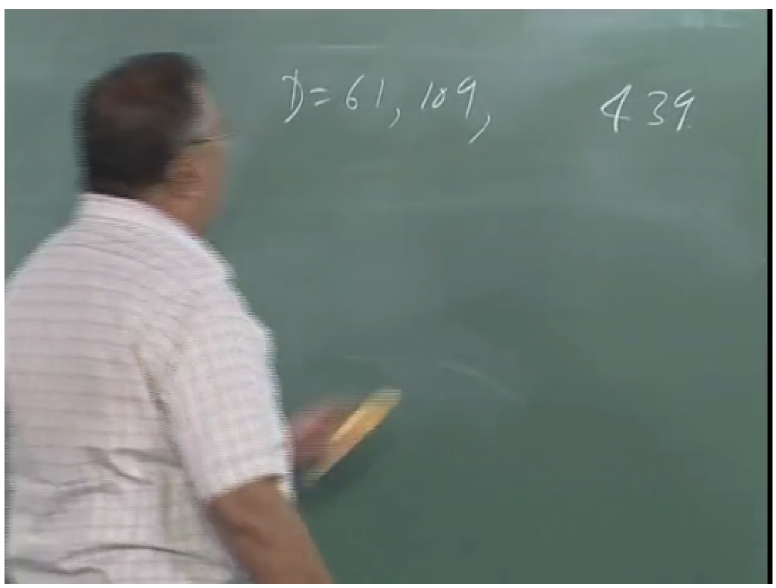

D =61, 109, 439 is a 433 and what is the other one 149 and 433.

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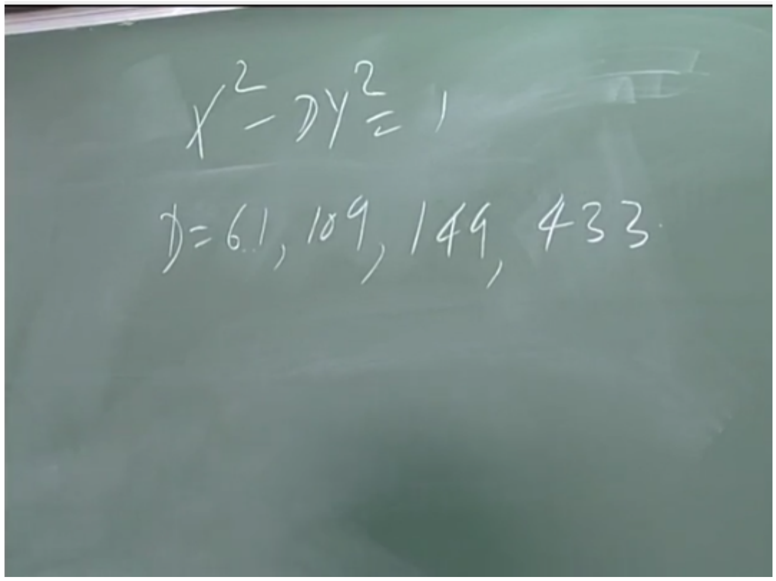

If you do that so just this we saw already gives you solution which goes to trillions the order on magnitude of x. So, the solution was communicated by (FL) in letters and then Wallis published all this in1657 in latin and later on he wrote a famous book an algebra in 1685 where this solution was put in that algebra equation came in let it is one of the earliest science books in English Wallis algebra written in 1685.

Now for now of course wrote 2 English mathematicians saying that he willingly and joyfully acknowledges the validity of their solutions but privately he complained to his friend Huygens that they cheated they had not given in a general proof that they were asking.

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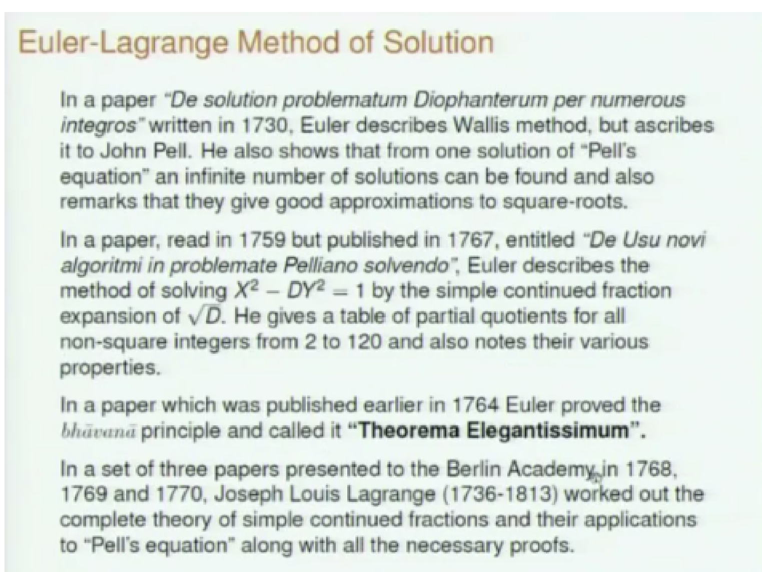

And he said this proof must be based upon the method of this end (FL) proved very little in his long carrier but he suggested lot of theorems for which he is wrote in the foot notes and I have a proof which is too long to be communicated here and many of them were based upon the method of design. Now Euler in 1730 starts again his journey into the same equation it is like (FL) starting from larger number coming down to smaller numbers (FL) is the standard method of design.

Euler in 1730 describes Wallis method and ascribes to John pell and so naming of this equation as pell's equation is due to Euler, Euler wrote a famous book on algebra and where he called it Pel's equation and that name as stuck and he shows first Euker start showing that there are infinite number of solutions then in 1757 he gives a method for solving based upon the simple continued fraction expansion of root D and he also gives a table of solutions from 2 to 120 for this.

And in 1764 Euler rediscover the (FL) principle he called elegant most elegant theorem elegant as elegantsium meaning (FL) as we say in sankrit. So, this is the most elegant theorem and in 1768 to 70 (FL) wrote a series of paper where he proved everything that Euler hide that, so the Euler method become well established and proved by 1770, so what is the method, so what is the simple continued traction.

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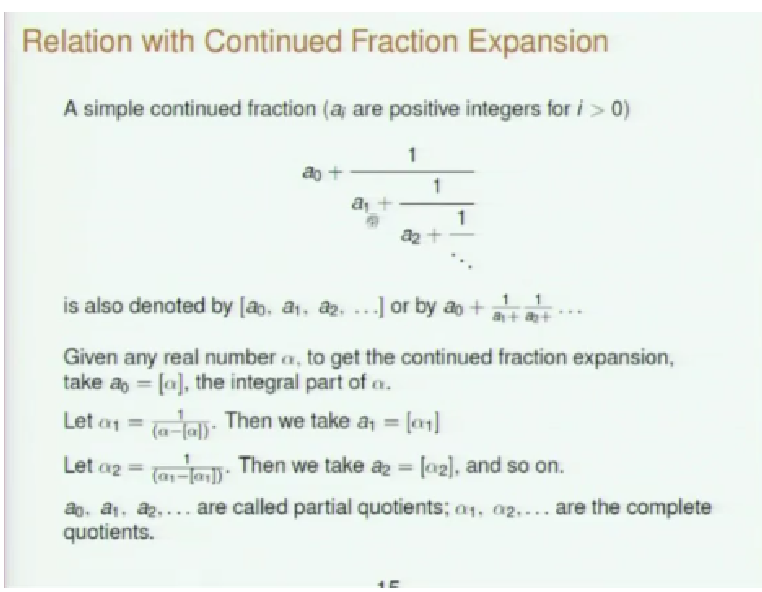

So, normally a0+1/a1+1+a2+1/ etc., further division like that is called continued traction it is a called a simple continued traction if only once appear here a0, a1, a2 are all integers they are positive at the the first one because integer can be negative. So, the various ways of denoting because this way of writing it will take too much space, so one were denoting it is by writing it this way another way of denoting it is by putting the + in the denominator.

So, hypographically inconvenient in typewriters but while writing it is quiet easy now how to get a continued traction associated with any real numbers just way pay some attention to this the first partial quotient these are called partial quotient, the first partial quotient a0 is just the integral part of alpha what is the integral part of alpha if the number is 1.3, integral part of that is 1, if the number is 1.9999 the integral part of that is still1.

