Measure and Integration

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Module No. # 02

Lecture No. # 08

Uniqueness Problem for Measure

Welcome to today's lecture on measure and integration. This is the eighth lecture on measure and integration. Today, we will be looking at a problem called the uniqueness of measures on algebras and sigma algebras. For this, we will need to define some terminology. Let us look at the uniqueness problem for the topic for today's lecture.

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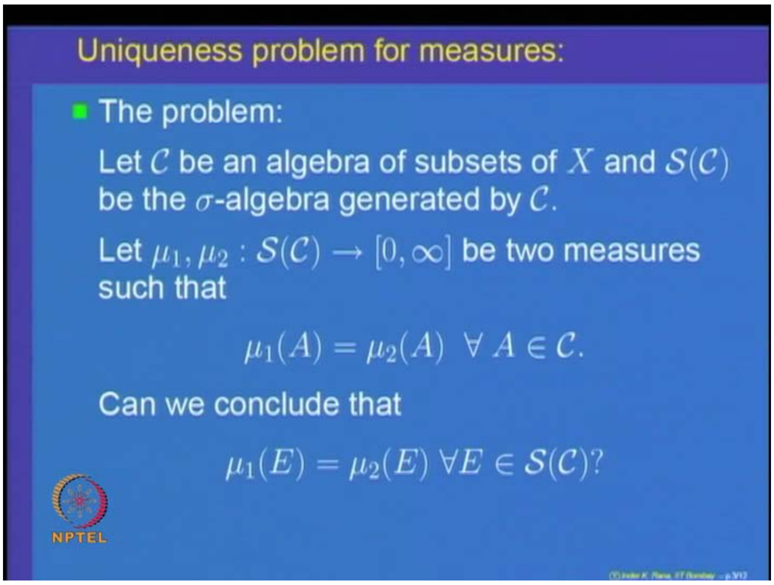

The problem is as follows. We are given C – an algebra of subsets of a set X; S of C is the sigma algebra generated by C. So, C is an algebra and S of C is the sigma algebra generated by C. We have got two measures mu_1 and mu_2 defined on the sigma algebra generated by C such that mu_1 of A is equal to mu_2 of A for every A belonging to C. For all elements in C, mu_1 and mu_2 agree.

The question is can we conclude that mu_1 of E is equal to mu_2 of E for every element in the sigma algebra generated by C? This is a general uniqueness problem which plays a role later on when we extend measures to general settings. To answer this question, let us make some definitions. First of all, this is not true in general for all measures; we have to put some conditions on the measures.

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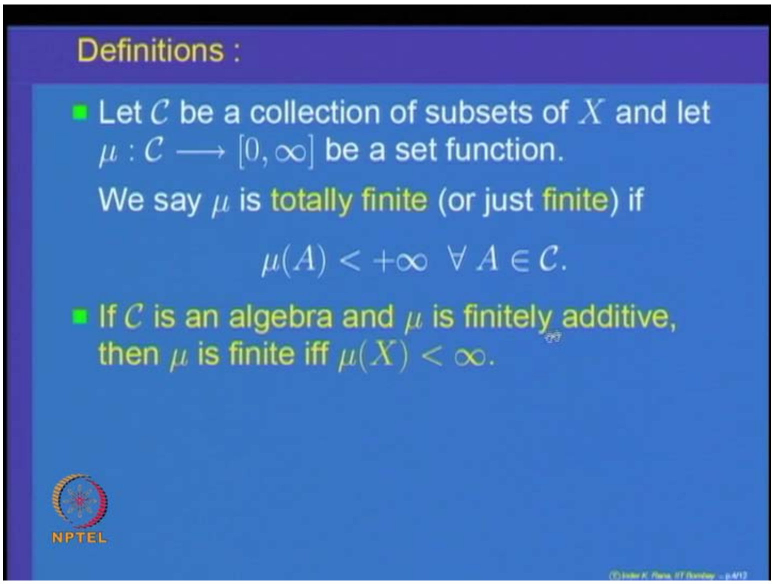

Let us look at what is called a totally finite measure. A measure C is called totally finite if mu of every subset A is finite in that domain of mu. C is a collection of subsets and mu is a set function. We say mu is totally finite or sometimes we also say it is finite if mu of A is less than plus infinity for all A belonging to C. Note that in case C is an algebra and mu is finitely additive, then mu is finite if and only if mu of X is finite.

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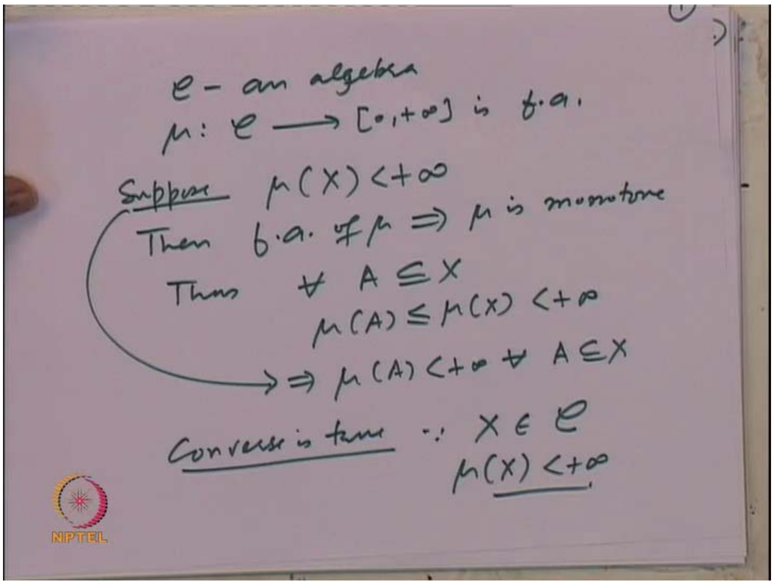

Let us assume C is an algebra and mu from C to 0 to infinity is finitely additive. Suppose mu of the whole space is finite. Note that the whole space X belongs to C because C is an algebra. We had seen earlier that finite additivity of mu implies mu is monotone. We had seen this property earlier; so, we will not go into the details of this again. Since mu is monotone, for every A contained in X, mu of A will be less than or equal to mu of X which is finite; this implies mu of A is finite for every A subset of X.

The converse is obviously true; converse is true because X belongs to C and so mu of X is finite. Whenever we are dealing with finitely additive set functions saying mu is totally finite, it is enough to say that mu of X is finite and as a consequence mu of every subset will be finite. This is an easy consequence of saying for a finitely additive set function on an algebra, mu of the whole space finite is the same as saying mu of every subset A of X, A in the algebra of course, is finite.

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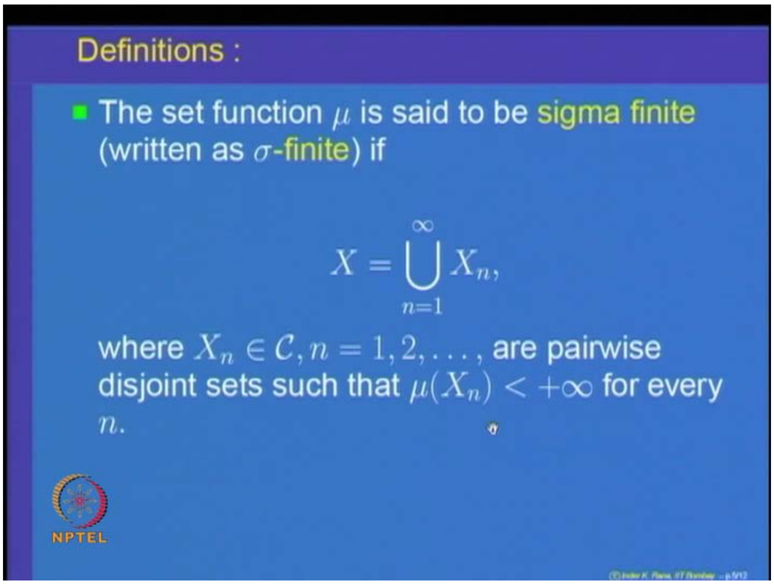

Let us look at the property. Let us take a set function mu and we say that it is sigma finite. We defined what is totally finite and now we define what is called sigma finite. mu is said to be sigma finite if we can write the whole space as a union of sets X_n n equal to 1 to infinity such that these sets are pairwise disjoint (we want these sets to be pairwise disjoint) and mu of each X_n to be finite.

Each X_n should be an element in the domain of mu in the class C and mu of X_n should be finite. We are essentially saying that for totally finite, mu of the whole space is finite.

sigma finite means what? X can be cut up into pieces X_1, X_2, X_n and so on and mu of each X_n is finite; this is what is called sigma finite set functions.

