Measure and Integration

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Lecture No. # 09

Extension of Measure

Welcome to lecture 9 on Measure and Integration. As you recall, we have been looking at classes of subset of a set X called semi-algebra, algebra, sigma algebra and so on. Then, we also looked at set functions defined on these classes with properties. So, in particular, we defined the concept of measure.

A measure is a set function defined on a collection of subsets, such that the measure of mu of the empty set is equal to 0 and mu is countably additive. Today, we are going to start the process, which is called extension process. So, the topic for today's discussion is going to be Extensions of Measures.

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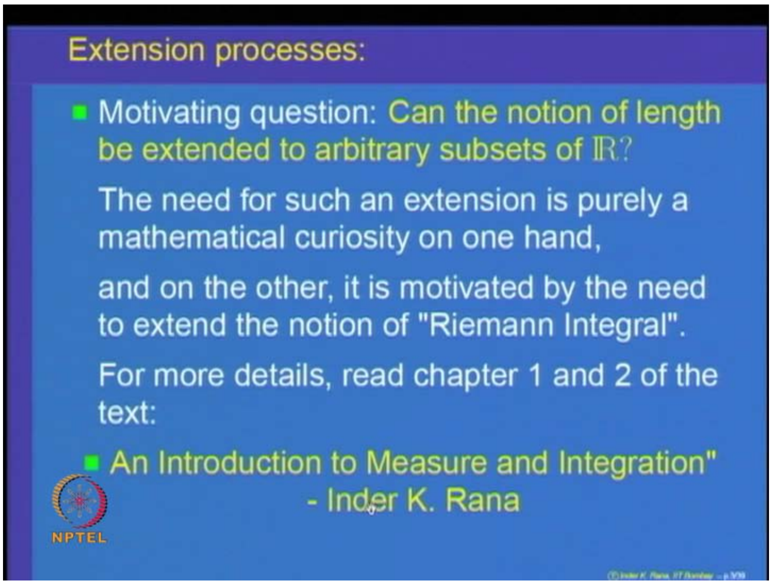

Basically, the question arises from some of the properties on the real line. Let us look at mathematically, the question; we know that notion of length is defined for all intervals;

so, the question is - can the notion of length be extended to arbitrary subsets of the real line? That means, can we define the notion of the length for an arbitrary subset of the real line? Of course, it should be compatible with the definition of length for the interval. So, the need for such an extension is – one, of course it is purely mathematical curiosity that we have the notion of length for an interval; can we define it for an arbitrary subset? Other reason which is more important is that, it arises from some problems in Riemann Integration.

The concept of Riemann Integral, which is defined for a class of functions, fails to satisfy some properties like the fundamental theorem of calculus does not hold for Riemann Integrable functions. So, in order to remove those difficulties, one started looking for an extension of Riemann Integral and that led to the problem of extending the notion of length from a class of subsets, that is, intervals to all subsets is possible. If you are interested in looking at more details about why Riemann Integral should be extended to a wider class of functions and how that leads to the concept of extending the notion of length to arbitrary subsets, read chapter 1 and 2 of the text book that we have mentioned earlier, namely: An Introduction to Measure and Integration by myself- Inder K. Rana

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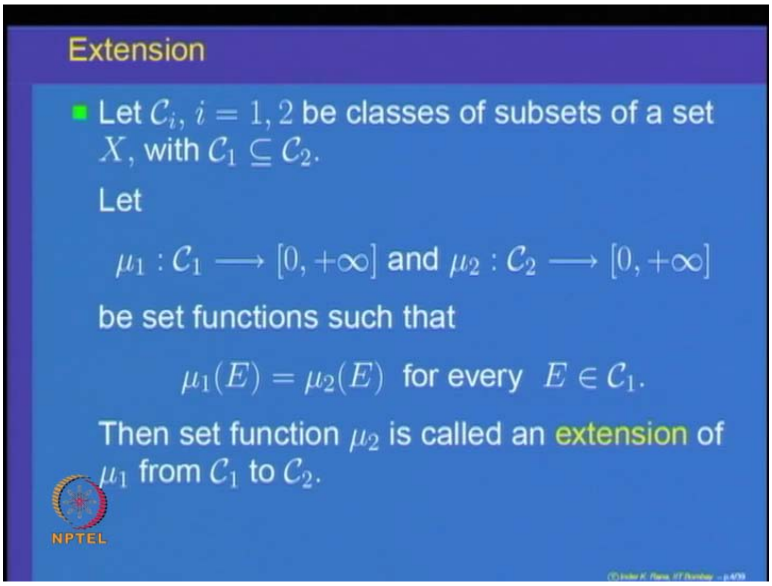

So, let us start with the question, what is an extension? So, let C_1 and C_2 be two classes of subset of a set X and let us assume C_1 is a subset of C_2. We have two measures or two set functions mu$_1$ and mu$_2$. mu$_1$ is defined on C_1 and mu$_2$ is defined on the collection C_2.

So, mu$_1$ and mu$_2$ are set functions. mu$_1$ is defined on C_1 and mu$_2$ is defined on C_2 with the property that mu$_1$ on C_1 is same as mu$_2$ for subsets of C_1. mu$_1$ and mu$_2$ agree on subsets of C_1.We call C_1 as a sub collection of C_2. So, in such a case, we call mu$_2$ as an extension of mu$_1$. On C_1, which is a smaller class, mu$_1$ and mu$_2$ are same. So, mu$_2$ is defined on a bigger class that is C_2. So, we say C_2 or mu$_2$ is an extension of the measure of the set function C_1.

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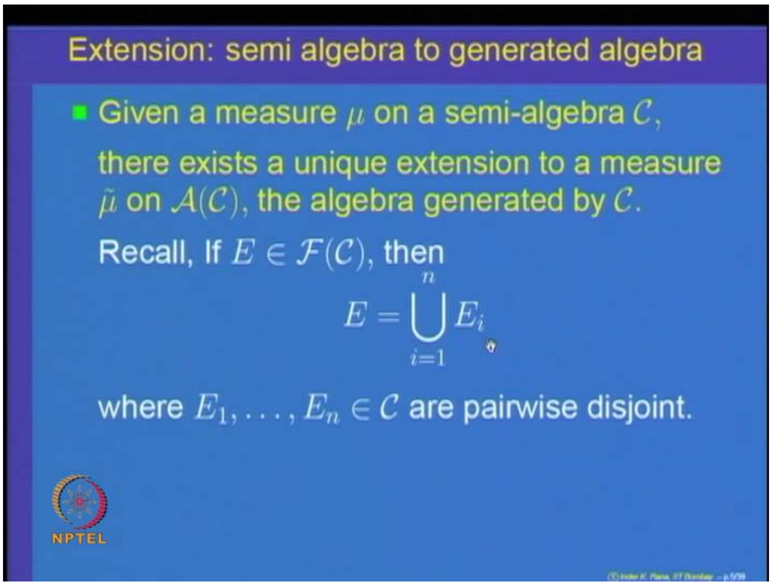

So, the problem is given a measure mu, we start with a measure mu on a semi-algebra C of subsets of a set X. We want to show that there exists a unique extension to a measure mu tilde on A of C, the algebra generated by it. So, this is going to be our first step of extension theory, namely: given a measure on a semi-algebra, we are going to extend it to a measure on the algebra generated by that semi-algebra.

Let us see, how this process is carried over. So, recall that a set E in the algebra is generated by a semi-algebra. We have characterized such sets and it can be given by a representation E is equal to union i from 1 to n E_i. So, every set in the algebra generated by a semi-algebra is a finite union of sets form the semi-algebra; in addition, they are pairwise disjoint. So, this was the result that we had proved. The algebra generated by a semi-algebra is nothing but all finite disjoint union of sets in the semi-algebra.

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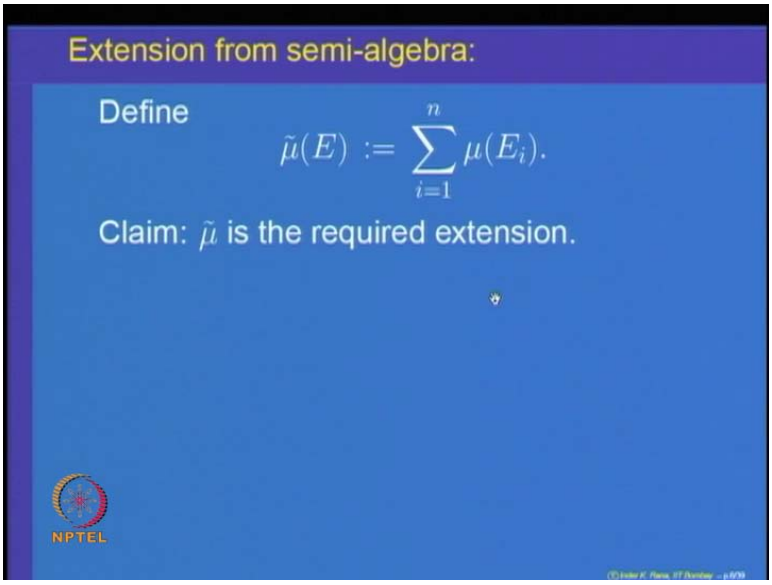

So, let us take any set E in the semi-algebra and define mu tilde of E to be a sigma i from 1 to n mu of E_i. The claim is that, this is the unique extension which we are looking for.