Now subtract integral part of alpha from alpha, so that will lie between 0 and 1 take 1 over that take the reciprocal of that that will much more than 1 that is called as alpha 1, then the integral part of alpha 1 call it a1 and now alpha 1-b integral part of alpha 1 take the reciprocal of it, this an number larger than 1.

Then the integral part of that alpha 2 that is a2, so that way you start getting a0, a1,2 etc., a0,a1, a2 are called partial quotient alpha 1, alpha 2 are called complete quotient.

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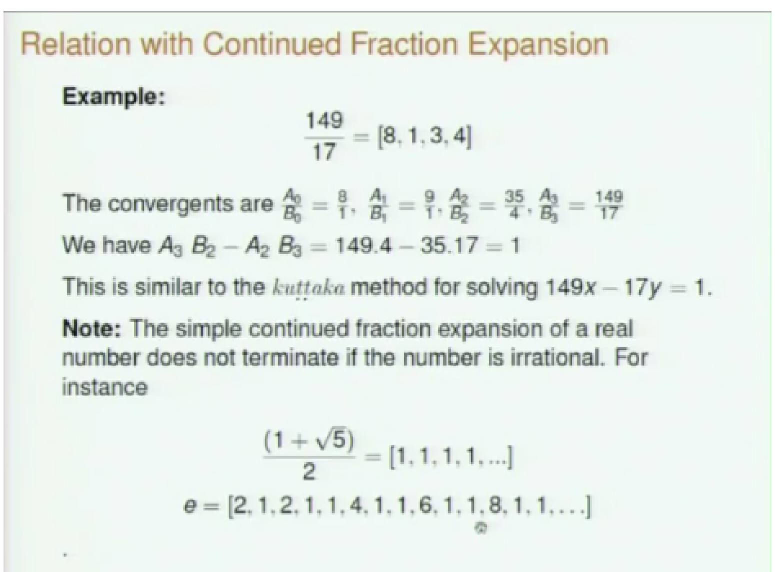

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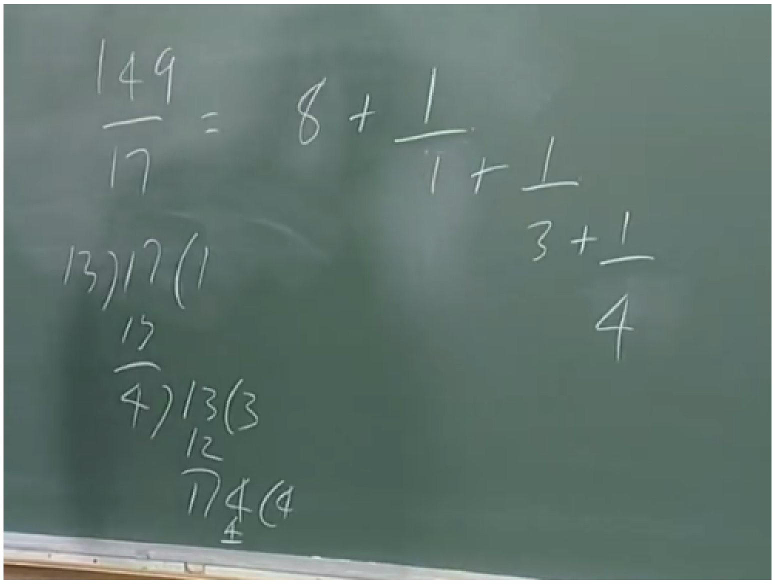

So, let us try this example 149/17 we want to express this as continued traction, so basically this is 8+1 over no no 9, 8 80 136, so 13/17, so now write 1 invert this, so this 1 over you are seeing some similarity with what we were doing earlier. So, this will be 8+1 over 1 is the 13+1+1 over 1+1 over 1 something like this 81 is 3, 4 things we getting where is the 4 oh yes.

So, this how it will look by mutual division you can get the continue traction for any rational number and if it is a rational number the continue traction terminates. And if it is an irrational it does not terminate here are 2 irrational numbers which has nice continue traction 1+root 5/2 called the golden ratio it is continue traction is 1, 1, 1, 1 e is the base of natural logarithms as a nice continue traction 2,1, 2, 1, 1, 4, 1, 1, 6, 1, 1, 8, 1, 1 etc.,

Unfortunately 5 does not have a nice continue fraction calculating the 10,000th partial quotient of the continued fraction of pie is as difficult of calculating the 10,000 decimal place of pie. So, they are equally random.

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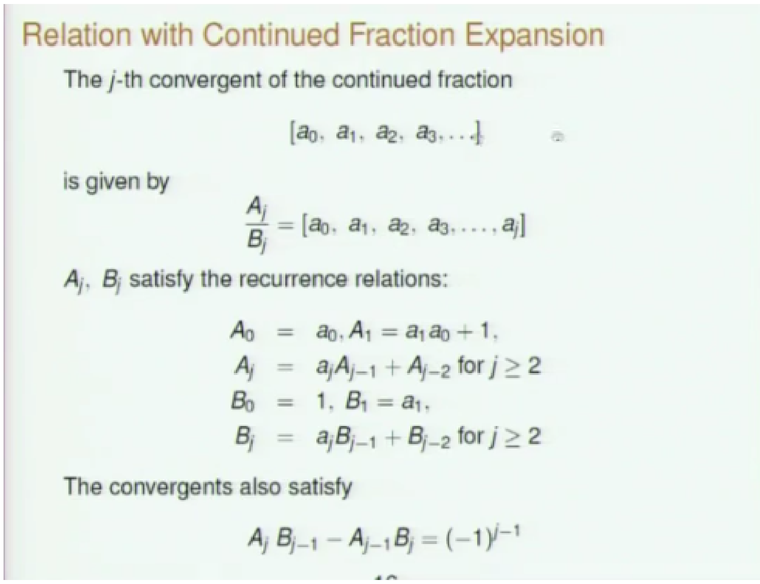

Now this you had continue fraction like this you stop at any point then that will become a rational number you calculate it, it can be written as Aj/Bj that is called convergent of the continue fraction. So, this conversion side this side recurrence relation these are essentially what is there in the (FL) process of Aryabhatta this calculating the convergent backwards is the (FL) is recurrence relation are the essentially the (FL) relations.

This a0, a1, a2 are the quotients that come in the mutual divisions of the (FL) and (FL) in (FL) and this convergent have to satisfy this property AjBj-1, Aj-1Bj is -1 to the power j+1. So, AjBj are obtained it will have a continue traction whether it is terminating or not terminating you terminate it any point you get the corresponding convergent. So, those will be rational approximations to the irrational number null number that you have.

They will optimal in some sense, so the conversions of this are so you can straight away see if you terminate it here it is 8, then 8+1 over 1.

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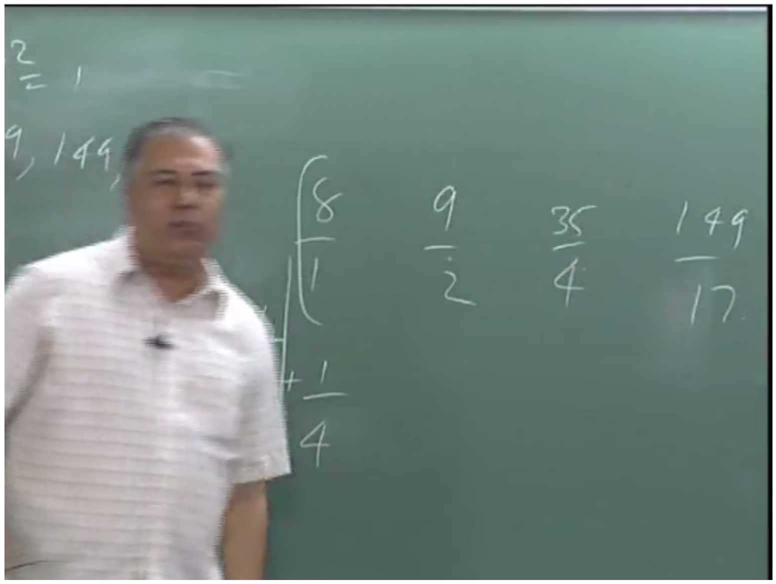

So, 8/1 is the first convergent, 9/1 is the next convergent if you terminate it here if you terminate it in the place 3 you will have 35/4 and if you go all the way it is 149/47 sorry whatever it is started thank you, so, these are approximations to 149/17when in some sense best approximations a later they will have book called (FL) gives when it is algorithm for doing this convergent and use as them as approximations.