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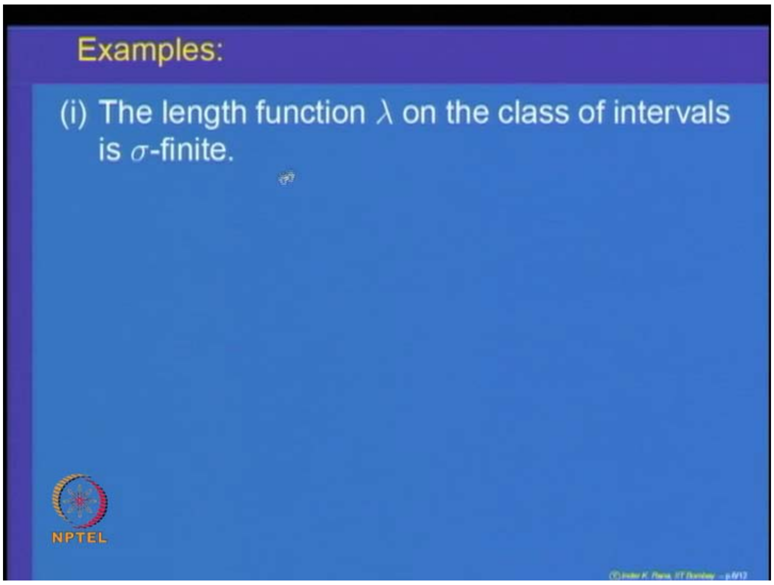

Let us look at some examples of set functions. The length function lambda on the class of all intervals is sigma finite. It is easy to see why a length function is sigma finite.

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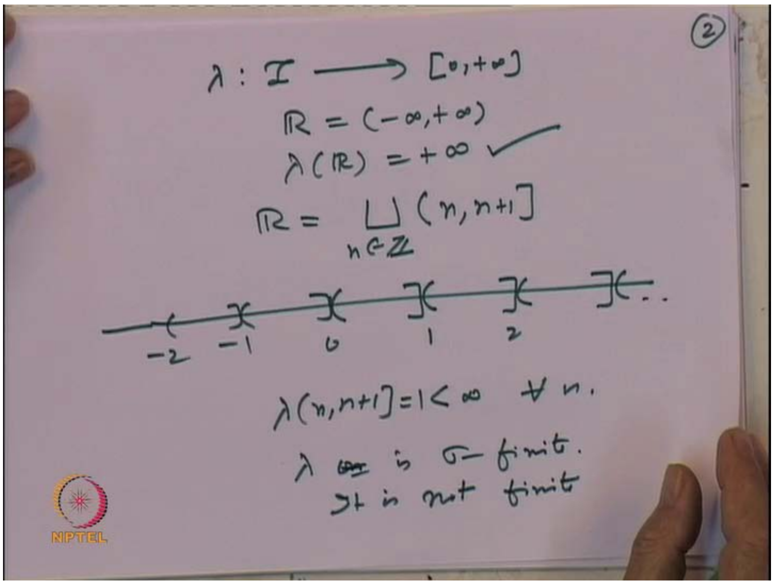

Here is the length function lambda on the class of all intervals taking values in 0 to infinity. The whole space, the real line, is of course the interval minus infinity to plus infinity. The length of R we know is not finite; it is equal to plus infinity but we can write R as a disjoint union of the intervals n to n plus 1, for example, n belonging to integers. The real line is cut up. Here is the real line; here is 0, 1; open 1, closed 2, open 2, closed 3 and so on; on this side, minus 1 and this is minus 2 and so on.

We have cut up the real line; we have divided the real line into many countable disjoint pieces; each one is an interval and length of n to n plus 1 for every n is equal to 1 which is less than, of course, infinity for every n. The whole real line is written as a countable disjoint union of intervals, each one having finite length; so, lambda on the class of intervals is sigma finite; of course, it is not finite because the length of the whole space is equal to plus infinity. This is an example of a set function which is sigma finite (Refer Slide Time: 07:35). We will give another example of set function which is totally finite.

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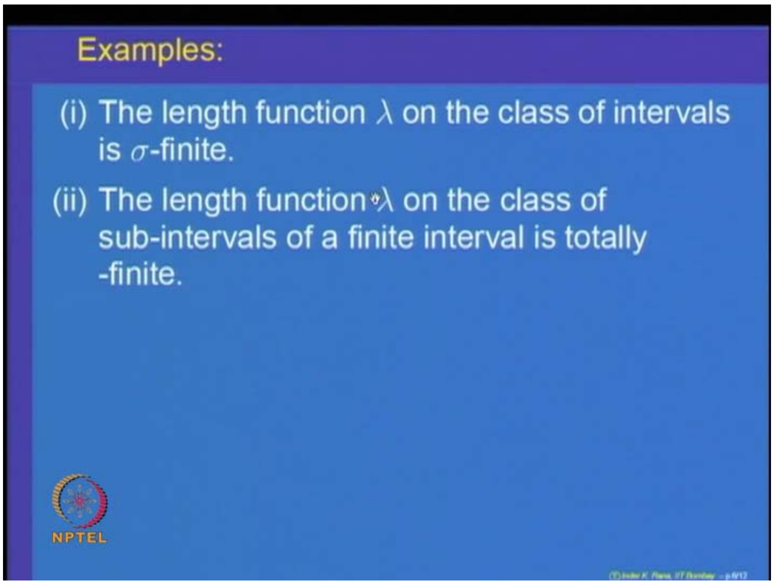

For that, let us look at the length function restricted to any finite interval.

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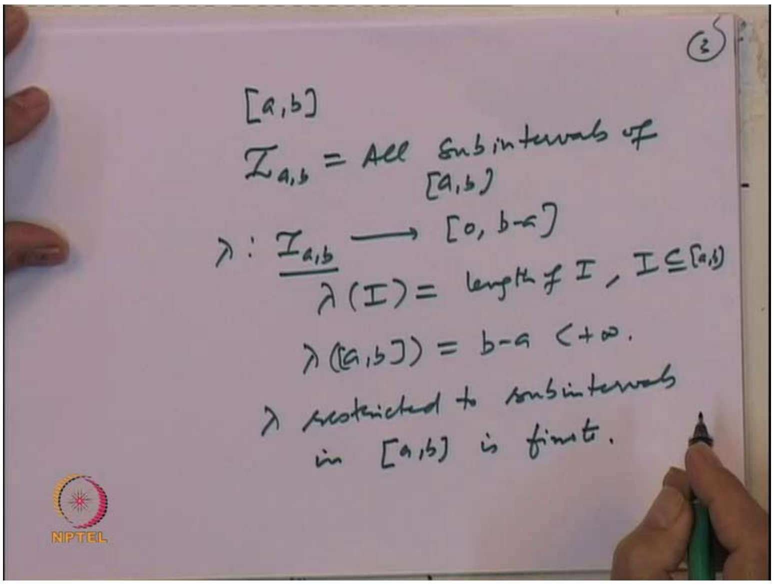

Let us look at, say for example, the interval a to b and let us look at all subintervals. Let us look at I tilde a, b to be all subintervals of a, b and define, of course, the length function as before. This will be a function from 0 to b minus a; length of I is equal to the usual definition of length of I for I contained in a, b. We know that length function is finitely additive, it is countably additive and so on; on these intervals also, it is countably additive. We know lambda of the interval a to b is equal to b minus a which is finite. One says lambda restricted to subintervals in a, b is finite or totally finite for every a and b; this is an example of a measure which is totally finite. Let us look at another example of a set function which is not…

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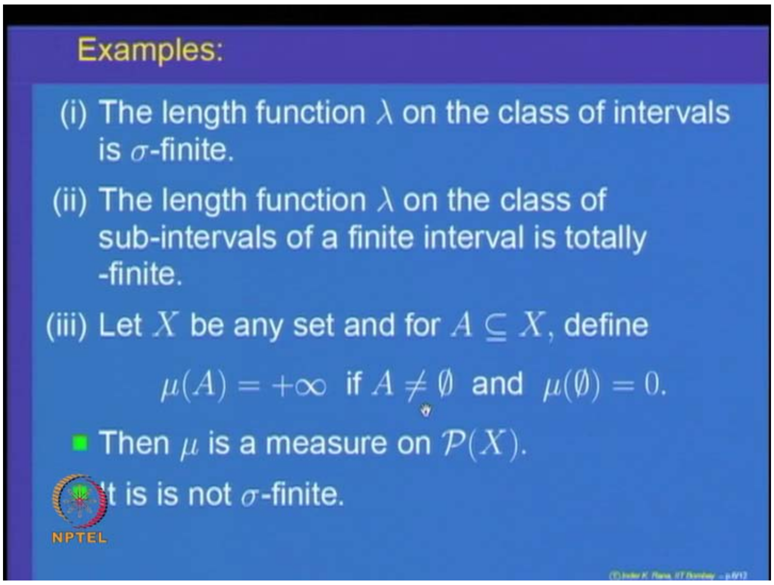

Let X be any set and for any subset A of X, let us define mu of A to be equal to plus infinity if A is nonempty and mu of 0 to be equal to 0. Then, obviously, this is a measure on P of X. This is a simple consequential property that one can easily check because mu of empty set is 0 is given and if A is any set which is a countable disjoint union or not, mu of the union will be equal to again plus infinity which is equal to sigma mu of A_is; at least one of them has to be nonempty. I hope it is clear that this mu is countably additive; this is a measure on P of X.

