**Measure and Integration**

**Prof. Inder K. Rana**

**Department of Mathematics**

**Indian Institute of Technology, Bombay**

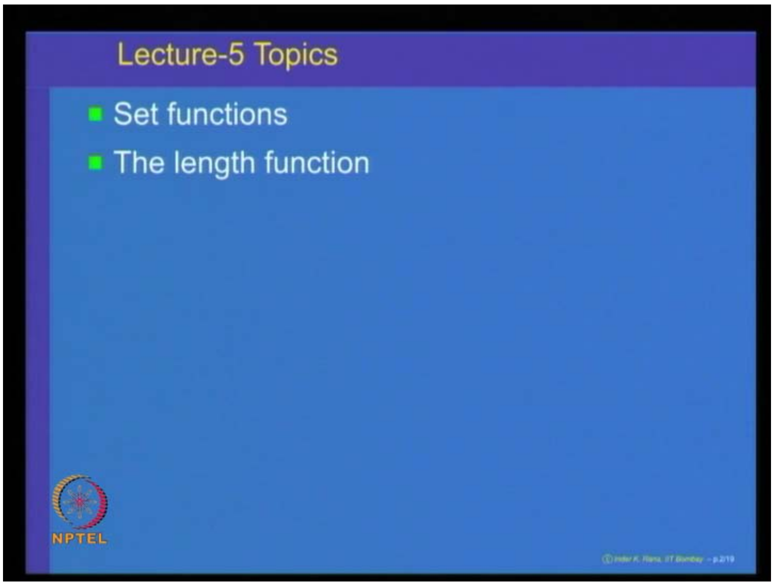
**Module No. # 02**

**Lecture No. # 05**

**Set Functions**

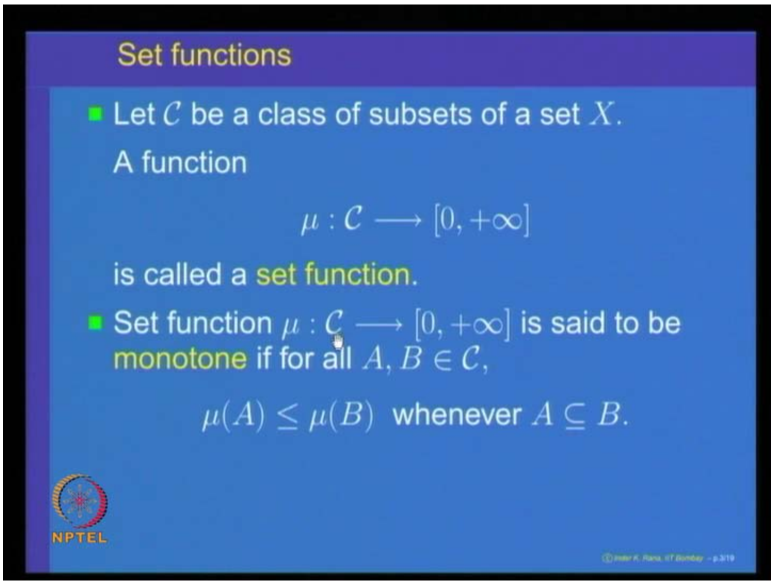
Welcome to lecture 5 on measure and integration. If you recall, in the previous lectures we have been looking at the various classes of subsets of a set X with various properties. We looked at what is an algebra and what is a sigma algebra and a monotone class. Today, we will start looking at functions defined on classes of subsets of a set X.

(Refer Slide Time: 00:48)




We will first look at what are called set functions and then we will look at a very important example of a step function, namely the length function. Let us start defining what are called set functions.

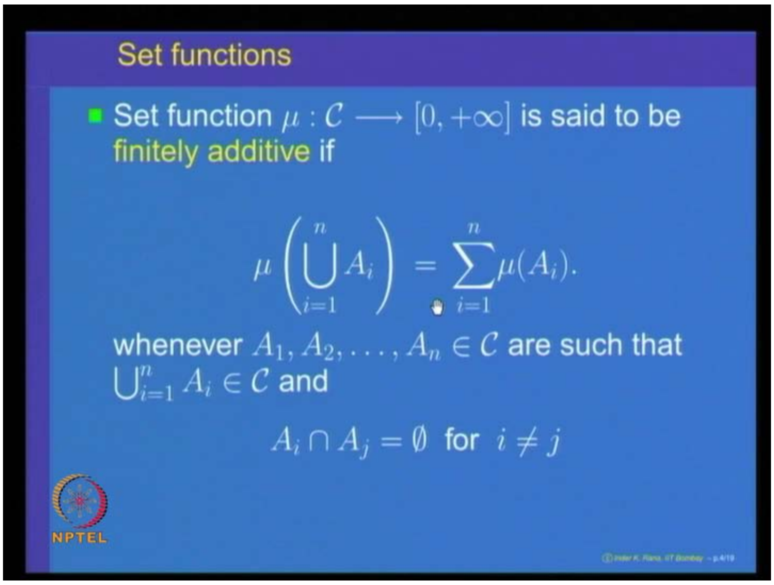
(Refer Slide Time: 00:59)




Let us start with C – a class of subsets of a set X. Any function mu (this is a Greek symbol called mu) is defined on the class of subsets C of a set X and taking nonnegative extended real-valued functions. This interval 0 to plus infinity, both included, denotes the set of all nonnegative extended real numbers. A function mu defined on this collection C of subsets of a set X taking values in nonnegative extended real numbers is going to be called a set function. It is a function whose domain is a collection of sets; that is why it is called a set function.

Next, we will be looking at some special properties; we will be analyzing such functions. Let us define a set function mu, of course, where C is a collection of subsets of a set X and 0 to plus infinity ((.)) the nonnegative extended real numbers. A set function mu is set to be monotone if it has the following property: for any two sets A and B in C, mu of A is less than or equal to mu of B whenever A is a subset of B. It is a monotone property that whenever A is a subset of B and both are in the collection C, we want that mu of A should be less than or equal to mu of B; this is called the monotone property.

(Refer Slide Time: 02:44)



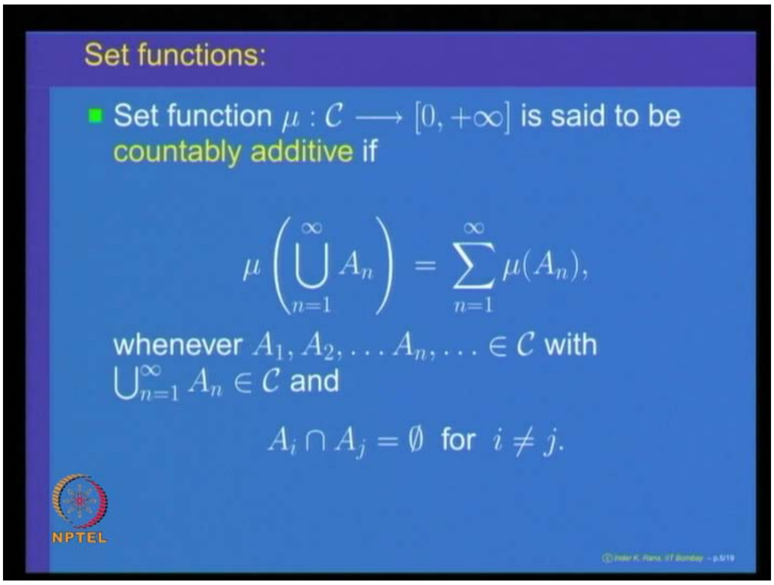

Next, we look at what is called finite additivity property of a set function mu. A set function mu is set to be finitely additive, I am emphasizing the point finitely and additive, if it has the following properties: mu of union of sets $A_i$, i equal to 1 to n, given any finite collection of sets $A_1$, $A_2$ up to $A_n$ in C, mu of the union of the sets is equal to mu of $A_i$s.

