Measure and Integration

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Module No. # 02

Lecture No. # 07

Countably Additive Set Functions on Intervals

Welcome to lecture 7 on measure and integration.

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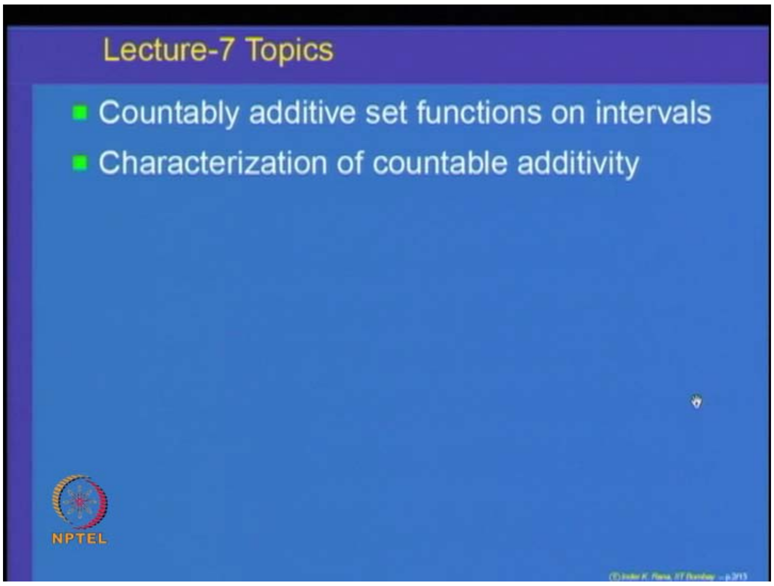

If you recall, in the previous lecture we had started looking at countably additive set functions on intervals and we proved some properties of such countably additive set functions. We will recall that theorem that we were proving and then continue the proof. If time permits, we will look at a characterization of countably additive set functions defined on algebras in the latter part of the lecture.

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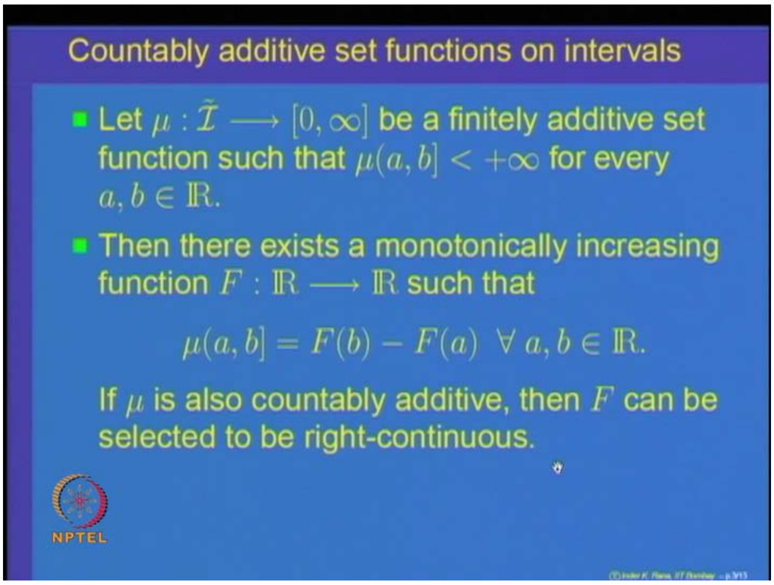

Let us just recall what were proving in the last lecture. We were trying to show that if mu is a finitely additive set function defined on the collection of all left-open right-closed intervals which was denoted by I tilde, if such a finitely additive set function is given with a property that mu of any finite interval is finite, mu of left-open right-closed interval a, b is finite for every a and b; then, we wanted to characterize such countably additive properties of such functions and relate it to a class of functions on the real line.

The claim of the theorem is that there exists a monotonically increasing function F from R to R such that the value mu of the left-open right-closed interval a, b is given by F of b minus F of a for every a and b belonging to R. We wanted to show that given a finitely additive set function on the class of all left-open right-closed intervals, it must arise from a monotonically increasing right-continuous function F with the relation that the value mu of a, b is given by the difference F of b minus F of a. Here, mu was only finitely additive. If we assume mu is countably additive, then this function F can be selected to be right-continuous.

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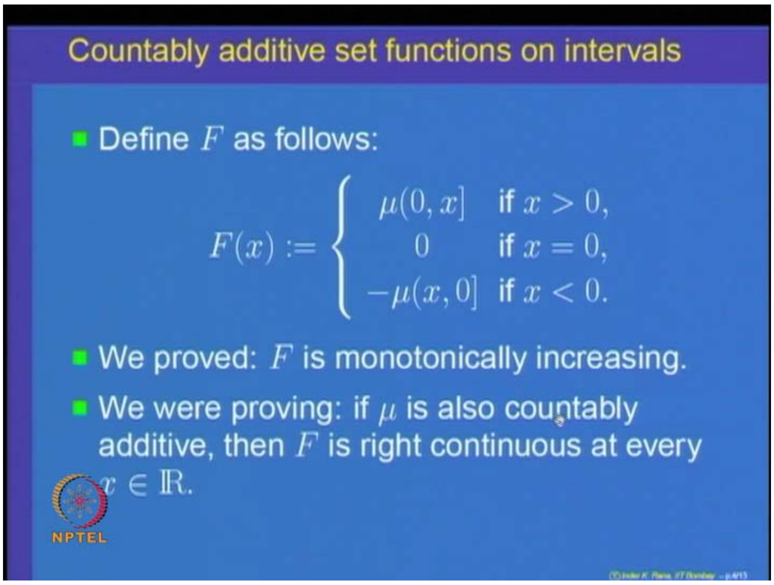

Let us just recall how we defined this function. We looked at the function F defined by F at a point x is defined as the measure mu of the interval 0 to x if x is bigger than 0; it is 0 if x is equal to 0 and is minus mu of x to 0, closed at 0 if x is less than 0. This was the definition of the function F. We proved the property that this function F is indeed monotonically increasing. For that, if you recall, we use the fact that F is the measure mu this mu is a countably additive is a finitely additive set function. Next, if we assume that mu is countably additive, we wanted to show that this function F is right-continuous at every point x in R.

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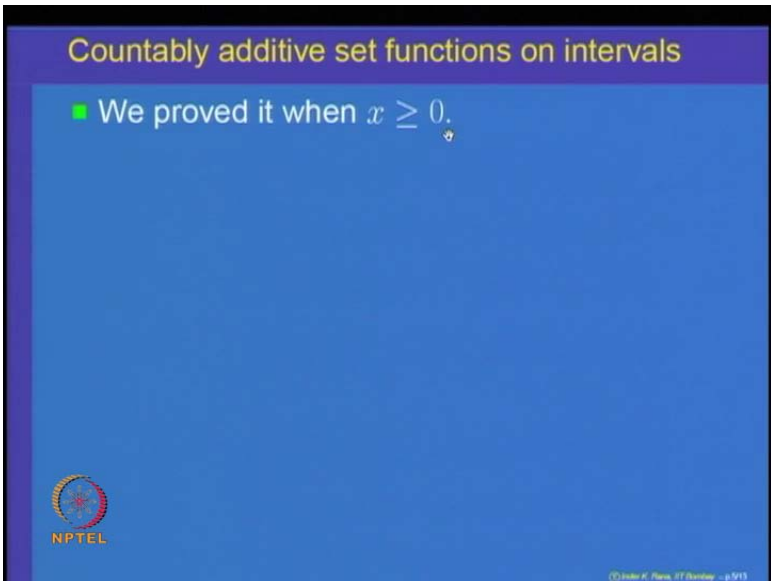

We had started looking at the proof when x is bigger than or equal to 0. We had proved that for any point x bigger than or equal to 0, F is right-continuous at the point x is equal to 0. Today, we will start with proving the remaining part of the proof – if x is less than 0, then also F is right-continuous at x.

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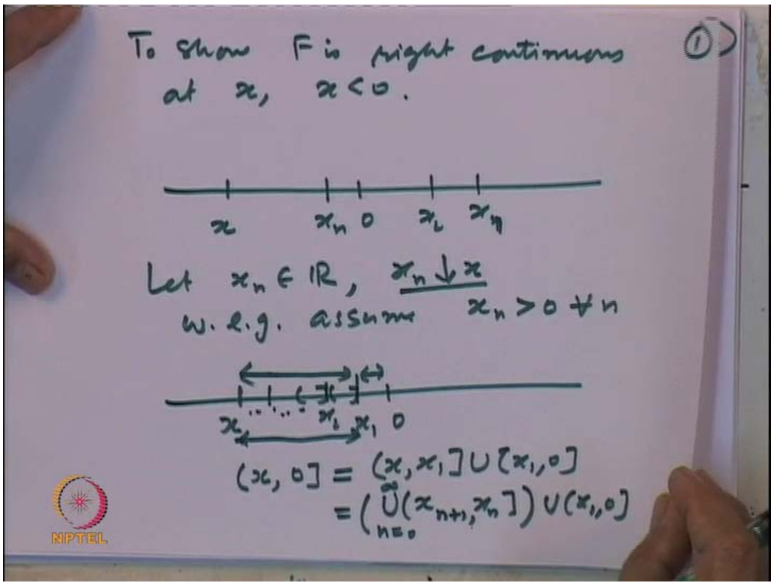

Let us look at the proof. We want to show F is right-continuous at a point x where x is less than 0. Here is the point 0 and here is x. To show right-continuity at the point x, let x_n belong to R. Let us a take x_n – a sequence in R such that x_n decreases to x; that means all the x_ns are on the right side of the x and are converging to x. All the points x_ns are on the right side and so here it may be x_1, here it may be x_2 and so on (Refer Slide Time: 05:07).