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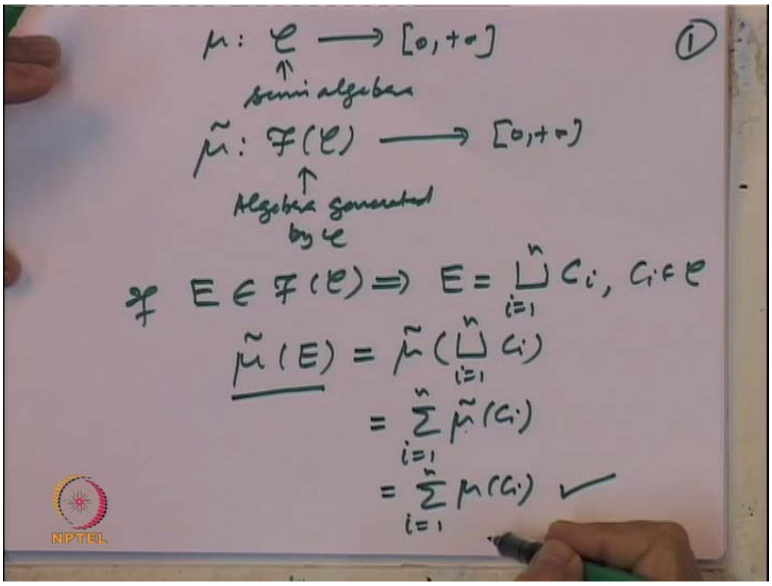

So, let us see, how do we do it? So, we have got mu on C and this is a semi-algebra. We define mu tilde on the algebra generated by C. So, this algebra (Refer Slide Time: 06:29) is generated by C and we want to define a function here, a set function, which should look like an extension. So, if E belongs to F of C, then we know this set E looks like a disjoint union of element C_i and i equal to 1 to n for sum n, C_i belonging to C. Now, why we defined it the way we have defined mu tilde? See, if mu tilde of E is going to be defined and it is going to be measured on the algebra F of C, then we know that every measure is also finitely additive.

So, by the finite additivity property of mu tilde which we have not yet defined, but the finite additivity property will say that this should be equal to mu tilde of the union C_i i equal to 1 to n. This being finitely additive, we should have i equal to 1 to n mu tilde of C_i, but mu tilde is going to be an extension. So, that means mu tilde on C_i is same as mu tilde $C_{i;}$ so, this is same as 1 to n of mu of C_i .So, that actually fixes what is going to be the definition of mu tilde of E. If E is a finite disjoint union of elements, which is C_i then mu tilde of E must be given by this (Refer Slide Time: 08:05). That also shows the uniqueness of the definition of mu tilde. So, mu tilde should be defined by this (Refer Slide Time: 08:14) and that is necessary. We will show that this definition also works.

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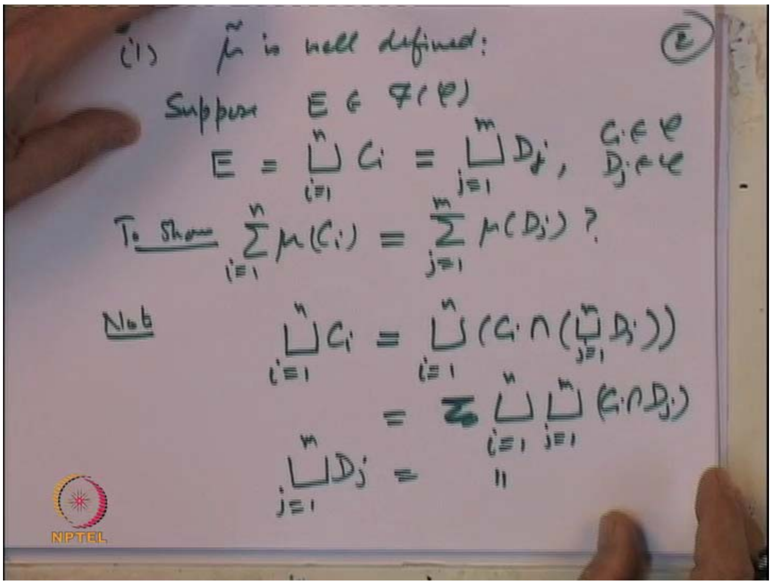

So, let us prove this property that mu tilde… .So first, we want to show that mu tilde is well defined. What does that mean? Suppose E is a set which is in F of C, then we know that E can be written as a finite union of set C_i is finite disjoint union of set C_i in C.

It is possible that it can have some other representation. So, it is possible that the results are represented as j equal to 1 to m of some sets D_i, where C_i's belong to C and D_j's also belong to C. So, to show them… because our definition is depended on the representation, we should show that mu of C_i summation i equal to 1 to n is same as summation mu of D_j j equal to 1 to m. This (Refer Slide Time: 09:28) we should show and then only we can claim that our function mu tilde is well defined.

So, let us show this. (Refer Slide Time: 09:43) Note, because E is given by this two different representations, I can write union C_i i equal to 1 to n also as union C_i intersection union D_j's j equal to 1 to m. So that is equal sigma; oh sorry, that is equal to union i equal to 1 to n union j equal to 1 to m C_i intersection D_j and similarly, union D_j's j equal to 1 to m is also represented by the same way because the two sets are same. So, it is the same representation.

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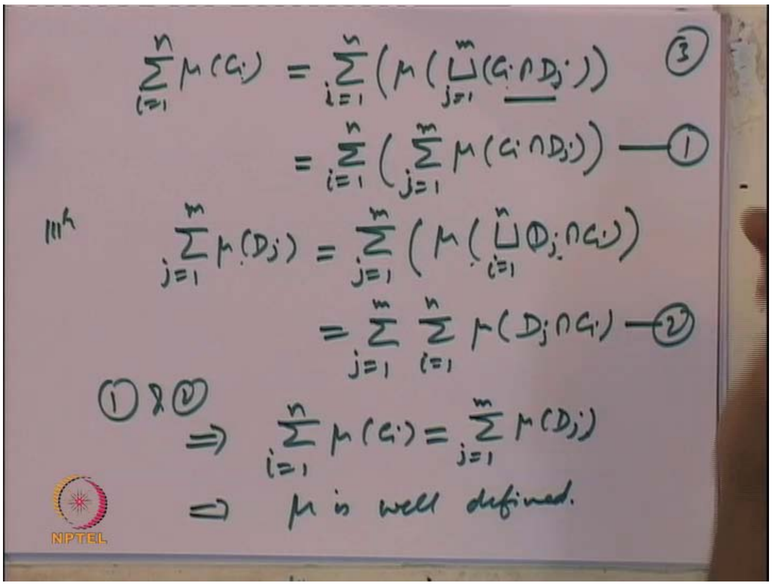

Now, let us compute. Sigma mu of C_i i equal to 1 to n. I can write it as sigma i equal to 1 to n and this mu of C_i is disjoint union of C_i intersection D_j. That is why, here we are using this representation (Refer Slide Time: 11:01) that we just now wrote j equal to 1 to m. This is a disjoint union; C_i's belong to the semi-algebra; D_j's belong to the semi-algebra. So, this intersection belongs to the semi-algebra and their union is C_i which is also in the semi-algebra and mu is a measure on the semi-algebra. So, this is also finitely additive and it is i equal to 1 to n.

So, I can write this sigma equal to j equal to 1 to m mu of C_i intersection D_j. Similarly, we can also write j equal to 1 to m mu of D_j to be equal to summation j equal to 1 to m and mu of D_j. So, that I can write as union of D_j intersection C_i i equal to 1 to n. Now, again by finite additivity property, this is (Refer Slide Time: 12:11) j equal to 1 to m sigma i equal to 1 to n mu of D_j intersection C_i.

So, look at this equation 1 (Refer Slide Time: 12:21) and look at this equation 2 (Refer Slide Time: 12:24). Equations 1 and 2 imply that sigma i equal to 1 to n mu of C_i is equal to sigma j equal to 1 to m mu of D_j and that implies mu is well defined. So, what we have shown is the following: If we take any set in the algebra generated by the semi-algebra and that has got a representation in terms of the elements of the semi-algebra. So, any element E in the algebra generated by the semi-algebra can be represented as a finite disjoint union of elements in the semi-algebra; Say C_i. So, pick many such representations and define mu tilde of E to be equal to sum of a mu's of this pc C_i sigma i equal to 1 to n.

It does not matter, which representation you choose. You will always get the same sum. So that means mu tilde of E is well defined. Now, let us look at the next property namely that mu tilde, which is defined on the algebra generated by the semi-algebra is finitely additive.

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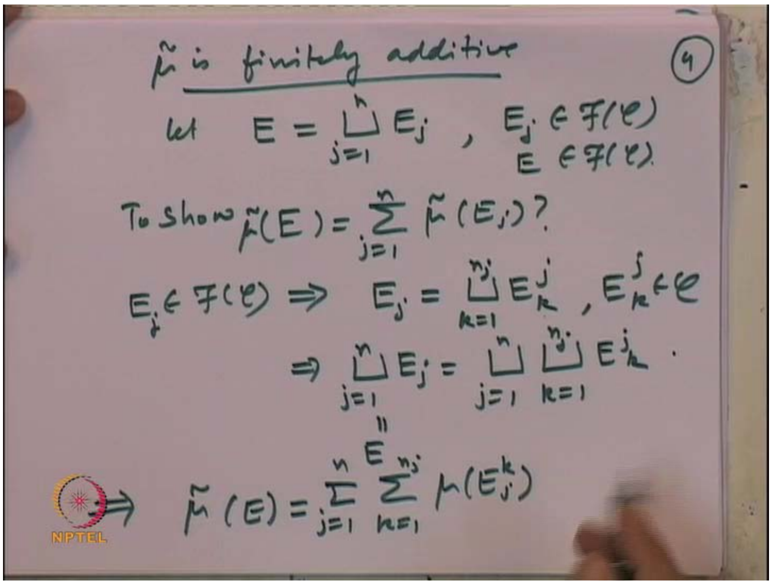

So, let us prove that property. So, we want to prove that mu tilde is finitely additive. So, to prove that what we have to show? So, let E be written as a union of E_j j equal to 1 to n, where each E_j belongs to the algebra generated by C and of course, E also belongs to the algebra generated by C. We want to show to that E mu tilde of E is equal to summation j equal to 1 to n mu tilde of E_j. so, this is what is to be shown (Refer Slide Time: 14:41)

Now, to show any such property, we have to go back to the definition of mu tilde of any set. Since, E belongs to the algebra generated by C and that implies… Let us write each E_j belongs to the algebra. So, each E_j can be written as a disjoint union of E_kj k equal to 1 to n_j, where E_kj belong to C for every j and k.