Of course, this will be whenever $A_1$, $A_2$, up to $A_n$ … This is a finite collection of sets in C such that their union also belongs to C, for otherwise this number on the left-hand side of this equation will not be defined. Further, the sets are pairwise disjoint; $A_i$ intersection $A_j$ is empty for i not equal to j. Once again, let us see what is finite additivity. Finite additivity means for any finite collection of sets in C, $A_1$, $A_2$, up to $A_n$ in C, such that their union is also an element in C. These sets are pairwise disjoint; for any such finite collection of sets, we want that mu of the union is equal to summation of mu of the individual $A_i$s.

Intuitively, keep in mind that mu in some sense is denoting the size of a set A and so we are saying mu of the union is equal to sum of the individual sizes whenever the sets $A_i$s are disjoint; we are requiring it for any finite collection i equal to 1 to n. If $A_1$, $A_2$, up to $A_n$ is any finite collection of sets in C which are pairwise disjoint such that their union is an element in C, mu of the union is equal to summation of mu of the individual $A_i$s.

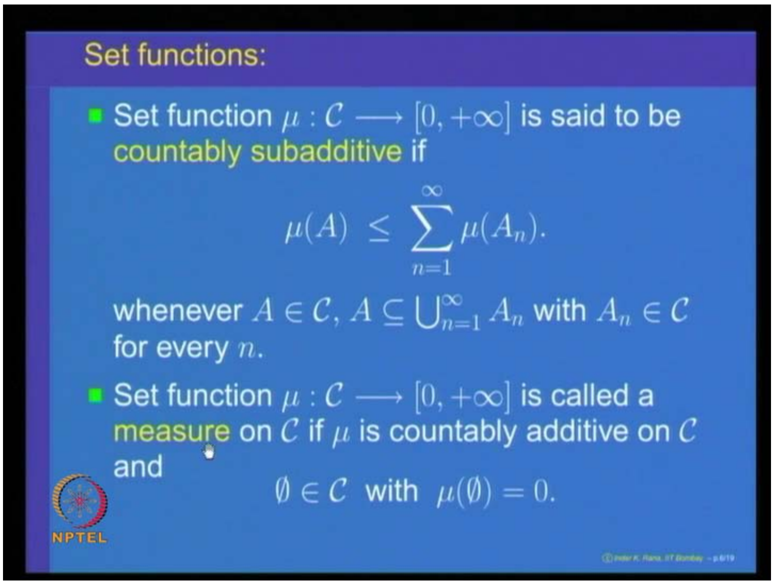
Such a property is called finite additivity property of mu or one says mu is finitely additive.

(Refer Slide Time: 05:02)




We can extend the generalization of this definition. We will say mu is countably additive, from finite we are going to countably additive, if mu of union $A_n$s 1 to infinity is equal to summation of mu of $A_n$s, of course, whenever $A_1$, $A_2$, up to $A_n$ is a sequence of sets in C such that the union is also an element of C and they are pairwise disjoint. Countable additivity is a property about a sequence of sets $A_1$, $A_2$, $A_n$ and so on in C which are pairwise disjoint and their union is an element in C; we want that for any such sequence of pairwise disjoint sets, mu of the union must be equal to summation of mu of $A_n$s, n equal to 1 to infinity.

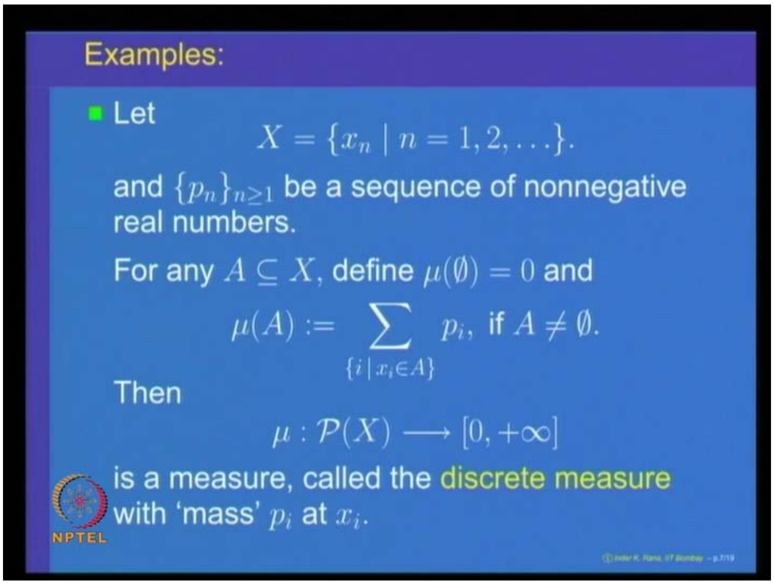
(Refer Slide Time: 05:59)




There is another notion of called countably subadditive if mu of A is less than or equal to summation 1 to infinity mu of $A_n$s whenever A is a set in C and A is contained in union of $A_n$s where $A_n$s are also in C for every n. In some sense, if A is covered by a union of sets $A_n$s, then we want the size – that is mu of A – to be less than or equal to summation mu of $A_n$s, n equal to 1 to infinity

This is called countable subadditivity because here we are just saying that mu of A is less than or equal to and we are not requiring that $A_n$s are pairwise disjoint; this is called countable subadditivity property of the set function ((.)). A set function mu is called a measure on C (C is a collection of subsets) if mu has the property that it is countably additive (it should be countably additive), the empty set belongs to C and with the property that mu of empty set is equal to 0. mu is defined on a collection C of subsets and we want the properties that the empty set belongs to C, mu of empty set should be 0 and mu on this collection should be countably additive; such a set function is going to be called a measure on C.

