Measure and Integration.

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Module No. # 02

Lecture No. # 06

The Length Functions and its Properties

Welcome to lecture 6 on measure and integration. If you recall, in the previous lecture, we had started looking at various properties of the length function. In today's lecture, we will continue looking at the properties of the length function and then we will try to characterize some other countably additive set functions on the class of all intervals in the real line.

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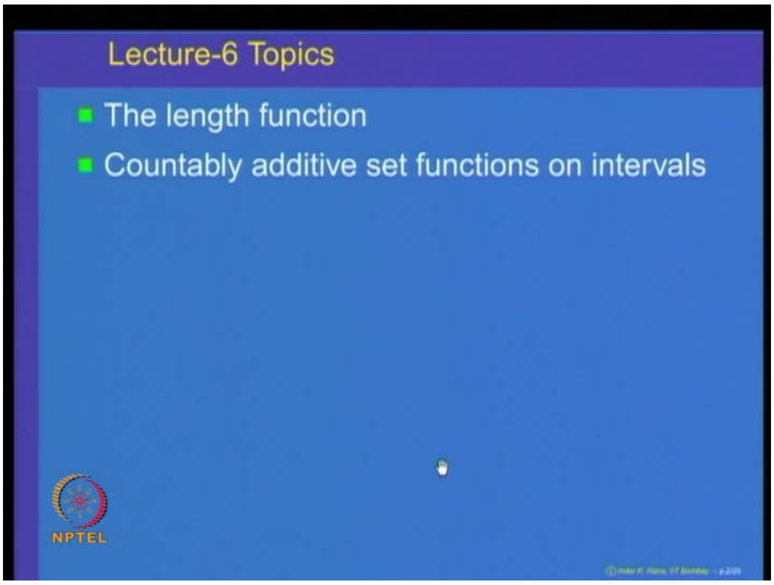

The first topic we will continue with is the length function and its properties and then countably addictive set functions on algebras.

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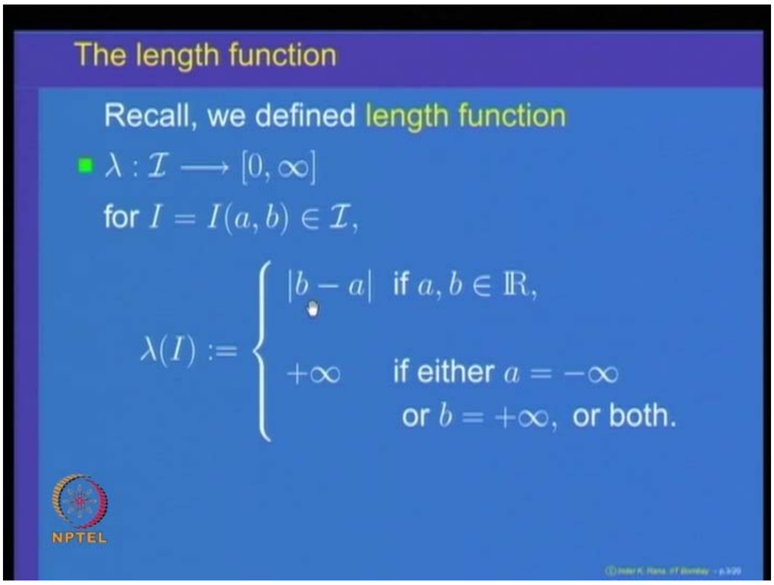

Let us just recall the properties of the length function that we have already proved. The length function was defined on the class of all intervals, that is, I and to every interval with left end point a and right end point b; it need not be left and right; normally, you will write the left end point a first and right end point b later. For an interval with end points a and b, we defined its length lambda of I as the absolute value of b minus a if a and b are real numbers; in case either of it is plus infinity or minus infinity, we defined the length to be infinite. For all finite intervals, the length is the usual concept of the difference between the values of the end points, that is, the absolute value of b minus a and plus infinity if the interval is infinite.

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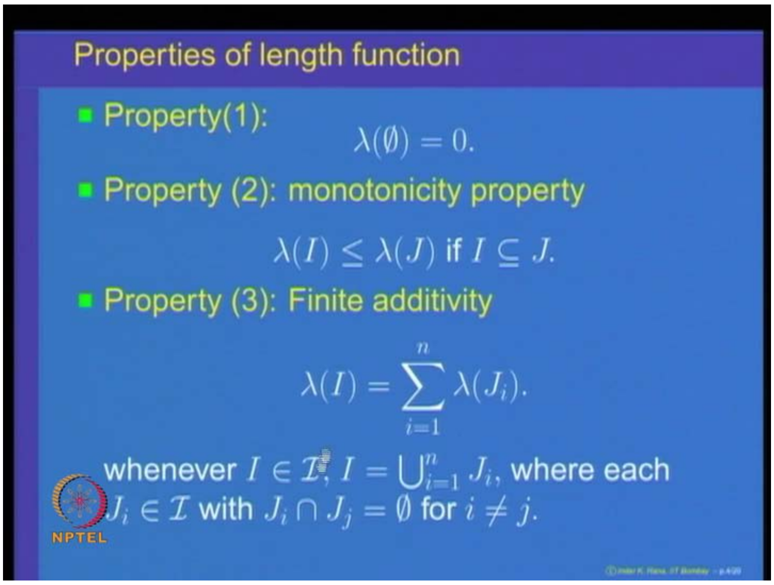

We proved the properties that the length of the empty set and that the interval is 0; then, we proved the monotone property of the length function, namely length of I is less than length of J if I is the interval which is inside the interval J; then, we proved the finite additivity property, namely if an interval I can be written as a finite disjoint union of intervals J_i, i equal to 1 to n, then the length of the interval I is the same as summation of lengths of the individual intervals. So, if I is a finite disjoint union of intervals, the length of I is summation over length of J_is.

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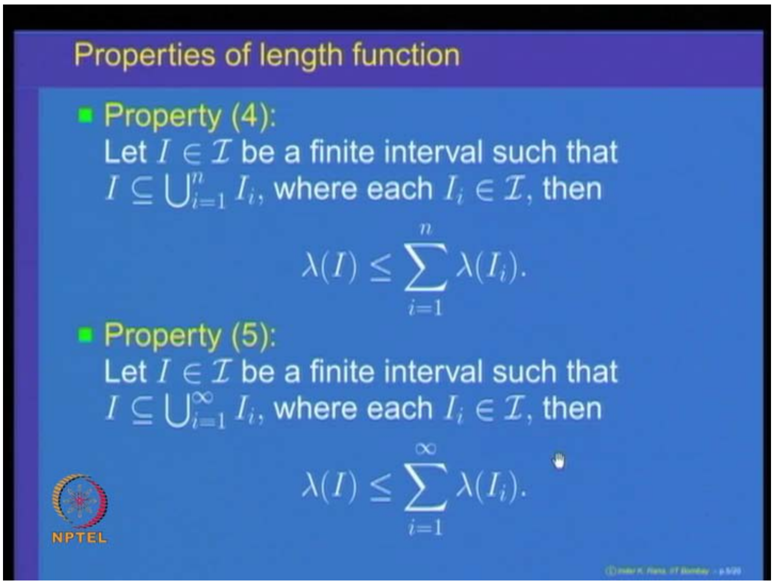

Then, we looked at a slight extension of this property, namely if I is a finite or an infinite interval (actually, we looked at that) and it is contained in a union of the intervals I_is, that is, I is covered by a finite union of intervals which need not be disjoint, then length of I is less than or equal to summation length of the intervals I_is 1 to n, the finite number of them.

I is covered by a finite union and then we extended this property to the arbitrary countable union. So, if I is an interval which is covered by a countable union of intervals I_i which need not be disjoint, then we proved that lambda of I is less than or equal to summation of length of the individual intervals. If you recall, this property used what is called the Heine–Borel property on real line.

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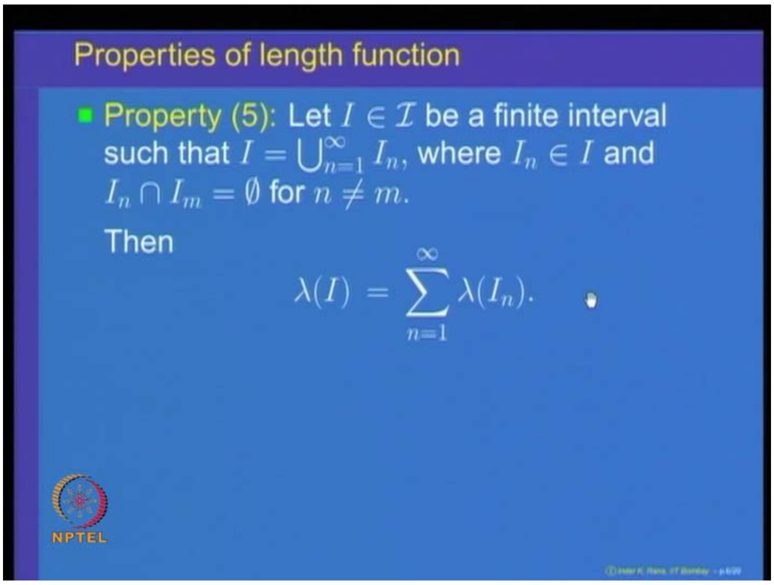

Let us continue our study. The next thing we want to prove is the following: if I is an interval which is any interval which is a finite interval, say, and it is a union of intervals I_ns, n equal to pairwise disjoint intervals, then the length of I is equal to sigma length of I_ns. This property in fact we had proved; let us prove it once again.