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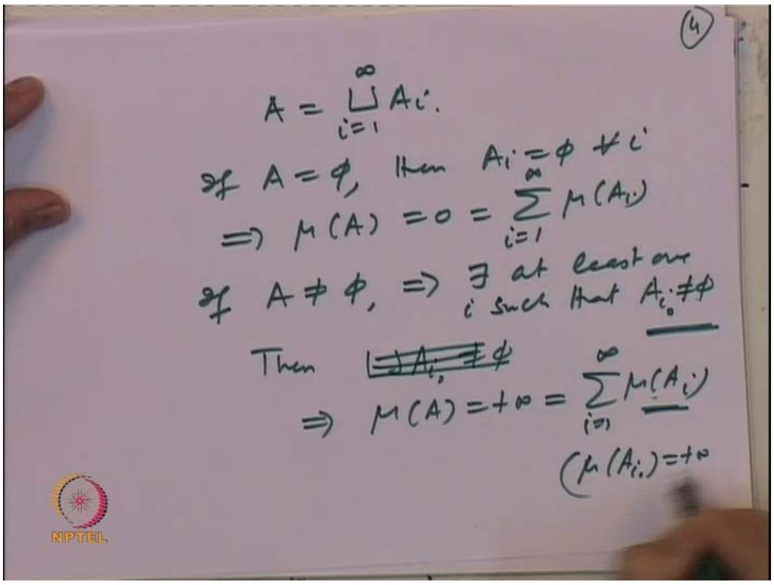

Let us suppose that A is equal to union of A_is, i equal to 1 to infinity. If A is equal to empty set, then A_i is equal to empty set for every i, implying mu of a which is 0 is the same as sigma mu of A_is, i equal to 1 to infinity. The second possibility is if A is not empty and A is equal to union. That implies there exists at least one i such that A_i is not empty. Then, let us say that such that i says that so let us say that is A_{i0}; there is at least one i; let us say that is i_0 (Refer Slide Time: 11:22).

Then, union A_{i0} is not empty; A is not empty anyway; this is not required (Refer Slide Time: 11:35). Then, mu of A is equal to plus infinity is equal to summation mu of A_is because at least one term here is not empty (Refer Slide Time: 11:47). So, that is equal to plus infinity; mu of A_{i0} is equal to plus infinity; this is also plus infinity; they are the same.

This is a measure on the class of all subsets (Refer Slide Time: 12:04). So, mu of A is plus infinity if A is not empty and mu of empty set is equal to 0 is a measure and this obviously is not sigma finite because there are no subsets anyway whose mu is finite. This is an example of a non-sigma finite measure.

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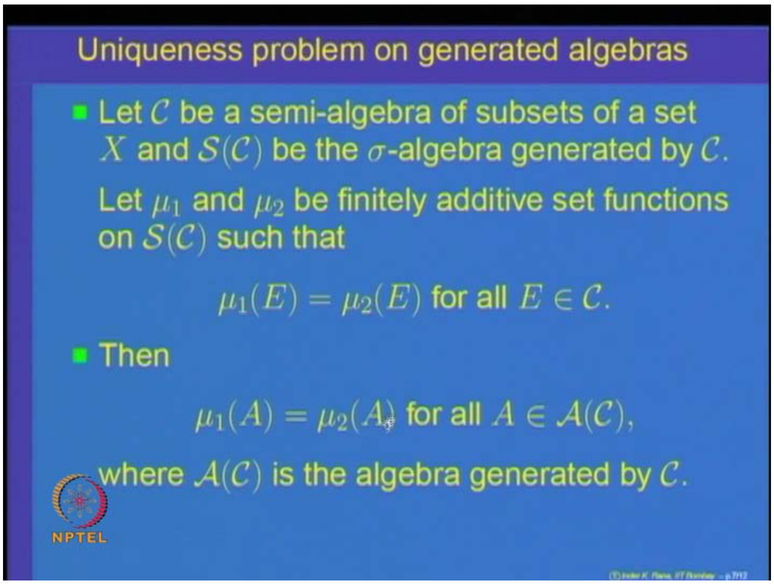

The theorem we want to prove today is the following. Let us take C, a semi-algebra of subsets of a set X and S of C to be the sigma algebra generated by C. Let mu_1 and mu_2 be two finitely additive set functions on S of C such that mu_1 of E is equal to mu_2 of E for all E belonging to C. Then, we want to show that mu_1 of A is equal to mu_2 of A for all A belonging to first A of C, where A of C is the sigma algebra generated by C. We are saying as a first step we are going to prove that if two measures mu_1 and mu_2 defined on a semi-algebra agree, then they also agree on the algebra generated by that semi-algebra. This is what we want to prove. Let us see the proof of that.

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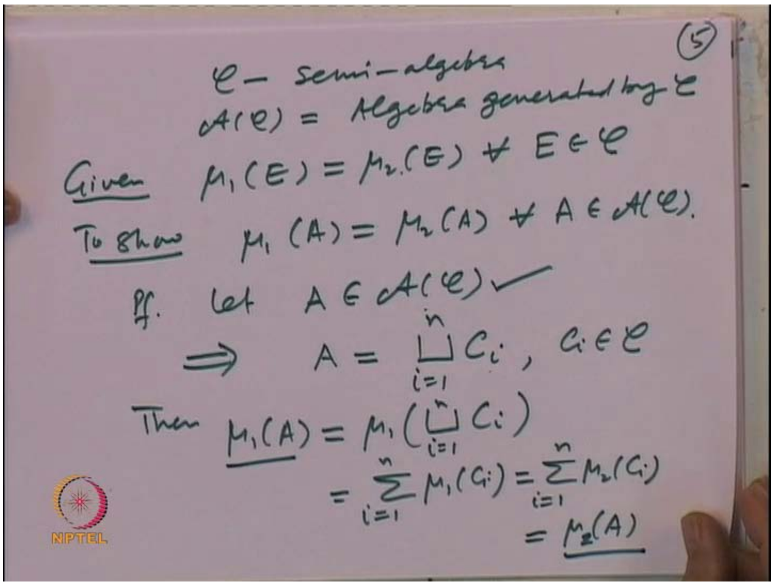

C is a semi-algebra; A of C is the algebra generated by C; we are given mu_1 of E is equal to mu_2 of E for every E belonging to C. We have to show mu_1 of A is equal to mu_2 of A for every A belonging to algebra generated by C. How do you prove it? Let us start. Let us take a set A which belongs to A of C. Recall we had shown characterizations of elements of the algebra generated by a semi-algebra.

We showed that if A is an element of the algebra generated by a semi-algebra, then this A must look like a finite disjoint union of elements C_i, i belonging to n where C_is belong to the semi-algebra C. Every element A in the algebra generated by a semi-algebra, we had shown, must have a representation like this, but then mu_1 of A is equal to mu_1 of this finite union. We know mu_1 is finitely additive. That implies this must be equal to sigma i equal to 1 to n mu_1 of C_i, but each mu_1 is equal to mu_2 on each element of C and C_is are elements of C; that implies that this must be equal to 1 to n mu_2 of C_is. Again by using mu_2 is finitely additive, I can write this as mu of A because A is a finite disjoint union of elements of this; so mu_1 of A is equal to mu_2 of A whenever A belongs to A of C (Refer Slide Time: 16:01).

This proves the theorem that whenever two finitely additive set functions mu$_1$ and mu$_2$ agree on a semi-algebra, then they also agree on the algebra generated by it(Refer Slide Time: 16:18). Let us go to the next step of this uniqueness problem.

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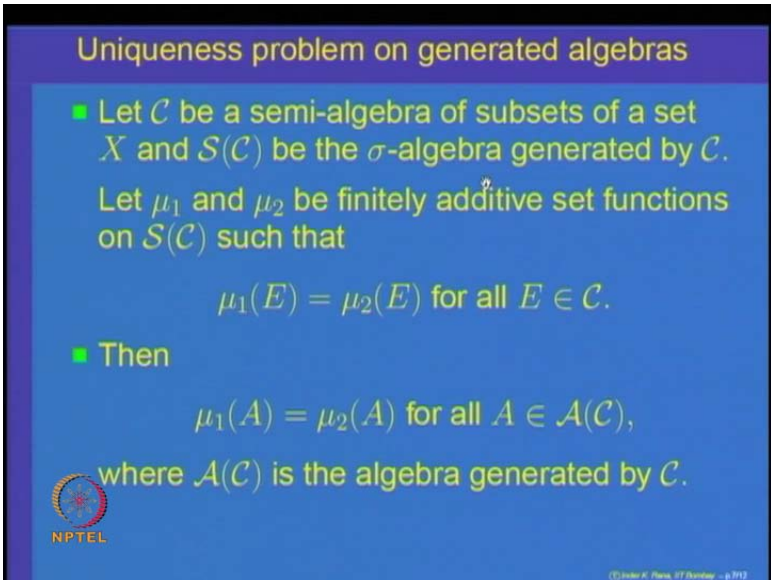

That is saying that let C be a semi-algebra of subsets of a set X once again and S of C be the sigma algebra generated by C. This we have already proved (Refer Slide Time: 16:39).