(Refer Slide Time: 07:32)

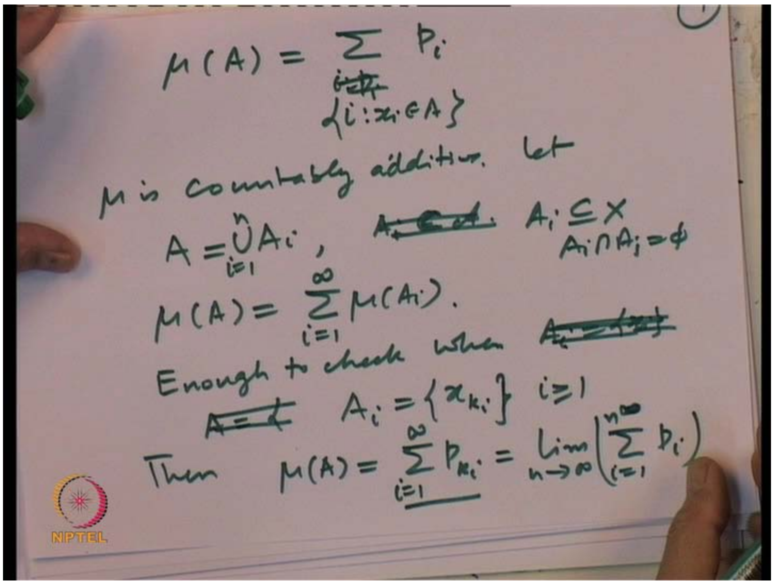



Let us look at some examples of set functions; let us start with a very simple one. Let us look at a set X which is a countable set; its elements are $x_1$, $x_2$, $x_3$ and so on. X is equal to $x_n$, n equal to 1, 2, 3 and so on. Let us fix $p_n$ – a sequence of nonnegative real numbers. X is a set which is a countable set with elements $x_1$, $x_2$, $x_3$ and so on and we are fixing arbitrarily some sequence of nonnegative real numbers.

For any subset A contained in X, let us define mu of the empty set to be equal to 0; for the set A if it is nonempty, let us define mu of A to be equal to summation over those $p_i$s such that $x_i$ belongs to A. A is a subset of X and so sum of the $x_i$s will belong to A. Look at those indices i such that $x_i$ belongs to A; pick up those $p_i$s from the given sequence $p_n$ and add them up; that is called mu of A. mu of A is defined as summation over those $p_i$s such that $x_i$ belongs to A.

We want to check that this is a measure on the collection of all subsets of the set X. That is quite obvious because mu of empty set is defined to be equal to 0. Let us observe that if A is a singleton set, then mu of the singleton set is going to be the number $p_i$; if a set A is a countable disjoint union of sets, let us check that this mu is a measure.

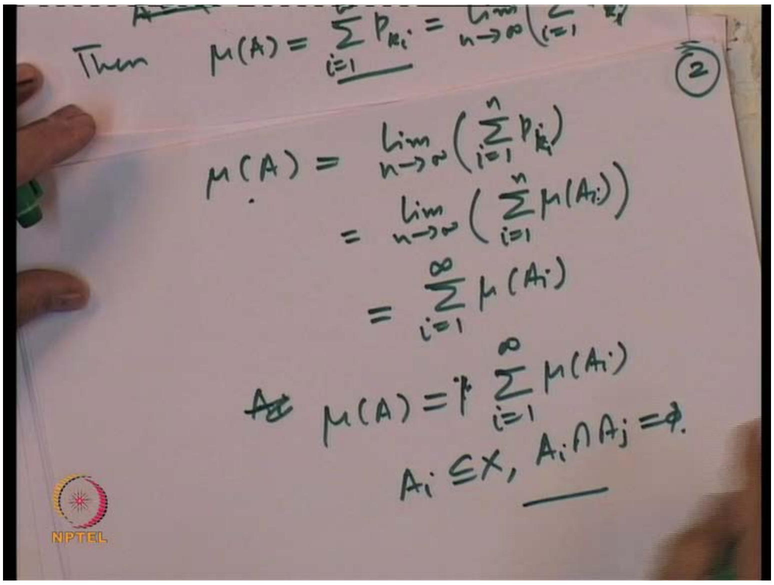
(Refer Slide Time: 09:32)




We are defining mu of A to be equal to summation $p_i$ where i is such that $x_i$ belongs to A. mu is countably additive. Let us take a set A which is union of $A_i$s, i equal to 1 to n and $A_i$s is a subset of A subset of where $A_i$ is any subset of X. We have to check that mu of A is equal to… We want $A_i$ intersection $A_j$ to be empty (Refer Slide Time: 10:29). We want this to be equal to mu of $A_i$s, i equal to 1 to infinity.

Let us observe; it is enough to check when each $A_i$ is a singleton $x_i$; let us check that case first. What is A? This is not $x_i$ because x itself is $x_1$, $x_2$, up to $x_n$ and so this will be the whole space. Let us look at the special case when $A_i$ is equal to some $x_{ki}$, I bigger than or equal to 1 (Refer Slide Time: 11:27). Then, the set mu of A is going to be equal to summation p of $k_i$, i equal to 1 to infinity. This can be written as limit n going to infinity i equal to 1 to infinity $p_i$ up to n. These are nonnegative numbers (Refer Slide Time: 12:05) and so this sum is nothing but the limit of the partial sums. These are nonnegative and so there is no problem in writing it that way.

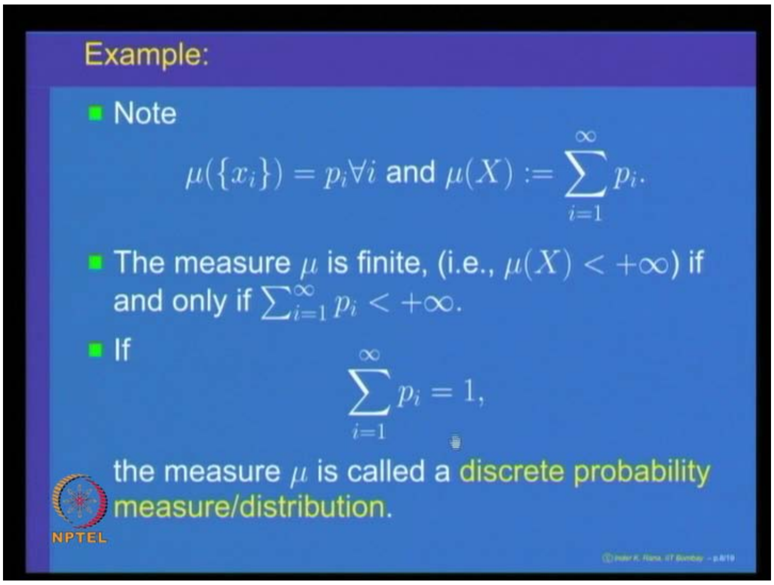
(Refer Slide Time: 12:17)




That means mu of A is equal to limit n going to infinity of sigma i equal to 1 to n of $p_i$. That means what we want to check? It is summation of mu of each $A_i$; so, this is limit n going to infinity of summation i equal to 1 to n mu of $A_i$ because each one is $p_i$. It is p of $k_i$, sorry (Refer Slide Time: 12:56); this summation p of $k_i$. This is $k_i$ and this is mu of $A_i$ (Refer Slide Time: 13:02).