So, every element E_j is in the algebra generated by C. so, it must be a finite disjoint union of elements of C. So that implies that the union E_j j equal to 1 to n is equal to union j equal to 1 to n union k equal to 1 to n_j of E_kj. This (Refer Slide Time: 15:58) is our set E. E is equal to union and so, we have represented E as a finite disjoint union of elements of C. So that implies that mu of E or mu tilde of E. You can choose any representation and in particular, this (Refer Slide Time: 16:16) so, it is equal to summation j is equal to 1 to n summation k equal to 1 to n_j of mu of E_kj .

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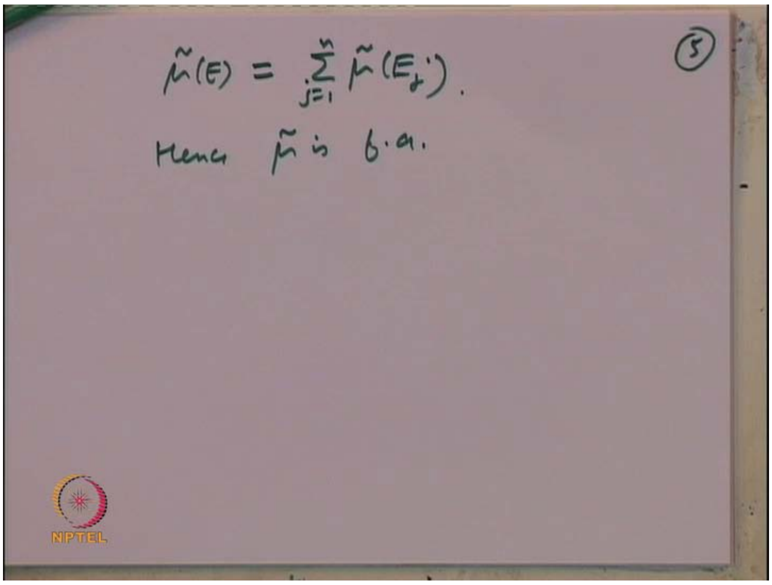

Now, using the finite additive property of a mu, we will write this. So, this is equal to… Look at this sum, (Refer Slide Time: 16:43) it is nothing but j equal to 1 to n mu tilde of E_j. That is by definition because E_j is union of E_kj over k. So, by definition, I can take that representation and say this is equal to this. So, that says mu tilde of E is equal to this (Refer Slide Time: 17:07) and hence mu tilde is finitely additive. We have proved that mu tilde is finitely additive and uniqueness, we have already shown. Thus, we have shown that a measure, which is defined on a semi-algebra can be in a unique way and can be extended to the algebra generated by it.

Basically, the idea is- because every element… Intuitively, keep in your mind that mu of a set is the size. So, any element in the algebra generated by the semi-algebra is a union of disjoint pieces in the semi-algebra and size of each of them is known. So, the size of the union must be equal to sum of the sizes of the individual pieces because they are disjoint. That was the idea and that helped us to extend a measure from a semi-algebra to the algebra generated by it. So that is the first step of the extension theory and as a consequence, the length function can be extended and that we have already shown.

Length function can be extended from the collection of all intervals to the collection of finite disjoint union of intervals, that is, the algebra generated by it. Now, we will go to the next step of the extension. So, we will start with a measure, which is defined on a algebra. We want to try extending it to the sigma algebra generated by it.

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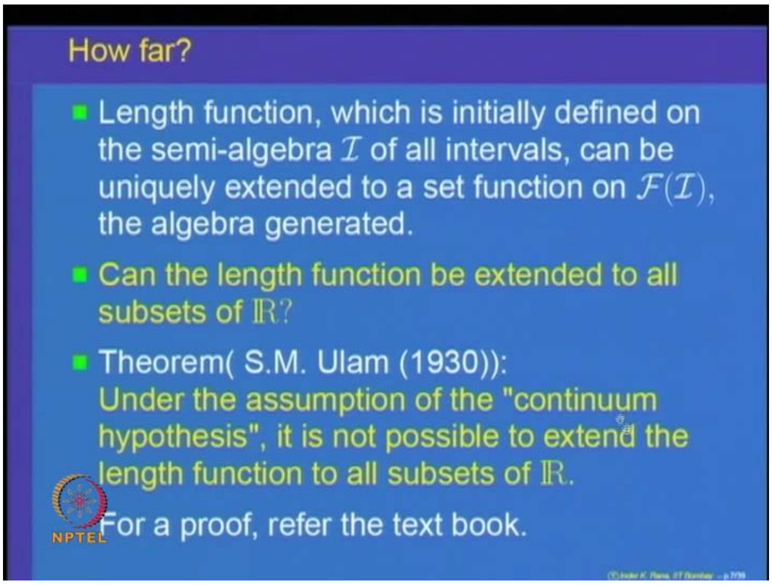

So, the next step in the extension theory is going to be- For example, we would like to say, can the length function can be extended to all subsets of the real line? We have done it from intervals to the algebra generated by intervals. There is a theorem by mathematician called S.M. Ulam and that theorem was proved in 1930. It says that under the assumption of "continuum hypothesis" and it is not possible to extend the notion length to all subsets of real line. This is a very important theorem. So, it uses two things:

namely, one is what is called continuum hypothesis. I will not go into the discussion of what is called continuum hypothesis at this stage. I would stress that one should read about this theorem from the text that we have just now mentioned, An Introduction to Measure and Integration. So, this is an important theorem, which says as a consequence, it is not possible to extend the length function to all subsets of real line.

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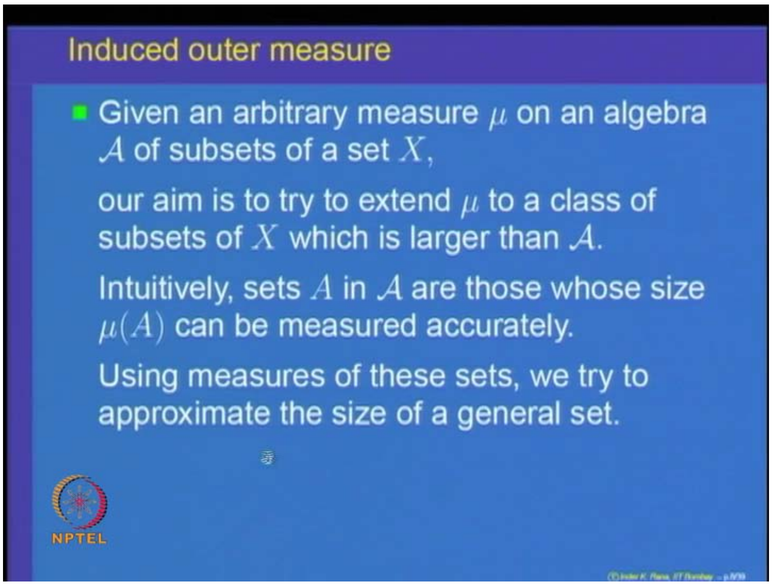

So, the question comes, if we cannot extend… so, in general, given a measure mu on a algebra of subsets of X, we would like to extend it to a bigger class than A. It cannot be done for all subsets, but let us try to intuitively follow our idea of measuring the size of an object. So, intuitively, given a measure mu on an algebra A, a collection of a subsets of a set A of set X, Mu of A is the size of the set A, which you can measure. Given an arbitrary set E, one may not be able to measure its size exactly using the mu, but we can at least try to approximate.

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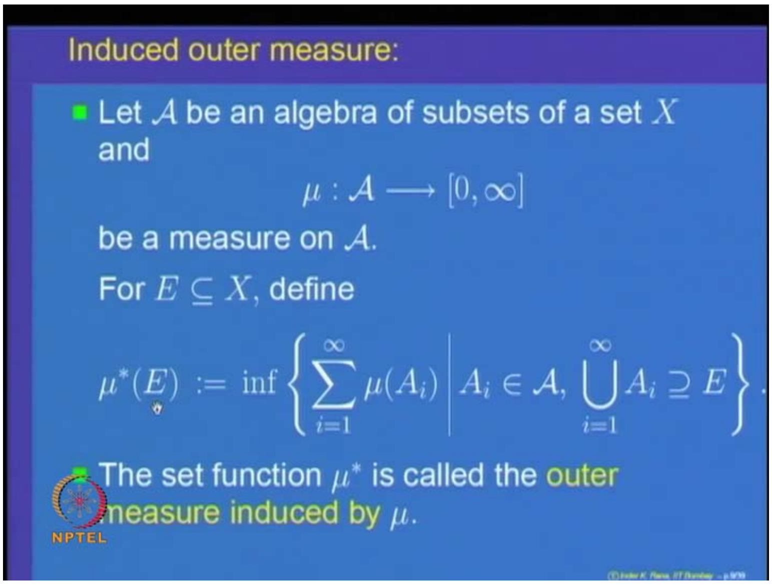

Let us define what is called as the outer measure induced by a measure? So, let us take mu - an algebra of subsets of a set X, A - an algebra of subsets of a set X, and mu - a measure defined on it. For any subset E in X, let us define, what is called mu star of E. so, what is mu star of E? What we do is given the set E, here is a set E and you cover it by sets A_i's in the algebra. You cover it by the sets in the algebra. Take a covering of E by the sets A_i's in the algebra. You know, what the size of the set Ai is. Let us take the size of the set A_i and add up all the sizes. So, what do you think this sum (Refer Slide Time: 21:48) will represent? In some sense, this sum will represent the approximate size of the set E. Of course, it depends on the covering A_i.