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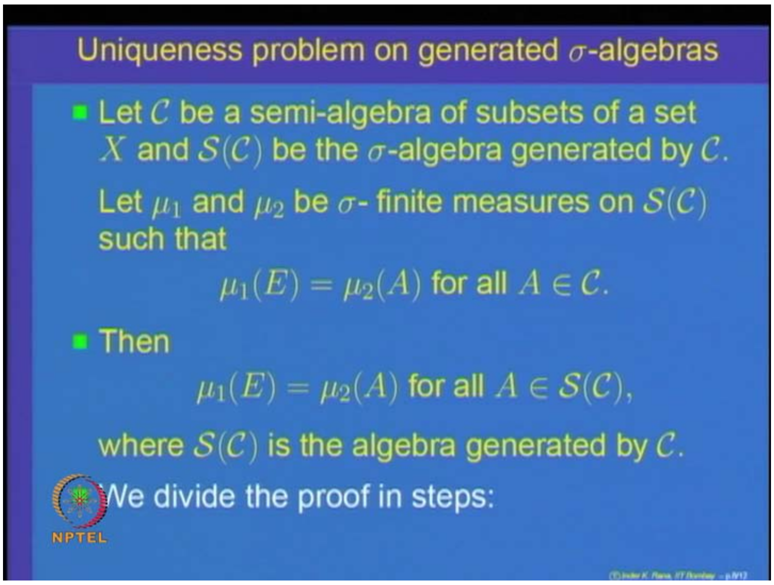

Let mu_1 and mu_2 be sigma finite measures on S of C such that mu_1 of E is equal to mu_2 of… this is a misprint mu_1 of E should be equal to mu_2 of E for all E in C. Then, mu_1 of E is equal to mu_2 of A for all A belonging to S of C where S of C is the sigma algebra generated by it. Let me state it; we will divide the proof into steps of course.

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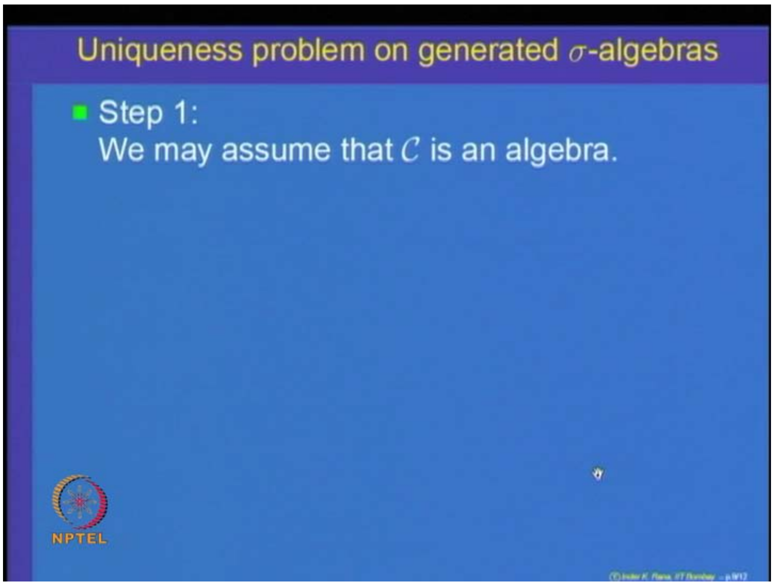

Let us look at the statement of the theorem once again.

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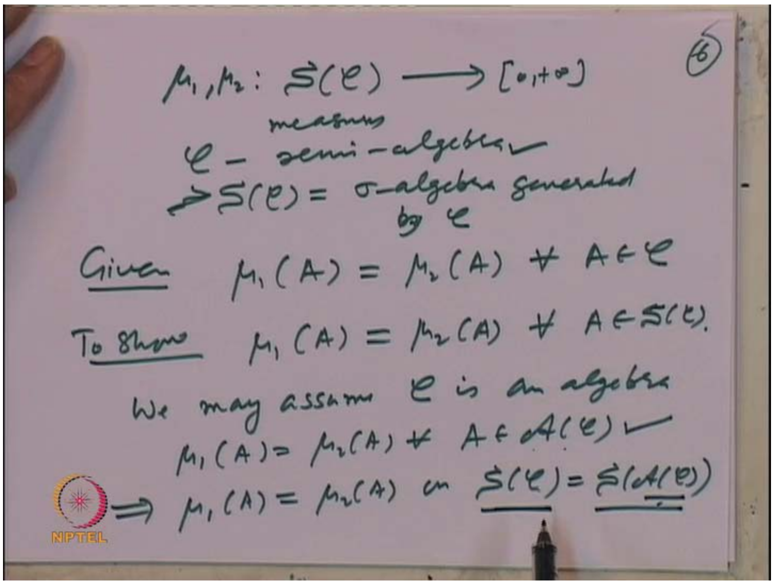

We are saying that let mu_1 and mu_2 be two measures which are sigma finite defined on the sigma algebra generated by a semi-algebra C. It measures C, the semi-algebra S of C, and this S of C is equal to the sigma algebra generated by C. Given mu_1 of A is equal to mu_2 of A for every A belonging to the semi-algebra. We have to show mu_1 of A is equal to mu_2 of A for every A in the sigma algebra generated by C; this is what we want to show.

Let us look at the first step (Refer Slide Time: 18:35). We may assume that C is an algebra. Here, we are given that C is a semi-algebra (Refer Slide Time: 18:38). Step 1 says we may assume that C is an algebra. That is because of the fact that we have just now shown that if mu_1 and mu_2 agree on the on a semi-algebra, then they also agree on the algebra generated by it. By the given hypothesis, mu_1 of A is equal to mu_2 of A for every A belonging to the algebra generated by C.

We already have mu_1 and mu_2 agreeing on the algebra generated by A of C. This implies mu_1 of A is equal to mu_2 of A on S of C, the sigma algebra generated by C, but note this is the same as the sigma algebra generated by A of C; that also we have shown. Given a semi-algebra, you can directly generate the sigma algebra or you can generate the algebra first and then generate the sigma algebra; both are same. Just now we showed that whenever two measures agree on a semi-algebra, they agree on the algebra generated by it.

mu_1 and mu_2 agree on the semi-algebra; therefore, they agree on the algebra generated by it. We want to show that they agree on the sigma algebra generated by it which is nothing but S of C. That proves the first step. As a first step in our proof, we are saying that the given class C on which mu_1 and mu_2 are defined is actually an algebra (Refer Slide Time: 20:27). That is the first simple equation in the proof – without loss of generality, we may assume that C is an algebra.

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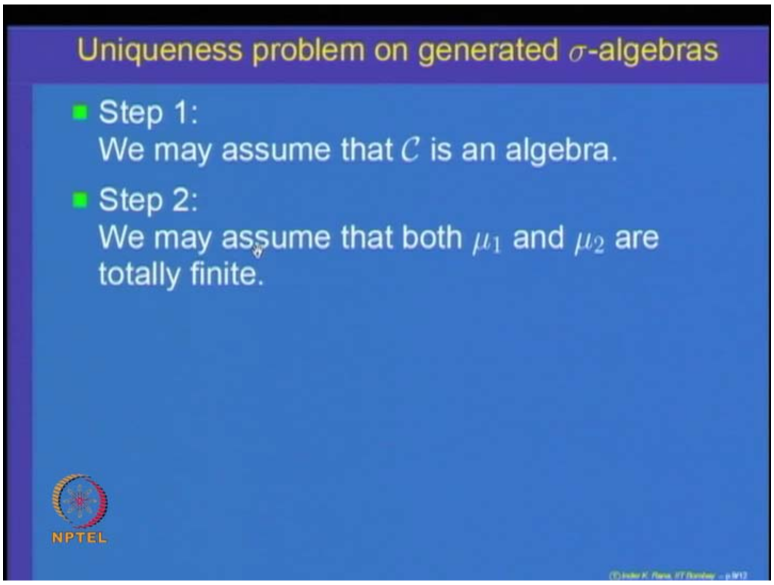

The next step says that we may assume that both mu$_1$ and mu$_2$ are totally finite. We are given mu$_1$ and mu$_2$ are sigma finite.

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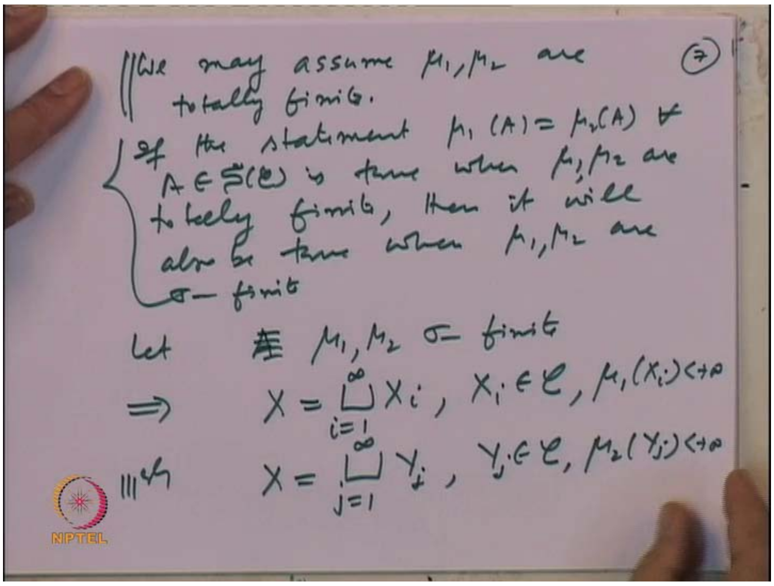

The next step is that we may assume that mu$_1$ and mu$_2$ are totally finite. What is the meaning of saying we may assume? This is the same as saying if the statement mu$_1$ of A equal to mu$_2$ of A for every A belonging to S of C is true when mu$_1$ and mu$_2$ are totally finite, then it will also be true when mu$_1$ and mu$_2$ are sigma finite. That is the meaning of saying that we may assume that mu_1 and mu_2 are totally finite. Let us see why that is the case.