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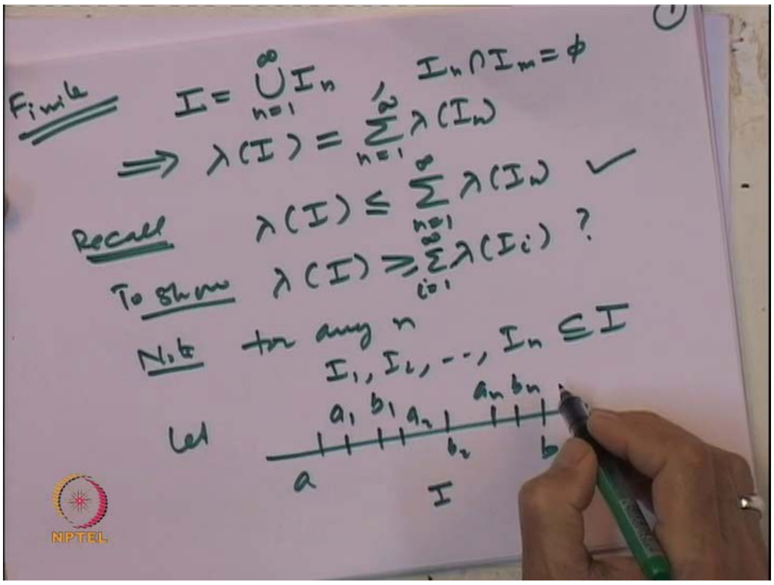

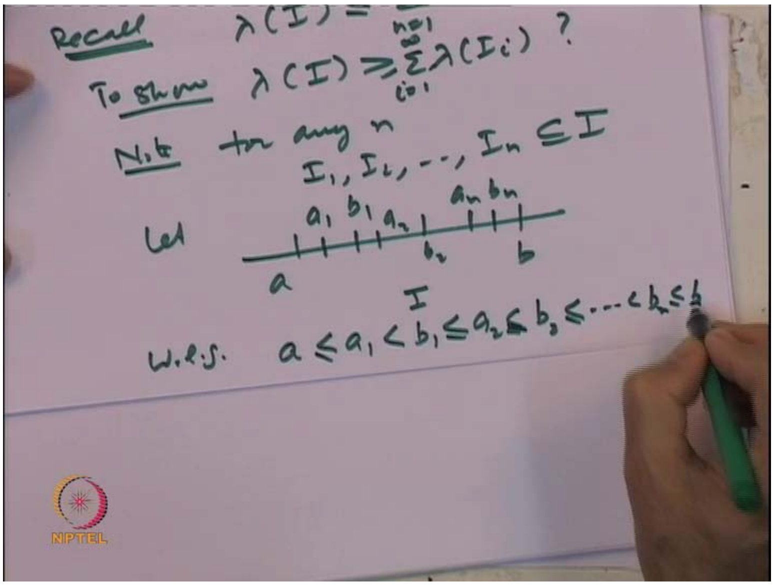

Let us look at this property. If I is an interval which is written as a union of intervals I_ns, n equal to 1 to infinity, I is finite (keep in mind we are keeping I as a finite interval) and I_n intersection I_m is equal to empty, then that implies the length of I is equal to sigma length of I_ns, 1 to infinity. Recall that we have already shown that length of I is less than or equal to sigma length of I_ns. That is because of the property that we just now proved: I is covered by a union of intervals and so length of I must be less than or equal to this (Refer Slide Time: 05:16).

To prove the other way round, we have to show that length of I is bigger than or equal to sigma I equal to 1 to infinity length of I_is; this is what is to be shown. Let us note that for any n, I_1 up to I_n are the intervals which are contained in I. I is finite; so, let us say this is the interval with end points a and b; that is I. I_1 is the interval which is inside a, b; so, it has end points say a_1 and b_1; I_2 has end points say a_2 and b_2; I_n has end points a_n and b_n.

But these being finite numbers and disjoint, we can arrange the intervals like a_1 here, b_1 here, maybe a_2 here, b_2 here and so on and a_n here and b_n here (Refer Slide Time: 06:28). What we are saying is we can assume without loss of generality and we can say that a is less than or equal to a_1 is less than b_1 less than or equal to a_2 less than a_2 less than b_2 and less than or equal to and so on and less than or equal to b_n less than or equal to b.

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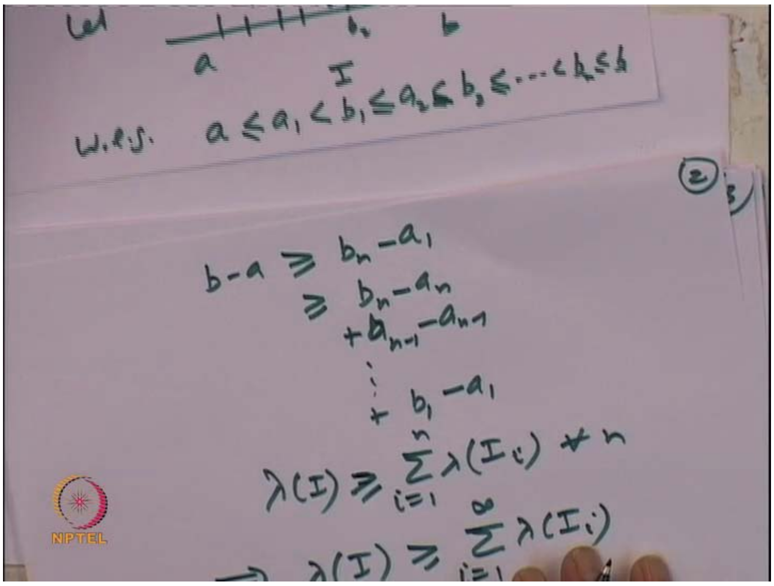

Once that property is true, we can ((.)) that b minus a is bigger than or equal to b_n minus a_1 which is bigger than or equal to… now, we can add and subtract consecutive terms; so, b_n minus a_n plus $b_{n \text{ minus } 1}$ minus $a_{n \text{ minus } 1}$ and so on plus b_1 minus a_1. So, add and subtract terms; subtract a bigger term and add a smaller term and so on. This is equal to sigma i equal to 1 to n length of I_is and this b minus a is length of I.

What we have gotten is true for every n. Sorry, this is bigger than or equal to (Refer Slide Time: 07:47). So, this is bigger than or equal to sigma i equal to 1 to infinity length of I_is, i equal to 1 to infinity. That proves the other way round inequality also (Refer Slide Time: 08:00). Hence, what we have shown is that the length function has the property that whenever a finite interval is written as a countable union of disjoint intervals, then the length of the interval I is equal to summation of the lengths of the individual intervals (Refer Slide Time: 08:02). We would like to extend this property not only to finite intervals but in fact to any interval. For that, we need a result.

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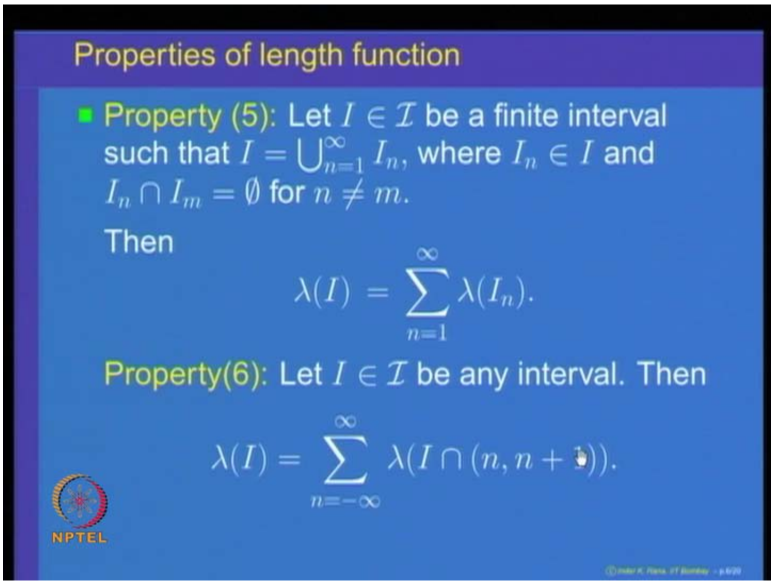

Suppose I is any interval, then we want to claim that the length of I is equal to summation lengths of I intersection the interval n to n plus 1; this is the property we would like to prove. In fact, one can have here the interval which is left-open n and right-closed n plus 1 because the end point is not going to matter. We want to prove that the length of an interval I is the same as the lengths of its pieces which lie inside the intervals n to n plus 1. To prove this property, let us observe the following.