Now, what we do is - we take the infimum of all these approximate sizes. That means, we take the infimum of these numbers or all possible coverings of set E and define that number as mu star of E. We will try to analyze what are the properties of this mu star of E? So, first of all, let us give it a name and this mu star of E is called the outer measure induced by mu. Why the outer? Because we are covering E by sets, these things cover E. (Refer Slide Time: 22:35) May be we are going outside E. So, this is an outer measure because we are trying to measure the size of this in terms of induced by mu because in terms of the known size is mu.

Once again, let us recall and look carefully at what this mu star is? Given a set E, arbitrary subset in X. Cover it by elements A_i, whose sizes you know. So, take a covering of E by the elements in the algebra. Look at the sizes of A_i's and add up all this. (Refer Slide Time: 23:17) That is the sum, mu A_i and that is the approximate size. Take the infimum of all these approximate sizes and that we are going to call it as outer measure induced by.

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So, the first property we want to say is - mu star is well defined. Well, what is the meaning of mu star is well defined? Let us go back to the definition

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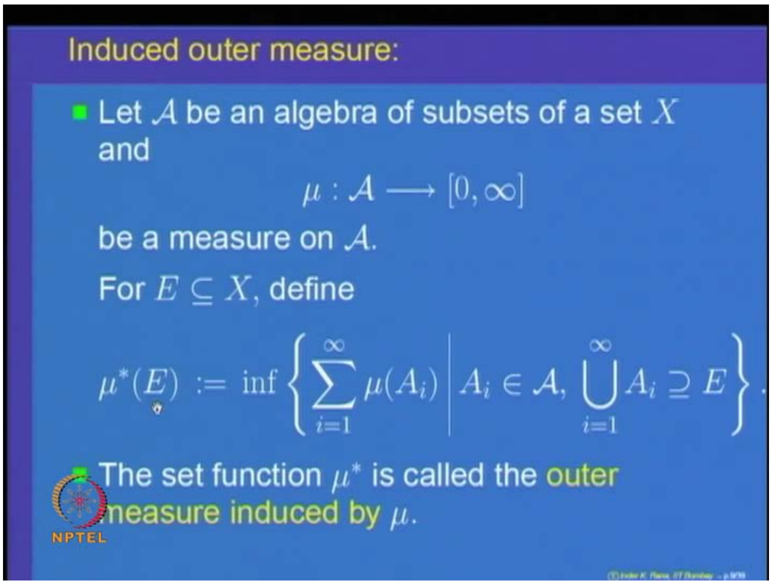

This is mu star, it is infimum of some numbers and infimum of a subset of numbers exist in the real line. If it is non-empty, it should be bounded below. Of course, all these numbers are going to be bounded below because all are non-negative numbers. So, it is bounded below by 0. Why is this collection non-empty? Because A is an algebra and so, the whole space belong to it. So, keep in mind that A is an algebra. In the definition of an algebra, the whole space X is an element. So, E is covered by X itself and X belongs to the algebra.

At least, there is one number in this collection over which you are taking infimum namely mu of X. It is a non-empty collection of extended real numbers. So, infimum always exist and hence, mu is a well-defined number. Of course, it could be equal to plus infinity. Keep in mind the numbers here; they are all extended real number. So, this set (Refer Slide Time: 24:56) is a collection of non-negative extended real number and their infimum always exist and infimum could be equal to plus infinity.

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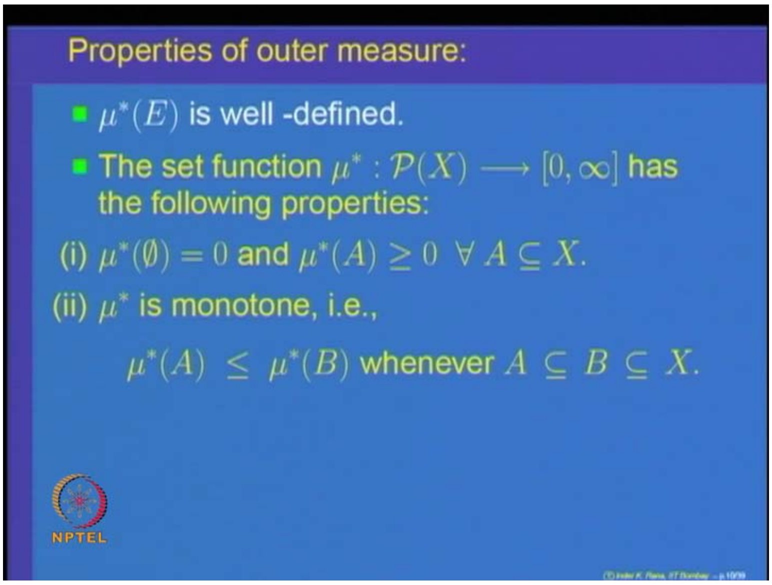

We have shown a mu tilde is mu star. The induced outer measure is well defined and so mu star is a well-defined set function on the class of all subsets of the set X. We want to show some properties of it: The first property is - mu star of empty set is equal to 0 and that is true because empty set belongs to the collection A in the algebra and mu star there is equal to mu of A and that is equal to 0 and for any set that is a infimum of non-negative numbers so, this infimum has to be bigger than or equal to 0. So, that first property is obvious. The Second property - we want to check that mu star is monotone. So, let us check that mu star is a monotone function.

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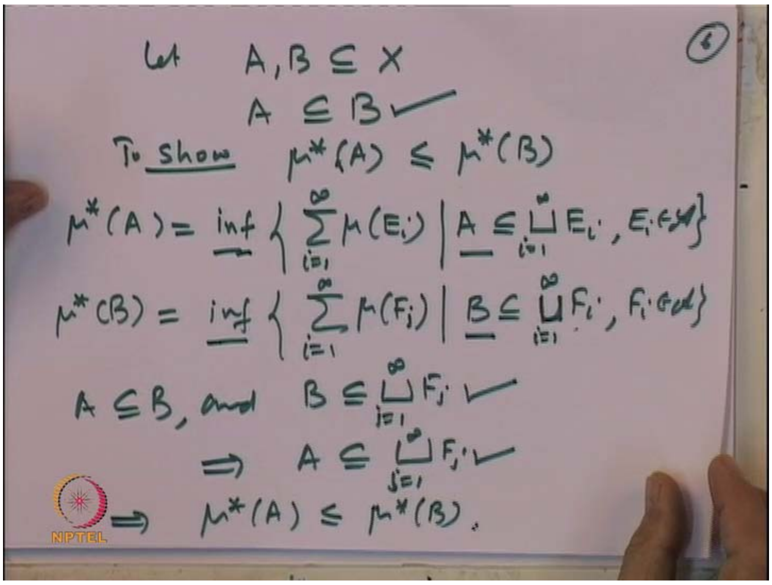

So, let A and B be subsets of X and A is a subset of B to show mu star of A is less than or equal to mu star of B. Now, what is mu star of A? In all these properties, we are going to use the definition of infimum, critically. So, what is mu star of A? Mu star of A is defined- as by our definition, it is a infimum over sigma mu of E_i‘s, say 1 to infinity, where this set A is contained in union of E_i‘s disjoint union. Of course, E_i‘s belong to the algebra, what is mu star of B? That is infimum i equal to 1 to infinity of mu of F_i‘s, where B is contained in union of F_i‘s i equal to 1 to infinity disjoint union, where F_i‘s also belong to the algebra A.

Now, if A is given to be a subset of B and B is covered by union j equal to 1 to infinity, then that implies A is also inside. So, this (Refer Slide Time: 27:55) is also inside F_j. So, what we are saying is every covering of B is also a covering of A. This is the infimum (Refer Slide Time: 28:07) over all possible coverings of B and this is the infimum over all possible coverings of A. Every covering of B is also a covering of A. (Refer Slide Time: 28:16) Here, we are taking infimum over a larger set and here we are looking at the infimum over a smaller collection of numbers. Whenever, you take infimum over a smaller collection of numbers, that is always bigger than or equal to infimum over a larger collection of numbers.

That is a simple property about infimum. If you are taking a infimum of a larger collection, then that tends to be smaller than the infimum over a smaller collection. So, that property implies that mu star of A has to be less than or equal to mu star of B. That is purely a property of infimum over what collection you are taking. Every covering of B is also a covering of A. so, coverings of B form a subset of coverings of A, and hence this property is true. That is the monotone property, namely mu star is monotone.