Because the number of civil days in a (FL) is in trillions the number of solar years in (FL) is also in 43 lakh 20,000 years into 1000 and you need ratios of this to calculate the positions of sun. So, those numbers fractions are very huge and you can always approximate them by fractions which smaller denominators and the best approximations are what are provided by the continue traction expansion of it.

And kind of a (FL) excels in giving this you make in use of this continue traction kind of results yes sir (()) (22:47) 9/2 or 9/1 oh 9/1 sir, 8+1/1, 1+8+1/1 whatever written in the slide is always more accurate than what I do on the board which is done at this for of the moment. Another interesting thing just we connection with (FL) this previous convergent we wrote this relation AjBj-1-Aj-1-Bj is -1 to the power j-1.

So, you have the relation you use this convergent 35/4,149 into 4-35 into 17 =1, so for this 149/17 and 35/4 this will be the 2 solutions for the (FL) involving 149 and 17. So, the entire continued fraction theory is as old as Aryabhatta's Aryabhatiya basically the entire thing is there. Now coming to our problem solution of the (FL) showed that solution of the (FL) can always be written as a periodic continue fraction.

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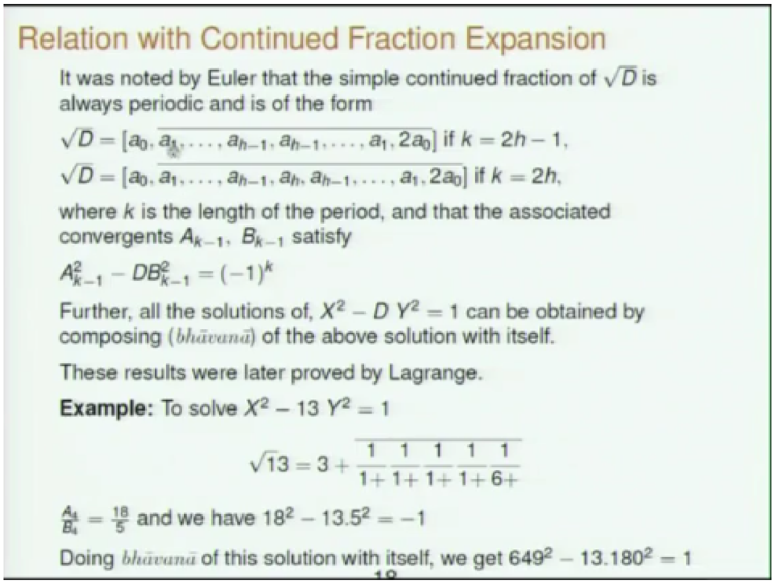

Periodic means at the end of this whatever I have put in the bar will keep repeating each other.

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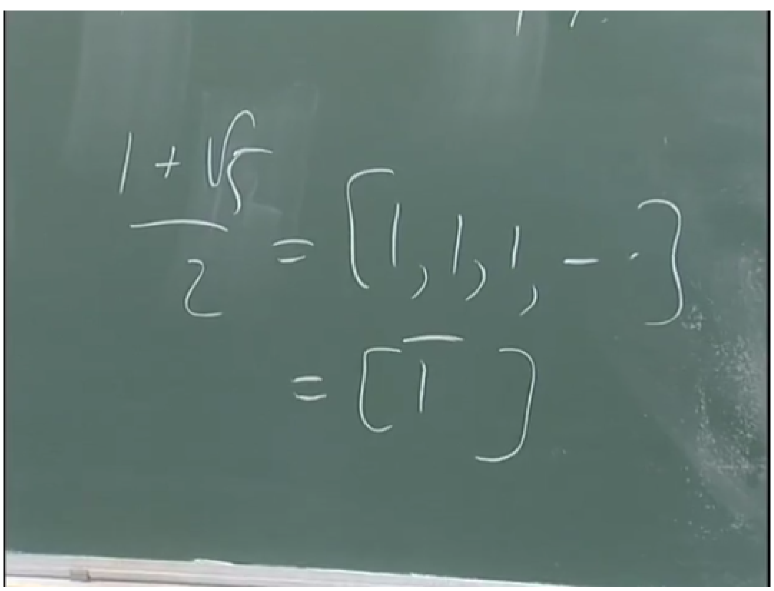

In fact these all 1 lines quadratic cell which had a periodic continue traction expansion, the period started right at the beginning right th=1,1, so it is nothing but 1 bar it is a period starts at right at the beginning. So, all quadratic sides have a periodic continued fraction and vice versa and root D is always of this forum with the last entry 2a0 if this is entry is a0, this called the period, the period can be odd, period can be even.

And the penultimate convergent here at the end of the period will give you a solution on -1, if the period is odd it will give the solution for +1, if the period is e1 if you obtain the solution for -1 you do (FL) and get the solution for +1, so if the period is odd then you will get the solution for x square-D y square =-1 at the end of the period, so let us take this example x square-13 y square =this is the given example given in all textbook.

Square root of 3 is 3+1over 1+ 1over 1+1 over1+1 over1+ 1 over 6 this bar means now again the same thing will start here it will go on repeating itself it is an irrational number so continue traction is in finite but it is periodic you have to take this convergent, convergent at this point 3+1/1+1/1+1/1+1/1 stop it here then you will have A4/B4 which is you can see to be 18/5 in fact we saw this solution yesterday while discussing the equation x square-13 y square =-1, the solution by Bhaskara was also this 18 and 5.

Now doing (FL) of this with itself you will obtain the solution for k=1, so this Euler method or we can call it Euler legrange method, legrange wrote down all the proofs.

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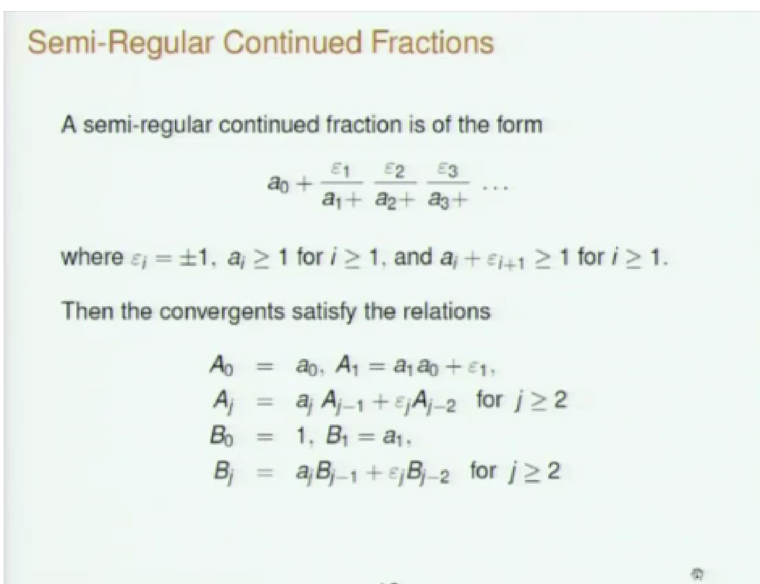

Now from phase simple kind continue traction you should go to something called semi regular one how was the different instead of only +1 is occurring here, +1 and -1 can both occur here, that is called a semi regular continued traction and certain mathematical conditions which we can ignore something like this should be satisfy. So, the convergent of the semi regular continue traction will involve this epsilons in their recurrence relations, we saw this recurrence relation in (FL) then itself.

