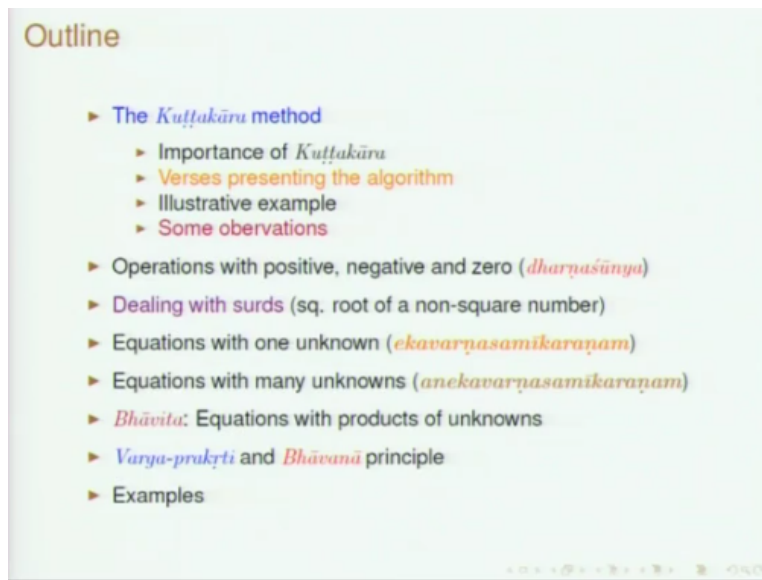


Mathematics in India: From Vedic Period To Modern Times
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Lecture-13
Brahmasphutasiddhanta of Brahmagupta-Part 3

So, earlier you had 2 lectures on Brahmasphutasiddhanta of Brahmagupta, so now is the third part which primarily deals with algebra. So, in Brahmagupta's Brahmasphutasiddhanta which is a (FL) book in 18 chapters is called (FL).

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So, these were also discussed by Aryabhata, so Brahmagupta starts this chapter which contains almost 100+ verses in the (FL) algorithm we briefly present the (FL) algorithm and then after making some of some observations we will proceed with what is known as (FL). So (FL) means a positive quantity (FL) means negative quantity (FL) is 0, so operations with positive, negative and 0, so this has been discussed in great detail in by Brahmagupta.

And then we will also discuss Brahmagupta's method for dealing with surds, so surd as you know is square root of a non square number and we will discuss (FL) and (FL) so (FL) here refers to an algebraic quantity say x so (FL) so you have equations on both sides, so with one unknown, so how do we equate them and solve them, so it is general approach and then discusses a few examples.

So, (FL) so is many unknowns so how do we handle equations with many unknowns and (FL) refers to the product of 2 unknowns and he also discusses the approach by which we will be able to solve see what is more interesting is to note that Brahmagupta gives general procedure see algebra was developed so we are talking about 6, 7 century and of course this is a very important problem, so (FL) and Bhavana principle.

(FL) is basically a second order indeterminate equation and Brahmagupta gives a very important principle called Bhavana principle and this has various applications it is also will be dealt with by professor Srinivas and then we will have a couple of examples.

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Importance of *Kuṭṭākāra*

- ▶ The *Kuṭṭakādhyāya* of *Brāhmasphuṭasiddhānta* consists of about 100 verses.
- ▶ Through the very first verse Brahmagupta conveys the importance of the technique of *kuṭṭaka* that is employed in solving a large class of problems in astronomy.

प्रायेण यतः प्रश्नाः कुट्टाकारदृते न शक्यन्ते ।
 ज्ञातुं वक्ष्यामि ततः कुट्टाकारं सह प्रश्नैः ॥ १ ॥

Mostly it may not be possible [for mathematicians] to solve problems without knowing the technique of *kuṭṭākāra*. Therefore, I am going to explain the *kuṭṭākāra* method, along with [a few] illustrative problems.

So, he starts this chapter with this (FL) giving the importance of Kuttakara, so (FL) so in fact triggers the kuttakara algorithm and then about 20, 25 verses. So, giving various kinds of problem with various examples from drawn from different disciplines, so (FL) means without (FL) so it is almost impossible to handle equations with unknowns and therefore he says I describe this (FL) with various illustrations.

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Gaining recognition among mathematician

Presenting a list of the topics, Brahmagupta mentions that one would gain recognition as 'acarya' amongst mathematicians only by gaining mastery over them.

कुट्टकखण्डन-अव्यक्तमध्यहरणैकवर्णभावितकैः ।
आचार्यस्तन्त्रविदां ज्ञातैः वर्गप्रकृत्या च ॥ २ ॥

कुट्टकेन	- by knowing to solve <i>kuttaka</i> problem
खण्डनेन	- operations with zero, -ve and +ve quantities
अव्यक्त सङ्कलनेन	- doing mathematical operation with unknowns
मध्यमाहरणेन	- elimination of middle term in a quadratic
एकवर्णसमीकरणेन	- dealing with equation with single unknown
भावितेन	- solving equations with products of unknowns
वर्गप्रकृत्या च	- solving second order indeterminate equation of the form $x^2 - Dy^2 = 1$

So, he says (FL) so this was basically summarises the various topic that he is going to discuss in this particular chapter which is called kuttakara (FL) so (FL) refers to the first order indeterminate equation and then as I was mentioning so (FL) so (FL) is 0 (FL) is negative (FL) is positive, so operation with all that (FL) so (FL) is basically an equation quadrilateral equation and how do we find (FL) is the middle term.

So, the coefficient of s and x both refers to as (FL) the context and we have to figure out the meaning (FL) and (FL) this of the various topics discussed in this chapter called kuttakara.

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Statement of the *kuttākāra* problem (+ terminologies)

- ▶ Suppose we have a number N , that satisfies the following equations
$$N = ax + r_a$$
$$= by + r_b.$$
- ▶ Here (a, b) are known as *chedas*¹ and (r_a, r_b) the *agras*, or *śeṣas*.
- ▶ If the remainder $r_a > r_b$, then,
$$a \rightarrow \text{अधिकाग्रहार (or) अधिकाग्रच्छेद}$$
$$b \rightarrow \text{ऊनाग्रहार (or) ऊनाग्रच्छेद}$$
otherwise, vice versa.
- ▶ The problem is to find N, x and y (all integers) given a, b, r_a and r_b (also integers)
- ▶ This forms an example of / order indeterminate equation, since we have 3 unknowns and only two equations.

¹They (a, b) are called *chedas* (divisors) in the sense that, when they divide N , they leave remainders r_1 and r_2 .