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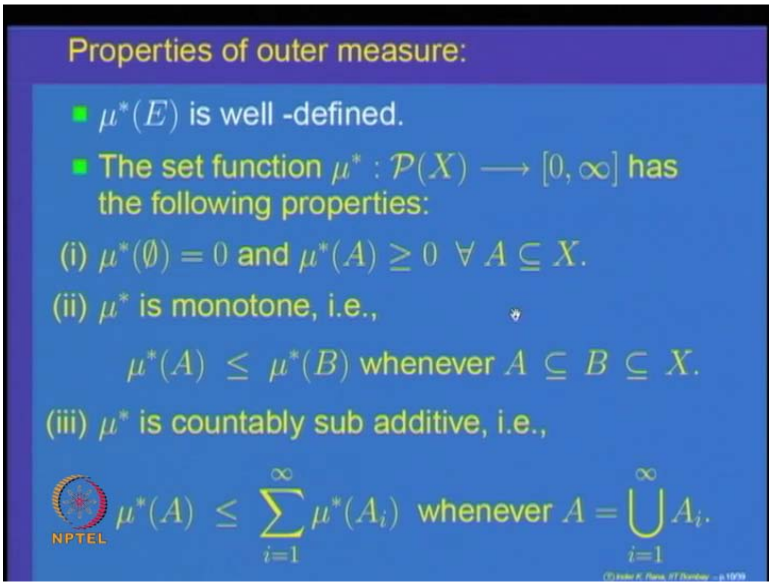

Let us look at the next property, namely; mu star is countably sub additive.

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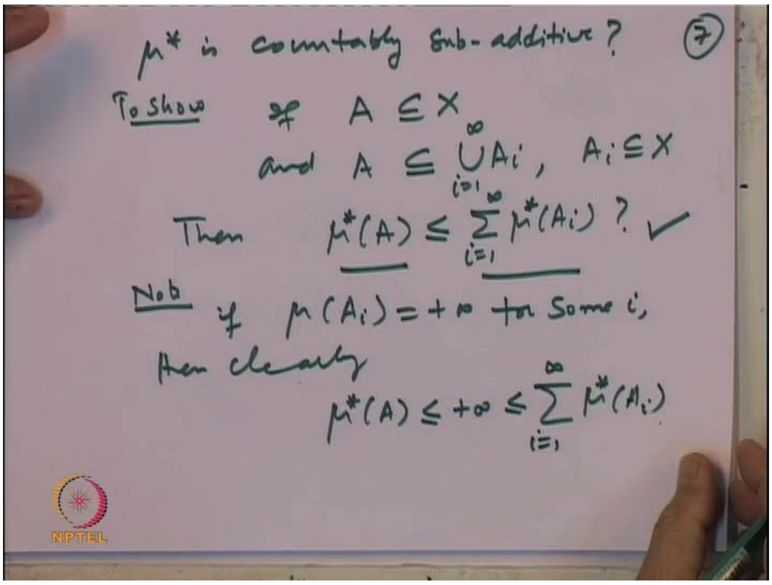

We want to prove mu star is countably sub additive. So, that means, we have to show that if A is a subset of X, A is contained in union of A_i's. A_i is also a subset of X and then we want to show that mu of A is less than or equal to summation mu of A_i's. So, this is what is to be shown. (Refer Slide Time: 30:08)

Now, let us observe the note. We want to show one number mu of A is less than or equal to sum of these numbers. If one of these numbers is equal to plus infinity, then obviously, this property is true. So, if mu of A_i is plus infinity for some i, then clearly mu star of A is a number, which is less than or equal to plus infinity, which is at least one of the mu A_i's. So, that is less than or equal to mu of… So, it is mu star we are looking at and let us just write mu star. We are trying to prove that mu star is countable. So, mu star of A_i is equal to 1 to infinity. So, what we are saying is this inequality is obvious, if one of the terms in this sum is equal to plus infinity.

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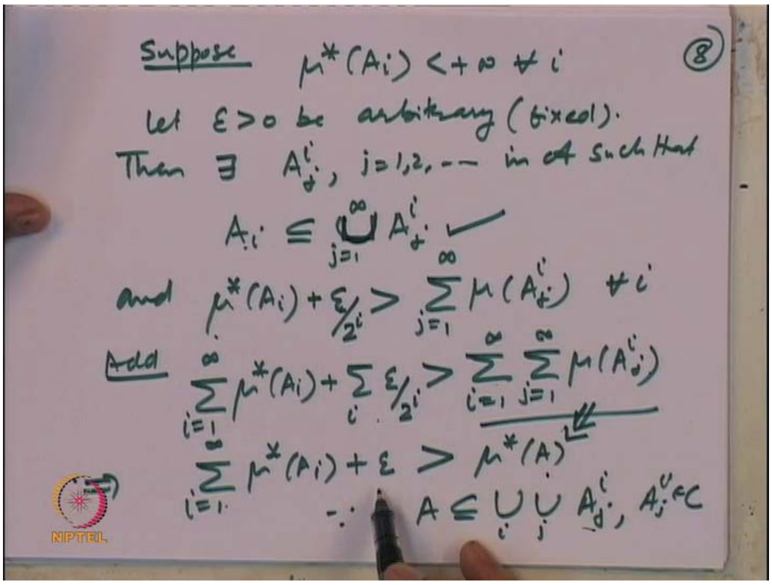

So, let us take the case when all of them are finite. Let us assume, suppose, mu star of each A_i is finite for every i. Now, what is mu star of A_i? Mu star of A_i is infimum for a certain collection. Here, we are going to use the properties of something, both being infimum and finite. So, let epsilon greater than 0 be arbitrary and of course fixed. You choose arbitrarily and fix it and mu star of A_i is the infimum for all summations in approximate sizes, then there exist at least one covering. So, there exist sets say, A_ji j equal to 1, 2 and so on. In the algebra A, such that this A_i is contained in this disjoint union of A_ji's.

So, mu star of A_i, which is infimum. If I add the small number epsilon to this, (Refer Slide Time: 32:38) it becomes bigger than summation mu of A_ji's j equal to 1 to infinity. Let me stress here, this is the kind of definition or this is the kind of analysis we will be coming across and we will be doing it again and again. So, let us be very clear about this. We have got some number, which is the infimum over some collection. If this infimum is finite, then, infimum plus a small quantity epsilon cannot be the infimum because that is on the right side of it. So, that cannot be the infimum of that collection.

Otherwise, Alpha plus… If alpha is infimum, then alpha plus epsilon will be the infimum, which contradicts the definition of the infimum. So, if alpha is the infimum, alpha plus any small number epsilon cannot be the infimum. What does that mean? That means, there must be a member of the collection over which you are taking infimum so that alpha plus epsilon becomes bigger than that number in the collection, over which you are taking infimum; that is why we are saying because mu star of A_i is finite, given epsilon, the infimum plus epsilon must be bigger that member of the collection over which you are taking infimum. (Refer Slide Time: 34:14) what is the collection that is obtained by taking a covering a disjoint covering of a disjoint covering not only disjoint actually any covering we are taking so any covering and such that this is true.

So, given epsilon, there exists a covering A_ji j equal to 1 to infinity of A_i such that mu star of A_i plus epsilon is bigger than this (Refer Slide Time: 34:39) and this happens for every i. So, if we add up these equation over i, summation over i is equal to 1 to infinity mu star of A_i plus sigma alpha over i is bigger than sigma over i equal to 1 to infinity sigma over j equal to 1 to infinity mu of A_ji and that is what we wanted. Mu star of A_i is bigger than something and we have got that kind of inequality.

Now, the problem is we are going to add epsilon infinite number of times. So, this will tend to become infinity and we do not want that. So, we will go back and refine our estimates. So, given epsilon bigger than 0 and this we can do it for any epsilon. In particular, whenever you are looking at A_i for a given epsilon, there should exist a covering, such that if we refine, it will make it 2 to the power i.

So, we will change our epsilon and that is true for every epsilon. In particular, it should be true for this (Refer Slide Time: 35:52). So, what we are saying is given epsilon there is a covering such that A_i is covered by that collection. mu star of A_i plus epsilon divided by 2 to the power i is bigger than the approximate size, that is, mu summation mu of A_ji and this is for every i. Now, if I add epsilon by 2 to the power i. So, that means, what we have got is now convergent and that means sigma i equal to 1 to infinity mu star of A_i plus epsilon is bigger than this sum.

Now, note that if i and j both vary, this is for every i (Refer Slide Time: 36:40). Now, if I take the union over i's and that will be union over this. So, I will get a covering of union A_j's, which will be covered by this. A is inside this and what we are claiming is; this is bigger than mu star of A because A is contained in union over i union over j A_ji and is A_ji belong to C. So, A is covered by this countable union and this (Refer Slide Time: 37:18) is one approximate size for mu of A and that is always bigger than or equal to mu star of A because of infimum. (Refer Slide Time: 37:26) So, this quantity implies that this is always bigger than this. So, I can claim that mu star of summation is bigger than this quantity.

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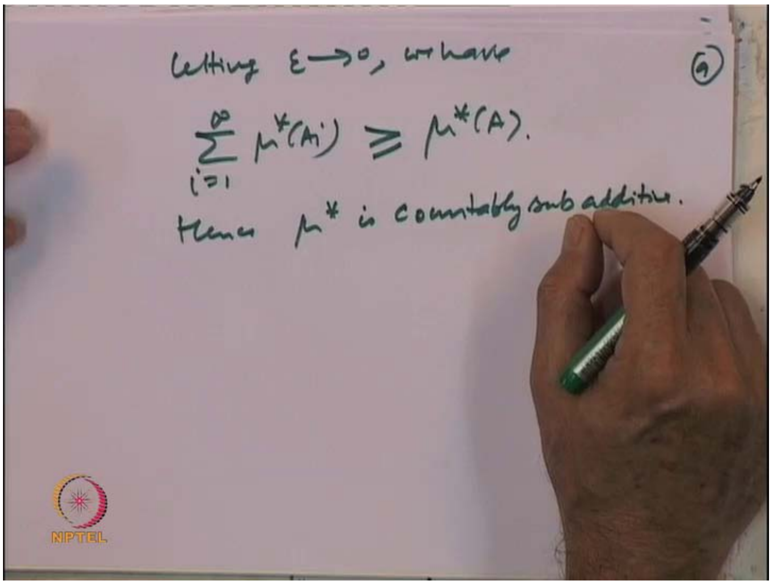

So, now this epsilon is arbitrary as it was fixed arbitrarily. So, I can let that go to infinity. One writes, let epsilon go to 0. We have sigma mu star of A_i i equal to 1 to infinity. This epsilon becomes zero, eventually. Now, I will write bigger than or equal to because in the limit, it can become bigger than equal to mu star of A. Hence, mu star is countably sub additive and so that we have proved countably sub additive. I just want to go through the proof of this once again because this is an important kind of analysis. We will be doing it again and again.