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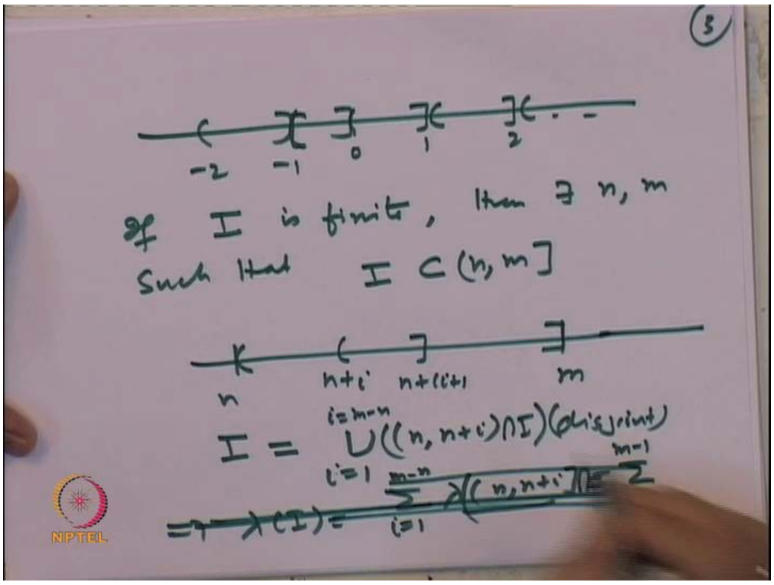

Let us observe; this is a real line; so, we can write it as the intervals, say, 0, 1, 2 and so on; on the other side, here is minus 1, minus 2 and so on. Let us take an interval I. If I is finite, if it is a finite interval, then obviously it will lie between some bounds. If finite, then there exists some n and m such that I is inside n and m. Here is some n and here is some m so that I is inside this (Refer Slide Time: 10:26).

Now, let us look at the pieces inside. This is n plus i and this is n plus i plus 1; so, intersection with this I. What we are saying is this I can be written as union of n to n plus i, i equal to, so 1 to up to up to n plus i equal to m, so that i equal to m minus n. Now, let us observe that these pieces are a disjoint union; this union is a disjoint union and a finite number of them; so, this will imply length of I is equal to summation i equal to 1 to m minus n lengths of n, n plus i.

This interval I does not intersect with any other interval which is bigger than m and which is less than n. So, for all those intervals, the intersection with I is empty. What I can write is this is the same of sigma of i equal to 1 to m minus n. I should have written the intersection with the interval I because the interval may start somewhere here; so, let us write this is intersection with the interval I (Refer Slide Time: 12:14). Let me write this again. This is intersection with I (Refer Slide Time: 12:21).

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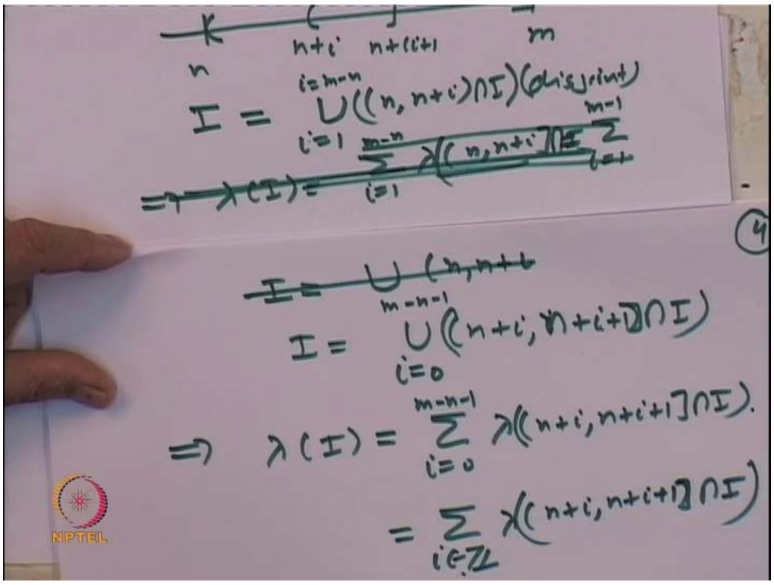

Let us write this again. The I is written. I can be written as a union n to n plus… Sorry, this is also not wrong ((.)). I can be written as union n plus i to n plus i plus 1 intersection I, i equal to starts with n, so 0 and goes up to when n plus i plus 1 is equal to, so that is m minus, we want n plus this is equal to m, so m minus n minus 1. That implies length of I and because this is a finite disjoint union, this is equal to summation i equal to 0 to m minus n minus 1 length of n plus i to n plus i plus 1 intersection I.

Now for the other parts ((.)) 0, I can write it as sigma over i belonging to integers length of n plus I, n plus i plus 1 intersection I over all integers i because the intersection with the other intervals is going to be empty and that is going to be 0. This proves that whenever I is finite, we are through; so, the I finite case is okay.

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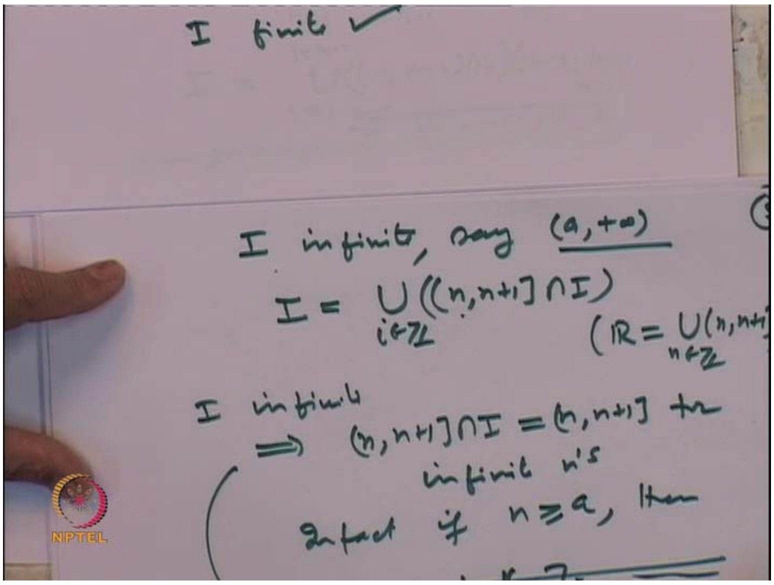

Now, let us prove the same thing when I is infinite. Let us take I as infinite. Then, I can write I is equal to union of same thing n to n plus 1 intersection I, i belonging to integers. This is keeping in mind that the real line is equal to union n to n plus 1, n belonging to integers; so, interval I is this intersection this (Refer Slide Time: 15:40). Now, because I is infinite, let us say it looks something like a to plus infinity.

I infinite implies that n to n plus 1 intersection I is equal to n to n plus 1 for infinite n_s. In fact, if n is bigger than or equal to a, then that is the interval (Refer Slide Time: 15:40); here is a and here is n; then, n to n plus 1 and so on are all going to be nonempty intersections, with the intersections being equal to n plus 1. This implies that sigma length of n to n plus 1 intersection I is going to be equal to plus infinity over all n belonging to Z; that is the same as length of I because I is an infinite interval. This proves the property that for any interval I, the length of the interval can be written as the length of its pieces I intersection n to n plus 1 (Refer Slide Time: 16:34).

This is an important property; it says length of any interval is a summation of lengths of its pieces. Note that length of each one of these pieces being a finite interval is a finite number. This says that any interval can be written as a countable disjoint union of intervals, each having finite length. This is an important property which is going to be called as sigma finiteness property of the real numbers of the length function.