After some stage, x_n has to cross over the point 0 (the value 0). What we are saying is this: without loss of generality, assume that all the x_ns are bigger than 0 for every n because x_n is going to converge to x and x is less than 0; so, at some stage it has to cross over. We can start analyzing the sequence from that point onwards. One writes this as without loss of generality the proof is not changed if we assume x_n is less than 0 for every n. Here is the situation; here is the point x, here is the point 0 and here is the point x_1 (Refer Slide Time: 05:53).

Here is x_2 and so on. Let us observe that the interval left-open right-closed at 0 can be written as x to x_1 union x_1 to 0. I can write this as from this point to x_1 and from this point onwards this one (Refer Slide Time: 06:24). Now, this interval x to x_1 I am going to split further into a union of intervals. My claim is that this x to x_1 is the same as x_1 to x_2 union x_2 to x_3 union x_3 to x_4 and so on.

The claim is that is the same as $x_{n \text{ plus } 1}$ comma x_n left-open right-closed union n equal to 0 to infinity union x_1, 0. The interval x to x_1 – this part (Refer Slide Time: 07:09) we are splitting it into left-open right-closed, left-open right-closed and so on; this is an equality because x_n is decreasing to x. At any point here if I take any point in between x and x_1, then that stage has to be crossed over by some x_n; that point will belong here. So, the interval x to x_1 is a union of the intervals left-open $x_{n \text{ plus } 1}$ to closed x_n, n equal to 0 to infinity.

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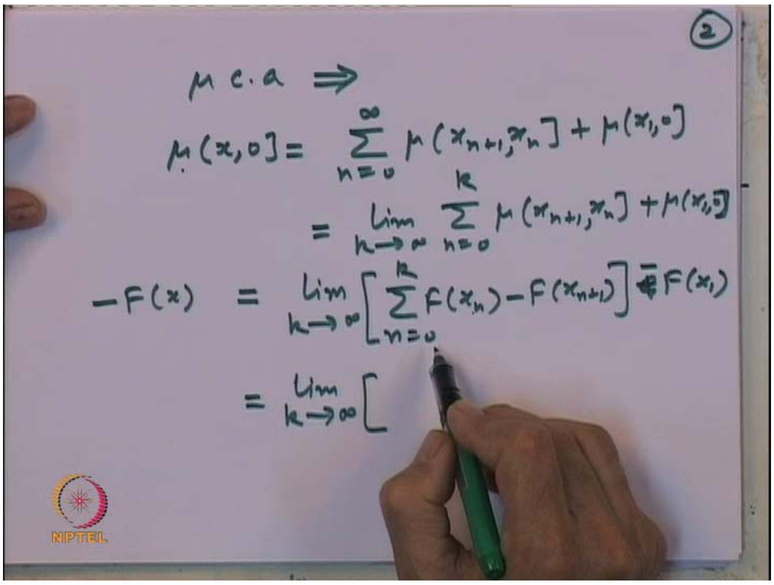

Also observe that these intervals are all disjoint. These are all disjoint intervals; so, I can write using countable additive property mu of the set function is equal to summation n equal to 0 to infinity mu of $x_{n \text{ plus } 1}$, x_n plus mu of x_1 to 0. Here, we have used the fact that mu countably additive implies this property is true. Now, this right-hand side is a sequence of nonnegative real numbers, possibly extended real numbers; I can write this as limit k going to infinity sigma n equal to 0 to k mu of $x_{n \text{ plus } 1}$, x_n plus mu of x_1 to 0, closed here at 0.

Now, we will write everything in terms of F. By definition, mu of x to 0 is minus F of x is equal to limit k going to infinity summation n equal to 0 to k. This is nothing but F of

x_n minus F of $x_{n \text{ plus } 1}$ plus F of F of x_1 to 0 so that is in fact minus F of x_1. Now, let us x note what this is. This is limit k going to infinity. What is this sum? This starts with n equal to n equal to 0 will give x 0. That is not so let us so. There was a mistake here.

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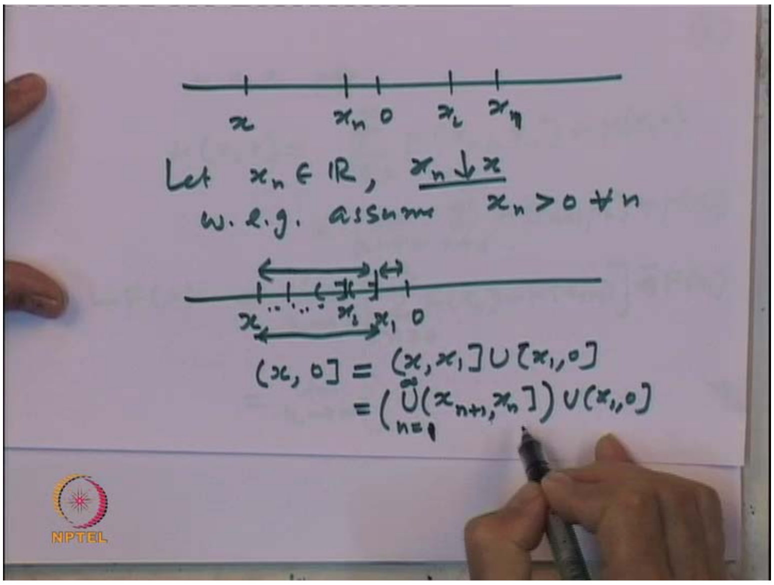

I should have written as union from n equal to 1 because it is 1 to 2 and so on. That was the mistake here

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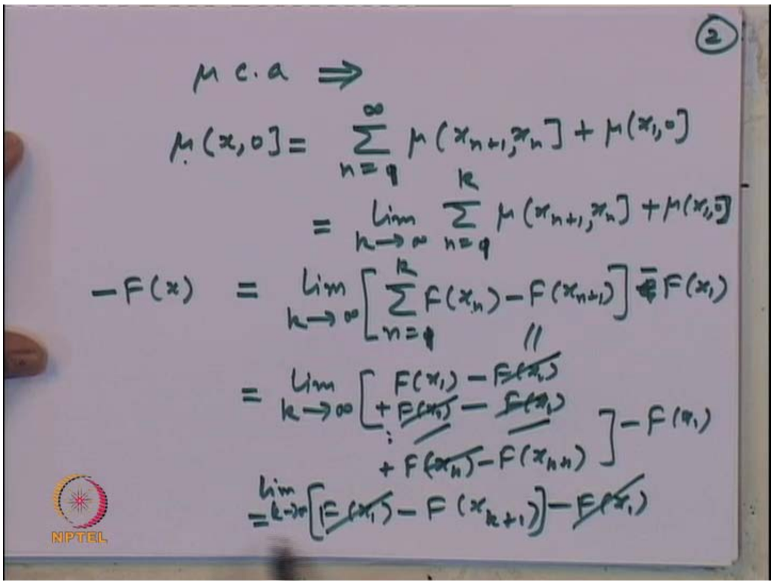

This sum is from n equal to 1 to, n equal to 1 to, n equal to 1 to k (Refer Slide Time: 10:12). What is this sum? n equal to 1 gives you F of x_1 minus F of x_2 plus F of x_2

minus F of x_3 and so on plus F of $x_{n \text{ equal to } k}$; so that is x_k minus F of $x_{k \text{ plus } 1}$. So, that is this part – this sum – and minus F of x_1 (Refer Slide Time: 10:42). We observe that in this x_2 and x_2 will cancel out; what was left is this is equal to F of x_1 minus F of $x_{k \text{ plus } 1}$ minus F of x_1. In this equation, this cancels with this; so, minus F of x. Sorry, there is a limit outside; so, limit of this k going to infinity.

