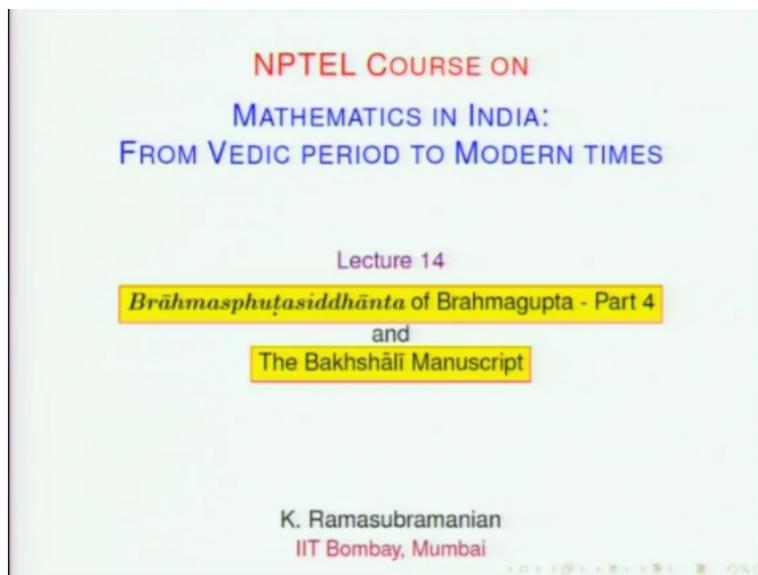


**Mathematics in India: From Vedic Period to Modern Times**  
**Prof. K. Ramasubramanian**  
**Indian Institute of Technology-Bombay, Mumbai**

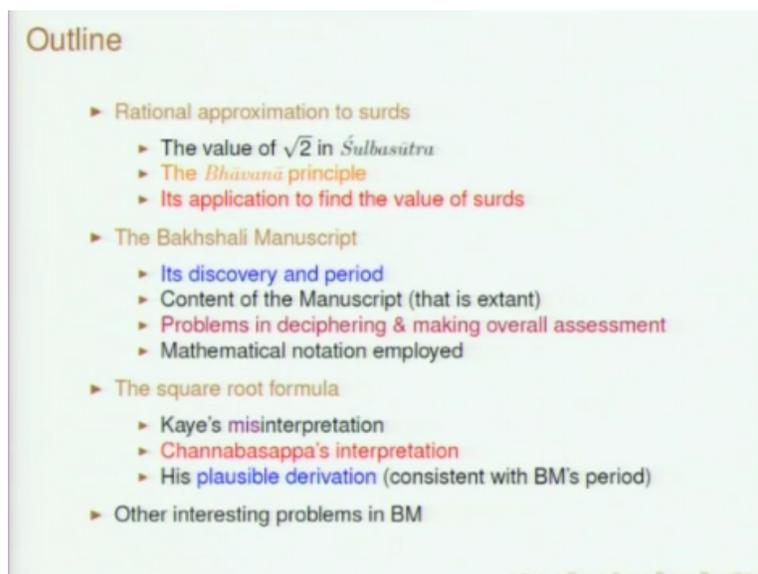
**Lecture-14**  
**Brahmasphutasiddhanta of Brahmagupta-part-4 and The Bakshali Manuscript**

So, in this lecture I will start with the portion which I want it to discuss in Brahmagupta's (FL) and then we will move on to discuss the content of a very important manuscript called the Bhakshali manuscript.

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So, in the last lecture, so I discussed about this (FL) problem and the Bhavana principle. So, now I am going to start with that so I will quickly recapitulate this (FL) and Bhavana, so and then move on to illustrate this principle has to how this can be used to derive a certain expression which we found in (FL) (FL) give a very interesting expression for the value of group 2. So, it was expressed in the form of a sum of rational numbers.

And we will be able to see now how that same expression can be arrive by the application of Bhavana principle. So, that is differs part of our discussion and then we will illustrate with one more example and then move on to the Bhakshali manuscript. Bhakshali manuscript so is something which is very very unusual both in it is content as well as the way it was discovered. There are so many manuscript that will find for various works.

So, in the form of formally in the form of paper in the brisk bark all that, so this is the only manuscript that is very, very ancient to the period is something which is estimated that to be around third century, fourth century by some people and some others have expressed into be around seventh century but most scholar agrees that it can be beyond seventh century based on the content as well as the language.

So, we will discuss little more about it does it proceed and one important result which is found in Bhakshali manuscript is something which is quite remarkable in terms of the sophistication that it has in obtaining the value of surds. So, it is a very interesting formula so that one finds and this formula has been sort of misrepresented by Kaye and it has been thoroughly analysed by Channabasappa in his paper in 1975.

In fact Channabasappa has written some 3, 4 articles so primarily on Bhakshali manuscript all of them are scholarly articles and he actually analyses the way a particular verse, so which he calls as Bhakshali sutra a like Aryabhatta sutra, so it is also in the form of very small sutra just a small couplet and it gives a very interesting expression for obtaining approximate value of square root of a non-square number.

This we will discuss in great detail and obviously we would not have any kind of evidence thus to how the author of Bhakshali would have obtained this result. So, it is anybody's guess but we will think of a possible derivation. So, with that more or less our lecture will be over.

(Refer Slide Time: 04:01)

How did *Sulvakāras* specify the value of  $\sqrt{2}$ ?

► The following *sūtra* gives an approximation to  $\sqrt{2}$ :

प्रमाणं तृतीयेन वर्धयेत्, तद्यतुर्थेन, आत्मयतुस्त्रिंशेनेन,  
सविशेषः। [BSS 2.12]

$$\sqrt{2} \approx 1 + \frac{1}{3} + \frac{1}{3 \times 4} \left(1 - \frac{1}{34}\right)$$

$$= \frac{577}{408} = 1.414215686$$

► What is noteworthy here is the use of the word सविशेषः in the *sūtra*, which literally means 'that which has some speciality' (speciality  $\equiv$  being approximate!)

► How did the *Sulvakāras* arrive at the above expression?

► As there are no evidences, it is anybody's guess! Presently our aim is to show how this can be arrived using the *bhāvanā* principle — enunciated by Brahmagupta much later!

So, to recall the expression which I want to do derive using Bhavana principle, so which has been stated in (FL) is the following (FL) So, this is the (FL) which presents the expression for the value of root 2, so he says (FL) so increase by 1/3 (FL) 1/4 of this means the previous value but this 1/4 he says (FL) is subtraction (FL) is 34, so 134 has to be removed from the previous value which amounts to this expression.

So, what does it amount to so it is 577/408 and this value is almost corrects to 6 decimal places for root 2. So, this is the (FL) so you may recall this (FL) was written so around 800BC or prior to that, that is the estimate. Most interesting thing about this is so here it is said (FL) (FL) essentially translates to not putting an equal sign but approximate okay. So, (FL) so that is okay how did (FL) arrive so we showed a certain method.

So, in geometrical construction by which one could get this expression but now we are going to even this expression one does not know because (FL) in text form and therefore no explanations say in this sutras they text themselves but coming to this come much later and there to we do not

find much of a explanation with reference to the approach which (FL) might have taken in arriving at this.

And therefore some 3,4 approaches have been suggested by scholars by which they would have arrived at this expression. Anyway so currently we will be trying to arrive at this expression using the Bhavana principle.