The very quickly recall so this is the kuttakara problem we have a number N and this number can be expressed in 2 forms $ax+ra$ and $b/+rb$, so ra and rb are called as (FL), so if ra is going to an rb a is called (FL) and b is called (FL), (FL) in the sense, so if you want to find out x you have to divide N/a and of course ra and rb are the remainders. This is just to get the terms clear, so when we see the verse.

So, a and b are refer to as (FL) and (FL) respectively, so the problem is to find out N , x , y given A ra , B rb so this is the problem, so we have a 3 unknowns and we have 2 equations and therefore this forms an example of indeterminate equation of the first order.

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The *kuttākāra* algorithm
 Suppose $N = ax + r_a = by + r_b$, with $r_a > r_b$. (a is *adhikāgrahāgāhāra*)

अधिकाग्रभागहात् ऊनाग्रभागहात् ग्रेषम्।
 यत् तत् परस्परद्वतं लब्धं अशेषः पृथक् स्थाप्यम् ॥ ३ ॥
 ग्रेषं तथेष्टगुणितं यथा अग्रयोरन्तरेण संयुक्तम्।
 शुद्धाति गुणकः स्थाप्यः लब्धं च अन्त्यादुपान्त्यगुणः ॥ ४ ॥
 स्वोर्ध्वेऽन्त्ययतो अग्रन्तो हीनाग्रच्छेदभाजितः ग्रेषम्।
 अधिकाग्रच्छेदहतं अधिकाग्रयुतं भवत्यग्रम् ॥ ५ ॥

- ▶ First $a \div b$. With the remainder and keep doing mutual division.
- ▶ Arrange the quotients one below the other.
- ▶ Multiply the last remainder (r_{2n}) by the desired number (*iṣṭagūṇitam*) ' t ' such that this plus $r_a \sim r_b$ in divisible by r_{2n-1} .
- ▶ The multiplier t has to be placed [below the quotients].
- ▶ And so too the *labdhi* ' t '.²

²Thus we have to form the *valli* with the quotients $(r_1, r_2, \dots) + t + k$

In fact the algorithm which has been presented by Brahmagupta in 3 versus or 4 more then what has been discussing, Aryabhata also discuss in 2 verse but it was a big task and here a beautifully laced down the algorithm (FL) so, (FL) means one below the other we need to place, so what is to be placed, so (FL) quantities which are mutually divided so, they have to be placed one below the other.

But then where do we start with so his he says (FL), a is (FL) and (FL) so, you start with a/b and then you proceed, so with the remainders you keep on dividing and all the quotients that you get are to be placed one below the other. Then comes in fact the term (FL) was used by Aryabhata to

find out a quantity or some stage of the division you stop and the remainder has to be multiplied by some quantity and then the difference has to be added or subtracted.

So, that this quantity is divisible by the previous remainder, so this has been very clearly stated here in Brahmagupta (FL) he calls it as (FL) so Aryabhata call it as (FL) see (FL) as I was mentioning refers to the remainder (FL) is difference (FL), so when it is added or subtracted (FL) means divides without any remainder that is what is meant by the (FL). Then he says that (FL) so whichever is a multiplier (FL) has to be placed below (FL) means this when this is divided so, whatever be the quotient that is called it as (FL) that also has to be placed.

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The *kuttakāra* algorithm: Example
 To solve $N = 34x + 2 = 13y + 10$; Here $a = 34, b = 13, r_a = 2, r_b = 10$

34) 13 (0
 0

 34 (2
 26

 8) 13 (1
 8
 5) 8 (1
 5
 3) 5 (1
 3
 2

Since the no. of quotients is **even**, t is to be chosen such that $(2 \times t + 8)$ should be divisible by 3. This $\Rightarrow t = 2$.
 $2 \times 2 + 8 = 12$. This divided by 3 gives the quotient *labhi* 4.

Valli

2	2	2	2	36	$(r_a - \text{urdhvarāṣi})$
1	1	1	14	14	$(r_b - \text{adhorāṣi})$
1	1	8	8		
1	6	6			
2	2				
4					

Having obtained the *valli*, the operations that remain are:

- ▶ $36 \div 34$, gives the remainder 2.
- ▶ $N = 2 \times 13 + 10 = 36$ is the desired number.

And then (FL) so, how do we so, having placed this only one below the other, so for instance if you see this problem, so 2, 1, 1, 1, 2, 4 so 2 is the (FL) and 4 is the (FL), so 2, 1, 1,1 so these are the quotients and they are placed one below the other what is to be done, so that operation is discussed in Brahmagupta here very clearly.

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The *kuttākāra* algorithm
 Suppose $N = ax + r_a = by + r_b$, with $r_a > r_b$. (a is *adhikāgrabhāgahāra*)

शुद्धाति, गुणकः स्थाप्यः लब्धं च अन्त्याद् उपान्त्यगुणः ।
 स्वोर्ध्वेऽन्त्ययुतो अशान्तो हीनाग्रच्छेदभाजितः शेषम् ।
 अधिकाग्रच्छेदहतं अधिकाग्रयुतं भवत्यग्रम् ॥ ५ ॥

- ▶ Starting from *antya*—which is *labdhi*—we have to carry out operations to complete the *valli* (अन्त्यादारभ्य कर्म कर्तव्यम्).
- ▶ The penultimate has to be multiplied with the one above and added to the ultimate (स्वोर्ध्वे उपान्त्यगुणः अन्त्ययुतः).
- ▶ This is to be carried out till we exhaust all the quotients, and we are left with only two numbers (*rāsīs*).
- ▶ At this stage, [the number at the top *ūrdhvarāsī*] has to be divided by *hināgraccheda*.
- ▶ This remainder (*s*) obtained is to be multiplied by *adhikāgraccheda*, and the result is to be added to *adhikāgra* to get N .

And this operation must missing in Aryabhatta's verses and it had to be supplied by the commentator Bhaskara, so here so he says (FL) see so, 2 into 1+4, so that has to be placed up, so that is 6 so this is the kind of description that Brahmagupta very clearly gives (FL) so, you go on and (FL), so this algorithm is more or less same which was explained while this Aryabhattiya was discussed.

So, 1 or 2 things which we are not very clearly stated then so I will just supply those details and then I will move on to other topics. So, here so let us choose the example $34x+2$ is equal to N and $13y+10$ is also equal to N , so here of the 2 remainders so, this 10 is larger and therefore 13 will be called as (FL), so 2 is (FL) and therefore it is called (FL), so the prescription was so (FL) had to be divided by (FL).

So, you divide this the quotient is 0 keep that so, you get 13 so and then you divide by 34 here after it is a mutual division, so this is dividing this you get 2 and 8 is the remainder, so this divides you get 1, so you get all the quotients, so you arrange them in this order one below the other. So, after the quotient see the last remainder is 2 here, and at this state what you need to do is so, you have to take a decision whether you have to add or subtract the difference of the remainders.