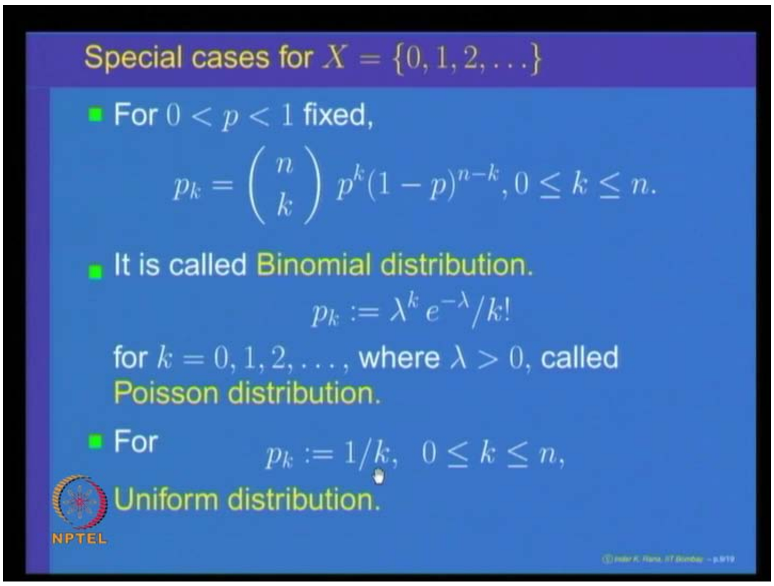
That is equal to i equal to 1 to infinity mu of $A_i$. mu of A is equal to summation mu of $A_i$s whenever $A_i$ is a singleton set; if not, it is a finite set; each finite is a union of finite sets. For a nonnegative series, you can add it anyway you like; it is easy to check that mu of A is equal to summation mu of $A_i$, i equal to 1 to infinity whenever $A_i$s are contained in X and $A_i$ intersection $A_j$ is empty. That says that this set function mu that we have defined is countably additive (Refer Slide Time: 13:55). This is what is called a discrete measure because it is given by a sequence and $p_i$ is called the mass at the point $x_i$.

(Refer Slide Time: 14:14)




As we observed, mu of the singleton $x_i$ is equal to $p_i$ for every i and mu of the whole space is equal to summation mu of the singletons is $p_i$; mu of X is equal to summation of $p_i$s. The obvious consequence of this is that mu of X is finite whenever this series is convergent. One says this discrete measure mu is finite; that is, mu of X is less than infinity; mu of the whole space is finite if and only if summation mu of $p_i$s is less than infinity. If this summation of $p_i$s – the series $p_i$ is convergent and its sum is equal to 1, then this measure mu is called a discrete probability distribution on the set X which is $x_1$, $x_2$, up to $x_n$.

(Refer Slide Time: 15:15)

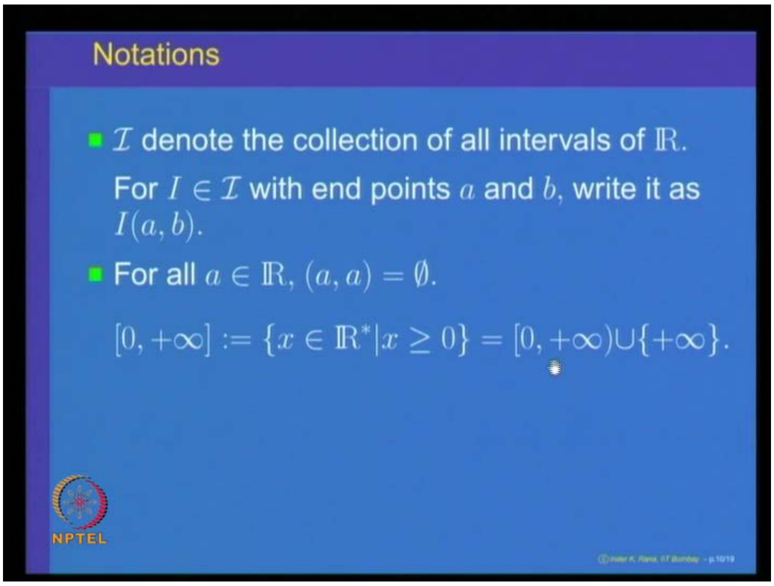


This is a very special case which plays an important role in the theory of probability and so on. X is the set of the numbers 0, 1, 2 and so on. Let us fix any number p which is between 0 and 1 and define $p_k$ to be equal to n ((.)) k (this is the binomial coefficient n ((.)) k) p to the power k into 1 minus p raised to the power n minus k, k between 0 and n. This is called the binomial distribution because of this binomial coefficient appearing in the definition of $p_k$.

It is quite easy to check that the summation of these $p_k$s is equal to 1. That is because summation of these $p_k$s is summation k equal to 0 to n and this side is nothing but p plus 1 minus p raised to power n and that is equal to 1. This is a distribution which plays a very important role in probability; this is a probability distribution. Supposing you have got a coin and you are tossing a coin with probability p for head appearing, then this $p_k$ represents the probability that in n tosses you will get k heads.

Another special case of this discrete distribution is called the Poisson distribution which is characterized by the definition that $p_k$ is equal to lambda to the power k into e raised to the power minus lambda divided by k factorial. This is called Poisson distribution; this is another important distribution in the theory of probability. Finally, when we take only a finite number of points 0, 1 up to n and $p_k$ is 1 over k and each point is given the same mass 1 over k, then this is called the uniform distribution. There are special cases of discrete probability distributions.

(Refer Slide Time: 17:32)

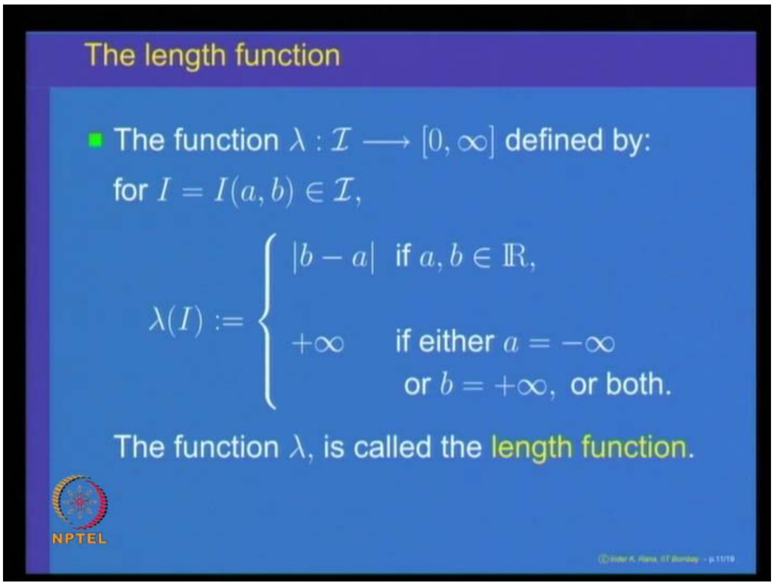


Next, we give an important example of a measure which is defined on the collection of all intervals in the real line. To do that, let us fix our notations. We will denote by I the collection of all intervals on the real line. For an interval I with end points a and b (the left end point being a and the right end point being b), we will write it as I of a comma b; a will denote the left end point and b will denote the right end point.