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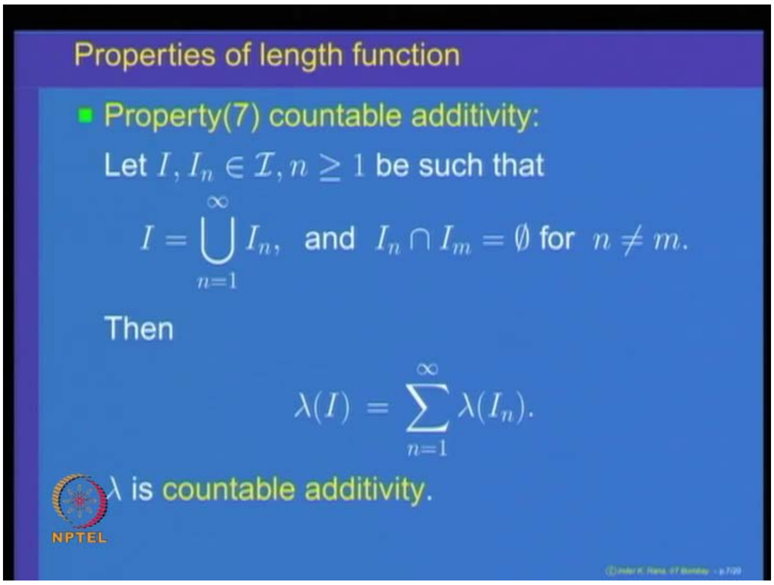

We are going to use this property to prove what is called countable additive property of the length function. That says that if an interval I is written as a countable disjoint union of intervals I_ns, then the length of the interval I is equal to summation length of I_ns.

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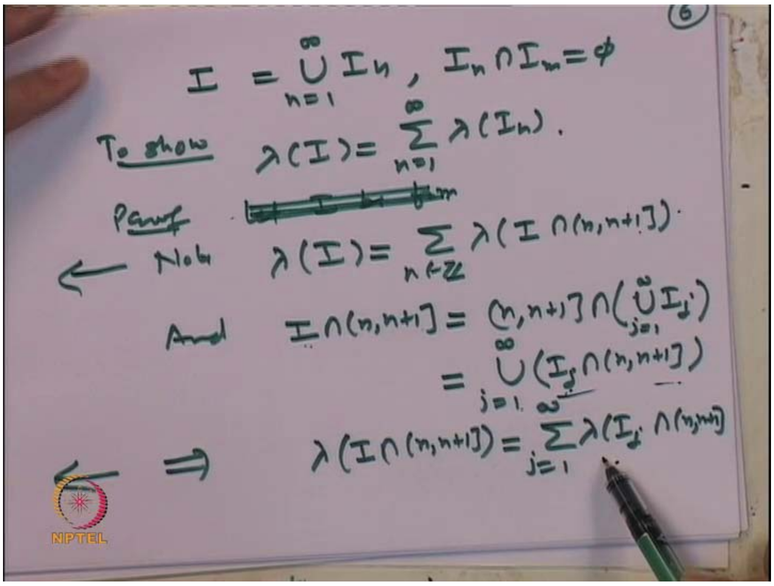

Let us start looking at the proof of this property. To prove this property, let us write I is an interval. I is an interval which is written as a union of I_ns, n equal to 1 to infinity where the intervals I_n intersection I_m is equal to empty. We have to show that length of I is equal to summation length of I_ns, n equal to 1 to infinity. Let us look at a proof of this.

Case I, let I be finite; actually, finite or infinite is not important; let us take a general case itself. Note that length of I is equal to summation length of I intersection n to n plus 1; this is because of the property that we have just now proved. Also note that I intersection n to n plus 1 can be written as shown here. This is a finite interval (Refer Slide Time: 19:07).

So, this is n to n plus 1 intersection… This interval I is a countable disjoint union; so, it is a union of I_j, j equal to 1 to infinity. We can write this as union j equal to 1 to infinity of I_j intersection n to n plus 1. Now, this is an equality for finite intervals only, because I intersection n to n plus 1 is a finite interval which is written as a; because I_js are disjoint, these intervals are disjoint and they are finite.

Thus, this implies, by the additive property for finite intervals which are disjoint, that lambda of I intersection n to n plus 1 is equal to summation j equal to 1 to infinity lambda of I_j intersection n to n plus 1. Here, we are using the fact that whenever an interval I is a finite interval which is a countable disjoint union of intervals 1 to infinity, then the length of I is equal to summation of length of this. Look at this equation here and look at this equation here (Refer Slide Show: 20:40). Length of I is equal to summation n over integers length of I intersection n to n plus 1 and that is computed to be equal to this (Refer Slide Show: 20:50).

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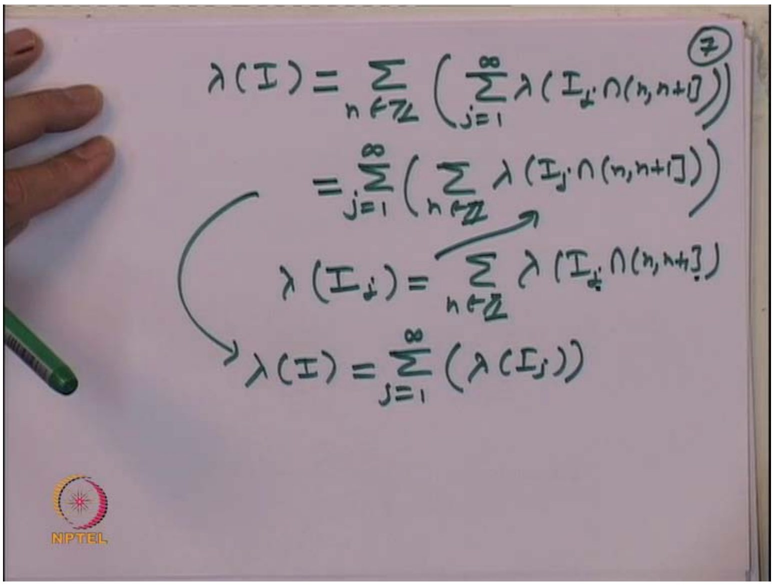

Combining these two, we get the property that length of I is equal to summation n belonging to Z of length of I intersection n to n plus 1 and that property we are going to put here; so, summation j equal to 1 to infinity lambda of I_j intersection n to n plus 1. Keep in mind that this is a double summation of the series and all of them are nonnegative; so, I can interchange the order of integration. I can write this as summation over j equal to 1 to infinity summation over n belonging to integers of length I_j intersection n to n plus 1.

Once again, I used the fact that length of interval I_j can be written as length of I_j intersected with n to n plus 1 summation over n belonging to Z; just now we have proved that fact – the sigma finiteness of the length function; any interval length I is summation length of its pieces inside n to n plus 1. This is here (Refer Slide Time: 22:22); this gives me the length of I is equal to summation j equal to 1 to infinity and this is length of I_j.

That proves the countable additive property of the length function that if an interval I is written as a countable disjoint union of intervals I_ns, then the length of I is equal to summation length of I_ns; whether I interval I is finite or infinite does not matter (Refer Slide Time: 23:04). The length function is countably additive; that is what we have proved; that is an important property of a length function.

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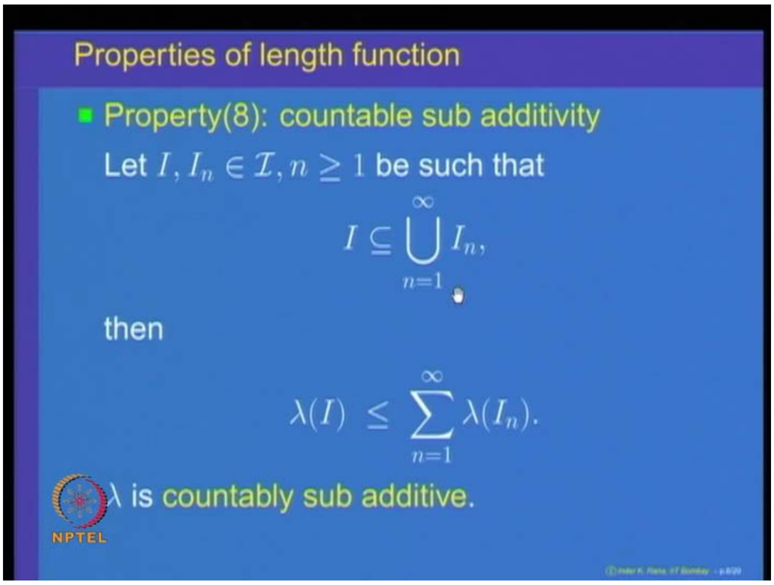

Let us extend this property to coverings which are not necessarily disjoint; that is called countable sub additivity. That says that if an interval I is such that I is contained in union of intervals I_ns, n equal to 1 to infinity which may not be disjoint, then obviously we should expect that the length of I is less than or equal to summation length of I_ns.