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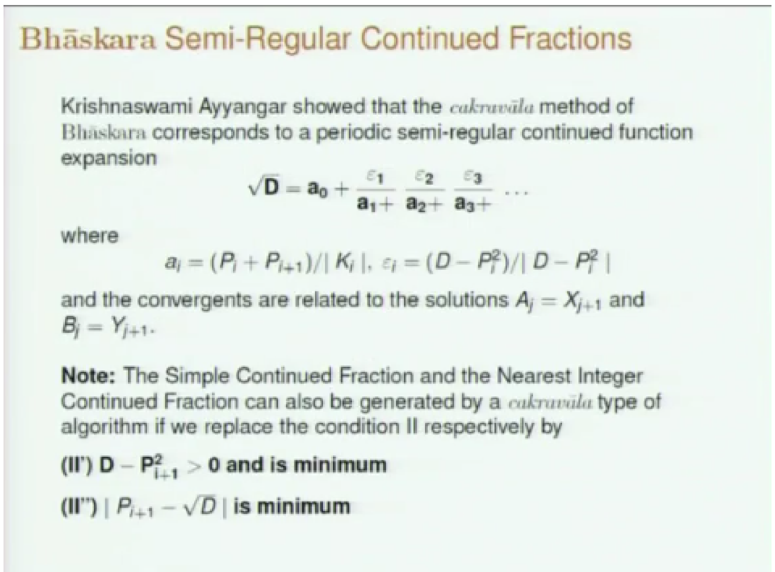

So, the first thing that (FL) did was to show that the Bhaskara method or the (FL) method at the time of Krishnaswami Ayyangar he was not aware that there were people prior to Bhaskara were done the (FL) method he of course mentions that Bhaskara is saying only that (FL). So, this must be some method earlier know to Indians, so he showed that the (FL) method corresponds to expanding root D as a semi regular continued traction a0, a1, a2 are all given by our quantities familiar Pi, Pi+1/mod Ki.

This epsilon i was also defined there, so you go back to that table you can immediately write down the.

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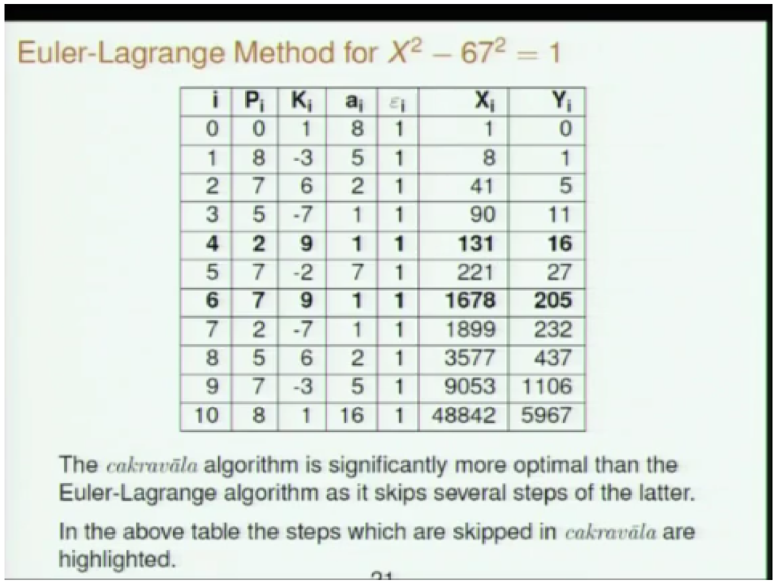

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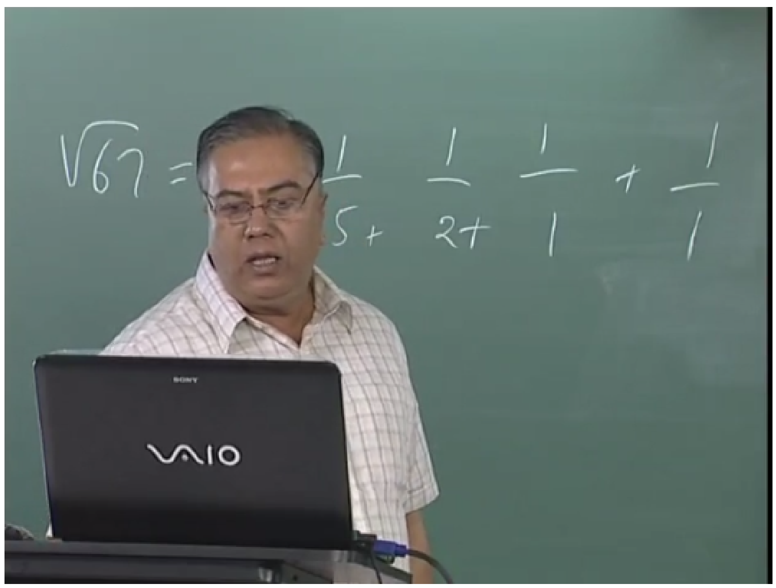

So, root 67the Bhaskara's continued traction, a0 is 8+1 over 5+1 over 2+1 over 1+ which column I am writing 8, 5, 2, 1, 1, 7 okay. Course is around 16 okay (()) (28:08). So, 8+1 over 5+1 over 2 +1 over 1 +1 over 1 oh I am writing down the Euler Largrance, so I am this is let go back (FL), mouse is refusing cooperate with me what give time.

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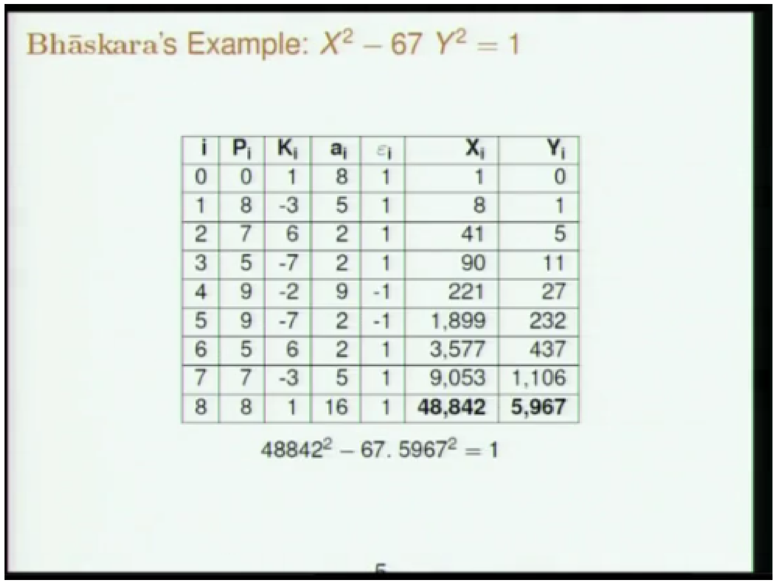

i	P_i	K_i	a_i	ε_i	X_i	Y_i
0	0	1	8	1	1	0
1	8	-3	5	1	8	1
2	7	6	2	1	41	5
3	5	-7	2	1	90	11
4	9	-2	9	-1	221	27
5	9	-7	2	-1	1,899	232
6	5	6	2	1	3,577	437
7	7	-3	5	1	9,053	1,106
8	8	1	16	1	**48,842**	**5,967**

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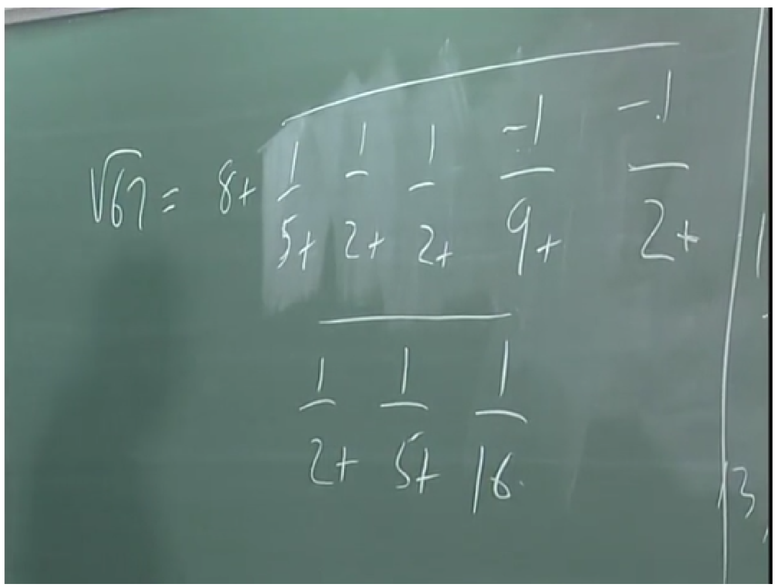

So, 8+1 over 5+1 over 2+ 1 over 2+ -1 over 9-1 over 2+ I will continue writing it here ½, 1/5, 1/16, so this is the semi regular continued fraction expansion notice right that the two -1's appearing here all of course this whole thing is periodic that will keep repeating itself, 5, 2, 2, 9, 2, 2, 5, 16. (()) (30:05).so, first thing Krishna swami Ayyangar's showed one that the (FL) method you can immediately write down this square root D as a continued fraction.