We are not saying that this is an open interval a comma b; we are just saying that it is an interval with left end point a and right end point b where the left or the right may or may not be or both may or may not be included in that interval. It is just an interval with end points a and b; the left end point is a and the right end point is b. On this collection of all intervals, we are going to define a function.

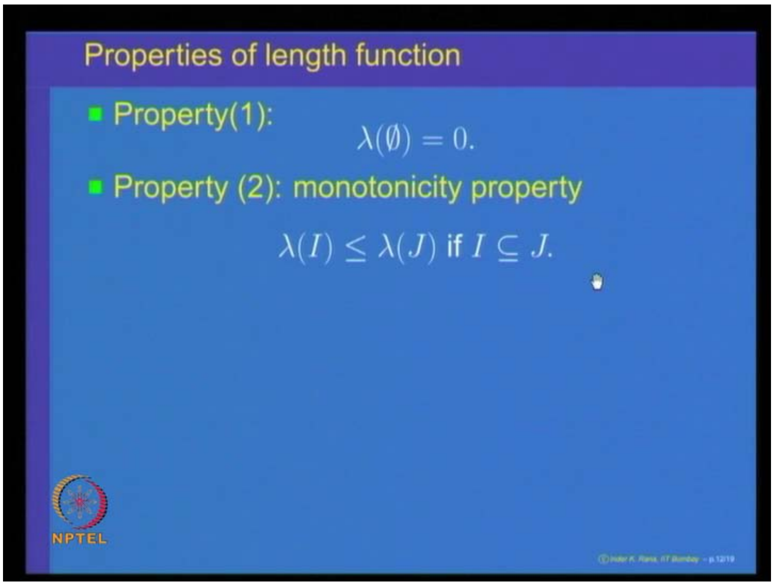
For example, recall that the open interval a comma a is the empty set (Refer Slide Time: 18:24). In the interval 0 to plus infinity, the square brackets indicate that we are including 0 and we are including plus infinity. This is a closed interval in R star, x belongs to R star – the extended real numbers, x bigger than or equal to 0; this is same as the open interval, closed on the left 0 and open on the right infinity in the real line, union the special symbol plus infinity that we had added in the extended real numbers.

(Refer Slide Time: 18:57)



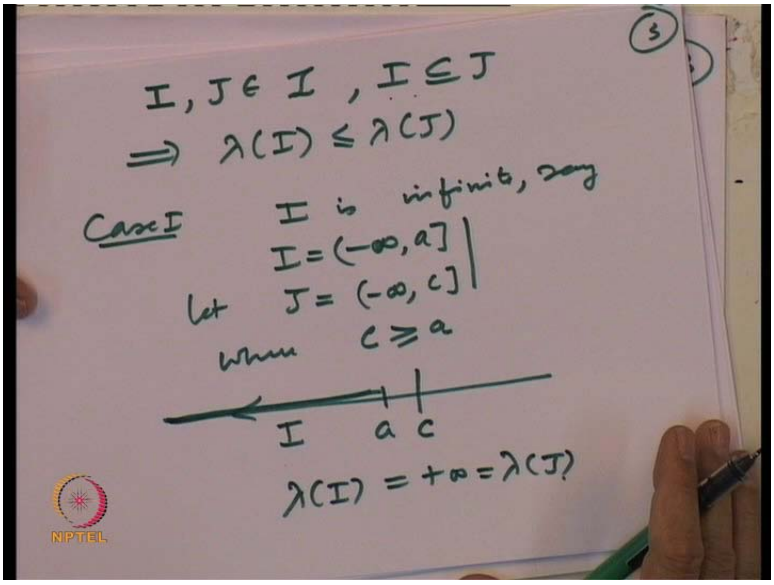

With these notations, we define what is called the length function on the class of intervals. It is a set function lambda defined on I taking values in 0 to infinity and is defined by take any interval I with left end point a and right end point b. We define it as the absolute value of b minus a if a and b are both real numbers. That means if the interval I is a finite interval with end points a and b, then its length is defined as b minus a and we define it equal to plus infinity in case either the left end point a is minus infinity or the right end point b is equal to plus infinity or both; length of I for an unbounded interval is defined as plus infinity. This function is called a length function on the class of all intervals. This length function is going to play an important role in our subject; let us study its properties.

(Refer Slide Time: 20:08)



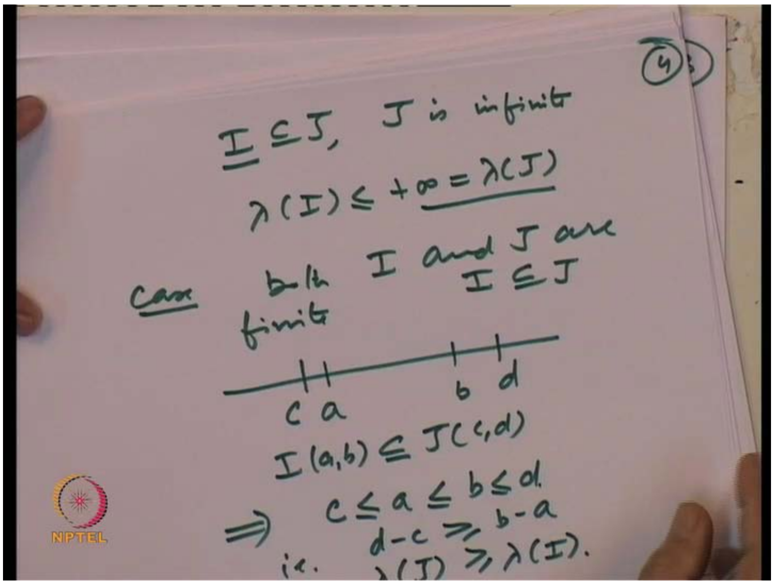

Next, we will be studying properties of this length function. The first property is that the length function has the property that lambda of the empty set is 0 because empty set is an open interval with left end point, say, a and right end point a; it is an open interval a comma a which is the empty set; by the very definition, that is equal to a minus a which is equal to 0. Next, let us check that this is a monotone set function, namely, length of I is less than or equal to length of J if I is a subset of J.

(Refer Slide Time: 20:48)




We want to check that whenever we have got intervals I comma J and I is a subset of J, this should imply that length of I is less than or equal to length of J. Since the intervals are characterized by the end points… Case I: let us say I is infinite, say I is equal to minus infinity to a. Since I is a subset of J, obviously J has to start with minus infinity and can go up to some point c where c is bigger than or equal to a. Essentially, what we are saying is if this is a and on this side all of it is the interval I and if J is to contain I, then J must be ending somewhere here – that is c (Refer Slide Time: 22:01). Clearly, both are infinite and so length of I is equal to plus infinity is length of J; that case is obvious.