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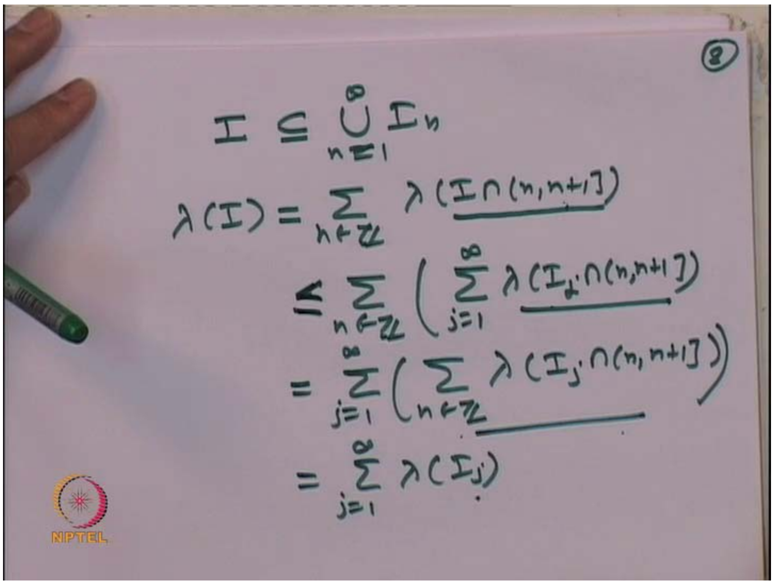

Its proof is very much similar to the earlier case. Let us just go through the proof again so that we understand how sigma finiteness of the length function is used. I is an interval which is contained in union of I_ns, n equal to 1 to infinity; these intervals I_ns may not be disjoint. Now, what we do is look at length of I; I can write this is equal to sigma n belonging to Z length of I intersection n to n plus 1; that is, sigma finiteness of the length function.

Now, the interval I intersection n to n plus 1 is inside because I can write this I as union over I_ns. Let me write this is less than or equal to sigma length over n; this thing is less than or equal to sigma length of I_j intersection n, n plus 1, j equal to 1 to infinity. Here, we have used the fact that this intersection this interval is covered by the union of these intervals. This is a finite interval (Refer Slide Time: 25:03). So, the length of this must be less than or equal to length of this.

Once again, it is a series of nonnegative numbers and I can interchange. So, this is equal to sigma j equal to 1 to infinity sigma n integers length of I_j intersection n to n plus 1. Once again, this is nothing but the length of the interval I_j by the sigma finiteness of the length function. Length of I is less than or equal to sigma length of I_js. This property is called countable subadditive property of the length function (Refer Slide Time: 25:52).

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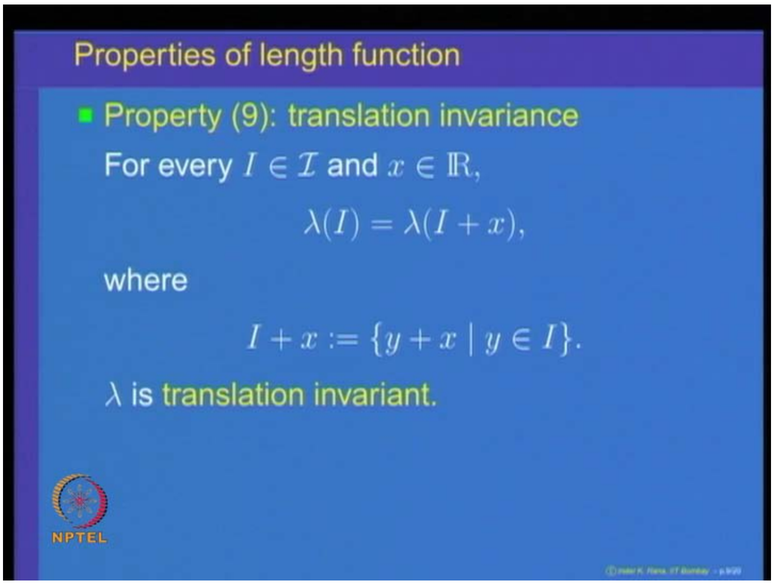

Here is a very important property of the length function which is called translation invariance. It says that if I take an interval I and translate it by some number x, then the length of it does not change. It says length of I is equal to length of I plus x when I take an interval I and translate. This is a translated set; just shift it or push it by a distance x. I plus x is all y plus x, y belonging to I. This property is obvious.

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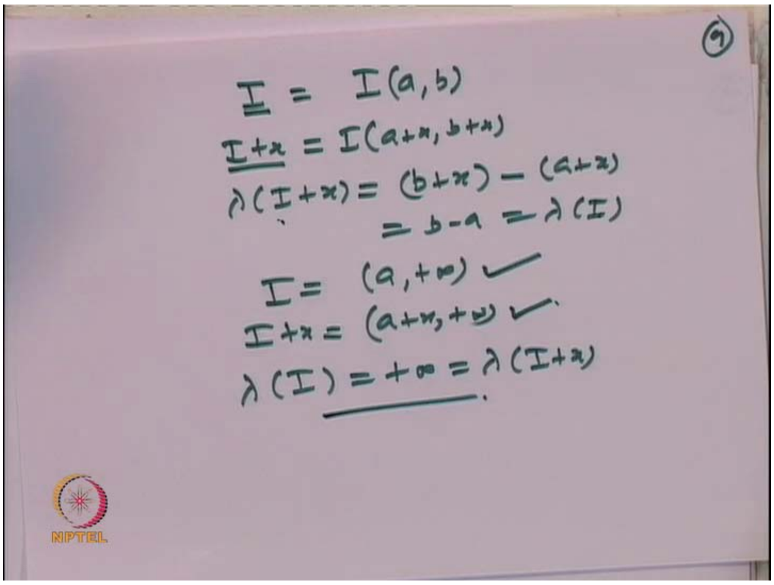

This property of translation invariance is quite obvious. Let us say it is an interval with left end point a and right end point b. Then, I plus x is the interval with the left end point a plus x and right end point b plus x. Length of I plus x is same as b plus x minus a plus x which is equal to b minus a which is equal to length of I. Length of I is same as length of I plus x; this is for finite.

The same proof ((.)) continues for infinite. For example, if I is equal to say a to infinity, then what is I plus x? I plus x is a plus x to plus infinity. In either case, length of I is equal to plus infinity which is the same as length of I plus x. We are basically observing that if I is an infinite interval, its translation remains an infinite interval; the values of both are equal to plus infinity. This is what is called the translation invariant property of length function (Refer Slide Time: 27:51).

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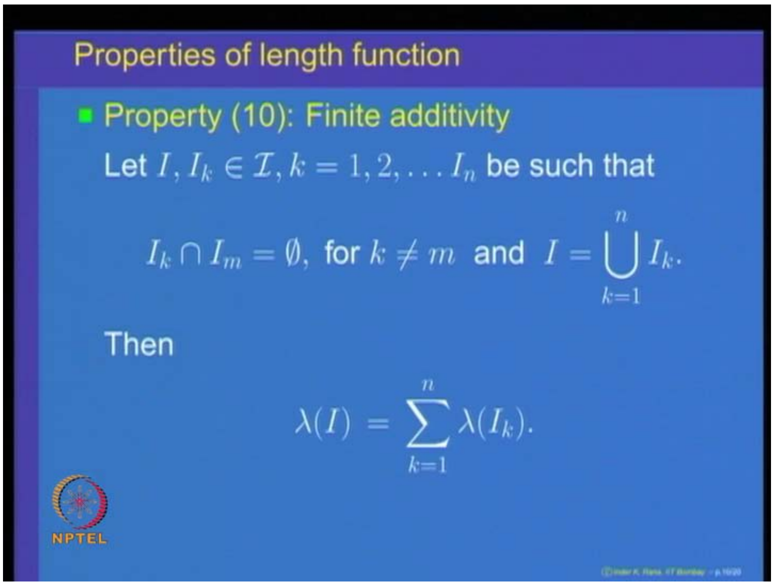

Finally, let us prove what is called the finite additivity property of the length function. We have used finite additive property of the length function for finite intervals and we proved countable additivity property for the length function. I just want to exhibit that the countable additivity implies finite additivity when we have the fact that the length of the empty set is equal to 0.