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**Vargaprakṛti and Bhāvanā principle**

- ▶ It was mentioned that one of the motivations for solving equation of the form
 
$$x^2 - D y^2 = K \quad (D > 0, \text{ a non-square integer})$$
 for integral values of  $x, y$  is to find rational approximation to  $\sqrt{D}$ .
- ▶ If  $x, y$  are integers such that  $x^2 - D y^2 = 1$ , then we have
 
$$\left| \sqrt{D} - \left( \frac{x}{y} \right) \right| \leq \frac{1}{2xy} < \frac{1}{2y^2} \quad (1)$$
- ▶ Also, using the *Bhāvanā* principle, from **one solution** of the equation  $x^2 - D y^2 = 1$ , an **infinite number of solutions** can be generated, via
 
$$(x, y) \rightarrow (x^2 + D y^2, 2xy) \quad (2)$$
- ▶ From the **inequality** given by (1), it can be seen that the successive solutions **generated by *bhāvanā*** given by (2), will yield better and better approximations for  $\sqrt{D}$ .

So, to recall the Vargaprakṛti and Bhavana so we have an equation of this form  $x^2 - D y^2 = K$  where  $D$  is the non square number greater than 0. So, whose approximation we want to find out, so  $K$  is some integer, so the idea behind posing this problem was to find  $x$  and  $y$  which are integers and it turns out that the motivation could be in trying to find out the value of root  $D$  one of the motivations for solving this.

So, this inequality shows that the successive values which can be generated from  $xy$  which satisfies this equation will be better and better approximation for root  $OD$ . So, this can be shown some unique quality this unique can also be worked out which is not very difficult to see. So, all that you need to remember is we move from  $xy$ ,  $xy$  is any guess so you given  $D$  given  $K$  you can always find some  $xy$  which will satisfy this equation.

So, and from that you move on to the other values by doing this Bhavana principle and we will see that we will get better and better approximation for OD, so this is the equation you start with so, to start with as you can see so what you can do is, so if x and y are sufficiently large you can easily see that D is going to be root D is going to be xy y, so this is how we start with in order to get better and better approximations.

So, even if it is not so after a few conversions you will see that it will be very close to root D okay.

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*Bhāvanā* and rational approximation for surds

► When we do *bhāvanā* of

$$x^2 - D y^2 = 1 \quad (3)$$

with itself, then as first approximation to  $\sqrt{D}$  we get,

$$\frac{x_1}{y_1} = \frac{(x^2 + D y^2)}{(2xy)} \quad (4)$$

► Now using (3) in (4), we have

$$\frac{x_1}{y_1} = \frac{(2x^2 - 1)}{(2xy)} = \left(\frac{x}{y}\right) - \frac{1}{y \cdot 2x}$$

► The second approximation to  $\sqrt{D}$  is obtained by doing *bhāvanā* of  $(x_1, y_1)$  with itself. That is,

$$\begin{aligned} \frac{x_2}{y_2} &= \left(\frac{x_1}{y_1}\right) - \frac{1}{y_1 \cdot 2x_1} \\ &= \left(\frac{x}{y}\right) - \left(\frac{1}{y \cdot 2x}\right) - \left[\frac{1}{y \cdot 2x \cdot (4x^2 - 2)}\right] \end{aligned}$$

So, let us start with the equation so first approximation for root D he take xy, and then what to do is, so this is the that root of x/y can be taken as 0 third of approximation okay for this equation. The first approximation is by doing Bhavana with itself and x square+Dy square by 2xy so this is the first thing, so remember so this is what we do. So, this how we this is the equation which we are going to repeatedly do okay.

Now using so 3 in 4 see so you will get 2x square -1/2xy and this can be express in this particular formula fine. So, second approximation so all that we need to do is so we have to plug in x1y1 so in this and it turns out to be this, so x1/y1 is this so this equation so you just have it and then you express this. So, now you get this pattern you can see so this expression as such repeats so in the second approximation fine, so now let us go and see.

(Refer Slide Time: 09:31)

How did *Sulvakāras* specify the value of  $\sqrt{2}$ ?

- ▶ The following *sūtra* gives an approximation to  $\sqrt{2}$ :  
 प्रमाणं तृतीयेन वर्धयेत्, तद्यतुर्धन, आत्मघतुस्त्रिंशेनेनेन,  
 सविशेषः। [BSS 2.12]

$$\sqrt{2} \approx 1 + \frac{1}{3} + \frac{1}{3 \times 4} \left(1 - \frac{1}{34}\right)$$

$$= \frac{577}{408} = 1.414215686$$

- ▶ What is noteworthy here is the use of the word सविशेषः in the *sūtra*, which literally means 'that which has some speciality' (speciality = being approximate!)
- ▶ How did the *Sulvakāras* arrive at the above expression?
- ▶ As there are no evidences, it is anybody's guess! Presently our aim is to show how this can be arrived using the *bhāvanā* principle — enunciated by Brahmagupta much later!

So, this approximation which was given in (FL) is  $1 + 1/3 + 1/3 \cdot 4$  and the next term will be  $-1/3 \cdot 4 \cdot 34$  in the denominator you will find the same terms appearing again. So, you will find see, so  $y$  then  $y \cdot 2x$ , then  $y$  into  $2x$  into one more term so you can see that you get that.

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*Bhāvanā* and rational approximation for surds

Thus, we have a series of better approximations that may be written as

$$\left(\frac{x_r}{y_r}\right) = \left(\frac{x}{y}\right) - \left(\frac{1}{y \cdot n_1}\right) - \left(\frac{1}{y \cdot n_1 \cdot n_2}\right) - \dots - \left(\frac{1}{y \cdot n_1 \cdot n_2 \dots n_r}\right), \quad (5)$$

where  $n_1 = 2x$  and  $n_i = n_{i-1}^2 - 2$ , for  $i = 2, 3, \dots, r$ .

**Example:** For  $D = 2$ , we start with  $x = 3$  and  $y = 2$ . we have

$$\frac{x_2}{y_2} = \left(\frac{3}{2}\right) - \frac{1}{2 \cdot 6} - \frac{1}{2 \cdot 6 \cdot (6^2 - 2)}$$

$$= \left(\frac{3}{2}\right) - \frac{1}{2 \cdot 6} - \frac{1}{2 \cdot 6 \cdot 34}$$

By re-writing the first two terms, the above approximation can be seen to be the same as in *Sūtra-sūtras*. Generating further terms in the series, we've

$$\sqrt{2} = 1 + \frac{1}{3} + \frac{1}{3 \cdot 4} - \frac{1}{3 \cdot 4 \cdot 34} - \frac{1}{3 \cdot 4 \cdot 34 \cdot 1154} - \frac{1}{3 \cdot 4 \cdot 34 \cdot 1154 \cdot 1331714} - \dots$$

where  $1154 = 34^2 - 2$ ,  $1331714 = 1154^2 - 2$ , and so on.

So, in general one can show that so the (FL) approximation will be of this form. So, we will start with this equation see this equation  $x^2 - Dy^2$  so suppose  $D$  is equal to 2, so this will be satisfied with  $x$  is equal to 3 and  $y$  is equal to 1. So, we get  $3/2$  and so by regrouping this, so we can show that, so this is the expression for root 2 and (FL) *sūtra* stopped here. So, they gave the first 4 terms so (FL) and it said (FL).

So, that actually gives a value correct to 6 decimal places almost and they stopped there and one can show that the next term is going to this and the next term and so on. So, this is something which can be straight away obtained by using this.