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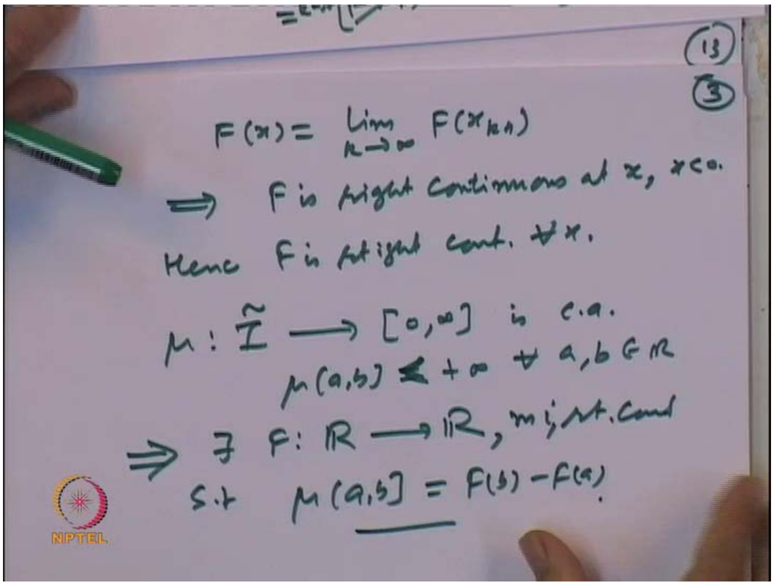

This gives us that F of x is equal to limit k going to infinity of F of $x_{k \text{ plus } 1}$. That proves the fact that F is right-continuous at x in the case when x was less than 0. Hence, F is right-continuous for every x. This proves the theorem that if mu on the class of all left-open right-closed intervals is countably additive with the property that mu of a, b is finite for every a, b in R, then this implies there exists a function F which is monotonically increasing and right-continuous such that mu of a, b is equal to F of b minus F of a. What we have shown is that to every countably additive set function mu on left-open right-closed intervals, you can associate a monotonically increasing right-continuous function. This is proved (Refer Slide Time: 13:06).

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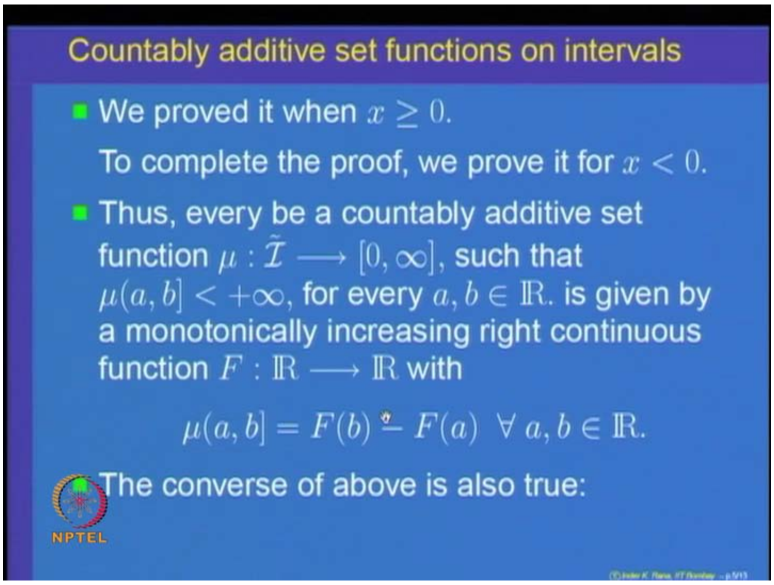

This completes the proof of the fact that to every countably additive set function on the class of intervals, we can associate a monotonically increasing right-continuous function with this property; in fact, the converse of this statement also holds. What will be the converse of such a statement? The converse of such a statement would be that if you are given a monotonically increasing right-continuous function F, then we can define a set function mu on left-open right-closed intervals in such a way that this relation is satisfied. That will prove that the only way we can construct countably additive set functions on the class of intervals is via monotonically increasing right-continuous functions.

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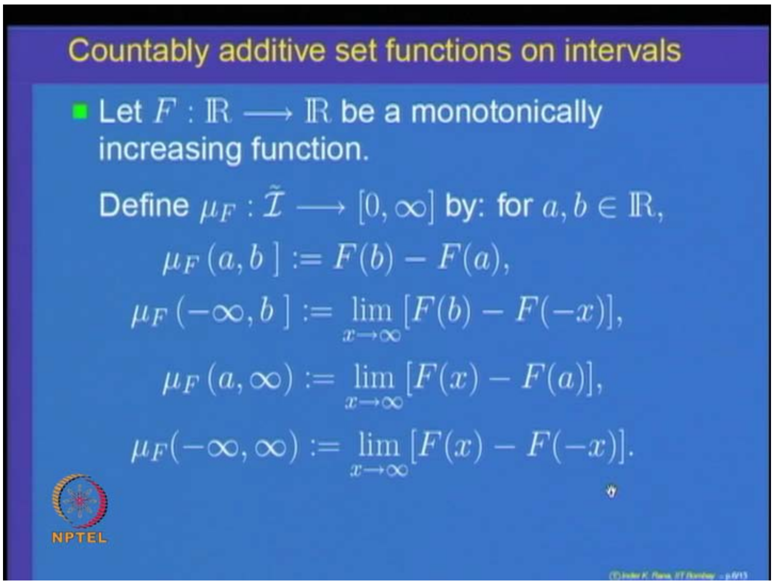

The converse part of the theorem says the following. Let F be a monotonically increasing function from R to R. Define mu_F – a set function on the class of all left-open right-closed intervals as follows. For any two real numbers a and b, we want to define what is mu_F of the left-open right-close intervals a, b. This is a property that has to be satisfied by F; that itself gives us the defining property of the set function mu. So, mu_F of the left-open right-close interval is defined as the difference F of b minus F of a for all real numbers a and b.

Now, the question comes: what happens if b is equal to plus infinity or a is equal to minus infinity or both of them? In that case, we write this as for mu_F of the infinite interval minus infinity to b. It is open on the left side and closed on the right side b; so, it is a left-open right-close interval on the real line. What we do is we take the definition as F of b minus F of minus x, x going to infinity.

As x goes to infinity, minus x will go to minus infinity; we are defining it via limits. Look at the interval minus x to b – left open; that is the value of the mu of F; then, take the limit of that as x goes to infinity. This is a definition of mu_F of minus infinity to b. Similarly, if it is on the right side, if a to infinity, we define it as take the interval a to closed x; then, the value of that will be F of x minus F of a and now take the limit of that as x goes to infinity.

The infinite interval unbounded on the right side, left-open right-closed, a to infinity is defined as limit x going to infinity F of x minus F of a. If it is the whole real line, then we define mu_F of the whole real line to be limit x going to infinity of F of x minus F of minus x. Look at the interval minus x to x and let both sides go to infinity. This is the way we define mu of F.

Note that this is a generalization of the length function. If F is the identity function namely F of x is equal to x, that is a monotonically increasing function, then this is nothing but b minus a; so mu of a, b is nothing but b minus a. This mu F is nothing but the length function when F is a monotonically increasing function. One can write down a proof of this on the lines of when we proved that the length function is countably additive.

On the same lines, one can write down the proof of the fact that this set function mu_F is also countably additive. One can wonder where one will be using the fact that F is right-continuous. Where we will be using the right continuity of this F is to prove that it is monotonically increasing – to prove that mu_F is countably additive. If this F is monotonically increasing, we can define this mu_F is finitely additive but to prove countably additive, we need F to be a right-continuous function.

If F is right-continuous, then one can write down a proof similar to that of the case of the length function. One uses the fact of right-continuity because one has to deal with the intervals which are left-open and right-closed. If you are keen to know a proof of this, you better write a proof yourself trying to see that the steps given for the proof of the length function is countable additive can be suitably modified to do this; we will leave it as an exercise. If you feel it is too tough an exercise, let us assume this and go ahead.

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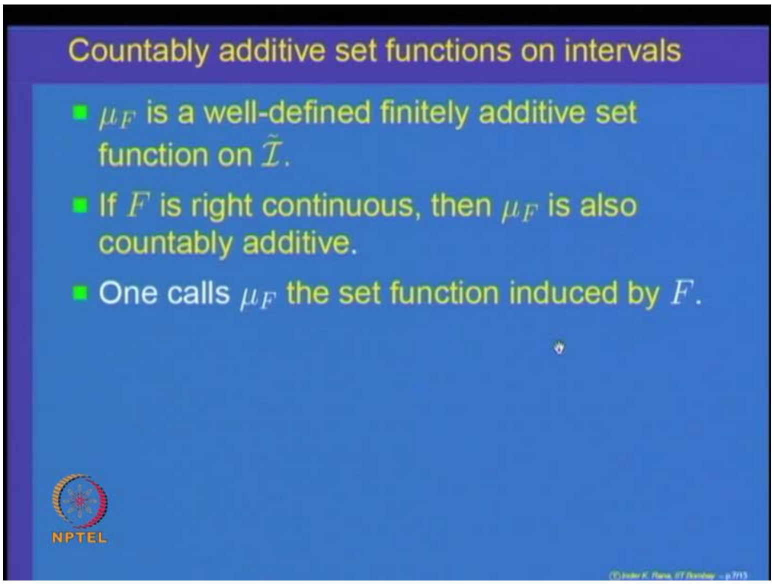

mu_F is a finitely additive set function and using if F is right-continuous, one proves that mu_F is also countably additive. This function mu_F is called the set function induced by the increasing function F.