So, the statement was it has to be added if it is even and it has to be subtracted if it is odd, so the number of quotients here is 1 and therefore so, you have to find out this T is referred to as (FL) here he calls it as (FL). So, the last remainder into $t + \text{distance of the remainder}$, so this should be such that t is chosen such that, so this divisible will be the previous remainder, so this is the prescription and obviously t is equal to 2 satisfies.

And what we get is 12 once you get this 12, so you have to divide by 3 and the (FL) is 4 that is placed here then it is **is** this so (FL) add to this you get 6 you complete the valli you read here. So, having reasons there so the 2 numbers one is called (FL) is number (FL) so all that that place to be done is the (FL) has to be taken and that should be divided by this (FL) fine. So, this 36 has to be divided by 34 the remainder is 2.

And this basically gives the see y so this into $2 + 10$ gives you 36, so that is the desired numbers. So, you just happen that is 36.

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The *kuttakāra* algorithm: Example

To solve $N = 34x + 2 = 13y + 10$; Here $a = 34, b = 13, r_a = 2, r_b = 10$

Since the no. of quotient is **odd**, t is to be chosen such that $(1 \times t - 8)$ should be divisible by 2. **This $\Rightarrow t = 10, 1 \times 10 - 8 = 2$.** This divided by 2 gives the quotient **labdhi 1**.

34) 13 (0
 $\frac{0}{13}$ 34 (2
 $\frac{26}{8}$ 13 (1
 $\frac{8}{5}$ 8 (1
 $\frac{5}{3}$ 5 (1
 $\frac{3}{2}$ 3 (1
 $\frac{2}{1}$ 1

2	2	2	2	2	138
1	1	1	1	1	53
1	1	1	1	1	32
1	1	21	21		
1	11	11			
10	10				
1					

- ▶ $138 \div 34$, gives the remainder 2.
- ▶ $2 \times 13 + 10 = 36$ is the desired number.
- ▶ It is noted that whether we carry on the division till we get the remainder as 1 or not, the process works.

So, okay one interesting thing that you need to do is to observe that this process can be terminated at whichever stage we like. So, in fact the rational will be explained professor Sriram and here the point that I want to make is so, this in this I have done the problem and I have at this stage we stop in the previous thing. So, now I am just continuing one more division and what you get is remainder 1, in fact Bhaskara prescribe that you can continue till 1.

But this can be terminated at any stage. So, if you continue this till 1, so the number of quotient that you obtain is add. So, therefore here we have to subtract the difference in the remainders 10 and 2 is 8 and t first we chosen again with the same principle and t is 10 and the (FL) is 1, so you place this and what you get here is, so the numbers of different here but again the same principle. So, you will be able to get the number 36.

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The *kuttakāra* algorithm: Example
 To solve $N = 34x + 2 = 13y + 10$; Here $a = 34, b = 13, r_a = 2, r_b = 10$

Suppose we terminate the division two steps ahead of reaching the remainder 1 in the above example.

$$\begin{array}{r}
 34) \quad 13 \quad (0 \\
 \underline{0} \\
 13) \quad 34 \quad (2 \\
 \underline{26} \\
 8) \quad 13 \quad (1 \\
 \underline{8} \\
 5) \quad 8 \quad (1 \\
 \underline{5} \\
 3
 \end{array}$$

Here again, since the no. of quotient is **odd**, $(3 \times t - 8)$ should be divisible by 5. This $\Rightarrow t = 6$.
 $3 \times 6 - 8 = 10$. This divided by 5 gives the quotient *labdhi* 2.

2	2	2	36
1	1	14	14
1	8	8	
6	6		
2			

What is the lesson?

- ▶ The mutual division process can be terminated at any stage of the division as we like. By choosing an appropriate '*mati*', the *valli* can be constructed.
- ▶ Though the *valli* may be different, it leads to the same solution.
- ▶ In fact, various artifices have been invented by later mathematicians like Mahāvīra, Bhāscaraçārya to achieve more simplification.

So, I do the other operation, so you stop before see in the previous thing so first I took with 2 then I added 1 more here I just stop at 3 itself, so you terminate at 3 and this become much simpler and the number of steps is less, so you will 36. So, the point is the mutual division can be terminate at any stage of the division and we choosing up that is that in fact explains why it is called (FL), so you can make it an intelligent choice at any point.

So, the basic point is you have to reduce the magnitude of the numbers and or some stage if you are able to see that you will be able to get an appropriate (FL) you can terminate the process and you can finish the problem okay. And as I was mentioning the algorithm has been described by later people so Mahaveera Bhaskara and so on and the basic thing which has been given by Aryabhata has been slightly modified by later people and various artifices have been invented by the later mathematicians, so which will be discussed in other lectures.

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Some general observations

- ▶ Āryabhaṭa seems to be the **first mathematician** to have provided **systematic procedure** for solving indeterminate equation of the first degree of the form

$$ax \pm c = by \quad (a, b, c \text{ are +ve integers})$$

- ▶ Considering his period, (5th cent CE), this indeed is a **landmark achievement** in the field of pure mathematics.
- ▶ Āryabhaṭa's verses (2) **are terse**, and hence require to be supplemented by commentary, whereas Brahmagupta's verses (3) are **comparatively easier**, and are complete by themselves.
- ▶ If (a, b) are not coprime, their common factor **should be a factor of c too**. Otherwise, the equation has **no solution**.
- ▶ On the other hand, if the equation has one solution $(x, y) = (\alpha, \beta)$, then $(x, y) = (\alpha + bm, \beta + am)$, is also a solution for **any integer 'm'**: Thus the existence of **one solution** \Rightarrow the **existence of infinite solutions**.

So, with this with only one more observation so I just proceed further, so this is something which is consider to be a landmark achievement in the field of pure mathematics solving an indeterminate equation and clearly laying out an algorithm for doing so. So, this is one important thing to know. The other is so if 1 solution is obtained some for instance if you look at this last point.

So, if you are able to find out xy is equal to alpha, beta is one solution, then x is $\alpha + bm$ and y is $\beta + am$ is also a solution for any integer m and therefore if you have obtain one solution you have obtain infinite number of solutions fine.

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Mathematics of positive, negative and zero

- ▶ *Brāhmasphuṭa-siddhānta* (c. 628 CE) is the **first available text** that discusses the mathematics of zero (*sūnya-parikarma*) along with operations with positives and negatives (*dhanarṇa*).³
- ▶ The first of the six verses presenting rules for *śaikalana* goes as:

धनयोर्धनम् ऋणमृणयोः धनर्णयोरन्तरं समैकां खम् ।

ऋणमैकां च धनमृणधनशून्ययोः⁴ शून्ययोः शून्यम् ॥ ३० ॥

positive + positive \rightarrow positive
negative + negative \rightarrow negative
positive + negative \rightarrow positive/negative
positive + negative \rightarrow zero (when of same magnitude (सम))
positive + zero \rightarrow positive
negative + zero \rightarrow negative
zero + zero \rightarrow zero

³ *Brāhmasphuṭasiddhānta* of Brahmagupta, Ed. with his own commentary by Sudhakara Dvivedi, Benaras 1902, verses 18.30–35, pp. 309–310.