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*Bhāvāna* and rational approximation for surds  
 Yet another example (non-textual)

- ▶ In an identical manner (to that of  $\sqrt{2}$ ), the series for  $\sqrt{3}$  can also be constructed.
- ▶ Substituting  $x \approx 7$  and  $y = 4$  in the वर्गप्रकृति equation, we have
 
$$7^2 - 3 \cdot 4^2 = 1$$
- ▶ Using *bhāvāna* principle as earlier we get
 
$$\sqrt{3} = \frac{7}{4} - \frac{1}{2.8} - \frac{1}{2.8.62} - \frac{1}{2.8.62.3842}$$
- ▶ This can be regrouped and written in the form
 
$$\sqrt{3} = 1 + \frac{3}{4} - \frac{1}{4.4} - \frac{1}{4.4.62} - \frac{1}{4.4.62.3842}$$

Now I will show one more example and then I will move onto so suppose you want to find out root 3 so you can start with this (FL) equation  $7^2 - 3 \cdot 4^2 = 1$  this satisfies this, so our initial choice is 7 and 4. So, then so this successive application of Bhavana principle leads to this equation and this if one word to write so with 1 is the first thing so 1 can regroup this and then write it in this particular form.

So that is why we see that as an application to (FL) and Bhavana principle one will be able to generate, so the expressions for surds in this particular forum and this is something which will go on. In fact here one should not have this follow equality it is not exactly equal because it is something which is going to done after each term we will see that the value increases the accuracy increases okay.

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## Discovery of the Bakhshali Manuscript (BM)

- ▶ BM was discovered—purely by a stroke of luck—by a farmer in the year 1881 CE as he was excavating the soil, in a place called Bakhshali.<sup>1</sup>
- ▶ It is in the form of birch bark, and only 70 folios are available. It is hard to estimate as what fraction would have got lost and what is available(?)
- ▶ Providentially the discovered manuscript reached the right hands, and after passing through several hands, finally reached F R Hoernle, an indologist who had interest in unearthing its contents—for whatever purposes!
- ▶ It was first edited and published in 1922 by G. R. Kaye.<sup>2</sup>
- ▶ Another edition has been brought out more recently by Takao Hayashi in 1995.

<sup>1</sup>This place is identified as a village  $\approx$  80 km from Peshawar (currently in Pakistan).

<sup>2</sup>It has been unambiguously shown by scholars (Datta and others) that the views expressed by Kaye were highly biased.

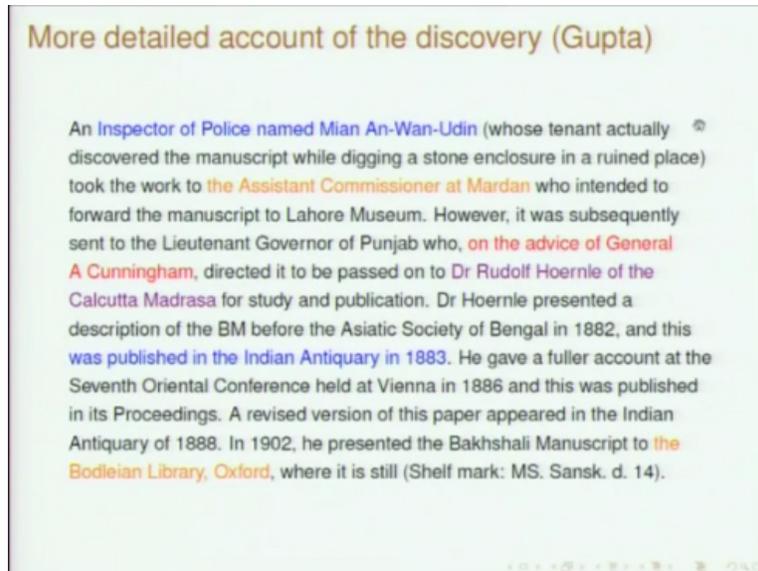
So, now I move on to the Bhakshali manuscript as I was mentioning this Bhakshali manuscript was something which was discovered by a stroke of luck apparently a former was digging somewhere and so he suddenly discovered a manuscript as he was excavating soil somewhere and this manuscript is in the form of bridge box. So, today it is very very hard to find manuscripts written in bridge bark but earlier that was one of the medium which was used for writing.

So, we can find manuscripts in palm leaves but bridge bark manuscripts are very very hard to see, so this is in bridge bark and they have found about 70 folios as you can easily understand, so if this has been lying under the soil for a quite long time, so it is something which is in a pretty damaged condition. So, anyway so that was discovered and this is in 1881 and this former so did not simply ignore it but took it meticulously to this person and the person took it to it.

They went into passed on into several hands and this was first published in 1922, so I was referring to kaye, so this kaye was the first person to publish it in the form of a text the manuscript is still there but it is there in London, so and recently it was edited once more and published by Hayashi, so Hayashi is a Japanese scholar and he has published quite a few works and one of them is Bhakshali manuscript.

And the place of discovery so they say it is close to Peshawar around the 80 kilometres and it is named after the village, so where it was discovered.

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The story goes like this an inspector of police so discovered the manuscript while digging, so who was and it was handed over to assistant commissioner and it further went on to the general Cunningham and it was passed on to Horenile and then Horenile so in fact for some got interested in the manuscript and curiosity he studied it, so he made a few presentation I think some 3, 4 presentations so as this narration goes in the very next year, so it was discovered in 81 and 82, so in Bengali .

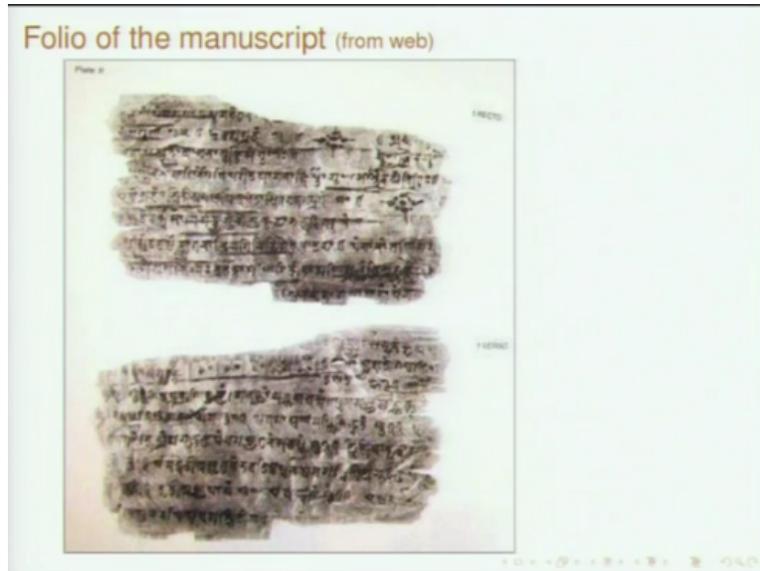
In Asiatic society of Bengal he made a presentation and then in 1883 and then once again in the oriental conference in Vienna in 1886 and revised version of the paper so were in 1889 in 1902 he presented this Bhakshali manuscript to Bodielian library and that where it is. So, the original manuscript is there in London fine. So, as regards the date of manuscript so as I was mentioning so there is a lot of uncertainty and kaye place it around 12 centuries CE.

Dhatta and others so place it around third century, so they placed between 200 to 400 Hayashi says it should be around 7 century, so all of them interestingly so say that we have arrived at this date based on the language which is that based on the content, which is that based on the script

which is founded. The script so scholars says there it is Sarada and the language this is Gatha, so it is form of (FL) and of course the content of the manuscript.