Next thing he pointed out that this simple continued traction algorithm can be viewed as a similar to (FL) algorithm only you change the condition of Bhaskara that modulus of P+1 square-D minimum you change it to the condition the D should be greater than Pi+1 square and D-Pi+1 square should be minimum, so the value of Pi+1 should be so chosen that it is less the square it is less than D and it is closest to the value of D.

And if you change it another condition Pi+1-root D is minimum you obtain what is called the nearest integer continue fraction we will see it later in the context of Narayana maybe if we have time. So, now I am doing the (FL) algorithm to reproduce the Euler lagrange method so, what I where is the differing to see where it is differing, it is differing these 2 places these 2 rows which are highlighted those 2 rows are skipped in (FL).

So, how do we see that, so you remember this in the step 3 you have 5-7 and the solution is 90, 11 right and now what does (FL) give us, 5-7, 90, 11 now took 5 you could have added 2 or 9 or 16 that in the Euler Lagrange process 9 is bar, 9 square it is 67 that is larger the 9 square it is 81 that is larger than 67 so you have to choose only 2 which is lesser than 2 square is lesser than 67, so amongst the allowed values, so here we have to put Pi+1 to be 2.

And here also 9 is appearing in this step, so these are 2 steps here in (FL) which are not allowed in the Euler Lagrance in both these cases epsilon is -1, whenever P square is greater than D epsilon will be -1, that is the sign that a step in which Euler lagrance is being skipped by (FL) at that point, so the -1's are hallmarks of the (FL) process. So, we remember there what was the next step in (FL) next step is 221 and the step after that is 1899, remember these 2.

We are going to see the Euler Lagrange process here, so 90 it comes 131 then goes to 221 then goes to 1678, then goes to 1899. So, 2 extra convergent are added 2 extra steps are added in the Euler Lagrange process after all historically he did come 500 years or 600 years apt of the (FL) process, so it has to have that much more weightage unit.

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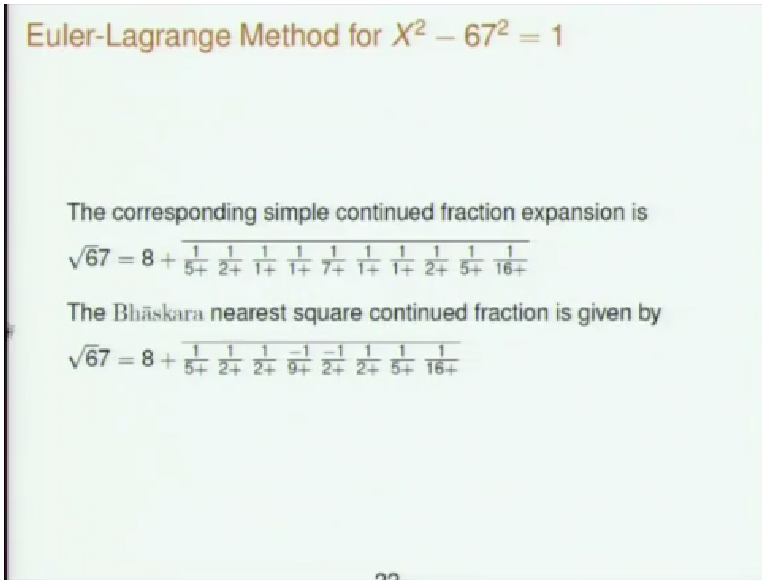

So, this sq root of 67 can be expressed as the simple continued traction it will look like this written as a semi regular continue traction to look like this, so the period here is 1, 2, 3, 4, 5,6, 7, 8, 9, 10 the period here 1, 2, 3, 4,5, 6, 7, 8, so 2 steps are eliminated in 67, so we can look at the great the old prime 61.

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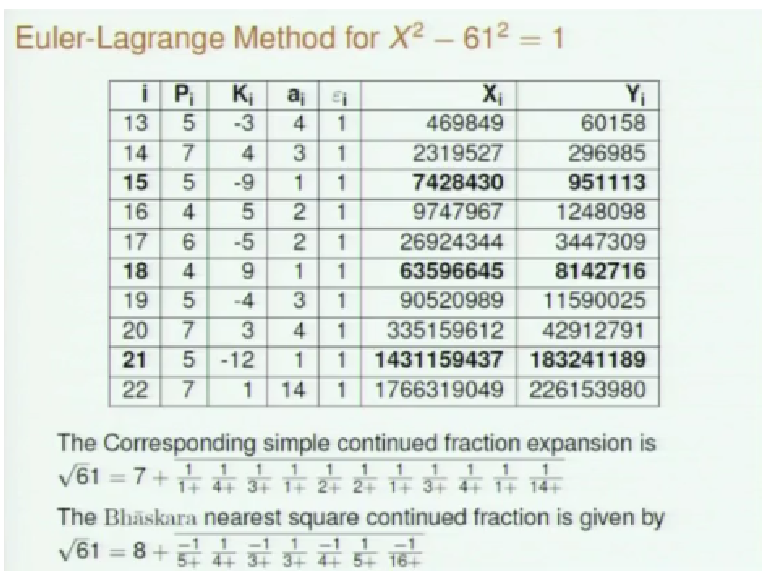

i	P_i	K_i	a_i	ε_i	X_i	Y_i
13	5	-3	4	1	469849	60158
14	7	4	3	1	2319527	296985
15	5	-9	1	1	**7428430**	**951113**
16	4	5	2	1	9747967	1248098
17	6	-5	2	1	26924344	3447309
18	4	9	1	1	**63596645**	**8142716**
19	5	-4	3	1	90520989	11590025
20	7	3	4	1	335159612	42912791
21	5	-12	1	1	**1431159437**	**183241189**
22	7	1	14	1	1766319049	226153980

So, in the Euler Lagrange process it will take 22 steps (FL) would have taken 14 steps .

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Euler-Lagrange Method for $X^2 - 61Y^2 = 1$

i	P_i	K_i	a_i	ε_i	X_i	Y_i
0	0	1	7	1	1	0
1	7	-12	1	1	**7**	**1**
2	5	3	4	1	8	1
3	7	-4	3	1	39	5
4	5	9	1	1	**125**	**16**
5	4	-5	2	1	164	21
6	6	5	2	1	453	58
7	4	-9	1	1	**1070**	**137**
8	5	4	3	1	1523	195
9	7	-3	4	1	5639	722
10	5	12	1	1	**24079**	**3083**
11	7	-1	14	1	29718	3805
12	7	12	1	1	**440131**	**56353**

The steps which are skipped in *cakravāla* are highlighted.

But actually you do not need to do all the 22 steps the simple continue traction will terminate here when you get -1, an =-1, so it is a square root of D which has n r the period. So, (FL) is indeed a hot subject, so so it has an odd period you are getting this -1 it will take 11 steps and to get that 11 steps 1, 2, 3, 4 extra steps Euler Lagrange adds (FL) will come in 7 steps instead of 11, so this we can see by looking at the 2 continue tractions.

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Bhāskara or Nearest Square Continued Fraction

In the continued fraction development of $\sqrt{D}$, the complete quotients are quadratic surds which may be expressed in the standard form $\frac{(P+\sqrt{D})}{Q}$, where P, Q and $\frac{(D-P^2)}{Q}$ are integers prime to each other.

If $a = \left[\frac{(P+\sqrt{D})}{Q}\right]$ is the integral part of $\frac{(P+\sqrt{D})}{Q}$, then we can have

$$\frac{(P+\sqrt{D})}{Q} = a + \frac{Q'}{(P'+\sqrt{D})} \qquad (1)$$

$$\frac{(P+\sqrt{D})}{Q} = (a+1) - \frac{Q''}{(P''+\sqrt{D})} \qquad (2)$$

where the surds in the rhs are also in the standard form.

The simple continued traction has 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 11 entries for 61 there is 1, 2, 3, 4, 5,6, 7 entries in the (FL) so it the period has 7 that is the seventh step this has 11 steps and these results 4-1's are appearing those are precisely the points where the Pi crossing root D Pi square is crossing root D okay. So, we are said enough about the comparison of the 2 we will say something in the end.

So, Krishnaswami Ayyangar then does various things that people normally do with continue tractions . he first showed that the Euler Lagrange formalism can be viewed as a modification of the (FL) and he also showed that (FL) can be viewed as a generalisation of the continue traction instead of using a simple continue traction you use a semi regular continue traction where +1 and -1 are the both the allowed.