⁴ ऋणशून्ययोः ऐकाम् ऋणम्, धनशून्ययोः ऐकाम् धनम् ।

Now I move on to the second section of this (FL) wherein we discuss the (FL), so (FL), so these are the 4 fundamental operations (FL) is addition (FL) is subtraction (FL) is multiplication and (FL) is division. So, when we have quantities which are positive negative and 0. So, how do we conduct this 4 basic operations. So, how do we understand that, in fact here very clearly lay down.

So, this has to be understood and appreciated so thinking of the times in which this text has been composed. So, very very clearly he gives out all the operations. So, let me give the one or two examples taking 1 or 2 verses and explain, so the rest I will leave it because we have to cover a few more things. So, for instance for the first verse of this section starts with the operation (FL) Brahmagupta says (FL).

So, (FL) is positive quantity (FL) is negative quantity (FL) so (FL) so when you have 2 positive quantities we add them you get positive, negative+negative is negative (FL) when you deal with 2 (FL) so the resultant will be (FL), so when you have positive and negative it can be positive or negative then he says so consider 0, so (FL) is 0, so when you deal with 0 so positive+0 is positive, negative+0 is negative and finally he says (FL),so this will look so trivial to us.

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Mathematics of positive, negative and zero

► The next couple of verses present rules for *vyavakalana*:

ऊनमधिकाद्विशोध्यं धनं धनादक्षणमृणादधिकमृणात्।
 व्यस्तं तदन्तरं स्यादृणं धनं धनमृणं भवति ॥ ३१ ॥
 शून्यविहीनमृणमृणं धनं धनं भवति शून्यमाकाशम्।
 शोध्यं यदा धनमृणादक्षणं धनाद्वा तदा क्षेप्यम् ॥ ३२ ॥

Positive – positive → positive
 Negative – negative → negative

positive – zero → positive
 negative – zero → negative
 zero – zero → zero

negative – positive → negative (simply to be added (क्षेप्यम्))
 positive – negative → positive (simply to be added (क्षेप्यम्))

But laying it down so when things were not formulated well, so is something which is interesting, so (FL) so, he devotes 2 verses (FL) is smaller quantity, so (FL) so, is larger (FL)

when you have a max larger quantity and you subtract the smaller one what you will get is positive. So, negative-negative will be negative and then so he goes on (FL), so (FL) is 0 (FL) a negative quantity so, (FL) so, what we will have is (FL) and (FL), so you will have (FL) positive.

And (FL) see (FL) as I was telling in (FL) refers to space and all synonyms will be employed to refer to 0. So, (FL) all will be employed, so (FL)-(FL) will be (FL) so, then (FL) is positive quantity (FL) if you want is subtract what you will get is (FL), so and (FL) so all that has been all verses have been dealt with in great detail.

(Refer Slide Time: 18:07)

The slide is titled "Mathematics of positive, negative and zero". It contains the following text:

► The next verse present rules for *gunana*:

ऋणमृणधनयोर्घातो धनमृणयोः धनवधो धनं भवति।
शून्यैर्णयोः स्वधनयोः स्वशून्ययोर्वा वधः शून्यम् ॥ ३३ ॥

negative × positive → negative
negative × negative → positive
positive × positive → positive
negative × zero → zero
positive × zero → zero
zero × zero → zero

Then the (FL) so, (FL) so, (FL) is negative (FL) see is if you have negative into negative, so that will be positive see (FL) will be the resultant if you deal with 2 negatives. So, where it is (FL) so all those things have been stated and so, if you look at this last part of this verse (FL) see (FL) is 0 and (FL) is also 0, so he says (FL) 0 into 0 is 0.

(Refer Slide Time: 19:01)

Mathematics of positive, negative and zero

► The next couple of verses present rules for *harana*:

धनभक्तं धनम् ऋणहतमृणं धनं भवति सं खभक्तं खम्।

भक्तमृणेन धनमृणं धनेन हतम् ऋणमृणं भवति ॥

खीदृतमृणं धनं वा तच्छेदं खमृणधनविभक्तं वा।

ऋणधनयोर्वर्गः स्वं सं खस्य पदं कृतिर्यत् तत् ॥

positive ÷ positive → zero

negative ÷ negative → positive

zero ÷ zero → zero (defined the undefined ?)

positive ÷ negative → negative

negative ÷ positive → negative

positive/negative ÷ zero → taccheda

zero ÷ positive/negative → taccheda/zero

(positive/negative)² → positive

Then the last part so, (FL) so, (FL) see (FL) so, (FL) see so, (FL) is positive, so (FL) positive I made a mistake here positive should may positive should positive here okay. So, **so** then negative by negative, so that is positive again, so 0 by 0, so this also has defined as (FL), so this we know today is undefined but does mean defined as 0. So, (FL) so, positive by negative, negative by positive, so all that has been stated, so then (FL) this is the very important thing.

So, he introduces a term called (FL), so (FL) is division, so when 0 becomes the deviser see (FL) means it refers to a quantity wherein we find the deviser to be 0 (FL) refers to 0, so it should be a (FL) compound (FL), so for which 0 is the deviser, so they are introduce the term called (FL) and so, the other part is 0 is the numerator and any other quantity is the denominator that also has been stated to be (FL).

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Operations with *karaṇī* or surds

- ▶ Brahmagupta devotes a few verses in the same section to discuss various operations with surds.
- ▶ In what follows we present one example.

इष्टोद्भूतकरणी पदयुतिकृतिः इष्टगुणिता अन्तरकृतिर्वा ।

गुण्यस्तिर्यगधो ऽथो गुणकसम्प्रदायः सहितः ॥

The surds being divided by a desired (i.e., suitable optional) number, the square of the sum of the square-roots of the quotients is multiplied by the desired number ...

$$\sqrt{a} \pm \sqrt{b} = \sqrt{c \left\{ \sqrt{\frac{a}{c}} \pm \sqrt{\frac{b}{c}} \right\}^2}$$

- ▶ Examples:

$$\begin{aligned} \sqrt{8} + \sqrt{2} &= \sqrt{18} \\ \text{and } \sqrt{8} - \sqrt{2} &= \sqrt{2}. \end{aligned}$$

So, so with this he basically describe all the operations which can be done with positive negative and 0. Then this (FL) refers to surd, so, operations with surds this also we discusses in great detail I will give you only one example here (FL) see this suppose you want to add or subtract 2 surds. So, he gives a interesting method he says (FL) means you can divide it by some choice so which will make the operation convenient to you.