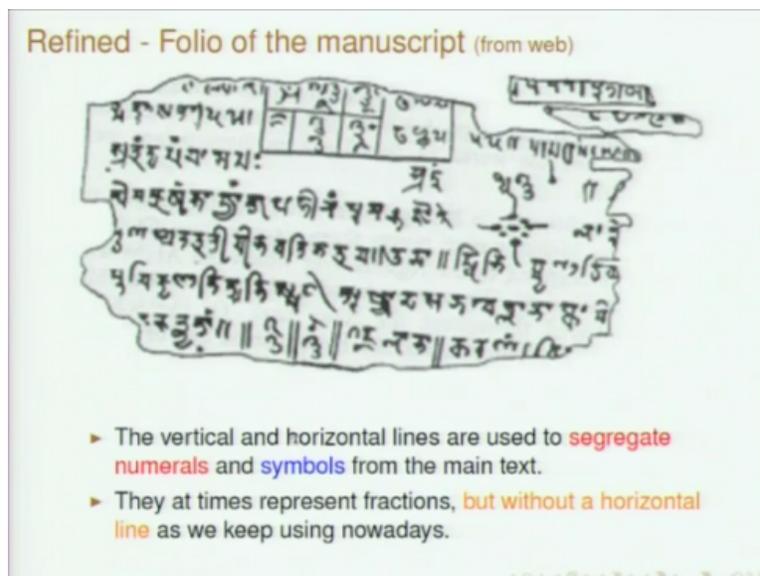
So, certain problems which are being discussed have also been analyse to estimate the date of the manuscript.

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So, this is how it roughly like when this is very poor photograph.

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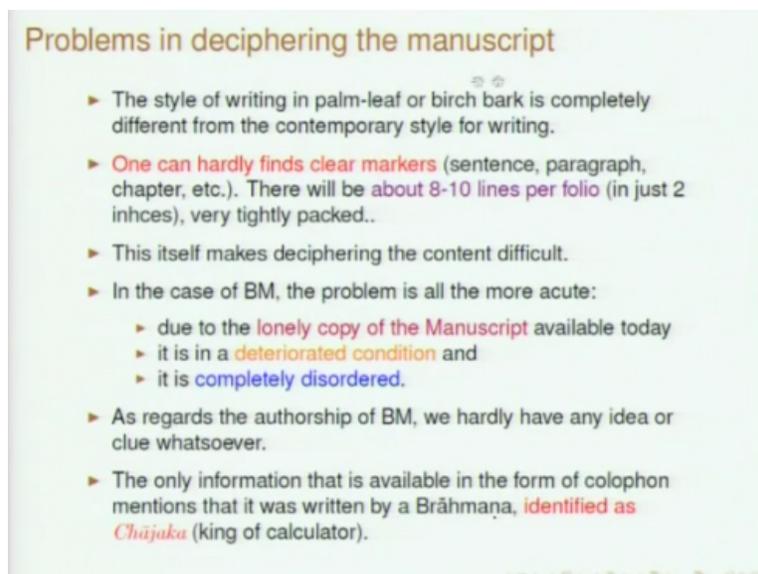


So, this is somewhat more refined, so this is how it a sort of a looks like a typical page of the manuscript. So, lot of things got damaged and whatever can be decipher in spite of this has been

attempted by various scholars. One interesting thing about this Bhakshali manuscript is so, it sort of segregates this numerals and any kind of calculation will be put in the form of some tables some lines are drawn.

So, when things are arranged like this see suppose there are number one below the other so they represent some kind of a fraction, suppose it is suppose this is 7 this is 5, so this will be sort of  $7/5$  something like this, so this is the convention and if you have 3 numbers arranged the first will be so the suppose we have 5, 3, 8 so it will be  $5+3/8$  that kind of a thing. So, it is a kind of convention. So, we find certain notations employed and the numbers are segregated by bars.

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**Problems in deciphering the manuscript**

- ▶ The style of writing in palm-leaf or birch bark is completely different from the contemporary style for writing.
- ▶ One can hardly find clear markers (sentence, paragraph, chapter, etc.). There will be about 8-10 lines per folio (in just 2 inches), very tightly packed..
- ▶ This itself makes deciphering the content difficult.
- ▶ In the case of BM, the problem is all the more acute:
  - ▶ due to the lonely copy of the Manuscript available today
  - ▶ it is in a deteriorated condition and
  - ▶ it is completely disordered.
- ▶ As regards the authorship of BM, we hardly have any idea or clue whatsoever.
- ▶ The only information that is available in the form of colophon mentions that it was written by a Brāhmaṇa, identified as *Chājaka* (king of calculator).

So, few remarks would be in place regarding the process of deciphering the contents of a manuscript. So, this style in which people have been writing in those days is far different from the style that is adopted by us today. So, if you look at any typical manuscript, so most of them are written in a very very very type manner, so what is written in something like 18 inches or 1 inches or 1 and half inches or 2 inches, so depending upon the palm leaves.

So, can we put in some 2 A4 pages, so that is the kind of very very tightly written and very very small, so it is extremely difficult unless the scholar is trained to decipher the content of a manuscript the second thing is there would be no clear markers, so the chapter or full stop etc.,

so they are not found and only in a few rare manuscript do you find all this markers, so which actually help us deciphering the content.

And particularly in my case of Bhakshali manuscript the problem is all the more acute for the following reason, suppose you want to edit a text Aryabhattiya so you will find some 20 manuscripts in various places 200 manuscripts sometimes depending upon the popularity of the text but in the case of Bhakshali manuscript this is the only thing that is available and therefore collaborating or collating it some other thing is impossible and it is in a retreated condition.

And it is also completely sort of jumble, so disorder as it was sort of discovered there and how much is lost how much is there available so this is also not known. So, that is a sort of colafen (( )) (19:41) so which has been identified by scholars that it is written by so this much is known by Chajaka whom they saw the cal the king of calculator.

**(Refer Slide Time: 19:54)**

**The square root formula**

- ▶ An interesting piece of mathematics found in BM concerns with the **formula for finding square root** of a non-square number.
- ▶ Any non-square number  $N$  may be expressed as  $\sqrt{A^2 + b}$ . The following formula is given in the manuscript

$$\sqrt{N} = \sqrt{A^2 + b} = A + \frac{b}{2A} - \frac{\left(\frac{b}{2A}\right)^2}{2\left(A + \frac{b}{2A}\right)} \quad (6)$$

- ▶ This is the famous **Bakhshali formula** about which we will discuss for a short while.
- ▶ The formula due to Heron<sup>5</sup> is:

$$\sqrt{N} = \sqrt{A^2 + b} = A + \frac{b}{2A} \quad (7)$$

- ▶ Evidently Bakhshali formula is an **improvised version** of Heron formula. What would have been the route taken by the author of BM to arrive at the formula?

<sup>5</sup>A Greek mathematician who lived in the later half of 1st century AD.

So, I was mentioning to you about an interesting formula presented in this Bhakshali manuscript, so the formula is to find out the square root of a non square number, so suppose we have a number n which can be expressed as A square+b, so let us assume A square is something which is close to n and less than n the formula which is given in the manuscript can be written in this form. So, root N is  $A + \frac{b}{2A} - \frac{b^2}{8A^3}$ .

So, this quantity actually sort of repeats itself here, so this is the famous Bhakshali formula and this particular form  $A+b/2A$  is something which has been presented by Heron also you have one more term added to that, so naively one feels that this should be a much better approximation than  $A+b/2A$  how did this Bhakshali author arrive at this formula. So, this will be discussing towards the end of our lecture.