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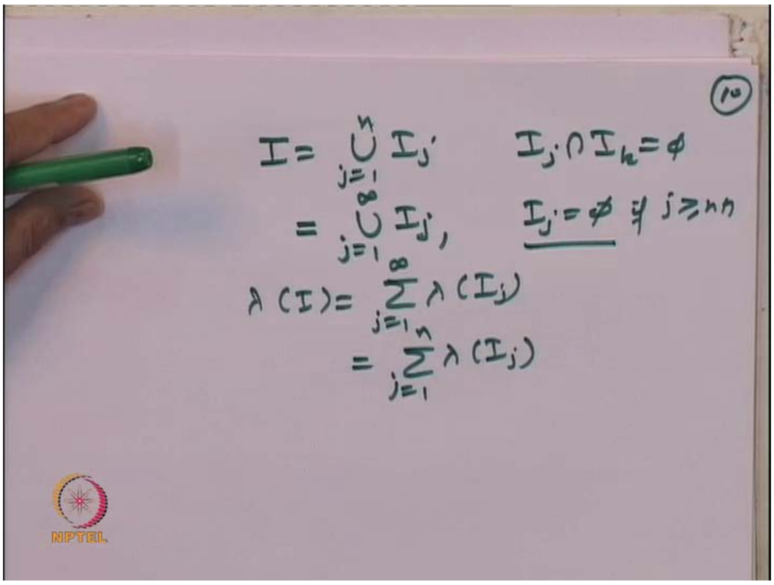

Basically, what we are going to say is if I is an interval which is the union of I_j, j equal to 1 to n where I_j intersection I_k is equal to empty, then I can also write it as union of j equal to 1 to infinity I_j where I can define I_j to be equal to empty set if j is bigger than n plus 1. From n plus 1 onwards, put everything equal to 0. Then, I is a countable disjoint union of intervals; so, length of I must be equal to summation length of I_j by countable additivity property.

That is the same as summation j equal to 1 to n length of I_j because from n plus onwards they are empty and so the length is equal to 0. Countable additivity implies finite additivity whenever length of the empty set I can put equal to 0 (Refer Slide Time: 29:27). Let us just recapitulate the various properties of the length function that we have proved, namely the length function is a set function defined on the class of all intervals in the real line with the properties that it is countably additive, countably subadditive, finitely additive, finitely subadditive, and translation invariant. The important property is that it is countably additive.

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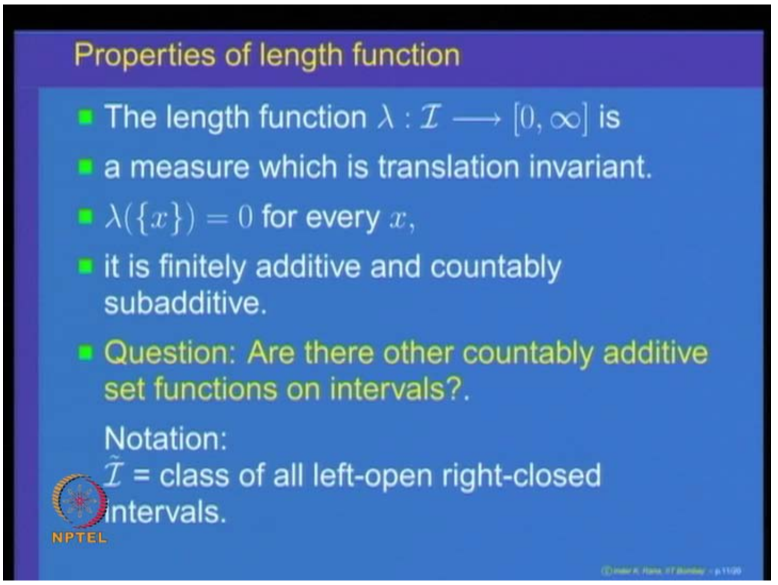

In view of this, the next question that arises is the following. The length function is a measure which is translation invariant because it is countably additive and length of the empty set is equal to 0. Also observe that the length of the singleton is equal to 0; this is also a property of the length function because the singleton set can be written as an open interval or a closed interval with the same end points and so the length will be equal to x minus x which is equal to 0.

It is finitely additive and countable subadditive – that we observed. Here is the question: are there other countably additive set function on the class of intervals? We would like to know. Length function which we just now proved is one such function which is countably additive set function on the class of all intervals. Are there other countable additive set functions on intervals?

To answer this question, let us make a notation. We will denote by I upper tilde; there is a wave kind of a sign; this symbol is called calligraphy I (Refer Slide Time: 31:25). The collection of all left-open right-closed intervals will be denoted by this symbol cal I, calligraphy I, with the upper tilde. This is a collection of all left-open right-closed intervals – intervals whose left end point is not included but right end point is included. Keep in mind that if it is infinite, then there is no right end point on the real line. This is a collection of all left-open right-closed intervals.

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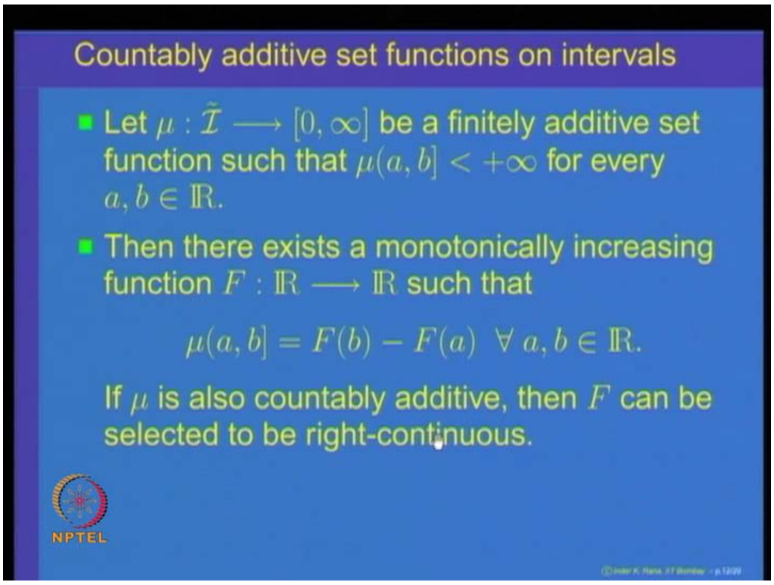

What we are going to prove is the following. Suppose we have got a set function mu on the class of all left-open right-closed intervals such that, let us say, it is finitely additive (this mu is given to be finitely additive) and also given the fact that mu for a finite interval is finite for every a and b. Then, we want to prove that this can be characterized by the distance of a monotonically increasing function F from R to R such that mu of the left-open right-closed interval a, b is given by F of b minus F of a for every a belonging to R.

What we want to show is that if mu is given to be a finitely additive set function on the class of all left-open right-closed intervals with the property that mu of finite interval is finite, then we want to show that this must be given by a monotonically increasing function F with a relation that mu of a, b is that nothing but F of b minus F of a. Keep in mind: if mu is the length function, then the obvious choice for F is the identity function y equal to x; then, it will be equal to b minus a.

In some sense, we are generalizing the length function; that means if mu is any finitely additive set function, then it must be given by this (Refer Slide Time: 33:47). To prove this, let us observe that mu of a, b is given by F of b minus F of a. That itself tells us what should be the definition of the function F. For example, if I fix here a point a, if a is fixed, that means F of a is fixed and then I can calculate F of b as equal to mu of a, b plus F of a.

This relation itself gives me a hint on how I should define the function F. Let us fix an a and the most convenient point is to fix a to be the origin. We will also show later on that if mu is countably additive, then this function can be chosen to be not only monotonically increasing but the right-continuous function.

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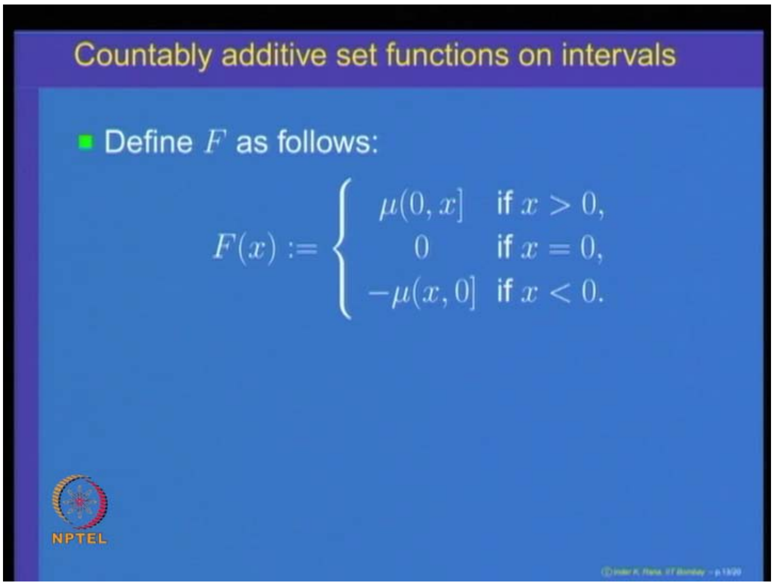

Let us define our function F from the real line. F at any point x in the real line is defined as mu of the open interval 0, closed at x – left-open, right-closed interval 0, x; size or mu of that if x is bigger than 0; it is defined as 0 if x is equal to 0 because that will mean mu of the empty set is equal to 0 – it is countably additive; mu of F of x to be equal to minus of mu of x, 0 if x is less than 0 because if F of x is less than 0, then this point x is going to be on the left side of 0; so, it is left-open right-closed interval. With this definition of mu, we want to claim that this function has the required properties. Let us check these properties of this function.