So, (FL) so divided by this (FL) is square root (FL) is add, so you have 2 numbers so AB o square root has to be added or subtracted. So, you divide by this and then he says take this square of that and then multiplied (FL) this quantity see whichever you chose you multiply and then take the whole square root. So, this is what I will be stated (FL), so and this will give you the difference or sum of the 2 surds with few more verses, so which deal with the square roots.

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Solution of linear and quadratic equations

- ▶ The section on arithmetical operations is followed by the section dealing with solutions of linear and quadratic equations (*ekavarṇasamikarṇam*).

- ▶ Brahmagupta first gives the general rule for obtaining solutions:

अव्यक्तान्तरभक्तं व्यस्तं रूपान्तरं समे, अव्यक्तः।

While dealing with linear equation in one unknown, the difference of the known terms taken in reverse order, divided by the difference [of the coefficients] of the unknown is the value of the unknown.

- ▶ In the verse above, the word *rūpa* (one with form) is used to refer to numbers of known magnitude (*vyaktāṅka*), as against *avyaktāṅka* ('formless' number, or number with 'unmanifest form').

- ▶ Suppose we have an equation,

$$Ax + C = Bx + D$$

then, $x = \frac{(D - C)}{(A - B)}$

Then I move on to the (FL) and (FL), so the general principle is laid down by Brahmagupta in the very first verse of this section. He says (FL) suppose you have an equation of this form, so $Ax+C$ is equal to $Bx+D$. So, this coefficient say I said x is (FL), so (FL) something to form is known sometimes he uses the word *rupa* sometimes he uses the word (FL). So, *rupa* means actually a constant so whose form is known see, so in contrast with (FL), so whose value is unmanifest *rupa* is manifest form which actually refers to the constant.

So, for instance D and c will be called as *rupa*, so that is why he says (FL) see the difference of the 2 *rupas* so $D-C$. So, (FL) in the sense so, this has to be taken the other order okay, so $D-C$ it is in this sense use of the word (FL), so (FL) is $A-B$, so the coefficient of the unknown quantity is also refer to as (FL) ,so I have seen. So, this in the verse (FL) refers to the difference between the coefficients of the unknown quantities $A-B$ (FL) is divided what is to be divided the (FL).

So, that gives the (FL) here refers to when you want to find equate this and then obtain the solution (FL) okay (FL) that is what it means.

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Solution of linear and quadratic equations

- Brahmagupta presents the solution to a quadratic as follows:

वर्गचतुर्विगितानां रूपानां मध्यवर्गसहितानाम् ।
मूलं मध्येनोऽनं वर्गाद्विगुणोद्धृतं मध्यः ॥ ४४ ॥

The absolute quantities multiplied by four times the coefficient of the square of the unknown are increased by the square of the co-efficient of the middle (i.e., unknown); the square root of the result being diminished by the coefficient of the middle term and divided by twice the coefficient of the square of the unknown (is the value of the) unknown.

- Suppose we conceive of quadratic written in the form (with constant on one side and the unknown on the other),

$$Ax^2 + Bx = C$$

then, $x = \frac{[\sqrt{(4AC + B^2)} - B]}{2A}$

Then when you deal with quadratic Brahmagupta gives this (FL) before he moves on to the example see lays down the principles, so what is to be done that is very interesting here (FL) so suppose you consider an equation like this, so $Ax^2 + Bx = C$, so he says he puts it on the other side. So, this is the general way in which the things have been handled. So, if you look at so the procedure that has been laid down.

So, you take rupa to one side and the unknowns on one side. So, and then handle this, so this C is refer to as rupa, so (FL) see so this (FL) here refers to the coefficient of x^2 okay. So, (FL) is multiplied 4, rupa, so what you get is basically $4AC$, so (FL) see (FL) is this middle term, so (FL) is square (FL) is this, so $4AC + B^2$ square (FL), so you have to take this square root (FL) so, again $-B$, so (FL) see (FL) see (FL) here again refers to A (FL) okay.

It is basically the formula $\frac{-B \pm \sqrt{B^2 - 4AC}}{2A}$, so, that is see away (()) (25:34). So, here so it has to be stated it in this particular forum and having laid down this here is a few examples.

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Example of *ekavarṇasamīkaraṇam* (I order)

द्व्यनमधिमसाशेषं त्रिहृतं समाधिकं द्विसङ्गणितम् ।
अधिमसाशेषतुल्यं यदा, तदा युगगतं कथय ॥ ४७ ॥

- ▶ The concept of *adhimāsa* is extremely important in Indian astronomy, as its computation plays a crucial role in our calendrical system.
- ▶ It x refers to *adhimāsaśeṣa*, then the content of the above verse, when expressed in modern notation amounts to solving the equation

$$2\left(\frac{x-2}{3}+7\right)=x$$

- ▶ This gives the solution $x = 38$

(FL) so, is very common problem which one encounters in astronomy, so the concept of (FL) so, which some of you would be familiar with, so we have to when we match this lunar and solar calendar, so the number of base which will constituted by 12 lunar months. So, will be falling sort about 11 days in a year and therefore, so every 2 and $\frac{3}{4}$ of the year we introduce a an average (FL), so it is has certain problem connected with that.

In that context he says (FL) means the remainder of a particular (FL) in a year (FL) so, he says (FL) means (FL) so, remove 2 $x-2$ (FL) divided by 3 (FL) so, $+7$ (FL) multiplied so (FL), so that itself is x , so what you get so is.

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Example of *ekavarṇasamīkaraṇam* (II order)

अधिमसाशेषपादात् त्र्युनात् वर्गोऽधिमसाशेषसम् ।
अवमावशेषतो वा अवमशेषसमः कदा भवति ॥ ५० ॥

- ▶ Let x refer to *adhimāsaśeṣa*. The problem posed in the verse when expressed in modern notation translates to

$$\left(\frac{x}{4}-3\right)^2=x$$

- ▶ Thus we have the following quadratic to be solved

$$x^2-40x+144=0$$

- ▶ This gives $x = 36$ or 4.

So, this is an example of (FL), so one more for (FL) see in fact this is second of (FL), so that is first order here again he says (FL), so (FL) is $1/4^{\text{th}}$ so, (FL) so (FL) -3, (FL) square (FL) it should be so, this is what it is (FL) so, how you will get, so you have a sort of quadratic and for which the solution is this, so one you choose 36.