Now to give direct algorithm for the he called this continue tractions that come in (FL) as Bhaskara continue tractions now a days they are called the nearest square continue tractions, so they can be called Bhaskara or the nearest square continued tractions and he gave a general rule that whenever you have quadratic cell I explained you the simple continued traction algorithm take the integral part subtract the integral part from the number take the reciprocal part.

Again take the integral part subtract that from the number, so like that go on that is the simple continued traction algorithm. So, for the Bhaskara continued traction he gave an algorithm not for a general real number he gave the algorithm for a quadratic cell because that is where the (FL) process is appearing it is appearing in the context of a root D. So, if you write a root D as a Bhaskara continue traction the total coefficient will come out to be quadratic cells like this.

Now see with the particular property that P and Q and D- P square by Q are integers which are prime to each other that is called the standard form of itself. So, at any step doing the Bhaskara process and doing the simple continued traction process how do they differ, so to show that you take this quadratic side writing this way where a is the integral part of P+ root D/Q and (()) (36:54) quadratic cell you write it as just write it as a+1 whatever in means you write it in the quadratic side form that is how it will look.

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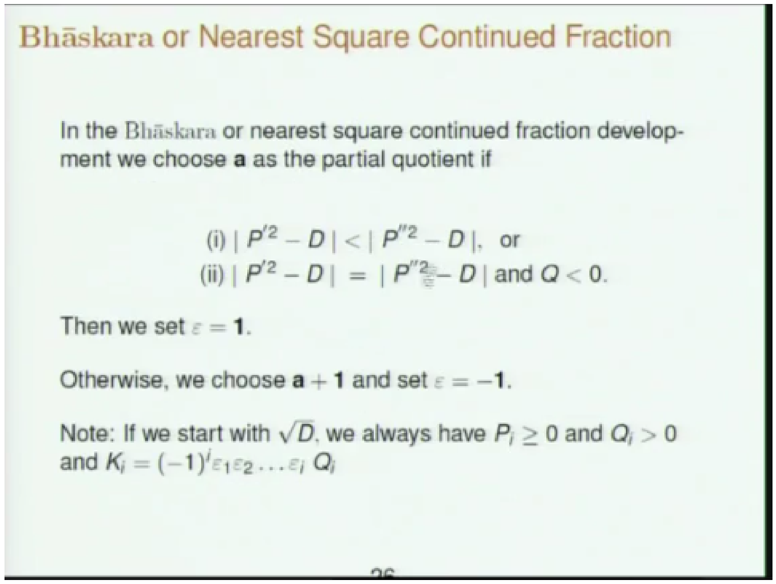

And so Krishnaswamy Ayyangar said if P prime square-P is less than P double prime square-D or when they are equal if Q is less than 0 choose a and set epsilon =1, in the other case choose a+1 and set epsilon =-1 and then he showed the relation between the quantities which appear in the (FL) algorithm and the quantities this k and the Q which appear in this continued traction expansion. So, you choose a when P prime square-D is less than p double prime square-D that is all that okay.

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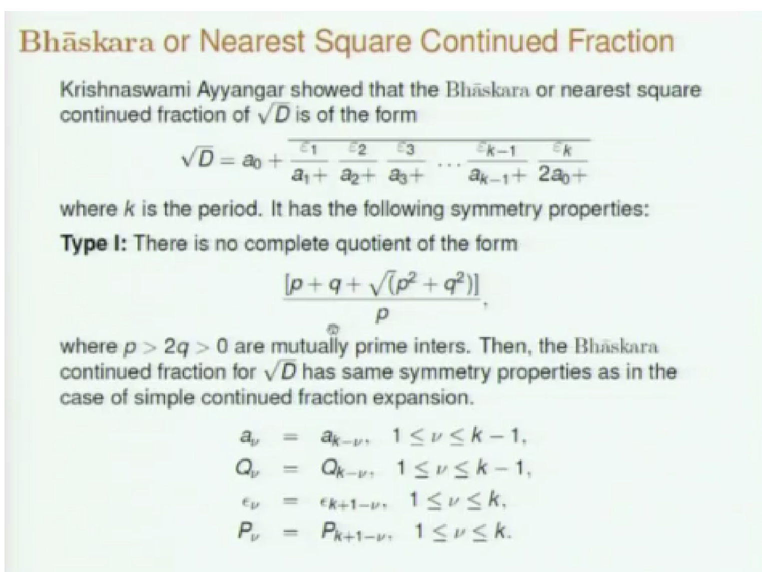

Next he also studied the periodicity properties of the continued traction, now just look at this let us look at the case of 61.

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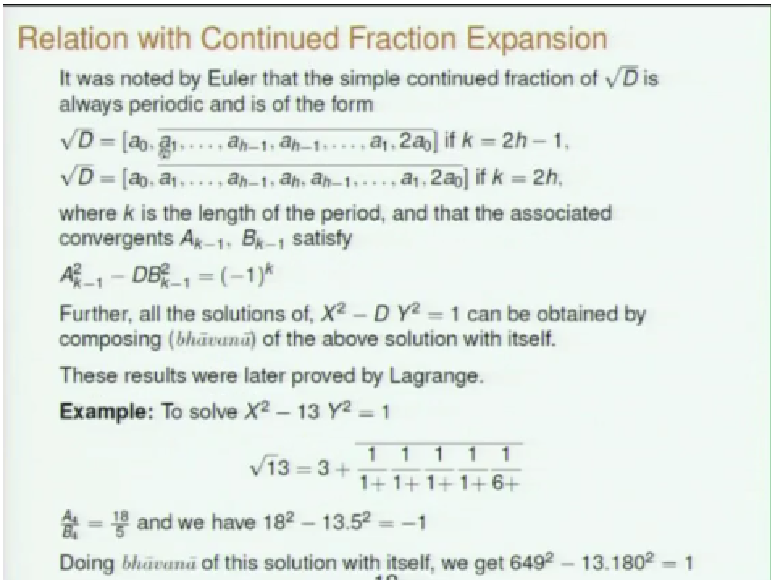

You see there is a complete periodicity here a1 to ah-1, ah-1 to a1 and a 2a0, a1 to ah-1, ah, ah-1 to an and a 2a0, so similar periodicity exist for the Bhaskara continued traction also

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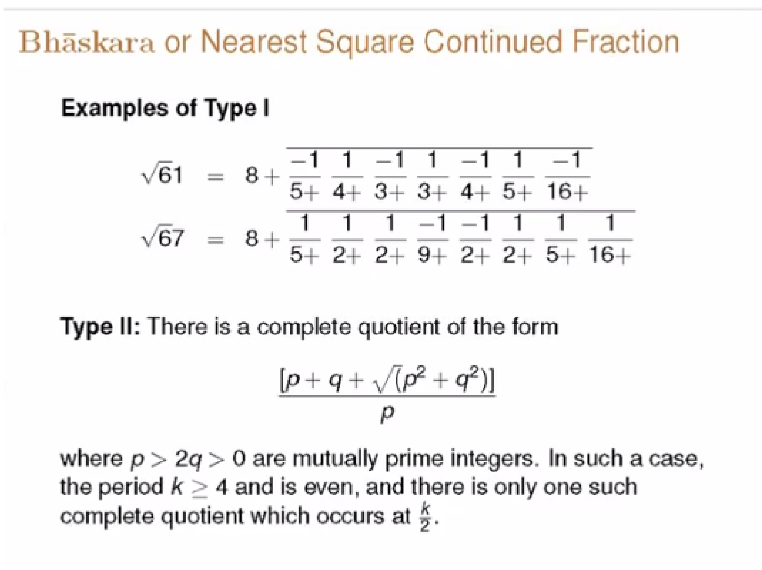

For most of the situations which Krishnaswami Ayyangar called as type 1 where certain condition does not happen certain kind of complete quotients do not occur the Bhaskara continued traction have the same symmetry. But in some particular cases they have a more complex symmetry.