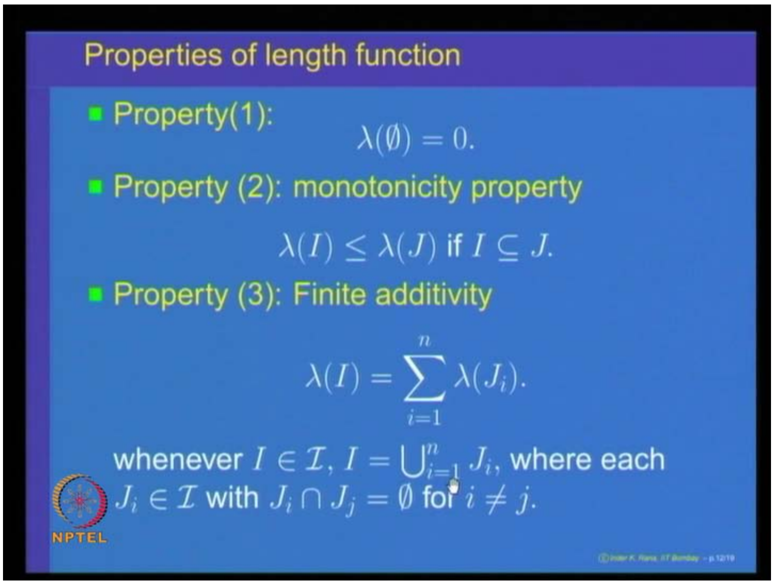
(Refer Slide Time: 22:15)




Let us look at the next case. I is a subset of J; J is infinite; whether I is infinite or not does not matter because the length of I is always less than or equal to plus infinity which is equal to length of J. J being infinite, its length is always going to be plus infinity and so this is obvious if this is the case. Finally, let us look at the case when both I and J are finite. Let us say I has got the end points a and b.

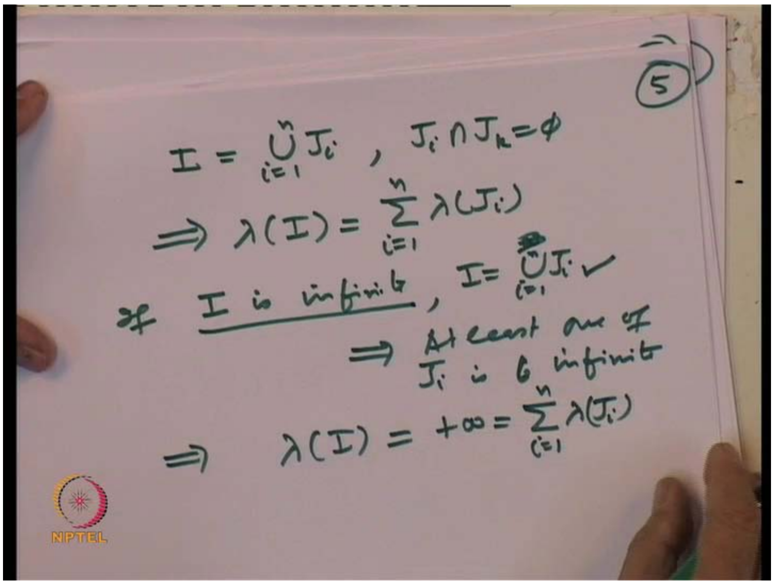
Now, I is subset of J and that means the end points of J have to be somewhere here and here (Refer Slide Time: 23:04). If I is with end points a, b and J is with end point c, d, then we should have c is less than or equal to a less than or equal to b is less than or equal to d. This implies this (Refer Slide Time: 23:21). That is same as saying that d minus c is bigger than or equal to b minus a and that is saying that the length of J is bigger than or equal to length of I. The monotone property is checked (Refer Slide Time: 23:40). The length function lambda is a monotone property; if I is a subset of J whenever an interval I is contained in another interval J, the length of I is less than or equal to length of J.

(Refer Slide Time: 23:54)




Next, let us look at another property; it is called the finite additivity property. What should be finite additivity property? Whenever an interval I is a disjoint union of some other intervals, then the length of I should be summation of length of Js. What we are saying is that if an interval I is written as a finite union of intervals $J_i$s, i equal to 1 to n where these $J_i$s are pairwise disjoint, then we want length of I to be equal to summation length of $J_i$s; that is going to be called finite additivity property.

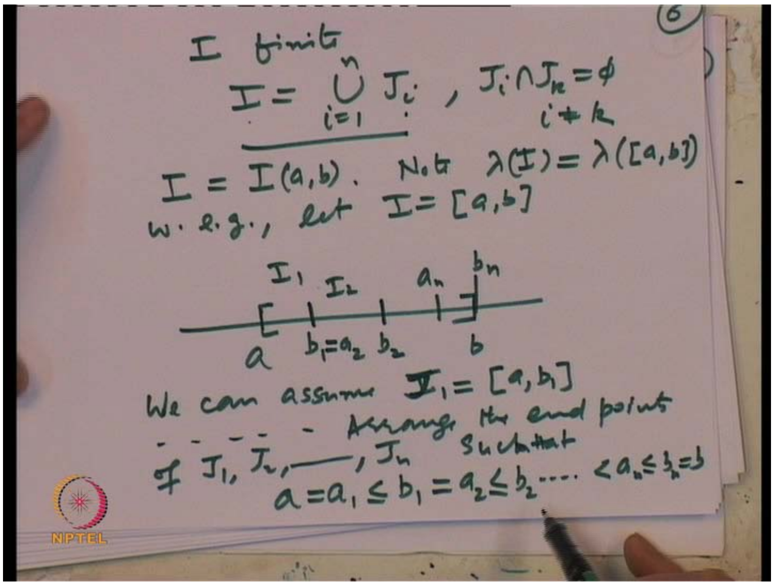
(Refer Slide Time: 24:40)



Let us check the finite additivity property. If I is equal to union of $J_i$s where all are intervals where $J_i$ intersection $J_k$ is empty, then that should imply that length of I is equal to summation length of $J_i$s, i equal to 1 to n. Let us assume that if I is infinite and I is equal to union of $J_i$s 1 to n, then that implies at least one of $J_i$s is infinite because if all of them are finite intervals, their union will be again a finite interval.

So, I infinite implies I equal to union of $J_i$s implies at least one of these $J_i$s has to be infinite (Refer Slide Time: 25:47). That implies lambda of I is equal to plus infinity is equal to summation lambda of $J_i$s, i equal to 1 to n because one of them is plus infinity. So the case when I is infinite is okay. Let us look at the case when I is finite.