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Example of *anekavarṇasamīkaraṇam*

गतभगणयुतात् दृग्णात् तच्छेषयुतात् तदैकसंयुक्तात्।
तदोगात् दृग्णं वा यः कथयति कुट्टकजः सः ॥

Let (N, A) → (revolution, civil days) in a *kalpa*
 (n', a) → (revolution, civil days) in a given period.

► Now the no. of revolution made by the planet in *aharyana* a is given by

$$n' = \frac{a \times N}{A} \quad (\text{integer} + \text{fraction})$$

► This may be rewritten as

$$n = \frac{(a \times N)}{A} + \frac{x}{A}$$

or $n + a = \frac{a(N + A) + x}{A} = y$

or $x = Ay - C,$

which is a *kuṭṭākāra* problem.

So, it is sums certain other considerations. Now I discuss this (FL), so you have a 2 variables, so we started with (FL) linear quadratic now (FL), so this again I just illustrate with 1 example (FL), so this problem will boil down to sort of (FL) but then so, presented in a context of finding the number of revolutions made by a planet in a given period and hence getting longitude.

So, here so, that is how it is post, suppose you have a very large period and this large period you call it civil days. So, in fact the Brahmagupta considers the period of (FL), so (FL) is 1000 times (FL) 1000 times 43 lakhs 20,000 years. So, he presents some huge numbers, so what is the number of revolution made by planet in this large period. So, that is how he goes about computing the planetary positions.

So, suppose N, A refer to the number of revolutions made by the planet in the total number of days in a kalpa. So, if n prime refers to the fraction of revolutions in a certain period called (FL), so (FL) is basically the count of days you start at a given period, so in this period this was the

position from this, so in (FL), so how many number of revolutions have been made it to be fractional number.

So, obviously by rule of 3 we will get n prime is a times N/A, so this will be I mean an integer part under fraction. So, you try to express it as this where N refers to the integral number of revolutions okay. So, this is x/A and you add A on both sides and you get an equation of this forum. So, this is a (FL) problem, so if 1 is able to get, so then he says he is a (FL).

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Bhāvita or equations with products of unknowns

भावितकरूपयुग्मना सा अव्यक्तवधा इष्टभाजिता इष्टान्योः ।
अल्पेऽधिकेऽधिकेऽल्पः श्रेष्ठः भावितद्वतौ व्यस्तम् ॥ ६० ॥

► In the verse above, Brahmagupta presents a rule for finding rational solutions to equations of the form

$$Axy = Bx + Cy + D$$

► The constants in the above equation are referred to as,

A → bhāvita or bhāvataka
D → rūpa
B, C → avyakta

► The first step in finding the solution to the above equation is to chose any number *m* (ishta), and obtain the quotient *q* (āpti).

$$\frac{(BC + AD)}{m} = q \text{ (āpti)}$$

► With *m*, *q*, *B* and *C*, we obtain the values of *x* and *y*. How?

Then Bhavita problem, so Bhavita as I was mentioning refers to the product of 2 unknowns. Here he lads down the rule for first (FL) so, suppose we have an equation of this forum, so this coefficient of the product of xy is refer to as Bhavitha, so recall the term rupa refers to the constant. So, what he says is (FL) is multiplication take product of A and D. So, (FL) so, (FL) so, here x is (FL) y is also (FL) and the co efficient also refer to as (FL), so (FL) is product of (FL).

So, B and c so, then what is to be done, see if you look at the forst top of the verse (FL), so (FL) BC (FL) I mean divided by (FL) see the point is so, you want to find out solution, so the prescription is the following, so you take the coefficient which are known, so take the product and divide by some (FL) means whatever you get as the quotient, so (FL) of the numbers Q and m so, (FL) so where is what he saying so, you have to basically add this to this and this to this.

So, q and c have to be added to B and C, so depending upon which is smaller which is larger and (FL), so if you divide by A you will get the required x and y.

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Bhāvita or equations with products of unknowns

- ▶ The equation $Axy = Bx + Cy + D$ to be solved may be written as

$$(Ax - C)y = Bx + D$$
 or

$$(Ay - B)x = Cy + D$$
- ▶ Multiplying the above and simplifying we get,

$$(Ax - C)(Ay - B) = AD + BC$$
- ▶ Now, by setting $(Ay - B) = m$, we have

$$(Ax - C) = \frac{(BC + AD)}{m} = q \text{ (āpti)}$$
- ▶ This straightaway gives the solution:

$$x = \frac{(C + q)}{A} \quad \text{and} \quad y = \frac{(B + m)}{A}$$
- ▶ Now, by setting $(Ax - C) = m$, we get the other solution

$$x = \frac{(C + m)}{A} \quad \text{and} \quad y = \frac{(B + q)}{A}$$

So, basically the principle that goes is the following see so, if you look at the C x is so, finally c+m/A and y is B+q/A or it is the other ways so, depending upon how you choose say basically this problem can be rewritten in this way and once you rewrite you get an equation of this form and this he writes it as so, q times m, so this is all it does and then so, basically adding you will get your x and y.

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Bhāvita problem: Illustrative example

भारते गण्यंश्वधात् त्रिचतुर्गुणितान् विशोध्य गण्यंज्ञान्।
नवति दृष्ट्वा सूर्यं कुर्वन्नावत्सगद्गणकः ॥ ६१ ॥

- ▶ The two variables in the problem are *rāsi* and *amśa* pertain to the sun.
- ▶ Now, the problem posed may be represented as follows:

$$xy - 3x - 4y = 90 \quad (A = 1, B = 3, C = 4, D = 90)$$
- ▶ The first step is to rewrite the equation in the form

$$(Ax - C)(Ay - B) = AD + BC$$
- ▶ Substituting the values we get

$$(x - 4)(y - 3) = 90 + 12 = 6 \times 17$$
- ▶ This gives the solution: $x = 10$ and $y = 20$.
- ▶ Obviously the solution is not unique. When we factor the RHS as 2×51 , we get $x = 6$ and $y = 54$.

So, an example so he says (FL) so, (FL) say this phrase (FL) is interesting phrase (FL) means an mathematician, so (FL) you know what (FL) so, this is how the sutra goes if you have (FL), so (FL) so, even if a follower to do this problem for 1 year and if he completes, then I would call him mathematician, so it is in this sense I mean the kind of (FL) statement then algebra was in it is initial phase.

So, this is how in most of the examples he says (FL) anyway what he says is (FL), so (FL) refers to 30 degree segment (FL) is degrees, so 30 degrees called the (FL) is degrees, so (FL) related to sun (FL), so this of the 2 quantities x and y, so one is (FL) y is (FL) okay (FL) so, xy take the product (FL) see so, 3(FL), so 3 and 4 (FL) multiplied 1 is (FL) the other is (FL) having subtracted (FL) so, this is equal to 90.