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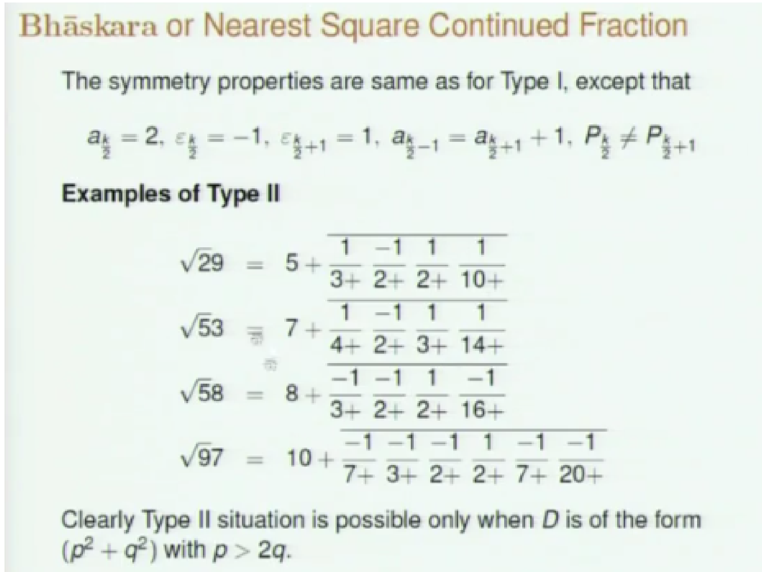

And these are examples where the symmetry is more complex 29, 53, 58, 97 the Bhaskara continue traction does not have the kind of symmetry that the simple continue traction has it has a more complex symmetry.

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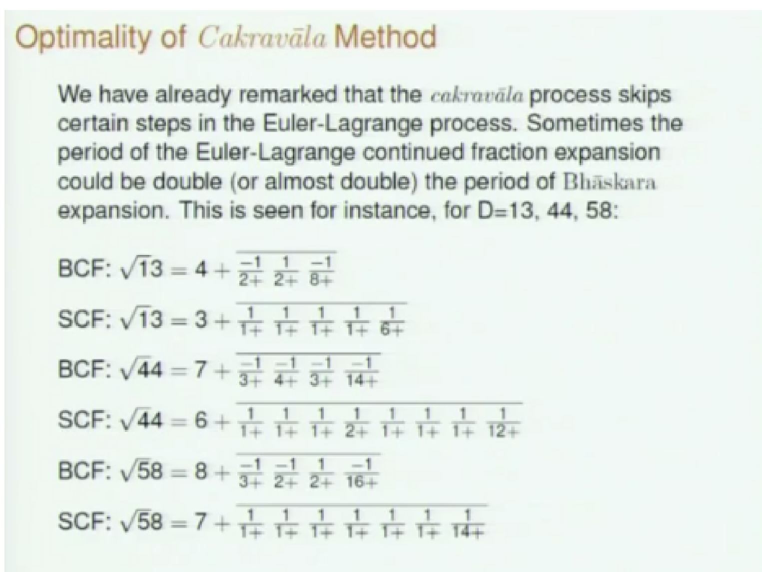

Similarly one can identify when you are doing the continue traction expansion when the midpoint occurs Euler had identified that . So, similar midpoint criteria have been found out in recent times Krishnaswami Ayyangar did not write it his time in the last 5, 6 years there are many others were working on this kind of problem they have identified the midpoint.

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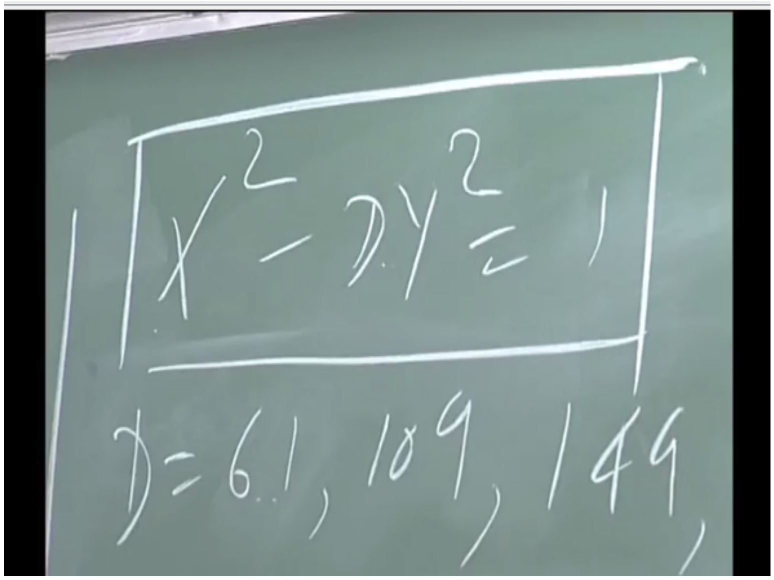

In fact this entire subject of x square-D y1 square is an equation which is of great interest even contemporarily one is it is a very is one of those algorithms which are non polynomial time that is the amount of time that is taken to solve this in general is of the order of D times D to the power square root of D, so it is a non polynomial kind of an algorithm. So, there is even now a quantum algorithm by Halgrim which will solve this in a polynomial time.

Of course that can be implemented only in a quantum computer which may come in the next 200 to 300 years. Then there were also many other interesting properties of this equation several books are written few of them do mention that brahmagupta and Bhaskara did solve and some people say partially fully some books do explain fairly nice way the way the equation was solved by (FL).

But the most important thing for us is that the old method was indeed the best method the (FL) method as I said showed you does keep various steps, so in that sense it is optimal whenever you have an algorithm one of the important thing you want to see is whether you are doing it in the minimal amount of number of steps and it happens in many cases that the period of the Euler Lagrange continue traction can even be double or almost double the period of the Bhaskara have given 5 cases.

Here is the case where the period is 3 there, period is 5 in the case of Euler Lagrange or Bhaskara the period is 3, a for 44 the Bhaskara is period is 4 the Euler Lagrange are the simple continue traction the period is 8 it is in the double the number of steps and if you see wherever this happening the number of negative signs are a indicator of the number of steps which are being skipped.

I am seeing you flight but I have to complete the yes sir (()) (41:21) the initial feasible point at which the Bhaskara's thing starts is closer, no both start from 10 not that way we when they take the first integer a no even that is not so the algorithm is more optimal it is allowing both Pi+1 larger than square root of D yes and Pi+1 lesser so allowing both the possibilities that it makes you go closer in a more nicer.

In fact even this (FL) even this you see in each of this gap what I was doing is I was taking the remainder instead of taken do the take the nearest integer then the number of steps needed in refusing even a rational number into a continue traction will be less it will be less than the number of steps needed when you are always taking lower they devisor 2 the remainder to be positive.

The remainder can be taken to be both positive and negative to reduce the number of steps, so this is what is being systematic but it cannot be arbitrary what (FL) was did was they try to + and – arbitrary Bhaskara or (FL) or whosesoever discover (FL) prior to him had a systematic algorithm for doing this. Now there is another interesting property discuss by a famous Swedish mathematician (FL).

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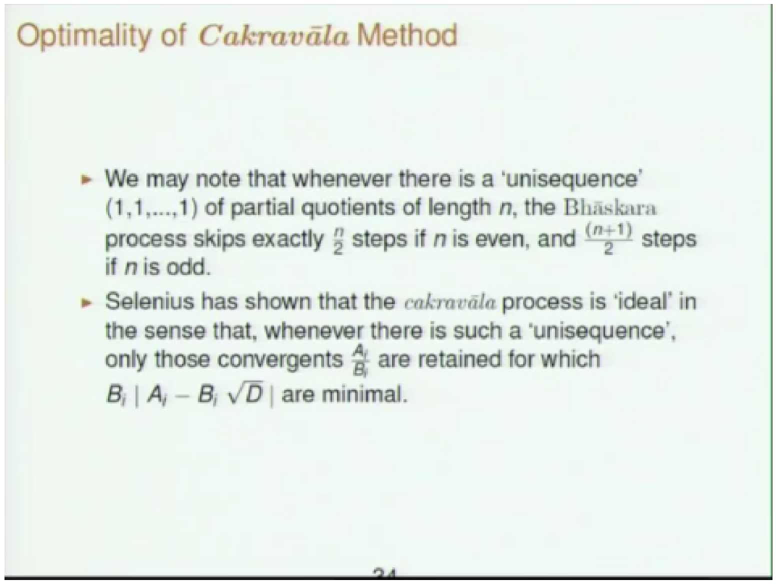

That the issue is this whenever there is a sequence of what are called the unit quotients will go to the previous step then you will see, the moment I am able to reach okay, take the simple continue traction you see there is a set of 1, 1, 1, 1 appearing here right whenever these are this partial quotients whenever such ones appear the (FL) get rids of, gets rid of them. So, these are uni sequences here there is a uni-sequence of length 4, here also there is a uni-sequence of length re followed by uni-sequence of length re.