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**Kaye's version of the square root formula**

- ▶ Kaye has reproduced the *sūtra* as follows:  
*akṛte śliṣṭa kṛtyunā śeṣa cchedo dvisamgūṇaḥ |*  
*tadvarga dala samśliṣṭa hṛti śuddhī kṛti kṣayaḥ ||*
- ▶ We write the above formula with a minor emendment in Devanāgarī:  
 अकृते श्लिष्टकृत्युनात् शेषच्छेदे द्विसंगुणः ।  
 तद्वर्गदल संश्लिष्टहृति शुद्धिकृति क्षयः ॥
- ▶ We accept the emendment for the following reasons:
  - ▶ in its feminine form it cannot go as an adjective ...
  - ▶ in yet another place in the BM it is found 'correctly'
  - ▶ by all probability the scribe might not have 'heard' properly and hence dropped स्.

But before that I wanted to give you the verse, so which present this formula and Kaye in his first edition so he has presented the verse like this, so there is a lot of as I was mentioning it is too difficult unless the person is sort of quite conversion to the subject. So, wherein he knows the content, so it is almost impossible to split the words appropriately and there could be a lot of errors which occur in doing so.

And there seems have been committed by Kaye 2 anyway a sort of improvised version of this is this (FL) if you see in his edition he simply calls (FL) this is the formula which in modern notation translates to this expression 6.

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**Kaye's (incorrect) version of translation**

- ▶ The correct rendering of the *sūtra* seems to be:  
 अकृते, श्लिष्टकृत्यनात् शेषच्छेदो द्विसंगुणः ।  
 तद्वर्गदलसंश्लिष्टद्विगुणः श्लिष्टकृतिः क्षयः ।
- ▶ The above *sūtra* has been translated by Kaye as  
 The mixed surd is lessened by the square portion and the difference divided by twice that. The difference is divided by the quantity and half that squared is the loss.
- ▶ Datta describes the above translation as 'wrong and meaningless' as this NO way leads to  $A + \frac{b}{2A}$ .
- ▶ However, Kaye somehow tries to map the description—rather unscrupulously—to the Heron formula.
- ▶ Why do we call it unscrupulous?

So, in fact after serious analysis so (FL) has arrived at the right formulation of the verse, so wherein this should be a sort of compound word (FL) so I will discuss in great detail has to how we can decipher this from this set of words which are found in this sutra. So, in fact it has been translated as the mixed surd is lessened by the square portion the difference divided by twice that and so on.

So, this is the kind of translation which has been presented by Kaye and Dhata who is a serious scholar, so to whom we owe a lot of our understanding to Indian mathematics he has said that this is all wrong and completely meaningless. So, to decipher the formula as simply  $A + \frac{b}{2A}$  and then say this is say must the Heron formula is something which is meaningless. It seems to be done rather unscrupulously that is what it amounts to because.

So there are various other evidences by which one will be able to clearly understand that this does not mean this. So, as has been clearly pointed out by Chanabhasapa why do we call it unscrupulous, so where a few reasons which I will quote 1 or 2.

**(Refer Slide Time: 23:52)**

### Channabasappa's observations (on Kaye's interpretation)

- ▶ The reasons have been neatly brought out by M N Channabasappa (MNC) in his **scholarly article** published in *Ganita Bhārati* in 1975.
- ▶ According to MNC, following the *Bakhshālī-sūtra* (BS) there is numerical example provided for illustration.

$$\sqrt{41} = 6 + \frac{5}{12} - \frac{\frac{25}{144}}{2(6 + \frac{5}{12})} \quad (8)$$

- ▶ Here MNC argues: If the author of the *Bakhshālī-sūtra* (BS) had Heron's formula in the mind, then why would he present a numerical example **that is disconnected from BS?**
- ▶ MNC also quoting Kaye's general observations on the nature of the text,

No general rule is preserved, but the solution itself indicates the rule

observes: "Kaye **fails to apply the above logic to BS** for square roots. He **thus violates his own norms** in compromising with his wrong translation of the *sūtra*".

So, this has been brought out in an article a scholarly article in Ganitha Bharati, so it is there mathematics journal in 1975 by channabasappa. So, he says see as would have seen even other lectures which are given by professor Sriram on brahmaputasiddhanta or Aryabhattiya. Aryabhattiya of course does not give various examples whereas Bhaskara, Brahmagupta they always so giving a certain general formulation they immediately give illustrative examples.

So, channabassapa says so soon after the sutra is presented we have a numerical example and the numerical example simply amounts to this if this amounts to this, then why would one simply says that the formula is only confining to the first root on from the series. So, this is the first objection, so nobody is going to give a certain illustration which is completely disconnect from the sutra which is stated before.

And he also says that so no general rule is preserved but the solution itself indicates the rule, so this is an observation which has been made by kaye and Channabasappa says that this violates his own observation okay fine.

**(Refer Slide Time: 24:59)**

**MNC's unconventional interpretation** (yet convincing!)  
Regarding the use of the word कृति

- ▶ MNC first points out that his discussion entirely rests upon his unconventional interpretation of the words कृति and ऊन.
- ▶ The line of argument goes as follows:
  1. The word *kṛti* generally refers to 'square'.<sup>6</sup>
  2. But the Bakhshali Author (BA) being prior to Āryabhaṭa, Brahmagupta, Bhāskara is **not compelled** to use it in the same sense **defined by the later authors**.
  3. BA uses the word **only in BS** and nowhere else in the text.
  4. For referring to 'squaring' – in half a dozen places – the BA consistently uses only **the word *varga***.
  5. Moreover, *kṛti* literally means '**a deed or process**', and hence the meaning 'square' is purely assigned one; there is nothing compelling to take it that way.
  6. Hence, based on the context he says, the word *kṛti* in BS should be taken to refer to **square root or *mūla***.
  7. He also **corroborates his thesis by citing *Śulbasūtra* texts** wherein we find the usage of the word *dvikarapī*, etc.

<sup>6</sup>Bhāskara defines: समद्विघातः कृतिरुच्यते।

So, with this I move on to the explanation of this verse this is quiet instructive and therefore I will thought I spend some time in going to explain to you the kind of approach that has been taken by Channabasappa in arriving at this is the very scholarly article, see here we have the verse (FL so this is sutra. Here the word (FL) appears see (FL) you understand.

So, these are the 3 terms that appear 3 words and once again we have the word (FL) appearing here (FL) usually the word (FL) is understood to be square. So, this as acquire the meaning of square so much after the time of Aryabhata, Bhaskara and so on wherein they very clearly says (FL) means square but (FL) as I was telling in the previous lecture the (FL) generally means a composition so (FL) right.

So, Kalidasa (FL), so (FL) is basically so it is derived from the root (FL) so (FL) means doing a certain process that is all it is, so and therefore it can mean the writing of a person it can mean the work done by a person, so in general in here in this context bassappa says that (FL) has to be not a square but a square root. So, (FL) therefore means fairly accurate square root fine, so (FL) so this is very interesting analysis which has been done.

So, that is nothing which prevents us from interpreting it as square root if we look at the derived meaning of the term because it has various other applications also in various other fields in various other senses, this is the first observation that he makes and the second observation that he

makes is the word (FL) (FL) normally means minus, subtraction okay. So, this lessened from that, that is how it means.