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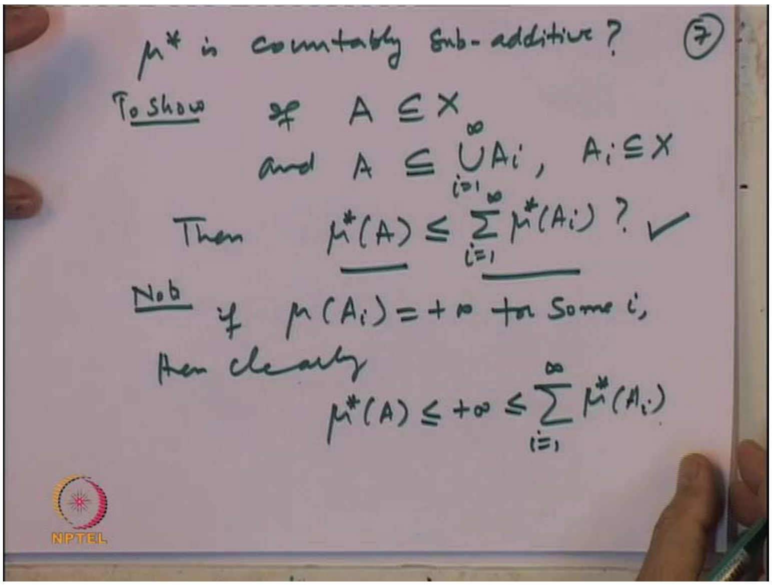

Let us just revise the proof once again. Mu star is countably sub additive. To show that mu star is countably sub additive, we have to show that if A is a subset of X and A is contained in union of A_i and A_i is contained in X then, I have to show that mu star of A is less than or equal to summation mu star of A_i. Now to show this, the first observation, which we should keep on mind- whenever, we are trying to show that one number is less than or equal to summation of a collection of numbers, then an obvious case may arise namely, one of the numbers may be equal to plus infinity. So, if mu of A_i is equal to plus infinity for some i, then clearly this side is equal to plus infinity and mu star of A is always less than or equal to plus infinity. So, we get mu star of A less than or equal to plus infinity and which is always less than this sum. So, it means that property is true.

So, the obvious case is- mu star and then mu star of A_i is finite for some i. So, what is the other possibility? Other is that mu star of A_i is finite for every i (Refer Slide Time: 39:52) Now, here is the main part of the construction that we are going to use, namely: it is an infimum, which is a real number. So, given epsilon is bigger than 0 arbitrary. We can find covering A_ji of the set A_i, such that mu star of A_i plus this small number and that small number will make it dependent on i. The stage at which you are doing epsilon divided by 2 to the power i bigger than the approximate sizes over which you are taking the infimum. So, once again the property of infimum being a real number is used here and nothing more than that. So, once that is done, you add both sides and this is for every i. Take the summation on both sides and so summation mu star of A_i plus summation of this over i is less the is bigger than summation of mu of A_ji. Now, this is a convergent series, (Refer Slide Time: 40:56) its sum is equal to epsilon. This is mu star of A_i summation plus epsilon and this is the quantity on the right hand side (Refer Slide Time: 47:06), is a approximate size of A, that is, this is bigger than or equal to mu star of A.

Mu star of A is infimum over all such numbers because A is covered by union over i union over j. A_i is covered by A_ji's. so, union over A_i covered by this union and A is inside it. So, this implies that summation mu star of mu of A_ji's summation over i and j is bigger than mu star of A. Once that is done, it means that we are letting epsilon go to 0. So, you get this quantity that says mu star is countably sub additive.

So, we have proved this property that mu star is countably sub additive. Now, the only thing left to be shown is that mu star actually is an extension otherwise, all this process will be a waste.

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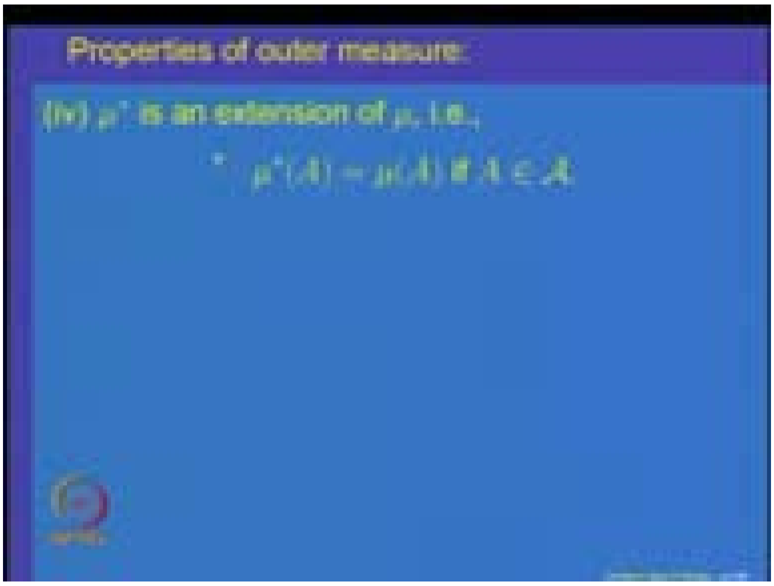

So, we want to claim that mu star is indeed an extension of mu. mu star is not countable additive, but at least we should check it is an extension and it is countably sub additive and that we have already checked. We want to check that mu star of A is equal to mu of A, if A is in A; to check that, let us look at the proper definition.

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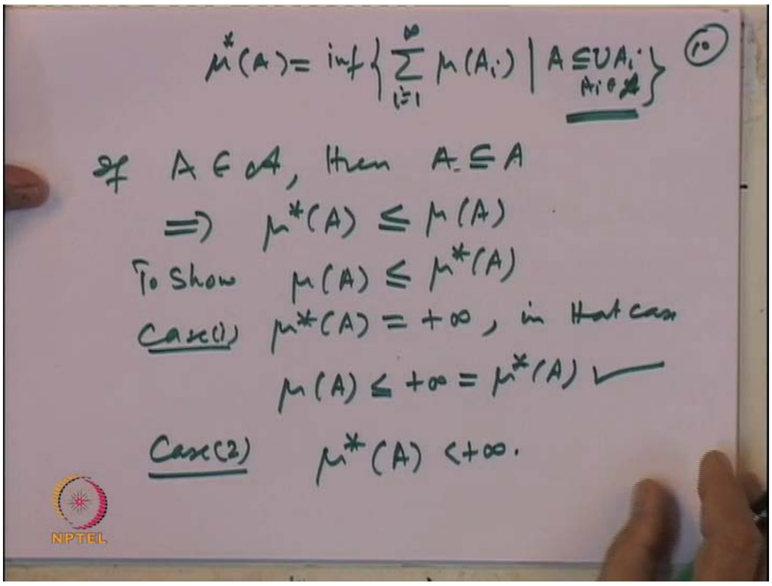

We had mu star of A is equal to infimum over summation mu of A_i i equal to 1 to infinity, where A is contained in union of A_i and A_i belong to the algebra A. If A belongs to the algebra, then A is actually equal to A. So, A is contained inside A and this is one of the elements here in this covering, A itself covers it. So, it will appear in one of the element over which you are going to take the infimum.

So, that implies mu star of A, which is the infimum less than or equal to mu of A and that property is by the shear fact, that A is covered by itself. A is in the algebra, that is, we want to prove the other way round, inequality to show that mu of A is less than or equal to mu star of A. once again, we want to show that one number is less than the other number. So, there is an obvious possibility of case one - mu star of A is equal to plus infinity. In that case, this is plus infinity and mu of A is always less than or equal to plus infinity, which is equal to mu star of A. So, the obvious case is when mu star of A is equal to plus infinity. Let us look at case two: mu star of A is finite.

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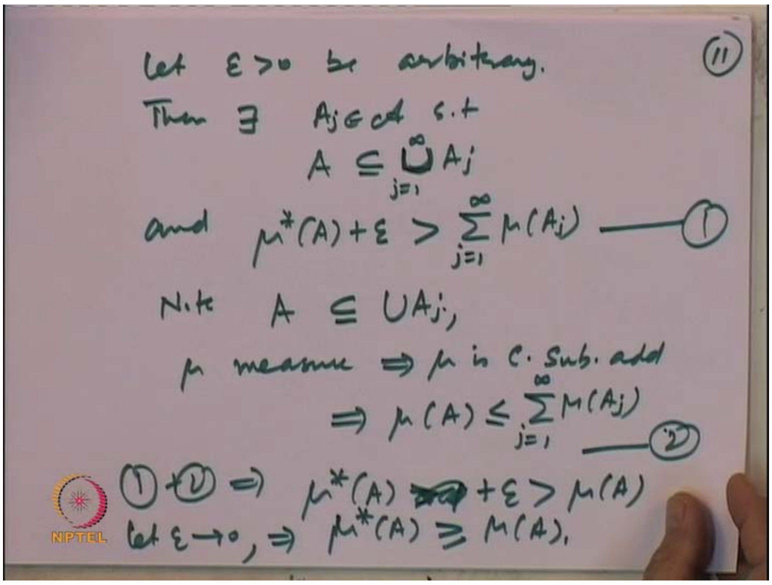

In that case, we are going to use the definition of infimum. So, mu star of A is the infimum of all possible approximate sizes, summation and so on. Let epsilon greater than 0 be arbitrary, then there exists a covering. So, there exist some sets A_j belonging to the algebra, such that A is contained in the disjoint union of A_j. The infimum says, the mu star of A plus epsilon cannot be the infimum and that has to be bigger than summation mu of A_j so that at least has one such covering possible. (Refer Slide Time: 45:39) This is infinity and it is not necessarily disjoint. We can make it and we will see it later on. So, this is finite. Now, note that A is contained in union of A_j and all of them are elements in the algebra. We assumed A is in the algebra and so everything is in the algebra. Mu is a measure and we showed every measure implies mu is countably sub additive. That implies mu of A is less than or equal to summation mu of A_j j equal to 1 to infinity.