Here there is a uni-sequence of length 6, so whenever there is a uni-sequence of parital quotients. So, whenever this a0, a1, a2 that you are see when they become 1 (FL) kills them that step. So, okay so, whenever you have a uni-sequence of partial quotients of length m the (FL) process keeps exactly n/2 steps if n is even and n+1/2 steps if n is odd. So, that is at the heart of the (FL) process.

And (FL) who sort of studied this uni-sequences in great detail a Swedish mathematician I was telling you . He showed that the remaining convergent are actually the optimal convergents in the sense that Bi modulus Ai-Bi root D they are minimal. So, of the n+1 convergence possible (FL) picks out the closest convergent and throws away the ones which are not that close.

Of course even simple can unit tractions all convergent are fairly close but (FL) picks up the ideal or the most optimal ones amongst them and that is what (FL) showed and he generalised the theory to what are called optimal continued tractions for arbitrary real numbers.

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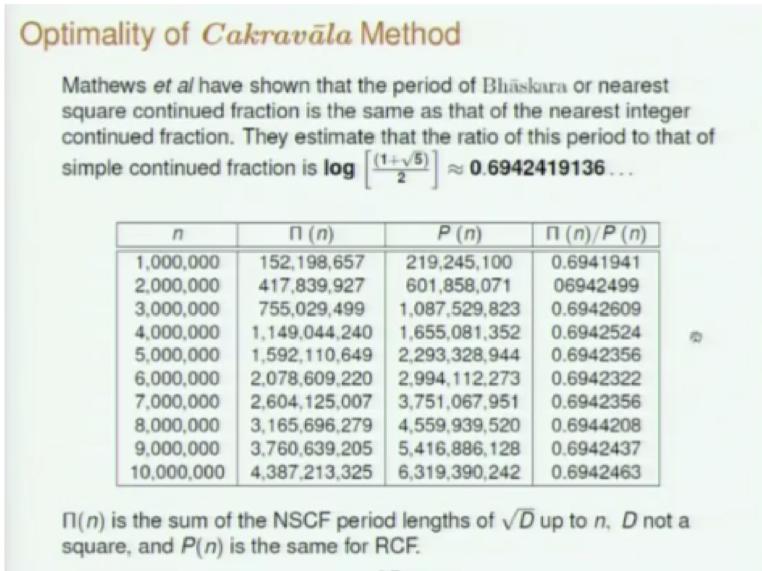

n	$\Pi(n)$	$P(n)$	$\Pi(n)/P(n)$
1,000,000	152,198,657	219,245,100	0.6941941
2,000,000	417,839,927	601,858,071	06942499
3,000,000	755,029,499	1,087,529,823	0.6942609
4,000,000	1,149,044,240	1,655,081,352	0.6942524
5,000,000	1,592,110,649	2,293,328,944	0.6942356
6,000,000	2,078,609,220	2,994,112,273	0.6942322
7,000,000	2,604,125,007	3,751,067,951	0.6942356
8,000,000	3,165,696,279	4,559,939,520	0.6944208
9,000,000	3,760,639,205	5,416,886,128	0.6942437
10,000,000	4,387,213,325	6,319,390,242	0.6942463

So, recently Mathews and that is a group in Australia were working (FL) systematically they were the people who showed the midpoint criteria and things like that, so they are try to estimate by computer simulation take the ratio of number of steps Euler lagrange and number of steps taken by the (FL) algorithm divide one by the other and the ratio is converging to something like log of 1+ root 5/2 or 0.694 etc.,

So, something like 30 % saving his what could be achieved there is a great mathematician called (FL) about 60, 70 years ago he was known of one of the pioneers in computational number theory. So, he wrote in mathematical reviews in 1940's early 41 or 42 a review of the paper of for Krishnaswami Ayyangar, incidentally Professor Krishnaswami Ayyangar is the father of the more well know figure A.K Ramanujan the English professor former professor English in Chicago.

So, he wrote that not only does it destroy the symmetry that simple continued traction have and the few steps that it seems to gain are not really worth the amount of distraction of the beautiful theory of the simple continued fractions that we have arrived with. In fact more or less a similar comment is made by the great mathematician (FL) in his book on history of number theory.

(FL) even says that this name pell's equation is very convenient because it is very definitive I do not know what it means and so we have to continue using it while referring to this beautiful equation anyway so this is the kind of estimate numerical estimate or else some theoretical estimates can also be made for the optimality.

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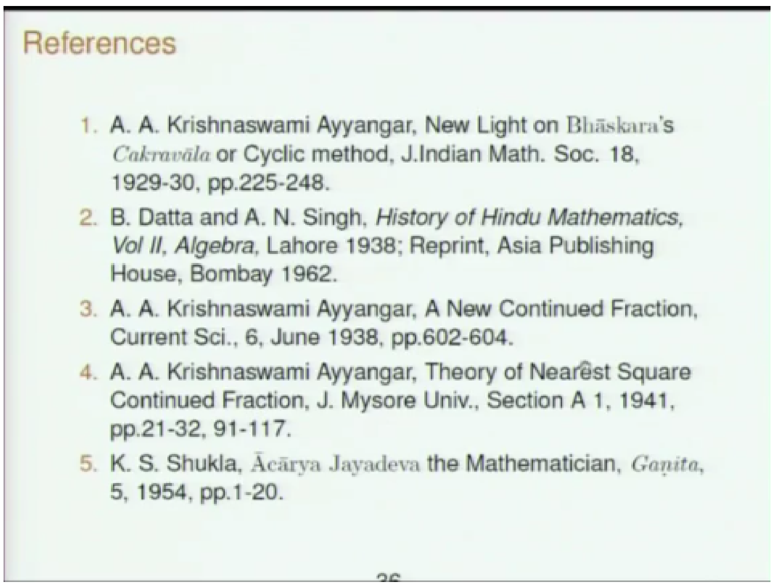

So, these are Krishnaswami Ayyangar's papers the last one appeared in the journal of mysore university in 1941 .

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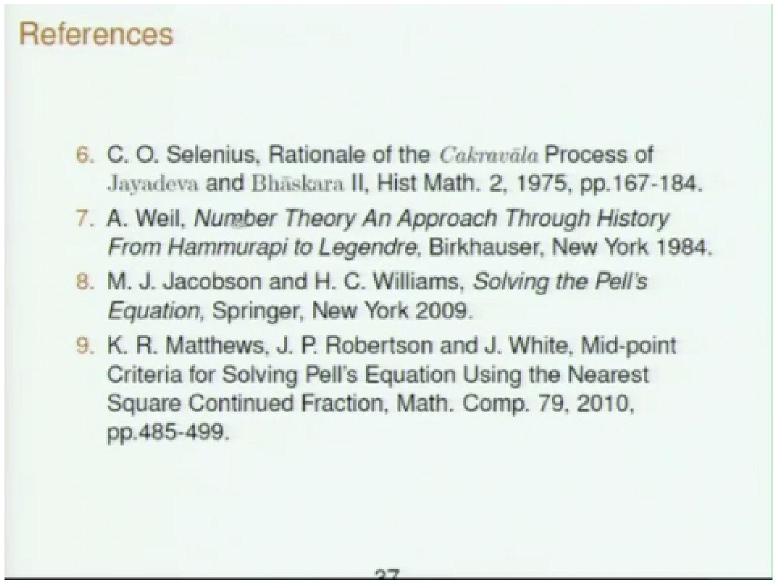

This (FL) monographs on history of number theory this is the famous paper of Selenius he has written 3, 4 papers in German and I think Swedish also earlier, this is the paper of Mathews, Robertson and white and this book has several papers on nearest square continue tractions and Jacobson and William are they are experts in the study of Pell's equation this is a book which runs above 700 pages in the discussion of Pell's equation.

So, it is a very interesting subject I have try to present you what is the mathematical sort of art of the (FL) process which goes back to (FL) and Bhaskara. Narayana also has something to say on me which I will briefly quiet you tell you during the talk on that thank you.