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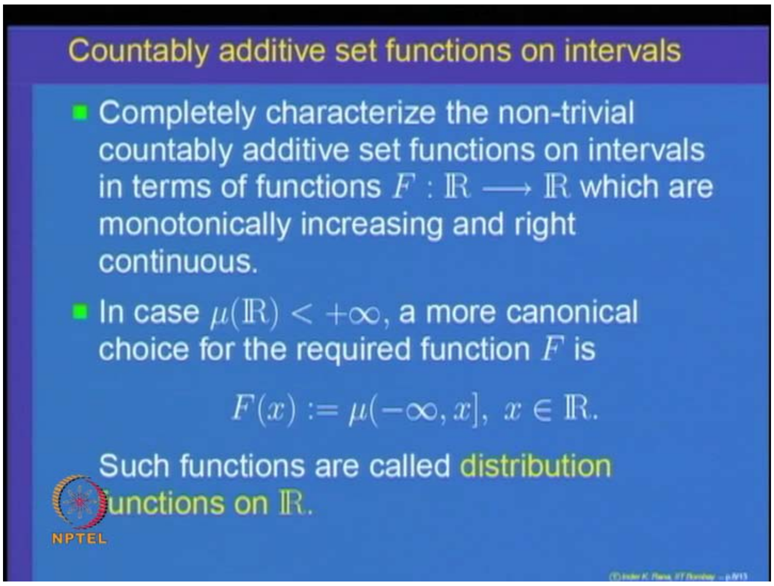

This gives us a complete characterization of nontrivial countably additive set functions. Why nontrivial? It is because we are looking at mu of the left-open right-closed interval a, b to be finite in terms of functions which are monotonically increasing and right-continuous. In some sense, there is a correspondence between measures on the class of all intervals and monotonically increasing right-continuous functions.

In case that countably additive set function mu has the property that mu of the whole real line is finite, then one can select this monotonically increasing function to be mu of minus infinity to x because than this is defined; we do not have to restrict the fact that mu of a, b is finite; that will be true anyway because this is finite. A more canonical choice for the monotonically increasing right-continuous function is mu of minus infinity to x when mu of the whole space R is finite.

In that case, this function F is called the distribution function on R. This plays a role in the theory of probability where monotonically increasing right-continuous functions are studied via what are called probability distributions. We will not go into that; we will just make a note of it in case we have a finite condition that mu of R is finite – we will take F of x to be this function (Refer Slide Time: 21:15). We have characterized all countably additive set functions on the class of all intervals.

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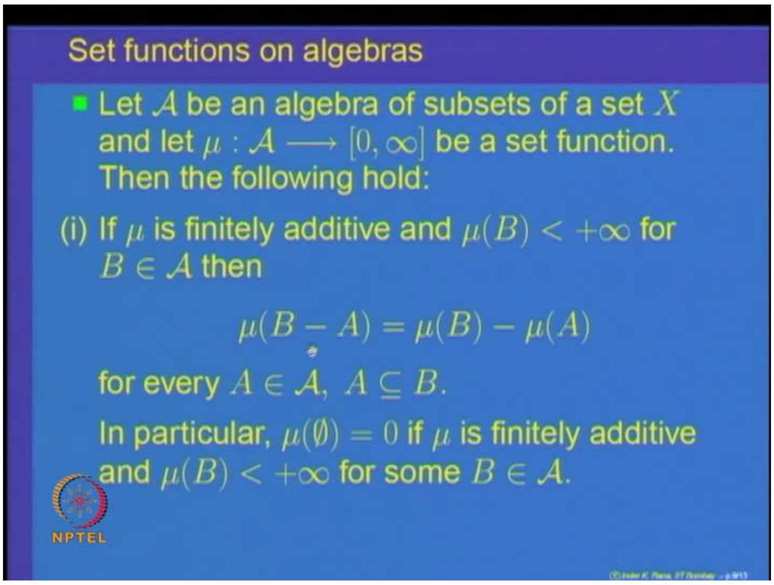

What we shall do next is the following: we will study what are called set functions on a general class of sets called algebras Let us start with looking at A – an algebra of subsets of a set X – and mu, a set function defined on this algebra taking nonnegative real values (taking values 0 to infinity). We want to show that the following holds: if mu is finitely additive and mu of the set B is finite for a set B in the algebra A, then mu of the difference B minus A is equal to mu of B minus mu of A whenever A is in the algebra and A is a subset of B. What we are saying is the following.

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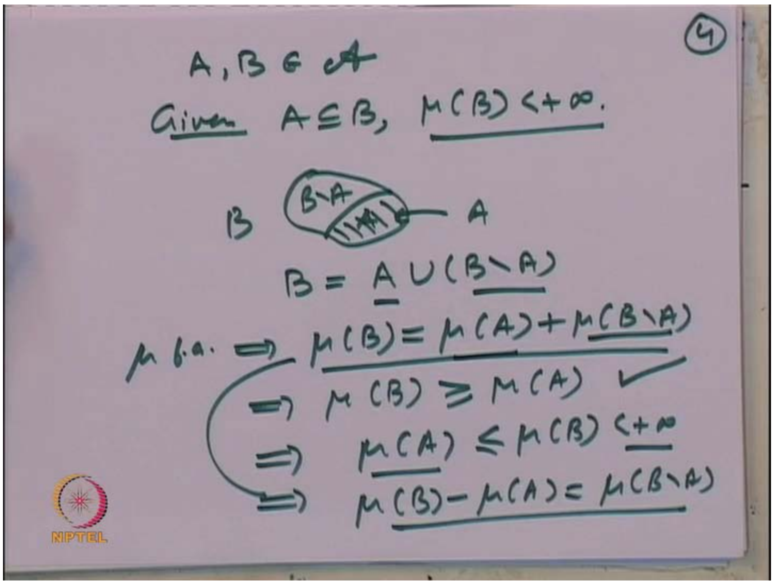

Let us take sets A and B belonging to the algebra A. We are given, of course, that A is a subset of B and mu of B is finite. Here is the set B and A is a part of it; this is B; that is A; this part is A (Refer Slide Time: 22:52). We can write B as A union B minus A; this is the part B minus A. Note that A and B minus A both are disjoint sets; B is written as a finite union – in fact, union of the two sets A and B minus A and they are pairwise disjoint.

mu finitely additive implies that mu of B is equal to mu of A plus mu of B minus A. Now, let us note that all these are real numbers; mu of B is a real number because it is finite; mu of A is a real number because A is a subset of B and mu of A will be less than or equal to mu of B; that is finite. This is an equation in real numbers anyway; that is not really important here but note that all are nonnegative quantities.

That implies that mu of B is bigger than or equal to a mu of A; that is one thing that we observe because this is nonnegative. This also implies that mu of A is less than or equal to mu of B which is finite. That implies that mu of A is finite. In this equation, I can say all are real numbers and so I can manipulate this as an equation in real numbers. This equation implies that if I take it on the other side, mu of B minus mu of A is equal to mu of B minus A.

That is what we wanted to prove. Note here we have used the fact mu of B is finite (Refer Slide Time: 24:54). Hence, mu of every subset of it is finite whenever that set is in the algebra. We can manipulate this as an equation only when they are real numbers; if they are equal to plus infinity at any one of them, then I cannot transpose them on the other side and write this equation.

We have used the fact that mu is finitely additive and mu of B is finite (Refer Slide Time: 25:20). That implies for every subset A of B which is in the algebra, mu of A is also finite and mu of B minus A is equal to mu of B minus mu of A. In particular, suppose I take B equal to A, this gives mu of empty set is equal to 0; in particular, mu of empty set is 0 if mu is finitely additive and mu for at least one set B is finite. These are consequences of a set function being finitely additive.

What we are trying to show is if a set function is finitely additive, what are the possible consequences? We showed finite additivity implies monotone; if B is finite, then I can interchange and write mu of B minus A to B equal to this. Finite additive plus mu of at least one set is finite implies mu of phi is equal to 0.