So, look at this equation 1 (Refer Slide Time: 46:26) and look at this equation 2. What does 1 and 2 imply? Mu star of A plus epsilon is bigger than this sum and that sum is bigger than mu of A. So, 1 and 2 imply that mu star of A plus epsilon is bigger than mu of A and epsilon is arbitrary. So, let epsilon go to 0 and that implies that mu star of A is bigger than or equal to mu of A. So, that proves the other way round inequality in the case when mu star of…

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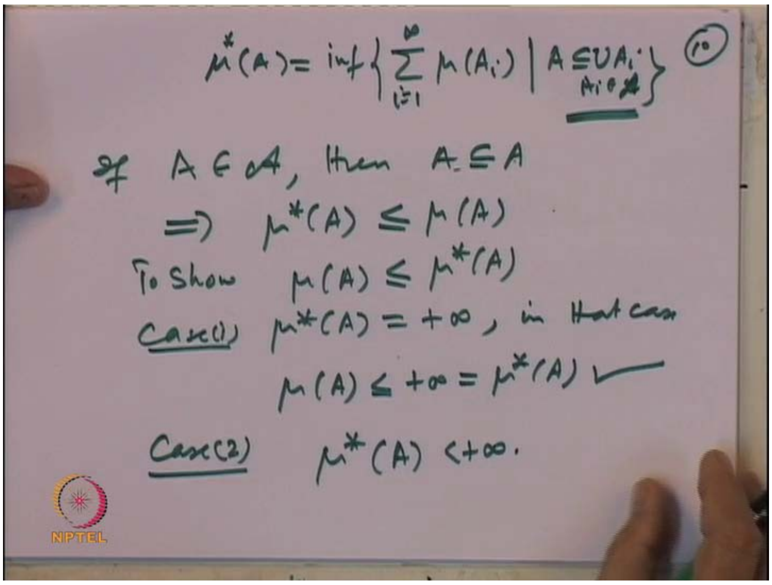

So, once again mu star of A is less than or equal to mu of A because A is one of the members which is covering it and mu of A is an element. So, mu star A is an infimum and that is less than or equal to… that is obvious property. To show that the case, when it is finite, we will look at the definition once again, (Refer Slide Time: 47:42) given epsilon is bigger than 0, there is a covering so that this holds infimum plus epsilon is bigger than one of the elements over which you are taking the covering. Now, using the fact that mu is countably sub additive, this is bigger than or equal to mu of A. Hence that proves the required property.

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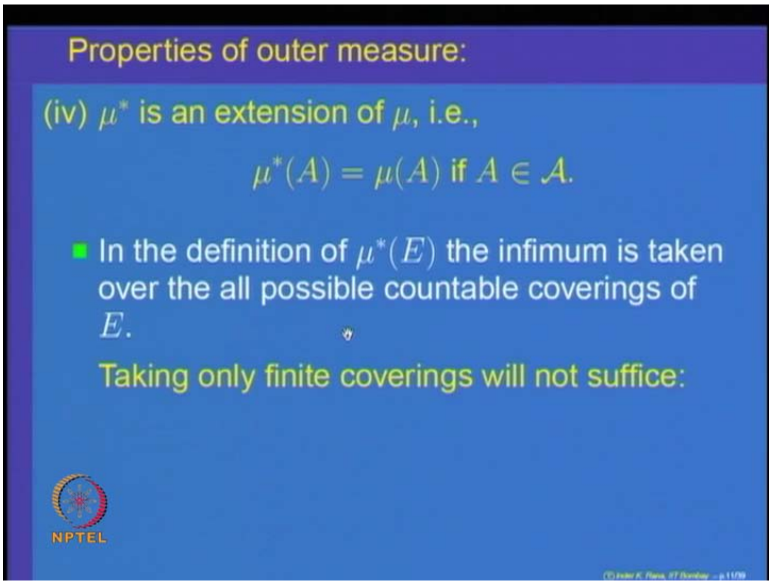

So, what we have shown is that mu star is indeed an extension of mu of A. So, let us go back and look at what we have done.

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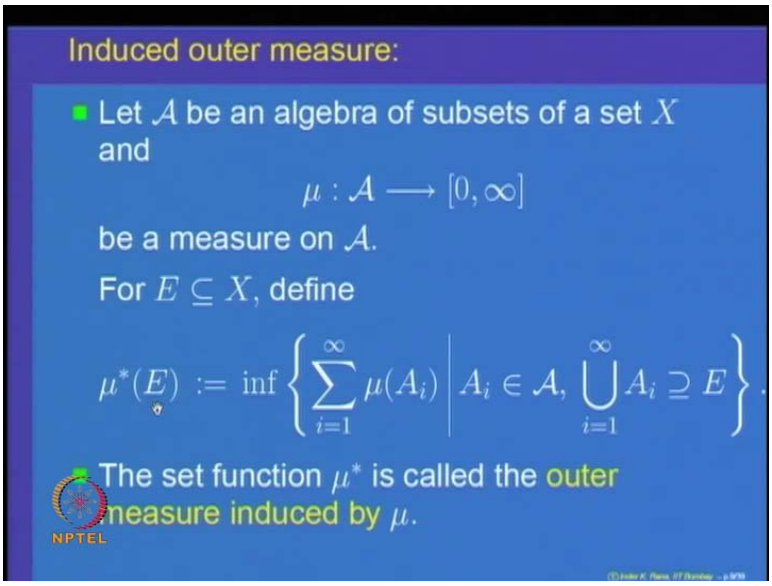

We started with a measure mu on the algebra A. Here, measure means, it is mu of empty set is equal to 0 and mu is countably additive. We are trying to extend it and so, we try to find out the size of any set by looking at sizes of sets in A. So, take any set E and cover it by elements in the algebra A. Look at the sizes of mu and they are called as mu of A_i so that the summation gives an approximate size of the set E. Look at the smallest possible of this numbers and it is called as the infimum. mu star of E, the induced outer measure is defined as the infimum over all this summations and this summations arise from coverings of E and so this is called an outer measure.

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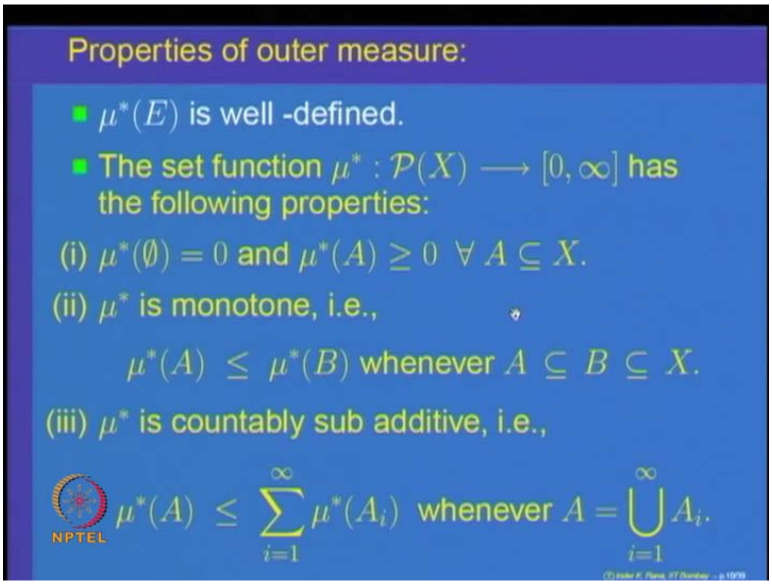

We showed it is well defined and we showed it is it has the obvious property, namely, mu of empty set is equal to 0 and mu star of A is bigger than or equal to 0. It is monotone and that means mu star of A is less than or equal to mu of B; Mu star is countably sub additive.

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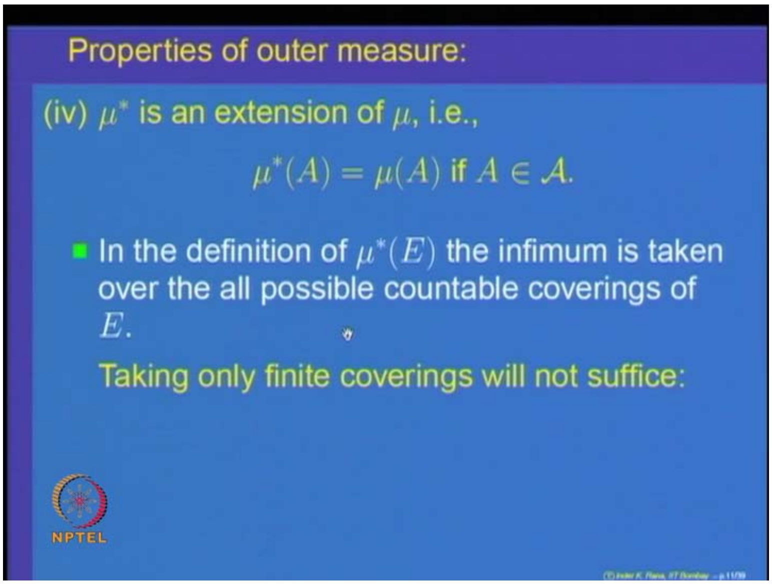

Finally, it is an extension. So, let me point it out that we have taken mu star of A as the infimum over those summations. We have taken the coverings which are countable in numbers. One can ask the question, Can't we take only finite coverings instead of countable coverings of it? So, let us give an example to show that that is not possible to do that; the finite covering will not suffice.