(Refer Slide Time: 26:09)



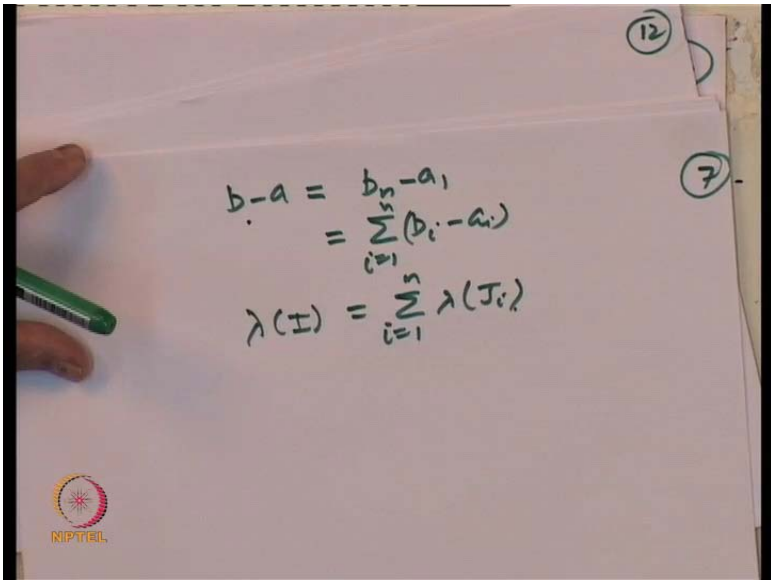

I is finite. I is equal to union of $J_i$s and $J_i$s are pairwise disjoint. Now, let us say the interval I has got end points a and b; let us say the left end point is a and the right endpoint is b; here is a and here is b (Refer Slide Time: 26:39). We want to compute the length of I. Note: the length of I is same as the length of the closed interval a comma b. I can include the end points in the interval I because the length depends only on the values of the end points; it does not matter whether the end points are inside or not. What we are saying is: without loss of generality, let I be equal to a comma b. This is the interval a comma b (Refer Slide Time: 27:18).

Now, I is equal to union of $J_i$s. The point a belongs to this union (Refer Slide Time: 27:29) and so it should belong to one of the intervals $J_i$s. It belongs to one of the interval

$J_i$s and actually it has to be end point of one of the intervals of $J_i$s because the interval cannot start somewhere else. Sorry, the interval $J_1$ is starting at a and ending somewhere let us call it as $b_1$; the end point may or may not be included.

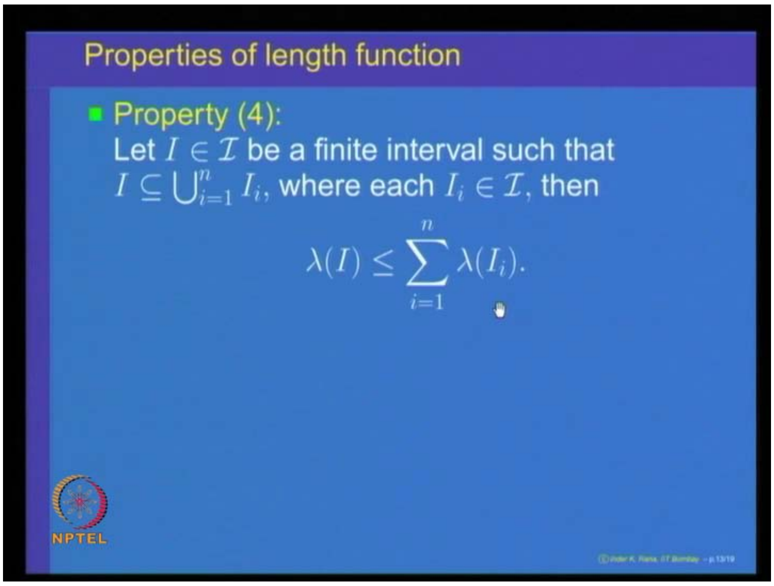
The first interval $I_1$ we can assume it starts here and ends somewhere here; that is, $b_1$ (Refer Slide Time: 28:15). Now, the point $b_1$ is again in that union (Refer Slide Time: 28:20). So, either it is already included in the interval $I_1$ or it should be an end point of another interval in the union $J_1$, $J_2$ up to $J_n$s. The second one must start here and end somewhere here; that is, $b_2$ (Refer Slide Time: 28:38). What we are saying is $I_2$ – some other interval; you can rename it as $I_2$; it should start somewhere again at $b_1$ and end somewhere here and so on. Here will be the last one $a_n$ and that should be $b_n$. What we are saying is this is going this way; we can arrange the end points of $J_1$, $J_2$ and $J_n$ such that such that a is same as $a_1$ less than or equal to $b_1$ is equal to $a_2$ less than or equal to $b_2$ and so on; so, $a_n$ less than or equal to $b_n$ which is equal to b. You can rearrange the end points of these intervals because this is a union and that is a disjoint union (Refer Slide Time: 29:51); this is what is possible for us to arrange.

(Refer Slide Time: 29:58)



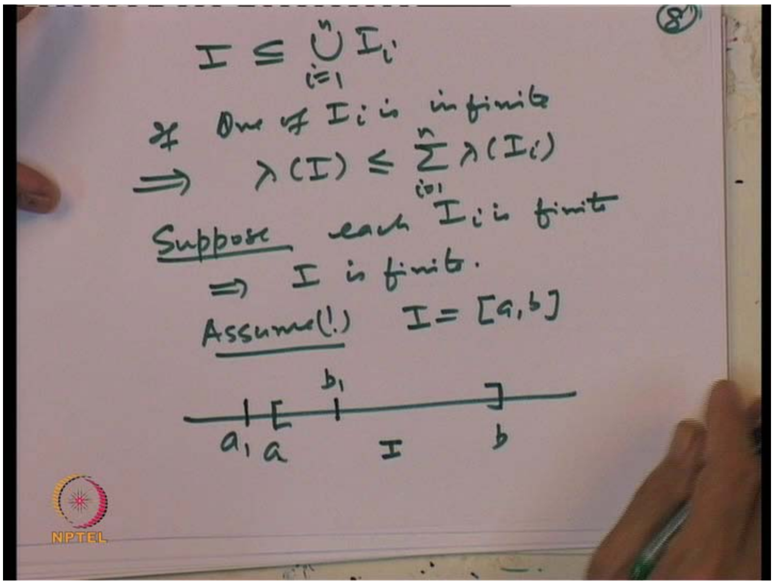

That clearly says that b minus a is equal to $b_n$ minus $a_1$; that is equal to summation $b_i$ minus $a_i$, i equal to 1 to n, adding and subtracting these terms in between; that is same as i equal to 1 to n lambda of $J_i$. Whenever i is a finite interval, i is equal to union of $J_i$s (they are pairwise disjoint), we have got that the length of this b minis a is the length of the interval I is equal to summation length of $J_i$s. That means that the length function lambda is finitely additive (Refer Slide Time: 30:42). This is the property of lambda – the length function being finitely additive; if an interval I is a finite union of pairwise disjoint intervals, then length of the interval I is equal to summation length of $J_i$s.