Now from this you have to obtain what is the actual position okay, so that is what he says. (FL) one is able to tell what sun is, so then this is the problem and one can show that one gets x is 10 and y is 20, so this is an example of (FL) problem.

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Introduction to *vargaprakṛti*

- ▶ Brahmagupta discusses the problem of solving for integral values of x, y for equation of the form

$$x^2 - D y^2 = K \quad (D > 0, \text{ a non-square integer})$$
- ▶ In equation of the above form,

$$\begin{array}{ll} x \rightarrow \text{jyestha-mula,} & y \rightarrow \text{kanistha-mula} \\ D \rightarrow \text{prakṛti} & \text{and } K \rightarrow \text{kṣepa} \end{array}$$
- ▶ One motivation for solving problem of this type is to find rational approximation to \sqrt{D} .
- ▶ If x, y are integers such that $x^2 - D y^2 = 1$, then we have

$$\left| \sqrt{D} - \left(\frac{x}{y} \right) \right| \leq \frac{1}{2xy} < \frac{1}{2y^2}$$
- ▶ The *Śulva-sūtra* approximation $\sqrt{2} = 1 + \frac{1}{3} + \frac{1}{374} - \frac{1}{37434} = \frac{577}{408}$ in fact, forms an example of $(577)^2 - 2(408)^2 = 1$.

Now I come to the most important part of this chapter, so which is almost the last (FL) verses, the first (FL) verses (FL), so and then we had this meaning with surds and then operation and then (FL) bhavita then goes on to (FL), so (FL) refers to the second order indeterminate equation.

So, in equation of this forum is known as (FL), so $x^2 - Dy^2 = K$, so given D and K, so we need to find x and y, so which are integers.

So, later Bhaskara has very clearly stated the terminology so, (FL) but Brahmagupta does not call them as (FL) in the verses which prescribe this but we just take at this will be convenient when we see Bhaskara also later during the lectures later. So, here so this problem gives rise to this kind of unique quality, so the one can show that this one of the motivations are solving this problem is to obtain the value of root D.

So, successive approximation of root D and we obtain using the Bhavana principle which has been given by Brahmagupta. So, we will demonstrate with the few examples but then so, just keep this in mind, so this is an any quality which will be satisfied root D see you can see that if x and y are large and suppose K is equal to 1, so then so, immediately you can get a first approximation, so root D is equal to, so y/x okay, so x/y root D is x/y .

So, this straight away you will get this, so root D $-x/y$ is less than $1/2xy$ and that is less than $1/2y$ square. In fact we will show little later perhaps in the next lecture that the approximation which has been given for root 2 in the (FL) sutra text, so which we discuss much earlier, so this approximation which was given kindly straight away obtain using this, so this is not to mention that this was the approach taken (FL) that is the very interesting way of obtaining this approximation.

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The *Bhāvanā* principle

मूलं द्विधेष्टवर्गाद् गुणकगुणादिष्टयुतविहीनाद्य।

आद्यवधो गुणकगुणः सहान्त्यघातेन कृतमन्त्यम् ॥

वज्रवधैकां प्रथमं प्रक्षेपः क्षेपवधतल्यः।

प्रक्षेपशोधकहते मूले प्रक्षेपके रूपे ॥

- If $(x_1^2 - D y_1^2) = K_1$ and $(x_2^2 - D y_2^2) = K_2$ then evidently

$$(x_1 x_2 \pm D y_1 y_2)^2 - D(x_1 y_2 \pm x_2 y_1)^2 = K_1 K_2$$

- In particular, given $x^2 - D y^2 = K$, we get the rational solution

$$\left[\frac{(x^2 + D y^2)}{K} \right]^2 - D \left[\frac{(2xy)}{K} \right]^2 = 1$$

- Also, if one solution of the equation $x^2 - D y^2 = 1$ is found, an infinite number of solutions can be found, via

$$(x, y) \rightarrow (x^2 + D y^2, 2xy)$$

So, this set of verses basically describe 3 important things, so 1 is defining the (FL) equation and then so, defining the second, third line actually give the Bhavana principle and of course a special case in the third one K is equal to 1, the fourth line basically gives that. so, let us try to understand the terms employed in this verse, so in fact if we consider $x_1 y_1$ and $x_2 y_2$ which satisfy equations with this (FL) K_1 or K_2 are called as (FL) see.

So, D is called (FL) K is called (FL), suppose we have an equation which satisfies this, so given D and K_1 you obtain x_1, y_1 , so given D and K_2 you obtain x_2, y_2 so, we have this 2 equations. Then to what is stated here is, so this equation will be satisfied and that is what has been stated in the second and third lines which you find here see (FL), so in fact Brahmagupta here uses the word (FL) and (FL) is multiplication (FL) so, this (FL) is the word which has been generally used.

And here he is use the word (FL) to refer to D, so (FL) here refers to this $y_1 y_2$ okay, so you have 2 equations. So, y_1 is (FL) and y_2 is also (FL) the product of y_1 and y_2 , the product of y_1 and y_2 multiplied by (FL). So, this is the equation which has been stated the D times $y_1 y_2$ (FL) so y is refer to as (FL) and x is refer to as (FL), so (FL) is product of (FL) say $x_1 x_2$, so (FL) means added to that see, so $x_1 x_2 + D y_1 y_2$ then he says (FL) is (FL) basically refers to cross multiplication.

Suppose if have 2 equations x_1y_2, y_2x_1 this is refer to as (FL), so that is what he say (FL) see K_1 and K_2 (FL) is product of that, so this equation is satisfied. So, this is what see this is what is basically Bhavana principle fine. So, having stated the Bhavana principle you take a particular you have x_1 is equal to x_2, y_1 is equal to y_2 . So which means you are Bhavana will get an equation of this forum and that is what is stated here.

And if one finds a solution to this, if suppose you find one solution with satisfies this equation x square- $D y$ square is equal to 1, then by Bhavana principle see what is most important to understand here is by employing this Bhavana principle you can move from xy to x square+ $D y$ square and $2 xy$. So, if one solution xy is known, so then all of Brahmagupta as approved. So, you have see if you do this also say this equation.

And as a particular case what one gets is $2xy$ and the x square+ $D y$ $D y$ square and you get infinite number of solutions. So, this is the most important aspect of Bhavana principle and using this principle one will be able to get successive values of (FL).