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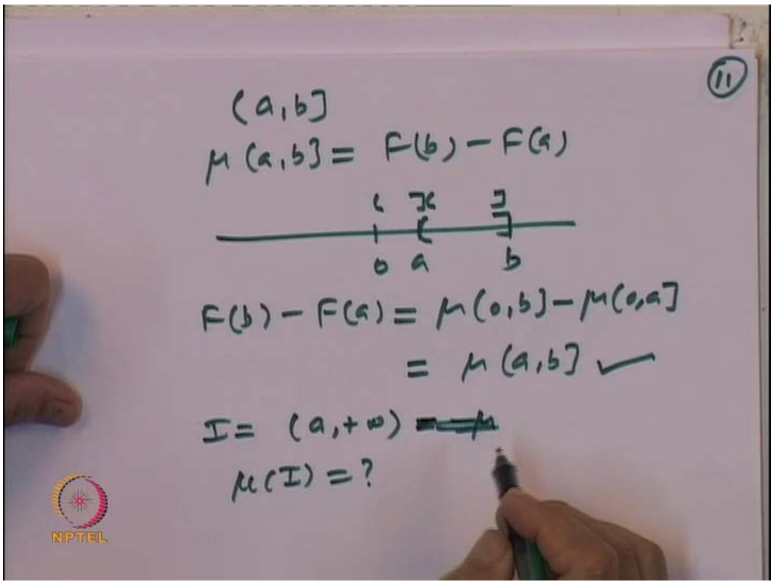

First, this satisfies the required equation. We want to check that for an interval a, b, mu of a to b is equal to F of b minus F of a. To check that, let us take this as the point 0. If a and b are both finite numbers, real numbers, let us say this is the interval a to b, then F of b minus F of a is equal to mu of 0, b minus mu of 0 to a. Let us observe that mu is given to be finitely additive. This is the same as mu of a to b because I can write 0 to b as union of 0 to a and union of 0 to b – disjoint intervals, disjoint pieces. Using that fact, this is just mu of a, b; that proves it (Refer Slide Time: 37:06).

Similarly if it is infinite, supposing the interval I is a to plus infinity, then I can write this as equal to… we have to… this is equal to mu of… it is finite. Let us observe one thing: if the interval I is infinite, then what is mu of I equal to? We have not defined what is the relation between this and the function. Keep in mind that we have defined F of x is equal to mu of 0 to F if x is equal to finite and this is equal to this if this is finite (Refer Slide Time: 38:06).

We want to check that this satisfies the required property; if I is equal to a to plus infinity, then I want to check that mu of I is a equal to F of... Sorry, this is only for finite intervals. I am sorry; we wanted to check only for finite intervals that this property is true. Whenever an interval I is a finite interval, then we know this is finite and this property is true (Refer Slide Time: 38:44).

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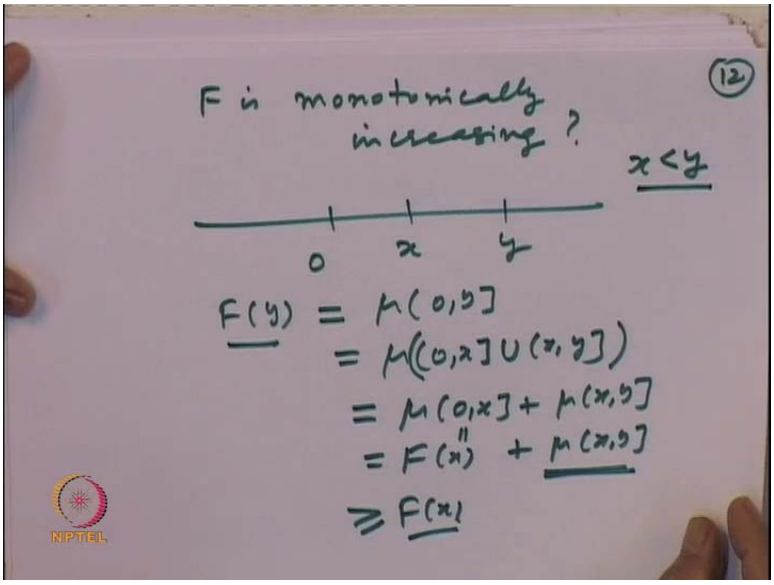

Now, let us check the next property, namely that F is monotonically increasing. Let us check this property. Let us take two points; let us take the case here is 0, here is x and here is y. We have got x less than y and we want to check F of y; we want to calculate F of y. What is F of y? By definition, it is mu of 0 to y and that I can write as mu of 0 to x using finite additive property I can write 0 to x union of x to y mu of that and that by finite additive property is mu of 0 to x plus mu of x to y.

Now, this is equal to F of x by definition (because this is equal to F of x) plus mu of x to y. This is some nonnegative quantity (Refer Slide Time: 39:56). So, we can write that this is bigger than or equal to F of x. If x is less than y, then F of x is less than F of y. That proves it is monotonically increasing in the case when both x and y are on the right side of it.

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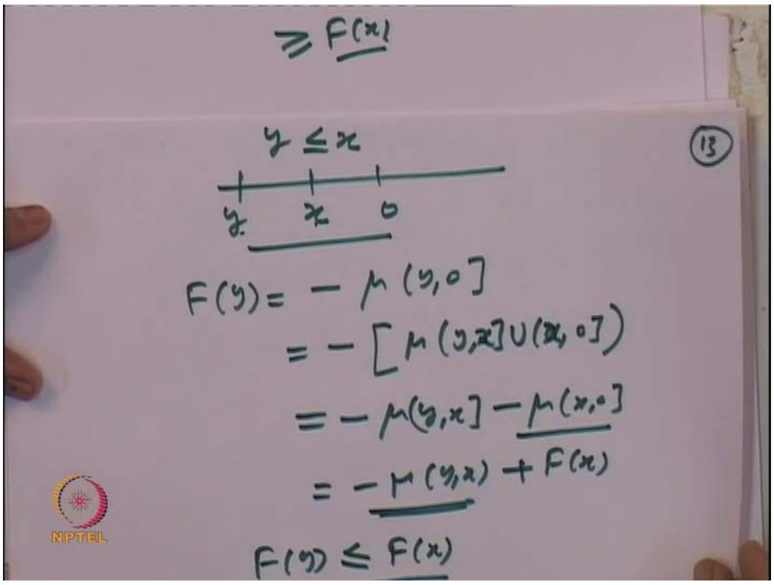

The same proof will work if it they are both on the left side of 0. Here is y and here is x. We have got y less than or equal to x. We want to look at what is F of y; that is equal to minus mu of y to 0 by definition; that is equal to minus y to 0. This I can write as mu of y to x union x to 0. That again by additive property is minus mu of y to x minus mu of x to 0; this is equal to minus mu of y, x plus F of x because F of x is defined as minus of this (Refer Slide Time: 41:14). This is a negative quantity and that means F of y is less than or equal to F of x. Once again, that property is true.

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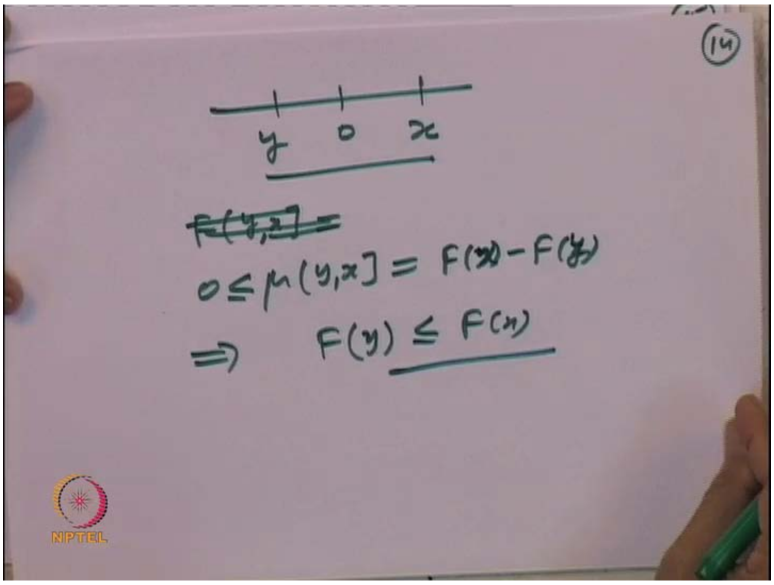

The third case: let us take it is 0 here, y here and x here. In that case, what is F of y? Let us look at y to x. Sorry. What is mu of y to x? From this, I can write it as equal to F of y minus F of x by definition. This is bigger than or equal to 0. Sorry, this is F of x minus F of y (Refer Slide Time: 42:16). That means F of y is less than or equal to F of x. Once again, in all possible cases, we have checked that F as defined above is a monotonically increasing function.