So, here he says that this (FL) has to be taken as division and not as subtraction and in fact he quotes pictures to prove the point that he tries to make, so with these 2 observations we move on to look at the arguments presented by Channabasappa as I was mentioning the word (FL) generally refers to square at the Bhakshali author so being prior to Aryabhata, so the first reason that he says is so this Bhakshali.

So, we do not very clearly know when it was composed but it might be following some tradition which has been much earlier also. So, according to bassappa this has been written even prior to Aryabhata, so around that period the (FL) term was reserved to refer to square if it occurs in a mathematical context. So, but prior to that the author is not compelled to use it in that sense the words pick up various senses at the front points of time.

And therefore this is the first reason that he says and he also says that if you look at the entire Bhakshali manuscript the word krti does not appear anywhere else but for this sutra. So, wherever the process of square had to be referred the author uses (FL), so **so** he says reserved the word krti to refer to square root and not square, so this is the second observation. So, it occurs in a dozens of other places where he only uses the word (FL).

So, as I was mentioning krti is a generic term which can refer to a deed or a process and therefore there is no compelling evidence for us to take only to refer to the process of squaring and for this particular verse he says it has to be taken as square root or (FL) and he also corroborates this with other evidences from other earlier text. So, suppose we place the Bhakshali manuscript prior to Aryabhatiya. So, the only other extent that we have is the class of text which are called (FL) text.

And in the (FL) text, so when we discuss that we saw, so I repeatedly hammered the word, so this (FL) refers to square root and that also stands from (FL) so (FL) so and so the (FL) refers to square root and therefore this is also derived krti is also derived from the same rule and therefore

it could refer to square root. So, in fact we have the word (FL) referring to square root of 2, square root of 3.

**(Refer Slide Time: 30:25)**

**MNC's unconventional interpretation (yet convincing!)**  
Regarding the use of the word ऊन

- ▶ MNC thesis is: the word *ūna* in BS should be taken to refer to the operation of 'division' though it is unconventional.
- ▶ The arguments presented in support of this are as follows:
  1. The word हरण and परिहा are derived from the same root ह.
  2. One of the *sūtras* of Pāṇini (3.3.29) explicitly states ऊन as synonym of परिहा.
  3. Hence, ऊन can be taken to refer to division.
- ▶ So much so, now कृत्युनात् means by dividing by approx. square root. In the earlier notation employed this translates to
 
$$\frac{A^2 + b}{A} = A + \frac{b}{A}.$$
- ▶ Now since शेष is  $\frac{b}{A}$ , the phrase शेषच्छेदो द्विसङ्गणः means divisor of  $\frac{b}{A}$  multiplied by 2. This translates to the expression
 
$$A + \frac{b}{2A}$$

So, now we move on to the other word una, una so he says it refers to division, so how do we substantiate this. So, here he says the word (FL) and (FL) both are derived from the root(FL), so (FL) means division, so (FL) (FL) so they are also so it is used in the sense of close to this okay, so sort of eliminated so taken away, so it is in that sense. So, in fact he codes certain (FL) sutra wherein (FL) says (FL) so that result is stated.

And therefore it can be taken to refer to something which is removed and removal is something which is by division process and therefore una can be taken to refer to division in this sutra fine. So, now with this, so framework so we had the word (FL) so this is also sutra begins, so (FL) so here he says so is by dividing by approximate square root, so (FL) square root una is division so (FL) is this, A is taken as the approximate square root of root N.

See the number is A square+b, so (FL), so krti here (FL) (FL) means well known (FL) is well established so this square which is well established before this root N is A square and therefore (FL) so is a it is referring to division by A and it comes to A+b/A. Then we have (FL) so is the first half of the verse (FL) (FL) I leave it now (FL) so, (FL) is b/A (FL) is deviser (FL) multiplied by 2.

So, we have  $b/2A$  you understand (FL) denominator multiplied by 2, so  $A+b/2A$  you have obtain this term.

(Refer Slide Time: 33:04)

**MNC's unconventional interpretation** (yet convincing!)  
 Corroborating evidences

- ▶ He corroborates this interpretation further by taking the notation employed in BM. For instance, in giving the expression for  $\sqrt{41}$ , the preliminary steps involved are represented as:

$$\begin{array}{|c|} \hline 6 \\ \hline 5 \\ \hline 6 \\ \hline \end{array} \quad \text{and} \quad \begin{array}{|c|} \hline 6 \\ \hline 5 \\ \hline 12 \\ \hline \end{array} \quad \text{means} \quad 6 + \frac{5}{6} \quad \text{and} \quad 6 + \frac{5}{12}$$

- ▶ Now, we present the meanings of few other words appearing in the verse

श्लिष्टकृतिः	means	approximate square root $A$
तद्वर्गदलः	means	half of the square of that $\frac{1}{2}(\frac{b}{2A})^2$
संश्लिष्टकृतिः	means	division by the composite $\div(A + \frac{b}{2A})$
(तस्य) क्षयः	means	subtraction of that
शुद्धिकृतिः	means	(is) the refined square root

- ▶ Thus, all the words in the verse **fit so well** to convey the intended meaning of the verse.

So, in fact as I was mentioning so you find this kind of markers in the manuscript, so suppose we find this number  $6/6$ . So, this is to be understood as  $6+5/6$  and then  $6512$ , so  $6+5/12$ . In fact the illustration which has been provided in Bhakshali manuscript is 1 of root 41, so root 41 can be express as  $36+5$ , so 6 square+5 so you have the same thing appearing see 6 is a and b is 5 and so in fact both of them are presented in this manuscript and he says  $6+5/12$ .

So, this is the reference in the manuscript has to how this thought with. So, to complete the analysis of the sutra so (FL) is A, so the word (FL) see the forum okay N, so (FL) see in fact see approximate I say because this is (FL) is something sort of kwon kind of a thing. So, that is why it is referring to A, so we have A square+b is the number and the closest square root is A okay, so it is in this sense.

So, then we had see (FL) the second half of the verse starts with (FL) see so (FL) refers to the previous quantity so what was the obtain is  $b/2A$  see (FL) so this  $b/2A$  (FL) refers to this square (FL) is half of it, so (FL) refers half of this  $b/2A$  the whole square then we have the word (FL) so which appears in the verse see I will just quickly highlight the verse see (FL) okay.

So, (FL) is sort of put together okay so, (FL) is  $A+b/2A$  (FL) means division so, (FL) see so, you divide by this see so, that is how it is. So, what is to be divided is this factor (FL) refers to so, this divide by this quantity. So, then you says (FL) so, (FL) is shadow subtraction okay. So, this quantity this ratio has to be taken as negative and negative to this number which you already obtained.

(Refer Slide Time: 36:08)

**MNC's unconventional interpretation** (yet convincing!)  
Regarding the use of the word ऊन

- ▶ MNC thesis is: the word *āna* in BS should be taken to refer to the operation of 'division' though it is unconventional.
- ▶ The arguments presented in support of this are as follows:
  1. The word हरण and परिहा are derived from the same root ह्.
  2. One of the *sūtras* of Pāṇini (3.3.29) explicitly states ऊन as synonym of परिहा.
  3. Hence, ऊन can be taken to refer to division.
- ▶ So much so, now कृत्यनात् means by dividing by approx. square root. In the earlier notation employed this translates to
 
$$\frac{A^2 + b}{A} = A + \frac{b}{A}$$
- ▶ Now since शेष is  $\frac{b}{A}$ , the phrase शेषच्छेदो द्विसङ्गणः means divisor of  $\frac{b}{A}$  multiplied by 2. This translates to the expression
 
$$A + \frac{b}{2A}$$

So, this is the quantity from which so, the ratio of this has to be remove and what to get is basically (FL) so, (FL) is (FL) square root so, a refined value of the square root ahh correct, that is how it is.