Let us take a set A contained in…. What we are given is mu_1 and mu_2 are sigma finite; mu_1 and mu_2 are sigma finite; mu_1 is sigma finite and so I can write X as union of X_is, i equal to 1 to infinity where X_is belong to C and mu_1 of each X_i is finite. Similarly, mu_2 is sigma finite; so, you can write X as union some j equal to 1 to infinity Y_j, where Y_js are subsets in C and mu_2 of each Y_j is finite.

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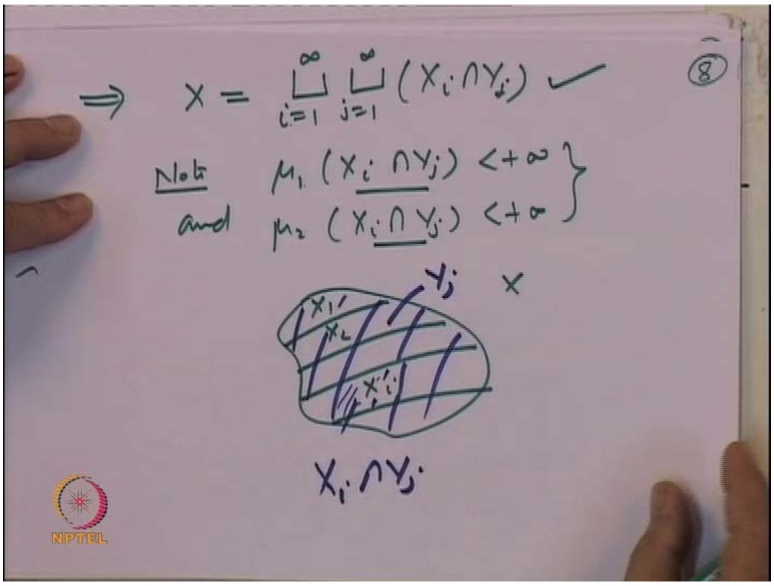

From both of these statements, I can write X as so this implies we can write X as union over i, 1 to infinity of X_is but that I can decompose into union of Y_js X_i intersection Y_j; I can write that. I can write this as a decomposition of X into subsets X_i intersection Y_j. Now, what we have achieved is the following: mu_1 of each X_i was finite and mu_2 of each Y_j have finite. This implies that mu_1 of X_i intersection Y_j is finite and mu_2 of X_i intersection Y_j is also finite; both mu_1 and mu_2 are finite on this ((.)).

In the picture, you can think of this as X; you divide; these are sets X_1, X_2, X_i and so on. You also have sets Y_js; they are decompositions like this (Refer Slide Time: 24:27). This piece is Y_j and this piece here is X_i intersection Y_j. The whole space is cut up into pieces; this is what this statement means where each one of them is finite (Refer Slide Time: 24:51).

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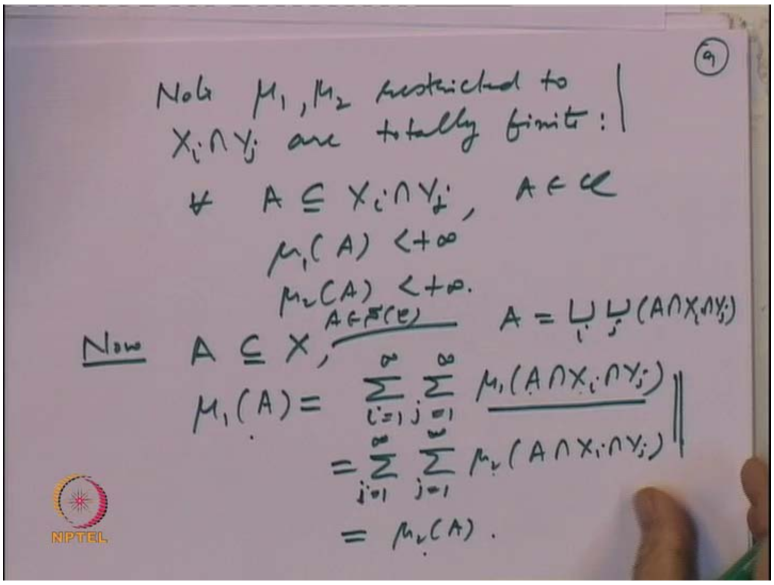

Here is an observation; note that mu_1 and mu_2 restricted to X_i intersection Y_j are totally finite. What is the meaning of this statement that they are restricted? That means if you look at the subsets, for every A contained in X_i intersection Y_j, A belonging to C, mu of A is finite and mu_2 of A is finite. For totally finite measures, we have already assumed that this statement is true and we are trying to show it for our sigma finite.

For any set A contained in X, mu_1 of A can be written as summation over i summation over j mu_1 of A intersection X_i intersection Y_j. That is because A is equal to union over i union over j A intersection X_i intersection Y_j. This is a countable disjoint union (Refer Slide Time: 26:26). mu_1 is a measure and so this must be true. Note that this A intersection is a set in the set X_i intersection Y_j where mu_1 is finite and then we know that there the statement is true. Here, A is contained in X, of course A belonging to S of C; so, the statement is true.

That means what? By the assumption that statement is true for finite measures we will conclude that this is same as mu_2 of A intersection X_i intersection Y_j and once again that is equal to mu_2 of A. The basic idea is for any set, we can bring it to the finite pieces (there, we know it is true) and go back to the original piece. This is the proof of the second step that we may assume without loss of generality – our measures mu_1 and mu_2 are both finite (Refer Slide Time: 27:32).

We have made two simplifications in our proof: the first one being we may assume that C is an algebra and the second one that mu_1 and mu_2 are totally finite. What do we want to prove now? We are only left with the case to prove that if C is an algebra, mu_1 and mu_2 are totally finite defined on the algebra C and if they agree on C, then they will agree on the sigma algebra generated by C. That is the next step we want to show.

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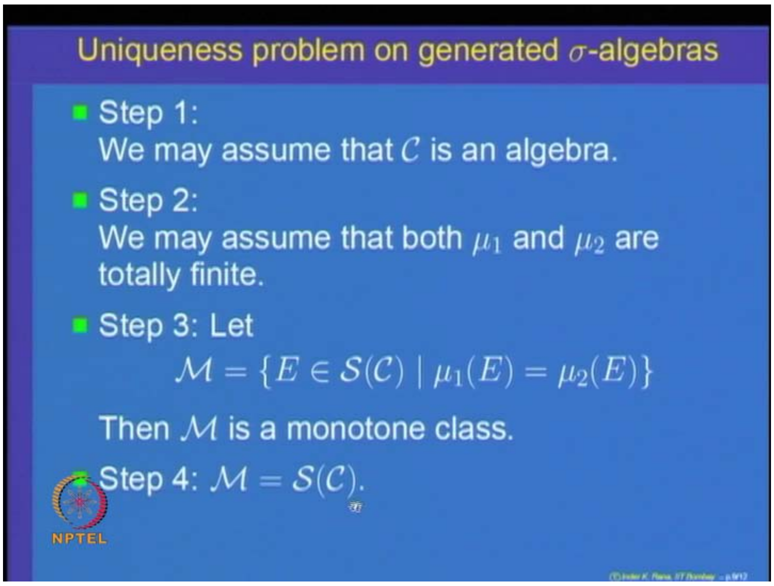

To prove the final step, let us write M to be the class of all those elements of S of C where mu_1 and mu_2 agree. What is the aim to prove? Our aim is to prove that this collection M is nothing but S of C. We are picking up subsets of S of C; M is a subclass of S of C; we want to prove that this is equal to S of C and that is proved as follows. First, we will observe then M is a monotone class; we will prove that. Once we have proved that M is a monotone class, we will also observe that we are given that mu_1 and mu_2 are equal on C; so, C is a subclass of M.

C is an algebra and C is contained in M. M is a monotone class; that will mean what? The monotone class generated by C must be inside M, but C is an algebra and the monotone class generated by an algebra is the sigma algebra generated by it. That also we have proved; that we will prove as step 4 – M is equal to S of C.