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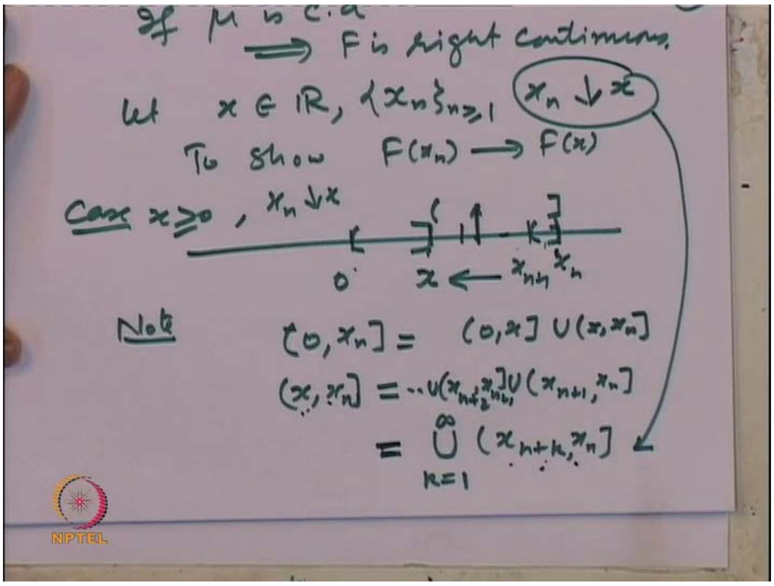

We want to check now that if mu is countably additive, then this implies F is right continuous – continuous from the right. The finite additivity property gave us that F is monotonically increasing and we are claiming that if mu is countably additive, then F must be right continuous. What is right continuity? Let us take a point x; let x belong to R; let us take a sequence x_n such that x_n decreases to a point x. We have to show that F of x_n converges to F of x; that is what we have to show – F of x_n converges to F of x.

Let us try to look at a picture. This is 0. Let us look at the case when x is bigger than 0. This is the case when x is bigger than or even equal to 0. Here is the sequence x_n decreasing to x. That means here is x_n, here is $x_{n \text{ plus } 1}$ and so on and that is decreasing to x. Let us observe in this case; look at the interval which is 0 to x_n. This interval 0 to x_n I can split as interval 0 to x left-open right-closed and then x to union of x to x_n (Refer Slide Time: 44:38).

Now, I want to split this portion also. So, the interval x to x_n, this interval (Refer Slide Time: 44:53) is same as… Let us start; this part is $x_{n \text{ plus } 1}$ comma x_n; the next part will be x_n plus 2 comma $x_{n \text{ plus } 1}$ and so on. I want to claim that this is union of $x_{n \text{ plus } k}$ to x_n, k equal to 1 to infinity. That is because if I take any point between x to x_n, take any point here (Refer Slide Time: 45:34), this x_n converges; so, it is going to cross over this point, any point inside the interval x to x_n. That means that it is going to fall inside one of these intervals and all these intervals are subsets of it.

It is quite easy to check that x to x_n, this interval, is a union of intervals $x_{n \text{ plus } k}$ to x_n, k equal to 1 to infinity. Here, we are using the fact that x_n decreases to x; this fact is being used here (Refer Slide Time: 46:11). Now, realize that x to x_n is a countable disjoint union of these intervals and mu is given to be countably additive; so, what we have is the following property.

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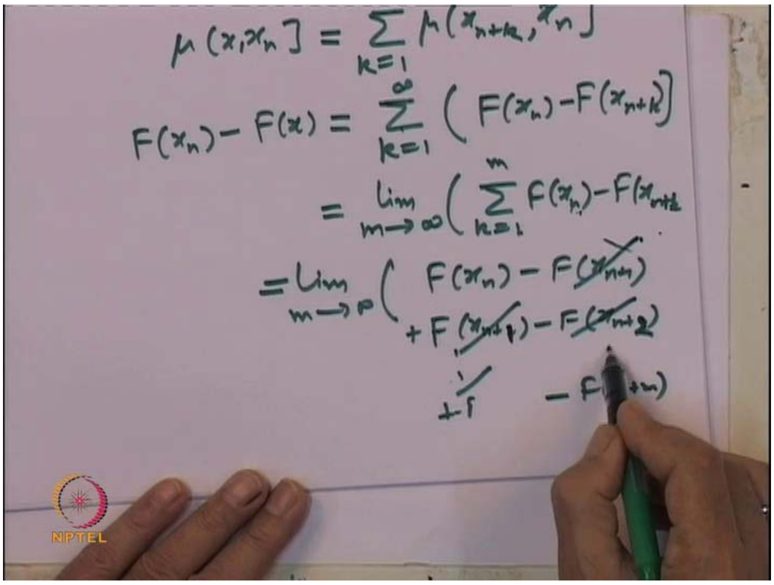

That says mu of x to x_n is equal to summation k equal to 1 to infinity of mu $x_{n \text{ plus } k}$ comma x_n. Let us write this in terms of F. That means F of x_n minus F of x is equal to summation k equal to 1 to infinity of F of x_n minus F of $x_{n \text{ plus } k}$. This is a series of nonnegative terms and so I can write as limit of the partial sums. Let us write limit of m going to infinity of summation k equal to 1 to m F of x_n minus F of $x_{n \text{ plus } k}$. This is equal to limit m going to infinity.

Now, this is a partial sum (Refer Slide Time: 47:35). What does that mean? This is k equal to 1; n to n plus k; so, n to n plus 1 ((.)). This is a sum where terms will cancel out; let me just write it. This is nothing but F of x_n minus F of $x_{n \text{ plus } 1}$. The next term will be plus F of $x_{n \text{ plus } 2}$ minus F of $x_{n \text{ plus } 3}$ and so on; it will be going up to m; so, k is equal to m; so, minus F of n plus m (Refer Slide Time: 48:26).

That means what? We are starting with k equal to 1 n to n plus 1; so, n plus 2 k plus 1. What are the terms which are cancelling? This says n plus 1, n plus 2, n n to so n plus. Sorry, it should be n plus 1 minus n plus 2 (Refer Slide Time: 49:06). When the next term comes, n plus 1, n plus 2 and so on, these terms cancel. What we will be left with is limit m going to infinity of F of x_n minus F of $x_{n \text{ plus } m}$. Sorry, this is x (Refer Slide Time: 49:27). That means what we get is the following. This is independent of m (Refer Slide Time: 49:35).

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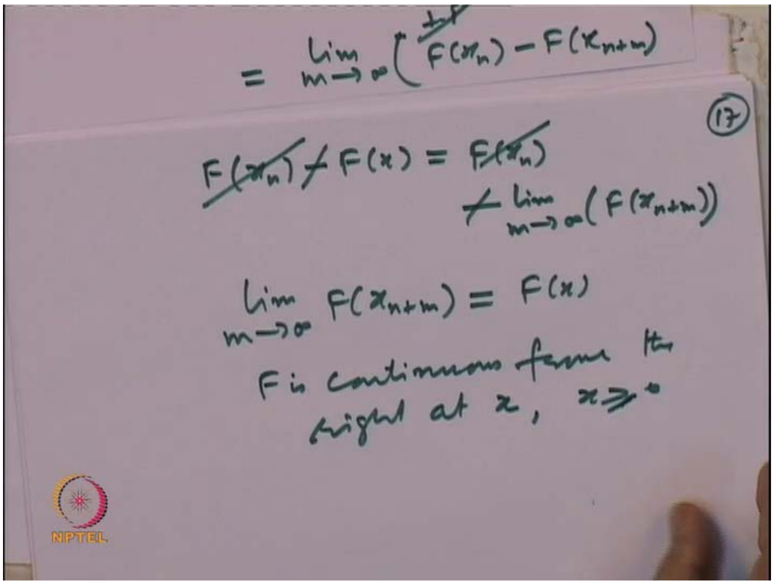

What we get is left-hand side was F of x_n minus F of x is equal to F of x_n minus limit m going to infinity of F of $x_{n \text{ plus } m}$. These cancel out; the negative sign cancels out (Refer Slide Time: 49:57). We get limit m going to infinity F of $x_{n \text{ plus } m}$ is equal to F of x. That is the same as saying that F is continuous from the right at x; that we have proved when x is bigger than or equal to 0; when x is bigger than or equal to 0, this is right continuous. The other case is when x is negative and still whenever x_n converges to 0 or x_n decreases to 0, we will show that F of x_n converges to F of x, showing that F is right continuous at x when x is negative also; we will do this in the next lecture. Thank you.