(Refer Slide Time: 36:32)

### Derivation of the formula

- ▶ Let  $N$  be the surd, whose approximate value is desired to be found. To begin with, we choose a number  $A$  such that  $A^2 < N$ , and as close as possible to  $N$ .<sup>7</sup>
- ▶ This number  $A$  is taken as the zeroth order approximation to  $\sqrt{N}$ . Now, the error ' $b$ ' is given by
 
$$b = N - A^2 \quad (9)$$
- ▶ The first order approximation is given by
 
$$A_1 = A + \frac{b}{2A} \quad (10)$$
- ▶ Now the error is given by
 
$$\begin{aligned}
 b_1 &= N - A_1^2 \\
 &= A^2 + b - \left(A + \frac{b}{2A}\right)^2 = \frac{-b^2}{4A^2} \quad (11)
 \end{aligned}$$
- ▶ Thus, we have moved from  $(A, b) \rightarrow (A_1, b_1)$ .

<sup>7</sup>In principle, it is not necessary that  $A$  should be an integer.

So, what is the way by which they might have arrived at so, a possible derivation is presented now suppose  $N$  is the surd and  $A$  is the nearest square which is less than  $N$ . So, we start with  $A$  and the error at this stage is  $N - A^2$  that is why we write an error at this stage is this. So, as a first order approximation so, we write  $A + \frac{b}{2A}$  is  $A_1$  so, this is like this see suppose you have root of  $1+x$  kind of a thing.

So, as a first order approximation  $1 + \frac{1}{2}x$  so, this is how we take so, that is how it is okay. The error so, at this stage is obtained by  $N - A_1^2$  square. So, 0th order it is and first order it is so,  $b_1$  is this. So, this  $A_1$  square see so,  $N$  is  $A^2 + b$  and  $A_1^2$  square we write. And what we get this  $-\frac{b^2}{4A^2}$  square. So, this is the error at this stage so, what we do is we start with the  $A$  and then we have moved to  $A_1, B_1$ . So, the first order approximation is this and this is the error.

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Derivation of the formula (contd.)

- ▶ In other words,

$$(A_1, b_1) \rightarrow \left( A + \frac{b}{2A}, -\left(\frac{b}{2A}\right)^2 \right) \quad (12)$$

- ▶ The second order approximation is obtained by

$$A_2 = A_1 + \frac{b_1}{2A_1} \quad (13)$$

- ▶ Using (12) in (13) we have

$$(A_2, b_2) \rightarrow \left( A + \frac{b}{2A} - \frac{\left(\frac{b}{2A}\right)^2}{2\left(A + \frac{b}{2A}\right)}, \dots \right) \quad (14)$$

- ▶ It is to be noted that the first term in the parenthesis of RHS of (14) is same as the Bakhshali formula for finding surds.
- ▶ How accurate is this approximation?

So, if you want to move to higher order. So, essentially so, this is what it is so, we have a started with this. So, this is  $A_1 b_1$  or this is the expression let we have  $A_1 b_1$ . So, second order approximation so, we use  $A_1$  so, instead of  $A$  so, use  $A_1$  here in this expression. So,  $A_1 + b_1/2A_1$  so, if plugging the value of  $A_1$  so,  $A + b/2A -$  so,  $b_1$  is this and so, this is how it is. So, this is basically the formula which has been stated in the Bakshali.

So, this is the shadow of approximation so, which one can keep on doing so, further and apparently stopped here whether the authors of Bakshali derived it this way is something which one cannot very clearly spell out. I mean it is only a possible derivation and given that so, Indians were sort of fond of this iterative process it is quite possible that this might have been the approach which they had taken to arrive at this expression.

**(Refer Slide Time: 39:52)**

**Numerical example**

- ▶ Let the surd  $N = 83$ . The integer whose square is closest to  $N$  is 9. Hence we choose  $A = 9$ . This  $\Rightarrow b = 2$ .
- ▶ The Bakhshali formula is

$$\sqrt{N} = \sqrt{A^2 + b} = A + \frac{b}{2A} - \frac{\left(\frac{b}{2A}\right)^2}{2\left(A + \frac{b}{2A}\right)} \quad (15)$$

- ▶ Substituting the values in the above formula we have

$$\sqrt{83} = \sqrt{9^2 + 2} = 9 + \frac{2}{2 \cdot 9} - \frac{\left(\frac{2}{2 \cdot 9}\right)^2}{2\left(9 + \frac{2}{2 \cdot 9}\right)} \quad (16)$$

$$= 9 + \frac{1}{9} - \frac{1}{18.82} = 9.110433604 \quad (17)$$

- ▶ The actual value is: 9.110433579, which is correct to 7 decimal places.
- ▶ The accuracy depends upon how close we choose the initial value  $A$  to be, or how small  $b$  is. However, even if  $b$  is large (within 'permissible limits'), successive iterates would lead to the exact value.

So, how accurate is this approximation so, we will take one numerical example so, however  $N$  is 83 so, the closest square is 9 square. So,  $A$  is 9 so, that implies  $b$  is 2 so, this is the formula so, we call  $A + b/2A$  so, - this square yeah and so, if you substitute this values. So, what one gets is so, 9.110433604 and you use modern calculators so, what we get is this expression. And you can straight away see that it is correct to 67 decimal places.

The suppose give where to choose so, 85, 87 so, the closer you take so, there is nothing which compulses to take  $A$  as even integers. So, this will work even otherwise the closer you take the higher the accuracy will be even if you take much so, it is basically a sort of zigzag function which will be close which will keep on converging to the actual value. I will quickly discuss one more problem. So, an interesting problem which has been presented in Bhakshali.

**(Refer Slide Time: 41:24)**

**Interesting problems**

पञ्चानां वणिजां मध्ये मणिविक्रीयते किल। तत्रोक्ता मणिविक्रीत्रा मणिमूल्यं  
कियद्ववेत् ॥...अथ विभाग पादोऽत्र पञ्चमान पदोऽत्र च।

A jewel is sold among five merchants together. The price of the jewel is equal to half the money possessed by the first together with the moneys possessed by the others, or  $\frac{1}{3}$ rd the money possessed by the second together with the moneys possessed by the others, or  $\frac{1}{4}$ th the money possessed by the third together with the moneys possessed by the others, or  $\frac{1}{5}$ th the money possessed by the fourth together with the moneys possessed by the others, or  $\frac{1}{6}$ th the money possessed by the fifth together with the moneys possessed by the others. Find the cost of the jewel, and the money possessed by each merchant.<sup>9</sup>

<sup>9</sup>Here it may be mentioned that though the solution to the problem is available in greater detail, the statement as such is not fully decipherable from the manuscript (see for instance, Hayashi, 174-75.), and hence what has been presented above is a partially—yet faithfully—reconstructed version of it given by CNS (38-39).

And then make a few observation before winding up this stack there are quite a few interesting example which have been presented so, what has been stated here is there are see I said so, he say only this formula has been stated here. So,  $A+b/2A$  is what is implied with the verse no one can get from this verse. So, whichever by you in fact the problem has been very beautifully analyse.