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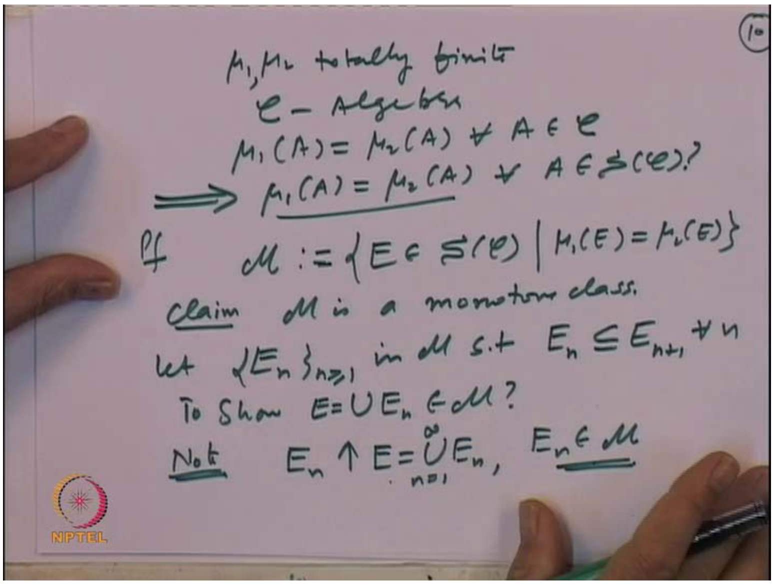

Let us prove step 3 and then conclude from it step 4. Step 3 we want to prove. We are given that mu$_1$ and mu$_2$ are totally finite; we are in this case; C is an algebra; mu$_1$ of A is equal to mu$_2$ of A for every A belonging to the algebra C. We have to show that mu$_1$ of A is equal to mu$_2$ of A for every A belonging to the sigma algebra generated by C; that is the question.

The proof: define M to be the class of all subsets belonging to S of C for which this property is true (Refer Slide Time: 30:21). That means mu$_1$ of E is equal to mu$_2$ of E. The claim is that this M is a monotone class. What is a monotone class? Recall that a monotone class is a collection of subsets of a set X which is closed under increasing unions and decreasing intersections. These two properties have to be checked; let us check them.

Let E_n be a sequence in M such that E_n is increasing; E_n is inside $E_{n\ plus\ 1}$ for every n. we have to show that union of E_ns belongs to M. Let us note that E_n is an increasing sequence; E_n increases to E which is union of E_ns. E_ns belong to M – keep that in mind. That is the set E (Refer Slide Time: 32:13). We want to show that E belongs to M; that means mu$_1$of E is equal to mu$_2$ of E.

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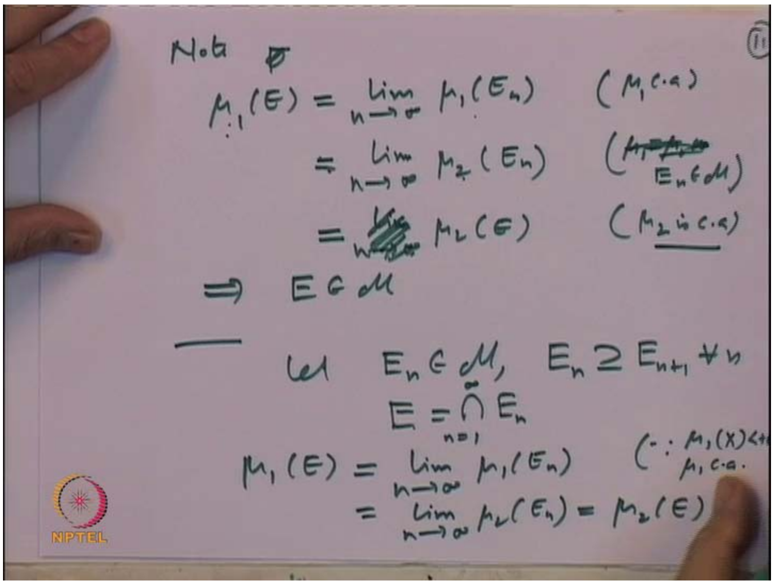

What is mu of E? How do we compute it? Let us observe that mu$_1$ of E is nothing but limit of mu$_1$ of Ens. Why is that? That is because mu$_1$ is a measure; it is countable additive; we had proved that countable additivity of the set function implies that whenever a sequence E_n increases to a set E, then mu of E must be limit of mu$_1$ of E_ns; that was the characterization property for countable additivity.

Go back and refer that was because of mu being countably additive (Refer Slide Time: 33:10); that is the property being used here – equivalent form of it. Now, each E_n belongs to M; that implies that even mu$_1$ of ((.)) is equal to mu$_2$ of E_n. This is equal to limit n going to infinity mu$_2$ of E_n; that is, we are using mu$_1$ equal to mu$_2$ on mu$_1$ mu$_2$ equal to mu$_1$ equal to mu$_2$ because sorry; this is because E_n belongs to M (Refer Slide Time: 33:46).

Once again, mu$_2$ is countably additive; there, mu$_1$ ((.)). That implies that this is mu$_2$ of E using the fact that mu$_2$ is countably additive. We have used a lot of things which we have proved earlier: mu$_1$ is a measure, E_n is increasing to E; so by countable additivity, mu$_1$ of E must be equal to limit n going to infinity mu$_1$ of E_n (Refer Slide Time: 34:19).

Now, each E_n belongs to M; E_n is a sequence in M (Refer Slide Time: 34:27). That means mu$_1$ of E_n is equal to mu$_2$ of E_n; this is equal to this (Refer Slide Time: 34:34). This is the second (( )). Once again, mu$_2$ is countably additive and E_n increases to E; so by countable additivity, this limit must be equal to mu$_2$ of E (Refer Slide Time: 34:47). It says mu_1 of E is equal to mu_2 of E. That implies that E belongs to M whenever E_n is a sequence which is increasing to M; this is for increasing.

The corresponding thing we have to prove when it is decreasing and that is where we are going to use the fact that mu_1 and mu_2 are totally finite. For the second case, let E_ns belong to M, E_n includes $E_{n \text{ plus } 1}$ for every n decreasing, and let E be equal to intersections of E_ns, n equal to 1 to infinity. We want to show that E also belongs to M. For that, once again, mu_1 of E is equal to limit n going to infinity mu_1 of E_n because mu_1 is totally finite, mu_1 of X is finite and mu_1 is countably additive. That, as observed earlier, is same as mu_2 of E_n because each E_n belongs to M and that is equal to mu_2 of E; once again, mu_2 is finite and E_n is decreasing to E; that proves this also; this proves that M is a monotone class (Refer Slide Time: 36:28).

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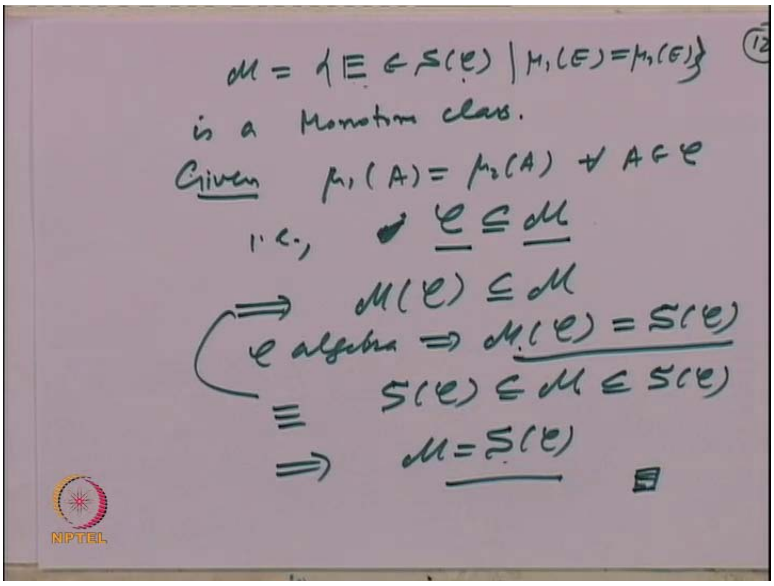

The class M which was equal to all subsets E belonging to S of C such that mu_1 of E is equal to mu_2 of E is a monotone class. We are given that mu_1 of A is equal to mu_2 of A for every A belonging to C. What does that mean? An equivalent way of stating that is saying that the collection C is inside the collection M. That is what it means by the very definition.

M is a monotone class and C is inside it; that implies that the monotone class generated by C must be inside M. Recall: what is monotone class generated by a collection of subsets of C? It is the smallest monotone class of subsets of X which include C; being the smallest, it must be inside it but note that C algebra implies M of C is equal to S of C. This is an important theorem which we had proved: if you take an algebra and generate a monotone class out of it, that is the same as generating the sigma algebra out of it.

This is the same as saying that S of C is contained in M but M is a collection of subsets of S of C; that is inside S of C; that is same as saying that M is equal to S of C (the sigma algebra generated by C). That means what? For all elements in S of C, mu_1 is equal to mu_2 of E. That proves the uniqueness theorem. We have finally proved in these four steps the theorem that if mu_1 and mu_2 are two measures defined on a semi-algebra of subsets of a set X and mu_1 and mu_2 are both sigma finite and they agree on the semi-algebra, then they also agree on the sigma algebra generated by C. This is an important theorem which we are going to use quite often. With this, we come to an end of a part of our course. This is probably the right stage to revise what all we have done till now. Let us revise what we have done till now.