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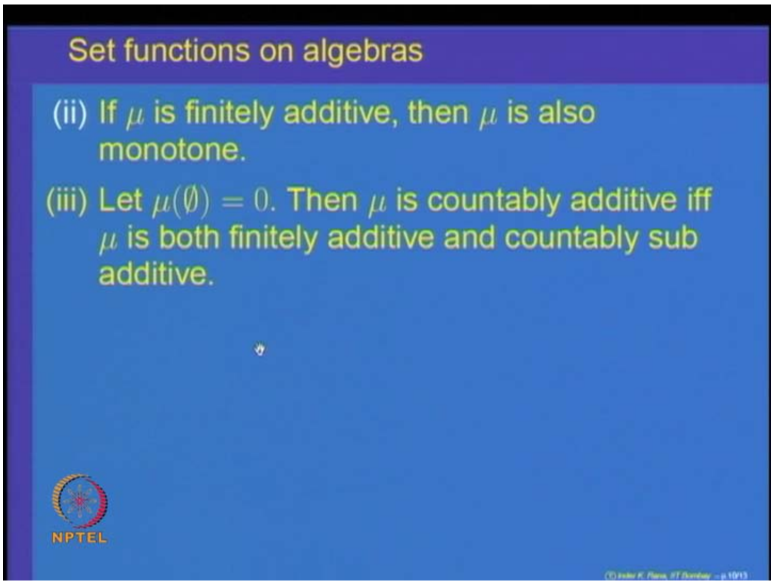

mu is monotone we have already shown. Let us look at the next property; that is a very important thing – characterization of countable additiveness of the set function. Suppose mu of phi is equal to 0, then we want to claim that mu is countably additive if and only if mu is both finitely additive and countably subadditive. We want to characterize the countable additive property of the set function defined on an algebra in terms of it being finitely additive and countably subadditive.

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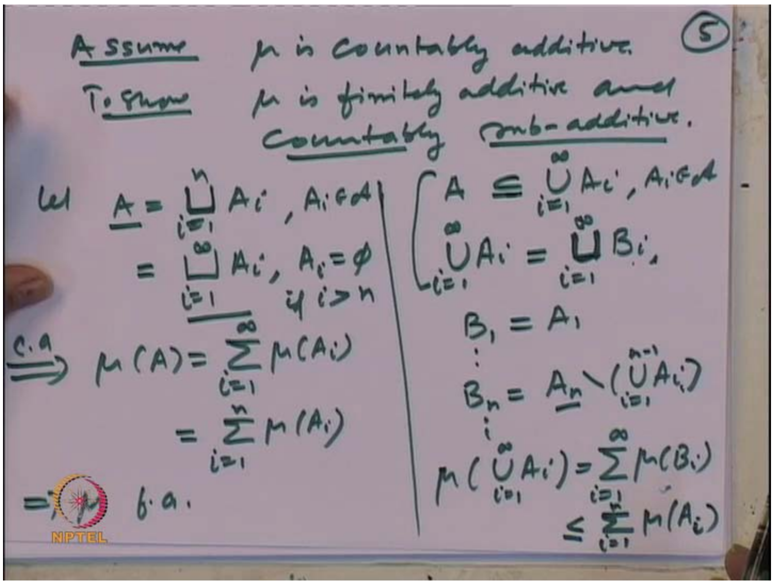

Let us prove these properties. Let us start by one way. Let us assume that mu is countably additive. We have to show mu is finitely additive and countably subadditive. Let us look at the first thing. To show it is finitely additive, what do we have to do? Let A be equal to a disjoint union A_i, i equal to 1 to n. Whenever the union is disjoint sets – pairwise disjoint, we will write it as a square union (a symbol for cup instead of writing it as usual) where A_is belong to the algebra A.

I can also write it as union of A_i, i equal to 1 to infinity where A_i is equal to empty set if i is bigger than n; from n onwards let us put them as empty sets. Then, A is a countable union of pairwise disjoint sets. This implies by countable additive property that mu of A is equal to summation mu of A_is, i equal to 1 to infinity, but that is same as sigma i equal to 1 to n mu of A_i because for i bigger than or equal to n plus 1, the sets are empty and mu of the empty set is given to be 0; therefore, it implies mu is finitely additive.

On the other side, let us try to prove that mu is countably subadditive. Let us take a set A in the algebra and let us say this is contained in union of A_is i equal to 1 to infinity. Now, let us observe the following: this union A_i, i equal to 1 to infinity where A_is are in the algebra A... (Refer Slide Time: 29:59). If you recall, we had shown that any countable union of sets in the algebra can be written as a countable union of disjoint sets in the algebra where again B_is are in the algebra but this is a disjoint union. How did we do that? Let us just recall that we defined B_1 to be equal to A_1 and in general B_n to be equal to A_n minus union A_i, i equal to 1 to n minus 1 and so on; that is how we had defined those sets B_i.

Note that at every stage B_1 is A_1 in the algebra; so B_1 is in the algebra. Similarly, B_n is A_n which is in the algebra; finite union A_i 1 to n minus 1 is in the algebra; the difference of the two sets in the algebra is again an algebra; so, each B_n is an element of the algebra. These are disjoint and their union because union B_1 up to B_n is the same as union up to A_1 to A_n and that is true for every n; so this is equal to true (Refer Slide Time: 31:14).

Using these two things, now let us write. A is a subset of this. This says mu of the union A_is, i equal to 1 to infinity will be equal to summation mu of B_is, i equal to 1 to infinity because this union A_i is the same as union B_is. Union of B_is is a disjoint union; by countable additive property, mu of the union is equal to this sum (Refer Slide Time: 32:00). Note that each B_n is a subset of A_n and by finite additive property – monotone property, this is less than mu of A_i, i equal to 1 to n.

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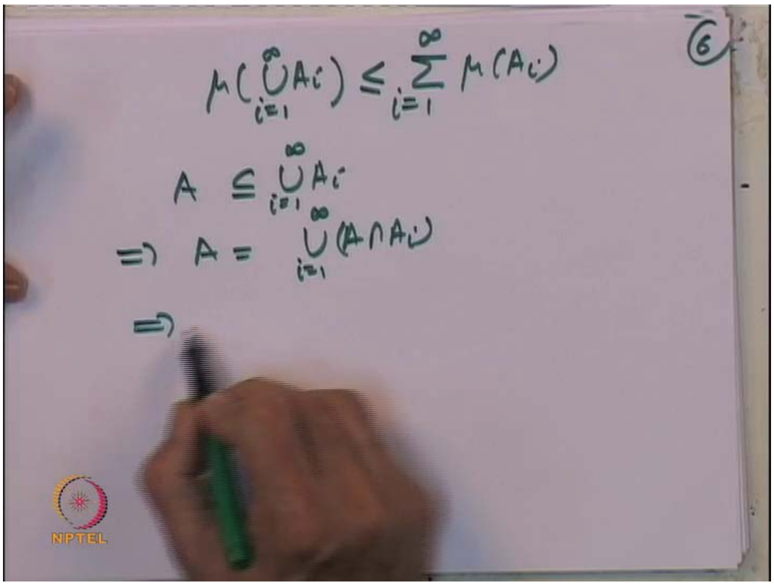

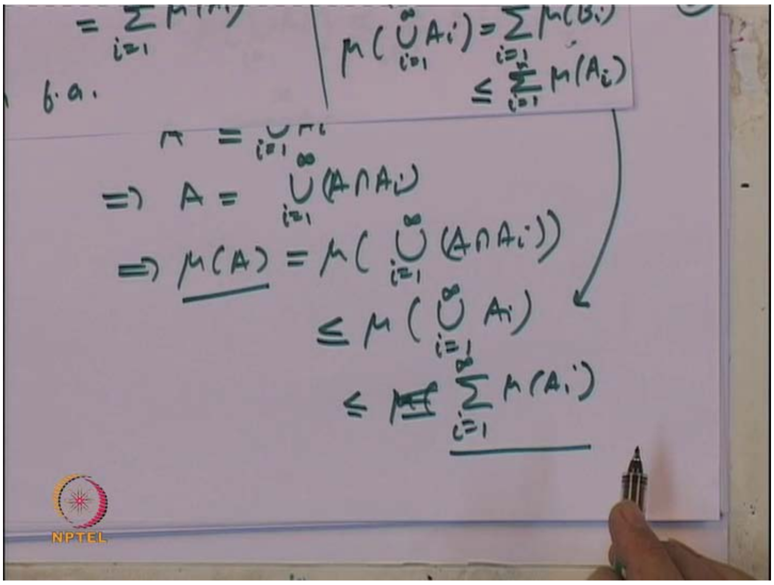

What we have shown is the following: mu of union A_i, i equal to 1 to infinity is less than or equal to sigma i equal to 1 to infinity mu of A_i. We just want to conclude that in fact mu of A is less than or equal to this quantity. Now, let us observe; A is a subset of union A_i. This implies that I can write A is equal to union of A intersection A_i, i equal to 1 to infinity; I can just intersect and then this is an equality.

That means mu of A is equal to mu of union i equal to 1 to infinity A intersection A_i. This is less than or equal to… because.… This union is a subset of the union; so, this is less than mu of union i equal to 1 to infinity of A_is because each one is a subset of this; so, this union is subset of this (Refer Slide Time: 33:46). From here, this is less than or equal to summation i equal to 1 to infinity of mu of A_i.

We have shown that whenever A is an element in the algebra is a subset of union of A_is, i equal to 1 to infinity (Refer Slide Time: 34:11), then mu of A is less than or equal to summation mu of A_is. That proves that mu is countably subadditive. We have shown if mu is countably additive, then this implies mu is finitely additive and also mu is countably subadditive (Refer Slide Time: 34:32). That completes one part of the proof; let us prove the other way around implication.