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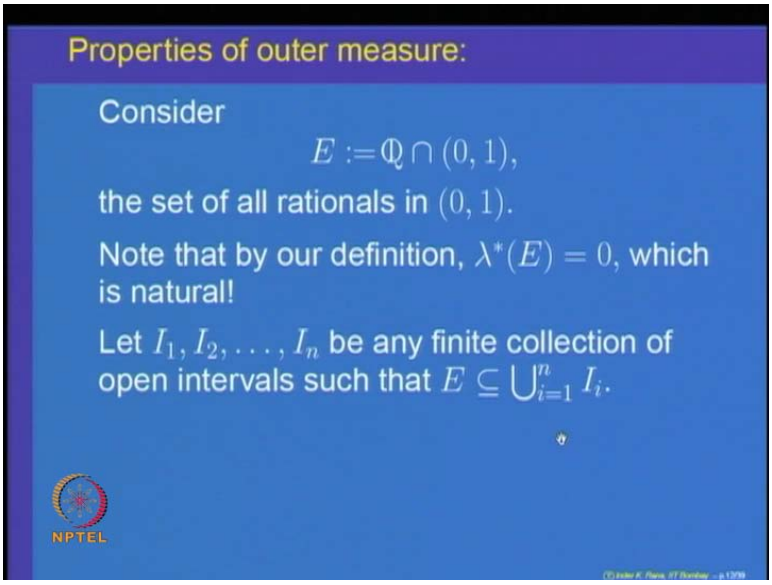

So, let us look at the set E - the case is the real line. We will look at the set E, which is rational and intersection with 0 to 1. So, we are looking at all the rationals in the set in the interval 0 to 1. Clearly, lambda star of E; we expect it to be equal to 0. Why we expect? Size of this… because it is a countable set and the length of each singleton is equal to 0. So, we expect the length of each, when added together also should remain small and lambda star of E is equal to 0; this is when lambda star is defined by taking countable coverings. Now, let us take a finite covering of E by intervals. So, E is covered by finite number of intervals union E_i.

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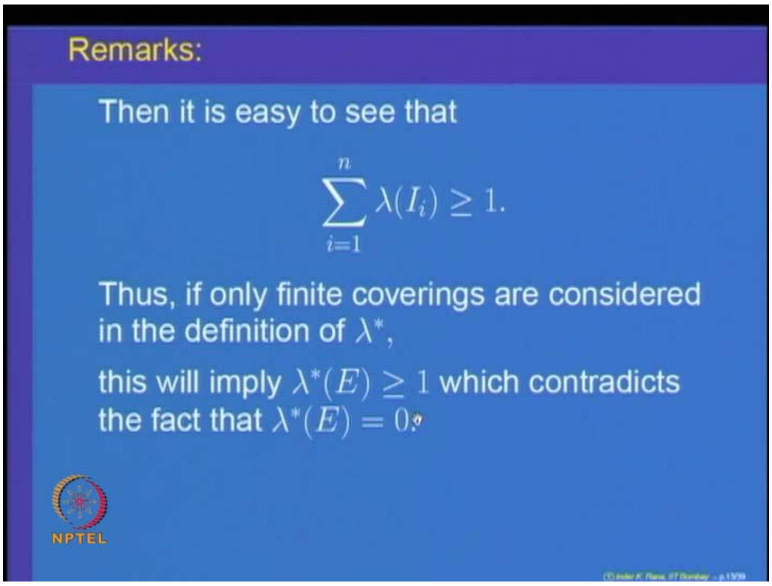

We claim that in that case, this number - the approximate size of E, will always will be bigger than or equal to 1 because of the following reason:

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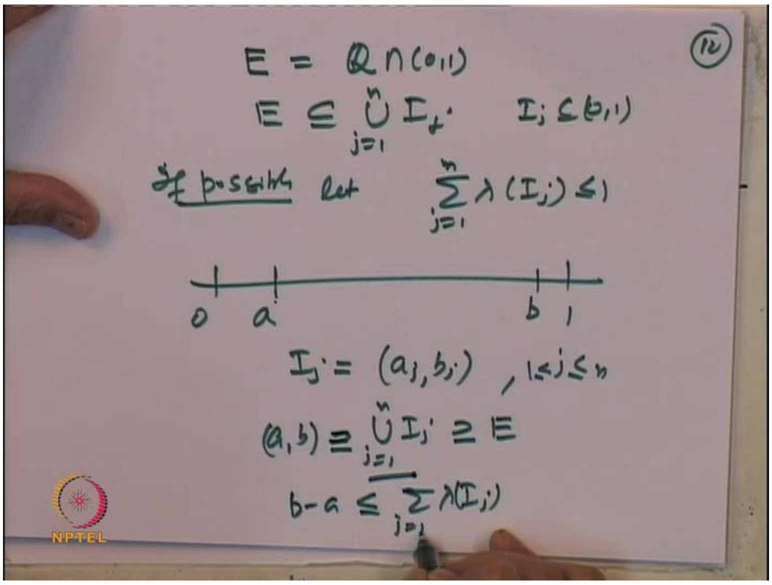

What is E? E is rationals inside 0, 1 and suppose E is covered by union I_j's j equal to 1 to n. If possible, let sigma lambda of I_j j equal to 1 to n be less than or equal to 1. so, these are finite collection. So, here is 0 and here is 1 and I_j's are intervals. Of course, intervals in 0, 1 which are I_j's of 0,1 covering set E and is rationals in 0, 1. So, now so let us say that I_j for the sake of just definition, it is a_j b_j and does not matter whether it is open or closed. You can just assume it to be open and it does not matter much, actually. Then we have got these numbers between a_j's and b_j's. so, look at all the left end points and look at the smallest of them and let us say the smallest is here is a.

So, what is a? a is the smallest of the number a_1 a_2 a_n. Look at the largest of b_j's and call that as b. Then, this a, b - may be closed, does not matter, is equal to union of I_j's or at least it will cover the union of I_j's j equal to 1 to n and they cover E. Now, that covers E and it this is less than or equal to… This is the smallest and that is the largest one, which is covering(Refer Slide Time: 53:06).

Now, b minus a is less than or equal to summation length of I_j's j equal to 1 to n. If that is less than or equal to 1, that means b minus a is strictly less than 1; that means, it has to be like this (Refer Slide Time: 53:31) but there is a rational here between 0 and 1, which belongs to E and E is inside a b. So, that is not possible and that will be a contradiction. So, this situation is not possible. (Refer Slide Time: 53:47)

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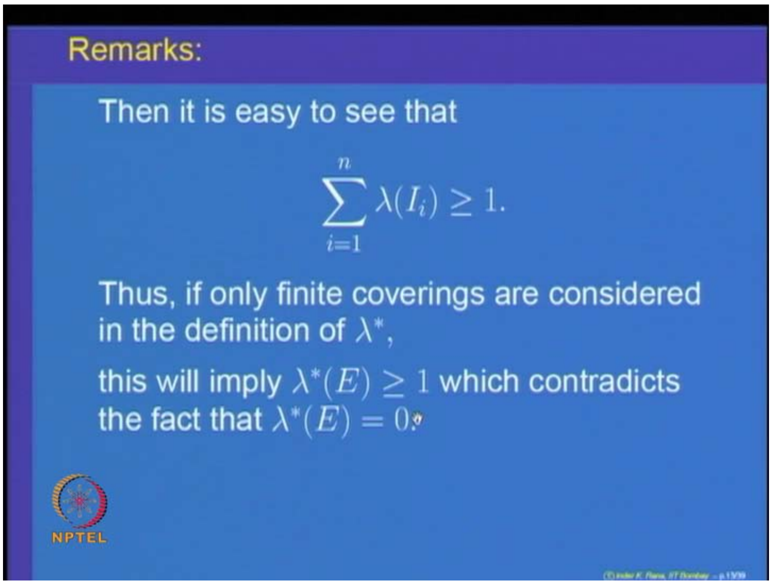

That means, whenever these are covering, we have to have lambda I_i is bigger than or equal to 1, but that means all approximate sizes is bigger than 1. That will imply lambda star of E is bigger than or equal to 1. It is not possible because we just now said lambda star of E should be equal to 0. So, in the definition of the outer measure, we cannot limit ourselves to only finite coverings. We have to allow all countable coverings also.

Today, we have tried to go beyond algebra. So, we started with a semi-algebra and a measure on it. We extended it to a measure on the algebra generated by it as a first step. As a next step, we started with a measure on an algebra. We showed that by an example, on the real line by Ulam's theorem that you cannot extend it to all subsets of real line.

Let us try to go as far as possible. So, we defined - given a measure mu on an algebra, we defined the notion of an outer measure for any subset a of that set X. We showed this outer measure has some nice properties. It extends the given measure and it is monotone, which is countably sub additive. So, in the next lecture, we will see how to get from it, an actual extension, which is a measure.

Thank you.