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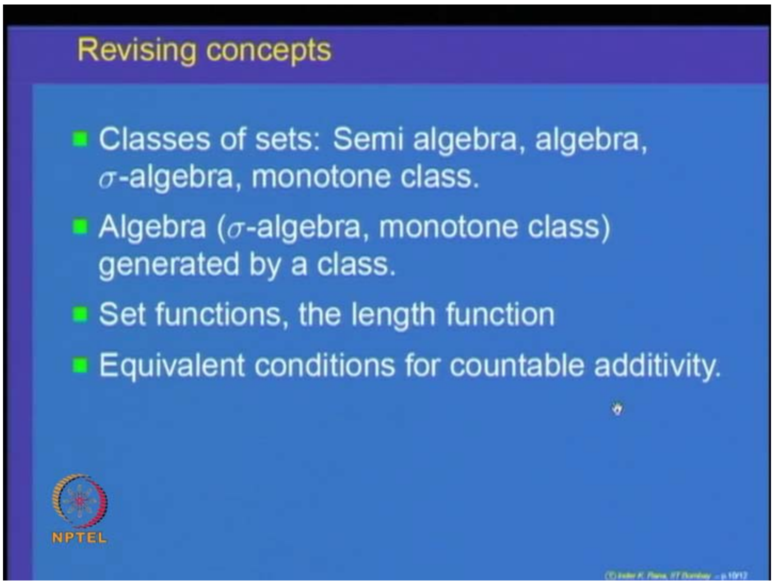

We started with looking at collections of subsets of a set X. We defined what is a semi-algebra. What was a semi-algebra? A semi-algebra was a collection of subsets of a set X with these properties: the whole space belongs to it, the empty set belongs to it, it is closed under intersections and the complement of a set is not necessarily inside it but can be represented.

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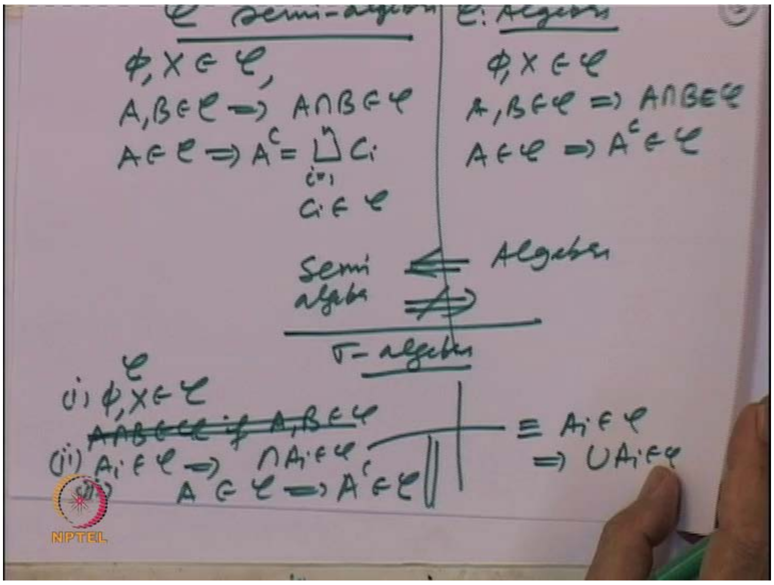

A semi-algebra C meant that empty set and the whole space belong to it – one property; the second one is A and B belonging to C should imply A intersection B belong to C; the third property is that A belonging to C implies A complement can be written as a finite disjoint union of elements of C for some C_is belonging to C; that is a semi-algebra.

Then, we defined what is called an algebra. A collection C is called an algebra with the first property as it is – empty set and the whole space belong to it; A and B belonging to C should imply it is closed under intersections and that also belongs to C; now we have something stronger; instead of just saying that A belongs to C, its complement is representable, actually we want that this complement also belongs to C; this is something stronger; we said this is a stronger property.

An algebra implies semi-algebra and the converse need not be true – that we have said. Then, we define what is called as sigma algebra. A collection C is a sigma algebra if, of course, it is an algebra first of all; it is phi and X belong to C; it is closed under intersection; so, A and B belong to C if A and B belong to C but this is not enough; actually, this should be true for any countable collection. Let us write whenever A_is belong to C, that should imply that intersection A_is belongs to C and because it is going to be closed under complements, this is property (i), this is the second and the third property is that A belongs to C should imply A complement belongs to C.

That automatically implies that C is also closed under… So, this property of countable intersections can be equivalently stated as, because of the complements, A_is belong to C imply union A_is also belong to C. A sigma algebra is a collection which is closed under countable unions and complements and, of course, the empty set in the whole space belongs to it. Then we define what is called a monotone class (Refer Slide Time: 42:40).

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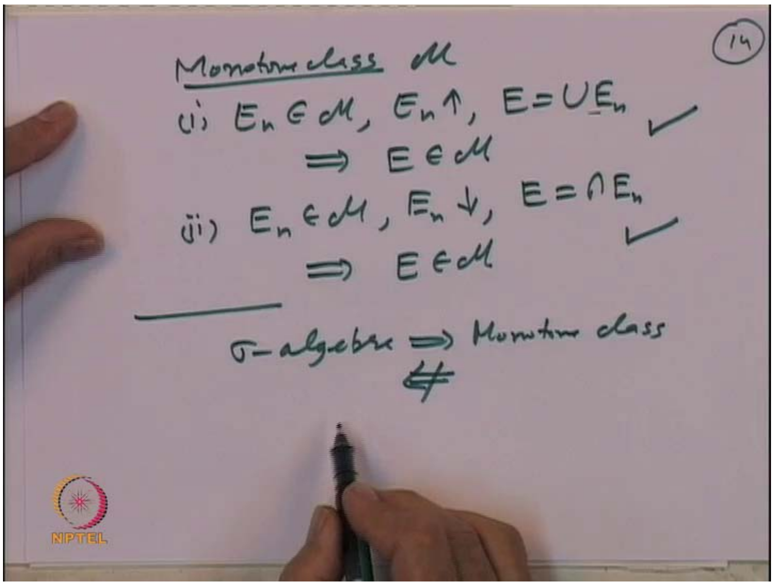

What was a monotone class? M was called a monotone class whenever a sequence E_n belongs to M, E_ns are increasing, E is equal to union E_ns should imply that E also belongs to M; this is one property. The second property we want that whenever E_ns belong to M, E_ns are decreasing and E is equal to intersection of E_ns should imply that E also belongs to M.

A monotone class is the collection of subsets of a set X with the property that it is closed under increasing unions and decreasing intersections. Of course, a sigma algebra implies a monotone class but the converse is not always true. These were the basic concepts of properties of collections of subsets of a set X (Refer Slide Time: 44:01) that we looked at.

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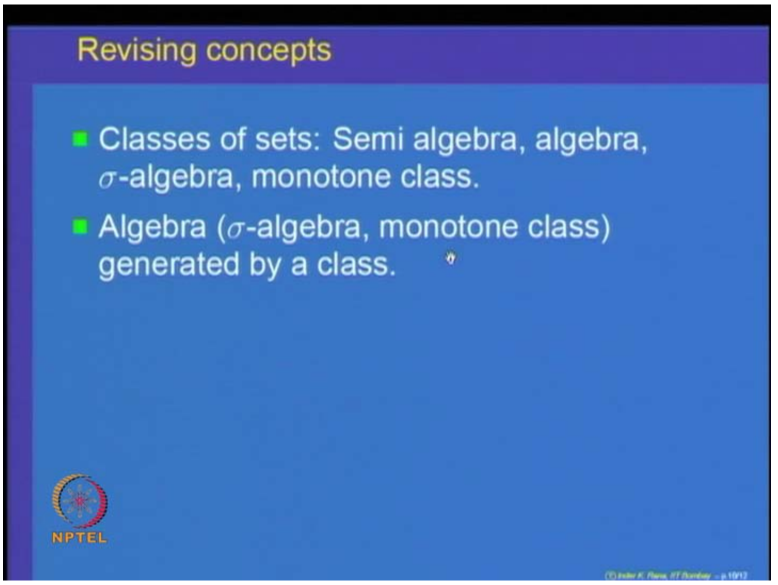

Then, we looked at what is called the algebra generated by a collection of subsets or the sigma algebra generated by a collection of subsets or the monotone class generated by a collection of subsets of a set X. In all these cases, we are basically given a collection C.