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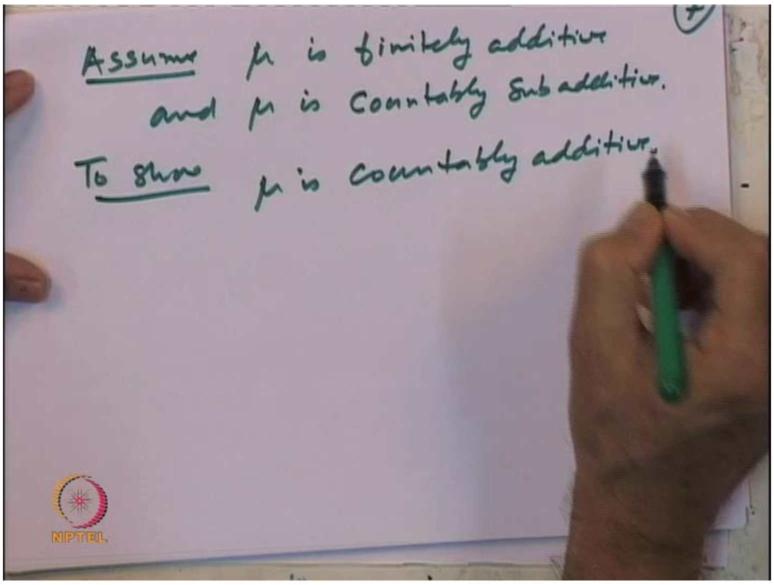

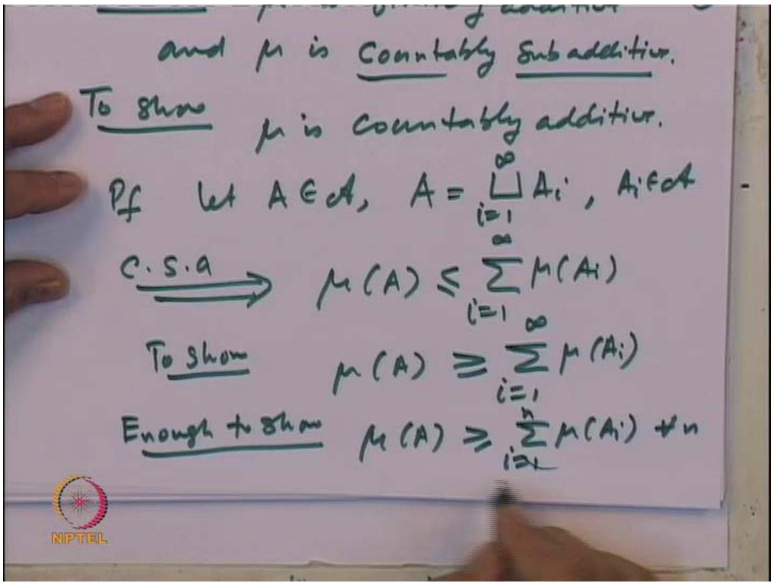

Assume mu is finitely additive and mu is countably subadditive. We have to show mu is countably additive. To prove countable additivity, what do we have to show? Let A belong to algebra and A be equal to disjoint union A_is, 1 to infinity and A_is belonging to algebra. We have to show mu of A_i is summation mu of A_is. Now, by countable subadditive property which is given to us, countable subadditive implies that mu of A is at least less than or equal to sigma i equal to 1 to infinity mu of A_is; countable subadditivity implies the fact that this is less than or equal to this (Refer Slide Time: 36:09).

We have to prove only the other way – show that mu of A is also greater than or equal to sigma i equal to 1 to infinity mu of A_i; this is what we have to show. Here is a small observation: to show this, it is enough to show that mu of A is bigger than or equal to sigma i equal to 1 to n mu of A_i for every n. If you can show for every n that mu of A is bigger than or equal to this, then it also will be true for i equal to 1 to infinity because this is nothing but limit of these partial sums; this is enough to show; we have to only show that mu of A is bigger than or equal to sigma mu of A_is, i equal to 1 to n.

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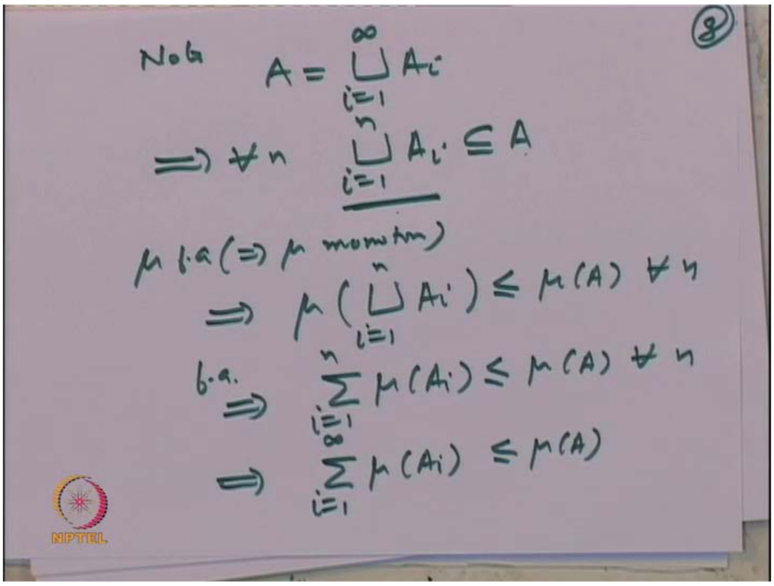

Note that A equal to union A_i, i equal to 1 to infinity implies for every n, the union A_i, i equal to 1 to n is a subset of A for every n. We are in algebra; so, this set is in the algebra; this is in the algebra (Refer Slide Time: 37:37). mu finitely additive implies mu monotone and hence implies that mu of union A_i, i equal to 1 to n will be less than or equal to mu of A for every n; again by finite additivity, this is nothing but sigma i equal to 1 to n mu of A_i is less than or equal to mu of A for every n; this is happening for every n. We can let n go to infinity and so i equal to 1 to infinity mu of A_i is less than or equal to mu of A.

That proves the other way around inequality also of the required thing; this proves this (Refer Slide Time: 38:31); that proves that mu is countably additive. What we have proved is the following (Refer Slide Time: 38:43). We have given a characterization of countable additive property of set functions which are finitely additive. If mu of empty set is equal to 0, then mu is countably additive if and only (note here the if and only if – we have proved both ways) mu is both finitely additive and countably subadditive. This is a characterization of countable additiveness of set functions, but, of course, the domain of the set function should be an algebra; that is important; this is a very useful criterion for countable additivity.

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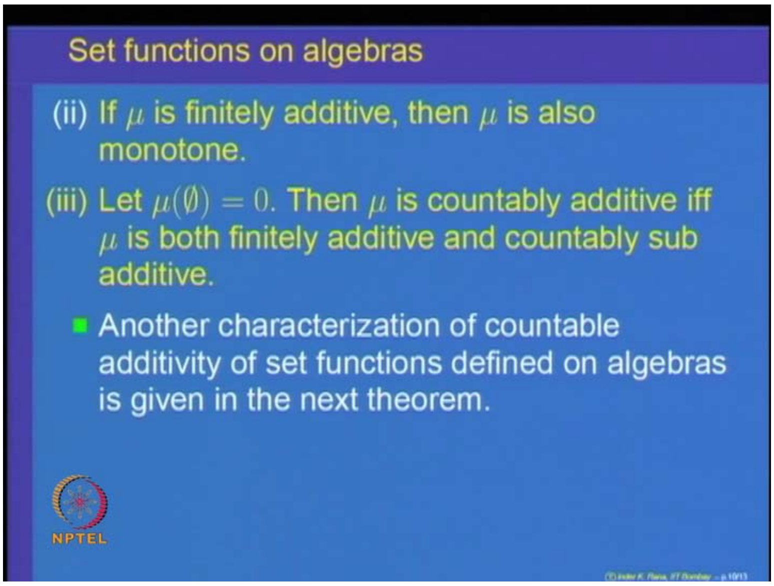

We will prove another characterization of countable additivity of set functions in terms of increasing and decreasing limits; that we will in the next theorem. That theorem is again about set functions defined on algebras.