And this unconventional meaning has been assigned to the word (FL) after lot of thought by (FL) and (FL) so, also tried to interpret the verse but one or two places you will find the little difficulty in trying to make various terms. So, go one with the other and finally give the meaning so, (FL) I think was in first century yeah. So, (FL) see there are as I told you there are lot of problems.

So, the this sense of the problem has been understood with the words that have been presented there but not every word s available for us because this manuscript has been damaged. So, it is said (FL) so, there were five merchants so, where willing to purchase something some jewel. So, he says a jewel is sold among five merchants the price of the jewel so, he stated in a complicated manner.

So, if he says so, it is exactly equal to half the money which is possessed by the first fellow+the sum of the other. So, one third the money possessed by this second fellow and the sum of the

others o, one third the money possessed by the other fellow and the sum of the others. So, one fourth and sum of the others so, what is the price of the jewel and what is the money possessed by this people. So, this is the problem fine so, (FL) one third (FL) so, one sixth so, this is how he takes the problem.

**(Refer Slide Time: 43:47)**

**Interesting problems**

*Solution:* If  $m_1, m_2, m_3, m_4, m_5$  be the money possessed by the five merchants, and  $p$  be the price of the jewel, then the given problem may be represented as

$$\begin{aligned} \frac{1}{2}m_1 + m_2 + m_3 + m_4 + m_5 &= m_1 + \frac{1}{3}m_2 + m_3 + m_4 + m_5 \\ &= m_1 + m_2 + \frac{1}{4}m_3 + m_4 + m_5 \\ &= m_1 + m_2 + m_3 + \frac{1}{5}m_4 + m_5 \\ &= m_1 + m_2 + m_3 + m_4 + \frac{1}{6}m_5 \\ &= p. \end{aligned}$$

Hence we have

$$\frac{1}{2}m_1 = \frac{2}{3}m_2 = \frac{3}{4}m_3 = \frac{4}{5}m_4 = \frac{5}{6}m_5 = q \text{ (say).}$$

Substituting this in any of the previous equations we get  $\frac{377}{60}q = p$ . For integral solutions we have to take  $p = 377r$  and  $q = 60r$ , where  $r$  is any integer. In fact, the answer provided in Bakhshālī manuscript is  $p = 377$  and  $m_1, m_2, m_3, m_4, m_5 = 120, 90, 80, 75, 72$  respectively.

And so, if somewhere to write it so, using what a notation so, this is what it amounts to see half of the first merchant+the sum of the others. So, this will be equal because that is the price of the jewel so, one third the money of the second merchant+the sum of the others. So, one fourth the money of the third person+and so on so, this will be let us say this is equal to p. So, this translates to this so, let us say like this is equal to something q.

So, then so, what one gets is basically an equation of this forum so,  $377/60q$  could be equal to p. So, this is what one gets and one can write it so, so this is  $377r$  q is equal to where r is any integer. So, bakshali actually gives the value p is equal to 377 and q is equal to 60. So, of course any multiple is also valid and so, this is the one of the typical problem which can be found him Bhakshali manuscript.

**(Refer Slide Time: 45:01)**

### Use of mathematical notations

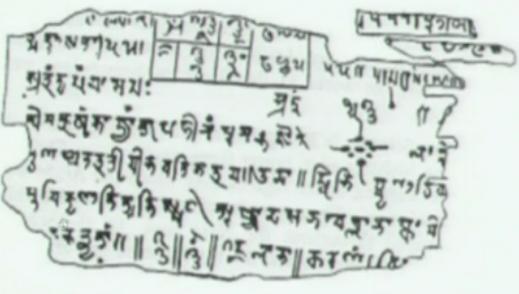
- ▶ BM is one the most important sources to know the kind of notations employed in those times.
- ▶ There are at least three different kinds of notations:
  - ▶ **Notation to represent fractions** – This is done by placing one number below the other without a horizontal bar.
  - ▶ **Notation to represent negative quantities** – A -ve quantity is denoted by a small 'cross' resembling the '+' sign to the right of it. This probably could be the deformed version of the character ऋ, used in Devanagari.
  - ▶ **Notation/Abbreviation for representing operations** – The operations line + √ are denoted by the characters such as य, म्, which are abbreviations of the words denoting those operations such as यति, मूल।
- ▶ In *Amarakosa* (c. 400 CE), we have the statement –
 

यदृच्छा विन्यसेत् शून्ये (place zero ...)
- ▶ Taking यदृच्छा = यावात्तावत्, we find '0' for unknowns (x).

This manuscript is also important so, as I showed you one of the slides earlier so, in connection with the kind of Notations.

(Refer Slide Time: 45:15)

### Refined - Folio of the manuscript (from web)



- ▶ The vertical and horizontal lines are used to **segregate numerals** and **symbols** from the main text.
- ▶ They at times represent fractions, **but without a horizontal line** as we keep using nowadays.

So, this is something which is frequently asked so, what was the kind of notations which might have been employed by those people in those days. So, this kind of Notation certain Notations are also available so, besides the kind of tabulation which one finds to segregate the numbers. So, how where the representing division how where the representing negative quantity how where the representing some unknown value.

So,  $m_1$ ,  $m_2$ ,  $m_3$  so, all of them are unknowns so, if one were to put in the form of notation, so what from the notation employed.

(Refer Slide Time: 45:56)

**Use of mathematical notations**

- ▶ BM is one the most important sources to know the kind of notations employed in those times.
- ▶ There are at least three different kinds of notations:
  - ▶ **Notation to represent fractions** – This is done by placing one number below the other without a horizontal bar.
  - ▶ **Notation to represent negative quantities** – A -ve quantity is denoted by a small 'cross' resembling the '+' sign to the right of it. This probably could be the deformed version of the character ३, used in Devanagari.
  - ▶ **Notation/Abbreviation for representing operations** – The operations line +  $\sqrt{\quad}$  are denoted by the characters such as य, म्, which are abbreviations of the words denoting those operations such as यति, मूल।
- ▶ In *Amarakośa* (c. 400 CE), we have the statement –  
यदृच्छा विन्यसेत् शून्ये (place zero ...)
- ▶ Taking यदृच्छा = यावात्तावत्, we find '0' for unknowns ( $x$ ).

Sp, to a certain extent we get clue form what is available there and he make a few remarks this connection scholars were identified that there are at least 3 kinds of an notations which have been employed one is a notation to represent fraction as I was mentioning. So, you place one number below the other, so this is understood as ratio, so this is one notation, so which is sort of horizontal bar and notation to represent negative quantities, so this is something which is been noted.

So, kind of +kind of a symbol, so if a quantity is negative, so you write the and then put a small kind of a mark to the right above it and some scholars have opine that this + kind of a symbol has to do with the word which they been employing to refer to a negative quantity (FL), so this (FL) might have been so represented with this kind of a symbol, so that is one suggestion which has been given.

And so this represent certain operations, so this also can been seen suppose you have to add 2 quantity in between, so this+symbol is not used for the they use the word (FL) see so (FL) to refer to and then sometimes (FL) to refer to the (FL) square root kind of the thing. So, these are certain notations which one can find out in this text and it is in this sense also, so this manuscript

is something which is quite unique and provides us lot of light on the kind of notations and the kinds of problems which have been attempting in those days.

So, in fact even (FL) so this is so (FL) so this is see that there are this quotation I have provided to the show that some kind of notational form has been employed to represent certain things even in those days though we do not have very clear evidences to what has been used.

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So, because this manuscript is handed from generation to generation and most of things done orally in this tradition so, with this few remarks, So, we conclude our discussion on Bhakshali manuscript thank you.