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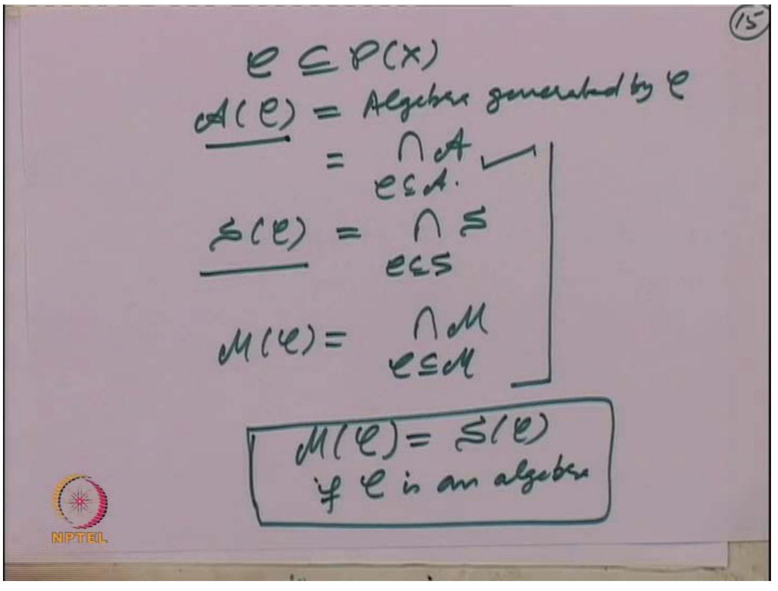

Let us just recall what was the meaning of saying generation. C is any collection of subsets of a set X. The algebra generated by C is the smallest one; it was the intersection of all the algebras which include C and we showed that such a thing exists. Similarly, the sigma algebra generated by C we said is nothing but look at all sigma algebras of subsets of X which include C and take their intersection; that is a sigma algebra; that is called the sigma algebra. Another way of saying is the algebra generated by C is the smallest algebra of subsets of a set X which include C.

Similarly, S of C is the smallest sigma algebra of subsets of C which includes C. Similarly, we have monotone class generated by C. It is the smallest monotone class of subsets of X which include C. We showed by these properties that such an object always exists. Then, we proved a very important theorem namely the monotone class generated by C is equal to the sigma algebra generated by C if C is an algebra; this was an important theorem that we had proved. These concepts were basically about collection of subsets of a set X.

Then, we looked at functions defined on such a collection of subsets of ((.)) set X and we called them as set functions. Set functions are functions defined on a collection of subsets of a set X.

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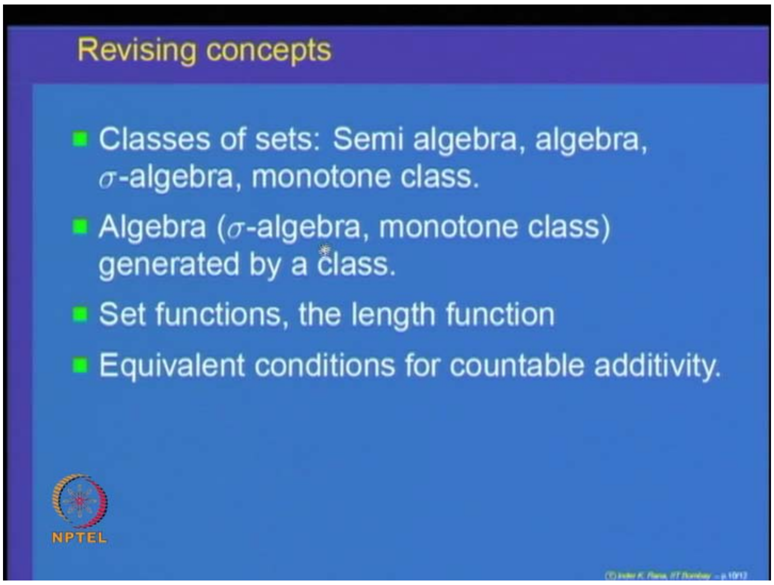

The important class of set functions was the length function. We showed that the length function had important properties: the length function which is defined on the class of all intervals in the real line was shown to be a countably additive set function which was also invariant under translations; that was an important property. Finally, we had proved some equivalent conditions for countable additivity and these conditions are very useful.

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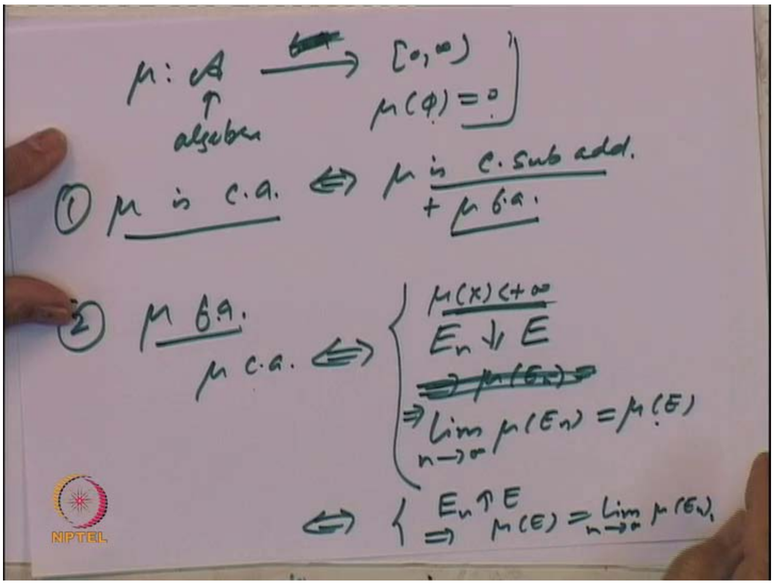

Let us just recall these equivalent conditions. We have used one of them today also. For example, if mu is defined on an algebra (this is an algebra) and it is finitely additive, then we said that mu is countably additive if and only if mu is countably subadditive. Let us write plus mu is finitely additive. Let us remove this condition (Refer Slide Time: 47:58). Let us write mu to be an algebra and mu of empty set equal to 0. mu is a set function defined on an algebra and mu of empty set is 0. Then, we proved that saying that mu is countably additive is equivalent to saying that mu is finitely additive and countably subadditive. This is quite useful in proving countable additivity of set functions. This was (1) (Refer Slide Time: 48:27).

Second: we proved that mu is finitely additive; we assumed that. Then, mu is countably additive if and only if whenever E_ns decrease of course under the condition A is an algebra to E; this should imply mu of E_ns is equal to the decrease so that is so let us write decrease to E then limit n going to infinity mu of E_n is equal to mu of E. We have to put an extra condition: mu of X is finite.

mu of X is finite and E_ns increase to E; that implies that limit of E_ns is equal to mu of E. This condition is equivalent to saying mu is countably additive when we have this (Refer Slide Time: 49:24). If you do not put this condition, then this may not be true, but then one can equivalently prove another thing: if E_ns increase to E, then that should imply mu of E is equal to limit n going to infinity mu of E_ns. This was the property of saying that ((.)) something countably additive (Refer Slide Time: 49:47).

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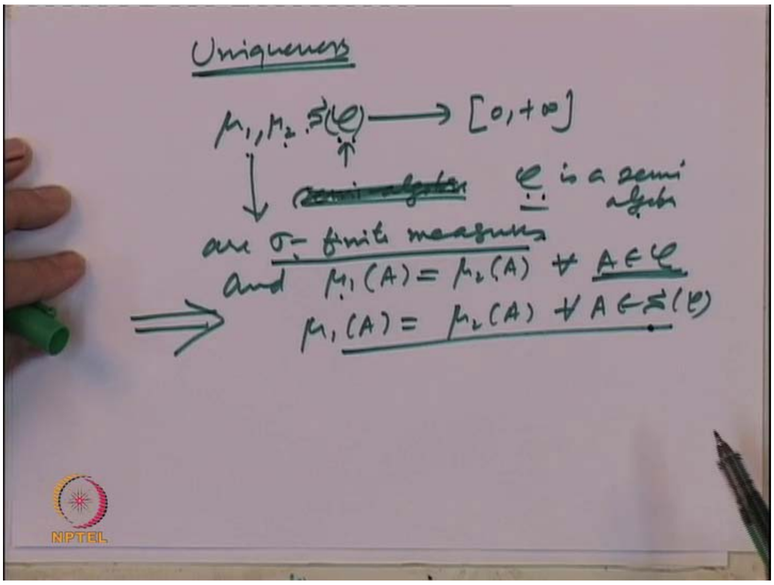

Finally today, we proved the uniqueness theorem saying that if mu$_1$ and mu$_2$ are two finite countably additive set functions defined on a semi-algebra (this is a semi-algebra, mu$_1$ and mu$_2$ are sigma finite measures and mu$_1$ of A is equal to mu$_2$ of A for every A in the semi-algebra), then this implies mu$_1$ of A is equal to mu$_2$ of A for every A belonging to the sigma algebra generated by C. mu$_1$ and mu$_2$ should already be defined; I am sorry; we should say they are already defined in S of C (Refer Slide Time: 50:59).

C is a semi-algebra. Let me state it once again; C is a semi-algebra; mu$_1$ and mu$_2$ are defined on the sigma algebra generated by C; both mu$_1$ and mu$_2$ are sigma finite and mu$_1$ and mu$_2$ agree on the semi-algebra; then they agree on the sigma algebra also; they agree on the whole domain. If they agree on the part of the domain which is the semi-algebra, then they agree on the whole of the sigma algebra also. That is the uniqueness result and that we proved under the condition that mu$_1$ and mu$_2$ are sigma finite measures. We will see how that is used in extension theory in the few lectures when we come to that.

We stop here today. In the next lecture, we will start a new topic called extension theory. We will like to extend a set function defined on a class to a bigger class; for example, on the real line, we have the notion of length defined on the collection of all intervals; we would like to define the notion of length for any set; that is the motivating thing for extension theory. We will use that and prove the theorems in the next lectures. Thank you.