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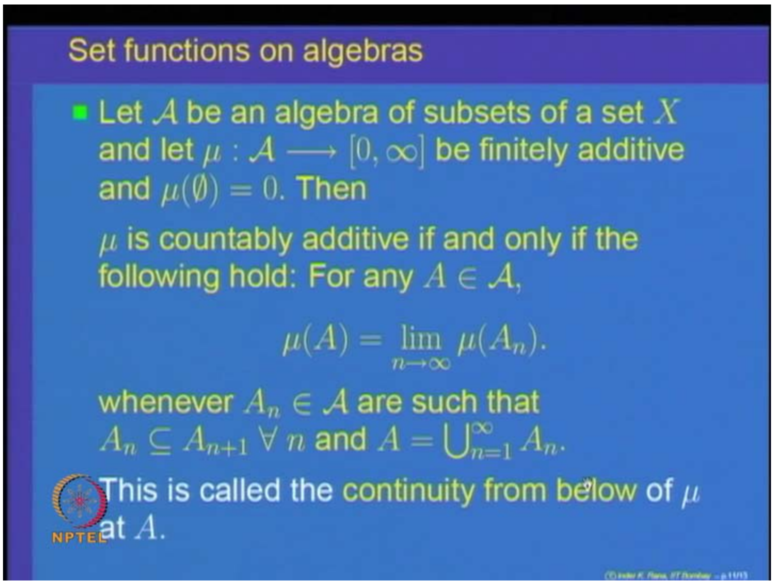

The theorem says the following: let A be an algebra of subsets of a set x and mu be finitely additive and with the property, of course, mu of empty set is equal to 0. Then, we want to prove that mu is countably additive if and only if, once again it is a characterization, the following property holds. The property says for any element A in the algebra A, we should have mu of A is limit of mu of A_ns. What are A_ns? Whenever A_n is a sequence of sets in the algebra which is decreasing – n is a subset of $A_{n \text{ plus } 1}$ for every n and the sequence A_n should be decreasing and A should be ((.)).

Sorry, A_n should be increasing; A_n is the subset of $A_{n \text{ plus } 1}$; that means A_ns are increasing sequence of sets in the algebra and A is the union of all these sets A_n. This is a characterization of countable additiveness of the set function mu provided one can prove the following: for any set A and for any sequence A_n of sets in the algebra which is increasing and A is the union, we should have mu of A equal to limit mu of A_ns. Let us prove this property once again.

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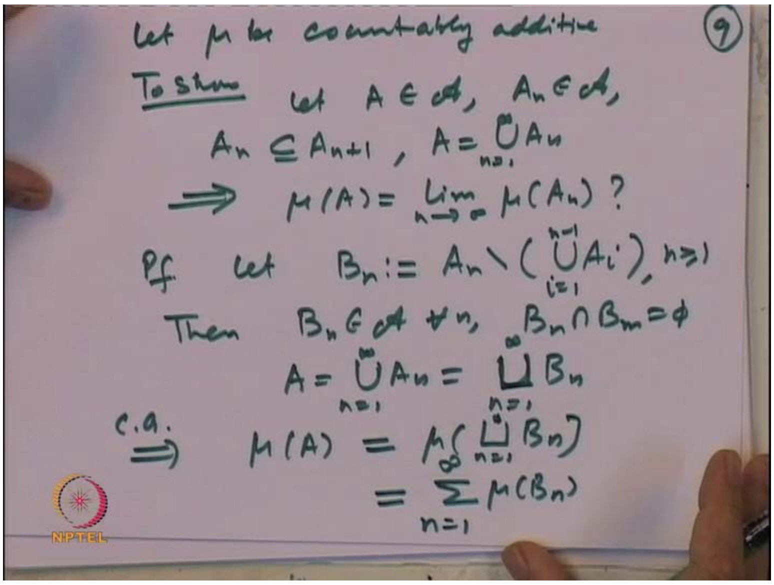

To prove this, what do we have to show? First, let mu be countably additive. We have to show the following. Take a set A belonging to the algebra, take a sequence A_ns belonging to the algebra such that A_n is subset of $A_{n \text{ plus } 1}$ and A is equal to union of A_ns. We should show that mu of A is equal to limit n going to infinity mu of A_ns; that is what is to be shown. Let us observe that A is union of A_ns and we are given something about countable additivity.

The obvious thing is try to write the union as a countable disjoint union; we do that. Proof: let B_n be defined as A_n minus union A_i, i equal to 1 to n minus 1 for every n bigger than or equal to 1. Then, as observed earlier, each B_n belongs to the algebra; B_ns are disjoint; and A which is union of A_ns is also equal to union of B_ns. Of course, this is disjoint; let me write that this is equal to this (Refer Slide Time: 43:16). It implies by countable additive property mu of A is equal to mu of this union B_ns and by countable additive property, that is summation n equal to 1 to infinity mu of B_ns; that is by countable additive property.

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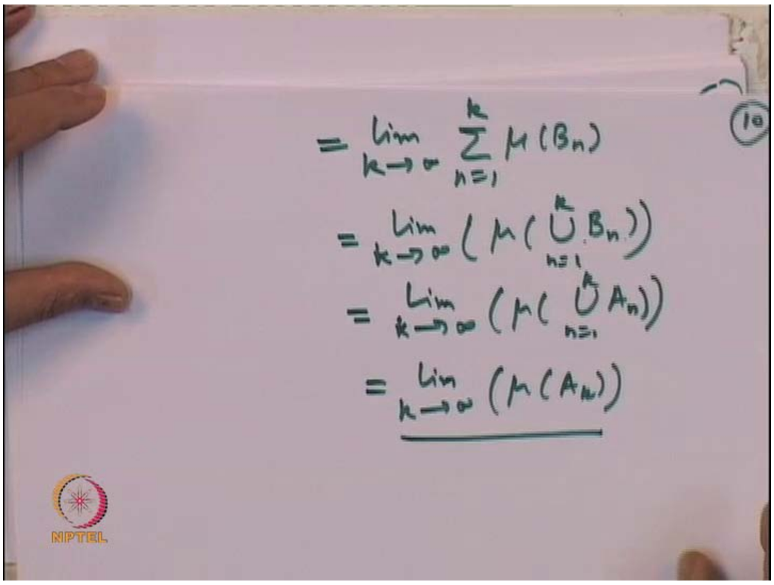

But we do not want B_ns; we want something in terms of A_ns; here is an observation. This summation I can write as limit k going to infinity of the partial sums so n equal to 1 to k of mu of B_ns but B_ns are disjoint; this is same as limit k going to infinity of mu of union B_n, n equal to 1 to k because B_ns are disjoint; by finite additive property, this must be true.

If you note, once again, mu is given to be countable additive and hence it is finite additive; by finite additive property, this is true. Now, the observation is that the union of B_ns n equal to 1 to k is same as the union of As. This is the same as k going to infinity mu of union A_n, n equal to 1 to k but note we have not used anywhere the fact that A_ns are increasing. Since A_ns are increasing, what is this union? This union is precisely mu of the largest set; that is A_k; so, that is mu of A_k (Refer Slide Time: 45:21).

We have shown mu of A is limit of mu of A_ks going to infinity whenever A_n is a sequence which is increasing (whenever A_ns are increasing) and A is equal to union (Refer Slide Time: 45:40). We have proved one way – countable additivity implies the required property. Let us look at the converse (Refer Slide Time: 45:47).

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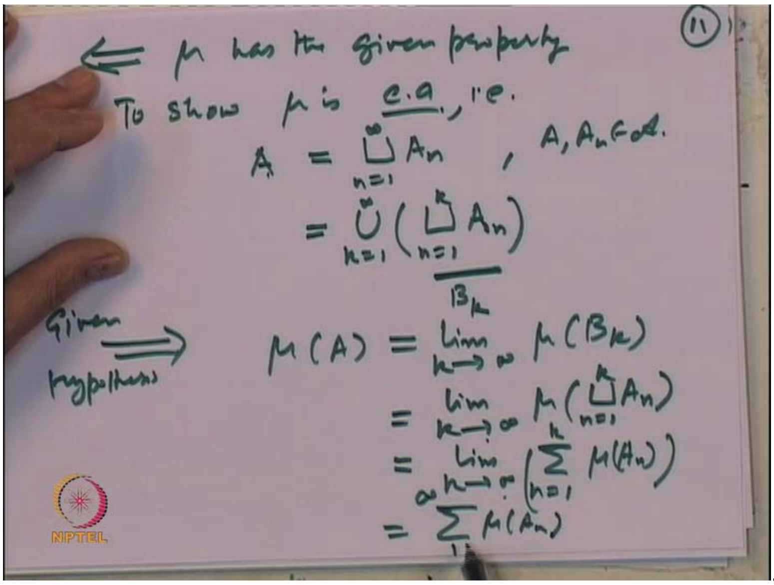

Conversely, let us assume mu has the given property. What is the given property? The given property says whenever a set A is written as union of A_ns and A_ns are increasing, then mu of A is equal to mu of the union. We want to show that mu is countably additive; that is, let us take a set A which is disjoint union of sets A_n, n equal to 1 to infinity where A and all A_ns are in the algebra.

We have to show that mu of A is equal to summation mu of A_ns but this I can write it as union over k 1 to infinity union A_n, n equal to 1 to k. Instead of taking n equal to 1 to infinity, take union of sets A_1, A_2 up to A_k and then take the union over k; both will be same. But the advantage of this way is that if we call this as B_k, then B_k is a set in the algebra because it is a finite union of sets in the algebra; B_k is increasing because we are taking union of more and more sets and their union is equal to A.