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Use of *Bhāvanā* when $K = -1, \pm 2, \pm 4$

The *bhāvanā* principle can be used to obtain a solution of equation

$$x^2 - D y^2 = 1, \quad \text{①}$$

if we have a solution of the equation

$$x_1^2 - D y_1^2 = K, \text{ for } K = -1, \pm 2, \pm 4.$$

$K = -1$: $x = x_1^2 + D y_1^2, y = 2x_1 y_1.$

$K = \pm 2$: $x = \frac{(x_1^2 + D y_1^2)}{2}, y = x_1 y_1.$

$K = -4$: $x = (x_1^2 + 2) \left[\frac{1}{2}(x_1^2 + 1)(x_1^2 + 3) - 1 \right],$
 $y = \frac{x_1 y_1 (x_1^2 + 1)(x_1^2 + 3)}{2}.$

$K = 4$: $x = \frac{(x_1^2 - 2)}{2}, y = \frac{x_1 y_1}{2}, \text{ if } x_1 \text{ is even,}$
 $x = \frac{x_1(x_1^2 - 3)}{2}, y = \frac{y_1(x_1^2 - 1)}{2}, \text{ if } x_1 \text{ is odd.}$

Brahmagupta also presents some other cases wherein K is $-1/+ 2 +/-4$, so all that has been stated by brahmagupta here so, this will be straight forward in certain examples, so you can simply guess x and y such that this equation is satisfied and once you know one solution then you keep on generating infinite number of solutions by using this Bhavana principle, so if you do not so, if

solving a certain problem in fact which will be discussed in great detail by professor Srinivas later.

So, if at a certain case certain stage of the problem, so you get K is equal to -1, +/-2 +/-4 then also you will be immediately able to solve the problem and this particular have been stated down what will be the xy values, so have also been laid down by Brahmagupta.

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Illustrative examples (given by Brahmagupta)

गणितकलाशेषकृतिं द्विनवतिगुणितां अष्टोतिगुणितां वा ।
 सैकं उदिने वर्गं कुर्वन्नावत्सगद्गणकः ॥ ७५ ॥

► The content of the verse may be translated into the following equations:

$$x^2 - 92y^2 = 1 \quad (1)$$

$$x^2 - 83y^2 = 1 \quad (2)$$

► Considering equation (1), we start with $(x, y) = (10, 1)$. This gives,

$$10^2 - 92 \cdot 1^2 = 8 \quad (3)$$

► Doing *bhāvanā*⁵ of (3) with itself amounts to

$$(10, 1) \rightarrow (10^2 + 92 \cdot 1^2, 2 \cdot 10 \cdot 1)$$

► Thus we have

$$192^2 - 92 \cdot 20^2 = 64 \quad (4)$$

► Dividing both sides by 64, we get

$$24^2 - 92 \cdot \left(\frac{5}{2}\right)^2 = 1 \quad (5)$$

⁵Recall: $(x, y) \rightarrow (x^2 + D y^2, 2xy)$

I take a couple of more minutes so, to present an example of this Bhavana principle, so which has been given by Brahmagupta himself. So, after stating this (FL) so this is how the section on (FL) commences okay (FL) is the word which refers to D but Brahmagupta is the word (FL) here and this has been said as (FL) so Bhaskara's versus they use the word (FL), so it is (FL) is basically the second order equation.

So, square and therefore it is refer to (FL) after presenting this what is (FL) equation and what is Bhavana principle and this particular case, so and then so how to find out, so even when K is not equal to 1 immediately jump into this solution, so after presenting what are the values of x1/1, so which will lead you to the solution, so he takes a example (FL) so (FL) refers to 30 degrees and (FL) is seconds so, (FL) refers to second sorry (FL) is minutes (FL) is seconds.

So, (FL) so, this is how it goes 30 degrees, degree, minutes and seconds, so (FL) see krti is use to refer to square here, so krti so, we will have instance to discuss on this later we will discuss this Bhakshali manuscript also, so in colloquial language krti actually refers to once on work our (FL) krti when you say composition of (FL) okay. So, krti mostly it is used to refer to square in many of this mathematical text.

And (FL) so krti square of so 2 things one is (FL) the other is kala okay, (FL) is 92 okay (FL) or (FL) you can take D to be 83, so either 92 or 83 (FL), so that quantity K is 1 okay, so (FL) see, so let us take first equation and here, so you can straight away see that 10 and 1, so x is 10 and y is 1 you will get 8. So, now use the bhavana, so recall so according to Bhavana principle, so if x y is a solution then x square+D y square and 2 xy is also a solution.

So, this is the Bhavana principle straight you can find out and convince yourself that , so here so 10 square+92 into 1 square so this is the mew x and this is the mew y okay 2xy 2 times xy so, this moves to this also is a solution and if you do this and then you divide both these sides 64 you get an equation of this forum. Here D is 92 that is what the that is the example that we are doing. So, 24 square-92 into 5/2 square, so is equal to 1, so this is like x square-D y square is equal to 1.

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Illustrative examples (given by Brahmagupta)

- ▶ Considering equation (5), and applying *bhāvanā*⁶ amounts to

$$\left(24, \left(\frac{5}{2}\right)\right) \rightarrow \left(24^2 + 92 \cdot \left(\frac{5}{2}\right)^2, 2 \cdot 24 \cdot \left(\frac{5}{2}\right)\right)$$
- ▶ Thus we have

$$1151^2 - 92 \cdot 120^2 = 1. \quad (6)$$
- ▶ Similarly, for the other problem $x^2 - 83 y^2 = 1$, we start with $(x, y) = (9, 1)$. This gives

$$9^2 - 83 \cdot 1^2 = -2 \quad (7)$$
- ▶ Doing the *bhāvanā* of the above with itself amounts to

$$(9, 1) \rightarrow (9^2 + 83 \cdot 1^2, 2 \cdot 9 \cdot 1) \quad (8)$$
- ▶ And hence we get

$$164^2 - 83 \cdot 18^2 = 4 \rightsquigarrow 82^2 - 83 \cdot 9^2 = 1$$

⁶Recall: $(x, y) \rightarrow (x^2 + D y^2, 2xy)$

So, once again do the Bhavana x those to x square+D times y square, so that is what it is and y goes to 2xy 2 times xy and you have this equation, so in the other problem so, this you have

basically reduce this and you can further keep on doing and all this will basically give you better and better approximations to root D, so $1151/120$ will be in approximate value of $92\sqrt{D}$ root of 92 will be this ratio.

If you do Bhavana once more with itself, so you will get better and better approximations. Similarly so if you choose the D equals 83, so then you have straight away you can see that so it is not necessary that K has to be always positive, so you can choose anything and so we get $9\sqrt{83} + 1$ and doing Bhavana of this with itself, so gives you this, so which when you divide by 2.

So, so actually $2\sqrt{D}$, so you can divide and you will get an equation of this form, so this is $x^2 - D y^2 = 1$ straight away. So, you can keep on doing Bhavana and you will be able to get better and better approximations for root D. So, with this we stop our discussion on Brahmagupta will continue with 1 or 2 more examples and then we will proceed to Bhakshali manuscript in the next lecture thank you.