By the given hypothesis, mu of A is equal to limit k going to infinity mu of B_k. Now, let us go back to represent B_k in terms of As; that is, limit k going to infinity mu of union A_n, n equal to 1 to k. We use the fact that mu is finitely additive; this is limit k going to infinity summation n equal to 1 to k of mu of A_ns, which is same as sigma 1 to infinity of mu of A_ns. That says whenever A is a disjoint union of countable disjoint union of sets in the algebra, mu of A is sigma mu of A_ns; that is the countable additive property of the set function.

We have proved the theorem completely, namely, if A is an algebra of subsets of a set x and mu is finitely additive with that property, then mu is countably additive if and only if mu has the property that mu of A is the limit of mu of A_ns whenever A_n is increasing and A_n is equal to union of the sets (Refer Slide Time: 48:59). This is characterizing countable additivity in terms of limits of increasing sequence of sets. This property says that mu is continuous from below at the point A; so countable additivity for a finitely additive set function is the same as saying they are continuous from below at the point A; from below because A is union of these sets. There is a corresponding result for sequences which are decreasing.

Let us state that result and prove it also (Refer Slide Time: 49:41). If A is an algebra of subsets of a set X and mu is finitely additive so that conditions are same as ((.)).We want the additional condition that mu of the whole space is finite; this is the additional condition put to state the result, namely mu of the whole space is finite. It says mu is countably additive if and only if the following holds: for any set A in A whenever mu of A is equal to limit n going to infinity mu of A_n and whenever A_ns are decreasing; $A_{n \text{ plus } 1}$ is subset of A_n and A is the intersection. Countable additivity is equal to saying for every set A in the algebra, if A is intersection of a decreasing sequence of sets A_ns, then mu of A must be equal to limit of A_ns.

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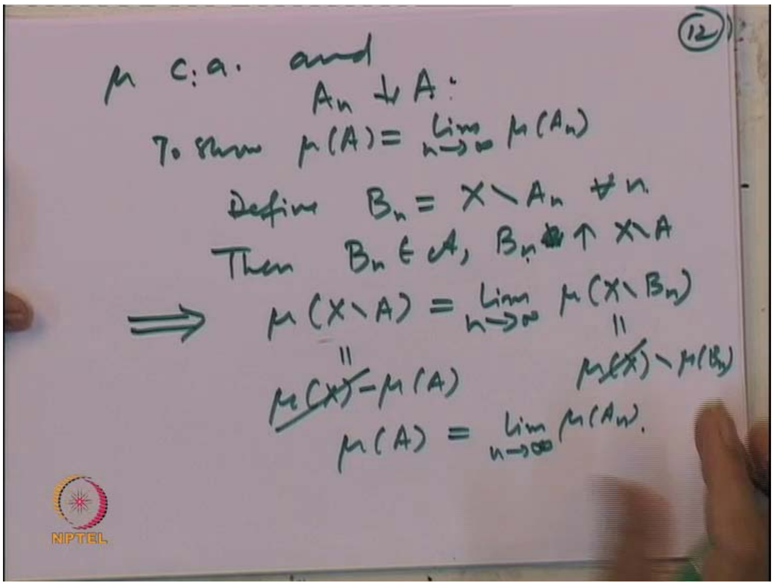

The proof of this uses the earlier theorem. Let us assume mu is countably additive and A_ns decrease to A, all in the algebra A. We want to show that mu of A is equal to mu of A_ns. We have to show mu of A is limit n going to infinity mu of A_n. We know something about increasing sequences. From decreasing, we want to manufacture an increasing sequence; that is done via complements. So, define B_n to be equal to X minus A_n for every n. Then, each B_n belongs to the algebra A; B_n is decreasing because A_ns are B_ns are sorry increasing as A_ns are decreasing. Where do they decrease? The B_ns increase to X minus A because A_ns are decreasing to A.

By the earlier theorem, countable additivity implies whenever a sequence is increasing mu of X minus A must be equal to limit n going to infinity mu of X minus B_n, but now we use the fact that mu of x is finite. This is same as mu of X minus mu of A (Refer Slide Time: 52:27). This thing is equal to mu of X minus mu of B_n and this is possible only because we have the fact that mu of the whole space is finite; so, everything is a finite quantity and we have already shown mu of the difference is equal to difference of mus provided the things are finite. So, this is equal to limit of this. Now, X cancels, negative sign, limit; so, mu of A is equal to limit mu of A_ns n going to infinity; countable additivity implies this.

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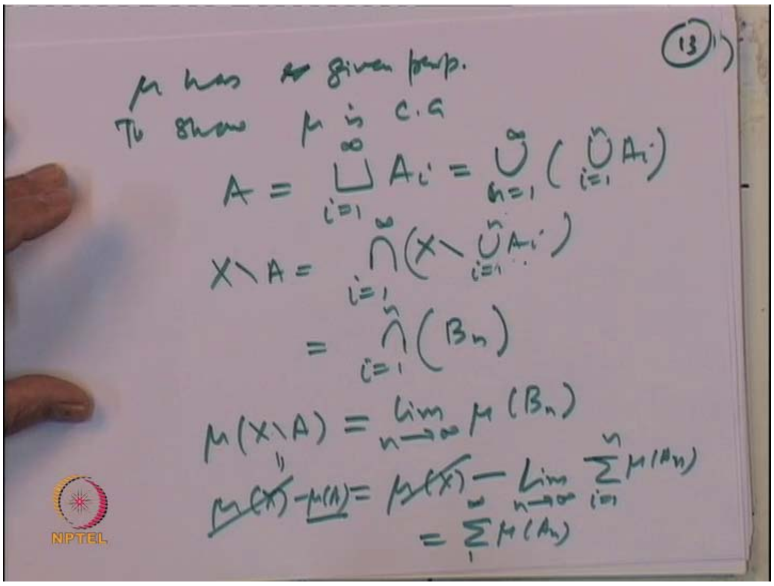

Let us assume this has that property that whenever A_ns increase… mu has the given property. We have to show mu is countably additive. Let us take A equal to a disjoint union A_is. What is X minus A? That is intersection of i equal to 1 to infinity X minus… let us cite this as union of n equal to 1 to infinity union of A_is, i equal to 1 to n. These are the ((.)). It is X minus union A_i, i equal to 1 to n; that is equal to intersection i equal to 1 to n; and this says nothing but, x minus so. Let us call this set as B_n.

Note that B_ns are decreasing and they are in the algebra because ((.)) the A_ns are union n, this will be increasing and so this will be decreasing; so, mu of x minus A by the given hypothesis is limit n going to infinity mu of B_ns. What is mu of B_n? mu of B n is X minus this; that is equal to mu of X minus limit n going to infinity mu of the union that is disjoint; so, summation i equal to 1 to n mu of A_ns. This is equal to mu of X minus mu of A because everything is finite. This cancels with this (Refer Slide Time: 55:13) and so mu of A is limit of this, which is equal to sigma 1 to infinity mu of A_ns. That proves countable additivity.

We have proved that mu is countably additive if and only if for a decreasing sequence of sets A_n A equal to this intersection mu of A is the limit under the condition mu of X is finite (Refer Slide Time: 55:39). This is important and this condition cannot be removed; this kind of thing is called continuity from above.

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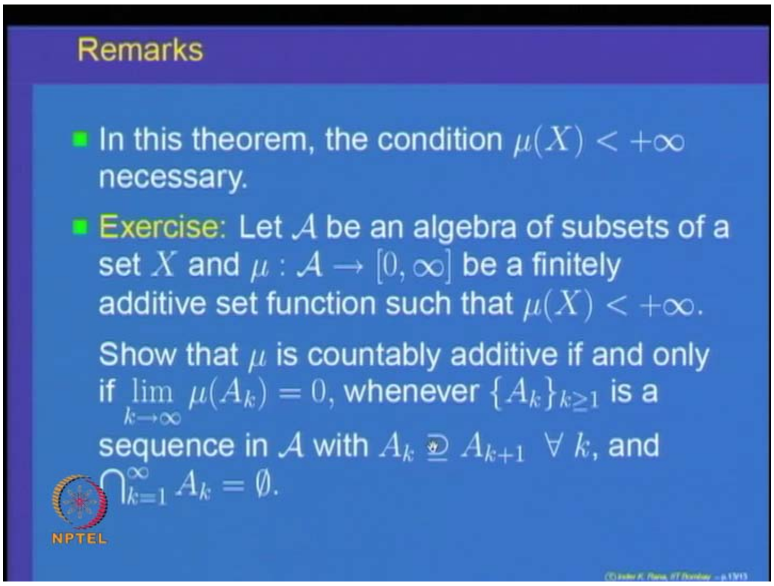

Here is a remark that the condition mu of X is finite is necessary in the second part and cannot be removed. We request you to construct an example; you can construct very easily an example on the real line with length function as the set function. Here is an exercise for you to do ((.)) such that mu is finite. Last part, we said A_n decreasing to A; that you can actually reduce a bit. It says whenever A_ns are decreasing to empty set; that is also equivalent to saying that mu is countably additive. These two parts we would like you to explore and understand and answer these questions. Thank you. Let us stop.