(Refer Slide Time: 30:59)




Next, let us look at another property. Supposing I is a finite interval such that I is contained in union 1 to n $I_i$s where a finite union of the intervals…, we are no longer saying that they are disjoint, then the claim is that length of I must be less than or equal to summation length of these intervals $I_i$s. If you drop the condition that these are pairwise disjoint, we are saying if an interval I is covered by a finite union of intervals, then the length of I must be less than or equal to summation of length of these intervals $I_i$s. Let us look at the proof of this. The proof of this is once again similar to the earlier properties.

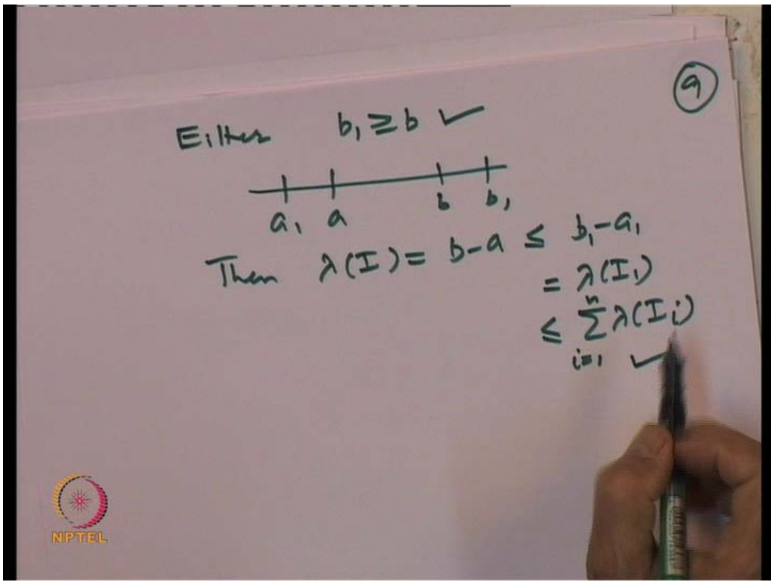
(Refer Slide Time: 31:52)




We are saying I is contained in union of $I_i$s, i equal to 1 to n. Obviously, if one of $I_i$ is infinite, clearly this implies length of I is less than or equal to summation length of $I_i$s; that is obvious because one of these terms on the right-hand side in the summation is plus infinity which is always greater than or equal to length of I, whatever be I. Let us suppose so that each $I_i$ is finite. This is a finite union (Refer Slide Time: 32:45) and that implies I is finite.

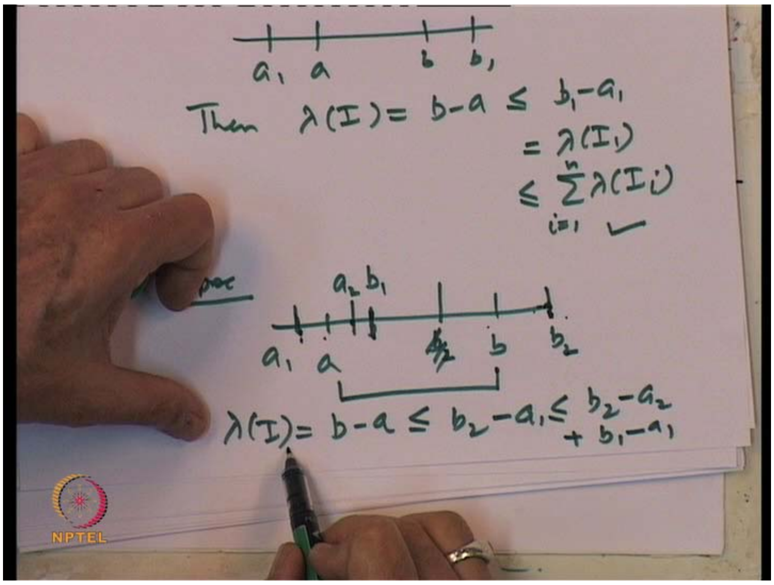
As before, without loss of generality, we assume that I is equal to a comma b; here is once again the same picture; here is a and here is b (Refer Slide Time: 33:13). The point a belongs to the interval I; this is my interval I and a belongs to I; that means it belongs to this union (Refer Slide Time: 33:24). It will belong to at least one of the intervals $I_i$s. Let us name any one of them which contains the point a to be $I_1$ and let us say the end points of that are $a_1$ and $b_1$. The point a belongs to one of the intervals $I_i$s because it is in the union and so it will belong to one of them, say $I_1$; let us say the end points of $I_1$ are $a_1$ and $b_1$. Here is the end point $a_1$ and here is the end point $b_1$. Now the possibility is this $b_1$ is on the right side of b; one possibility is it is on the right side of b.

(Refer Slide Time: 34:13)




Let us write either $b_1$ is bigger than or equal to b; that means my picture looks like this; here is $a_1$, here is a, here is b and here is $b_1$ (Refer Slide Time: 34:28). Then, length of I which is equal to b minus a is less than or equal to $b_1$ minus $a_1$, which is equal to length of $I_1$ and is obviously less than or equal to summation length of $I_i$s, i equal to 1 to n. In case $b_1$ is on the right side, we are obviously through by this case (Refer Slide Time: 34:54).

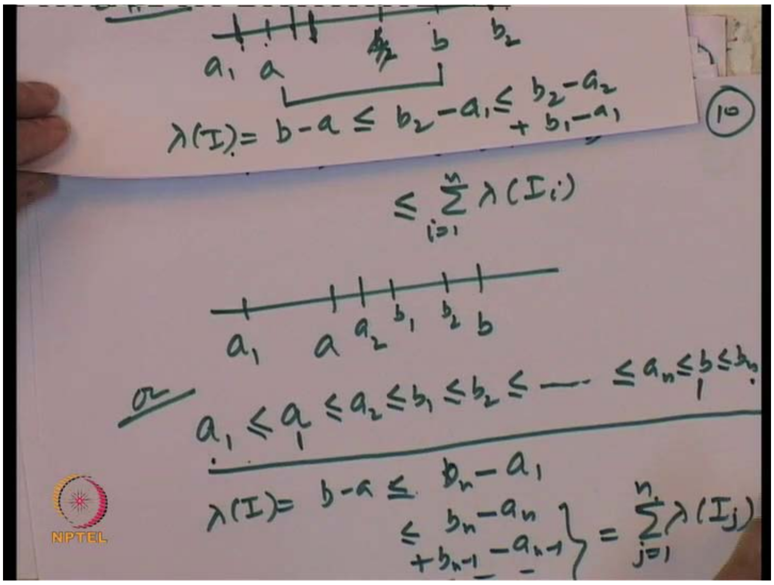
(Refer Slide Time: 34:59)



What is the other possibility? Case two. This is the picture (Refer Slide Time: 35:06). We have got a, we have got b, here is $a_1$ and $b_1$ is not on the right side but on the left side of b; let us take that as the picture. In that case, the point $b_1$ belongs to that union. $b_1$ is in the interval a, b and so it will belong to that union. $b_1$ belongs to I and so it belongs to the union (Refer Slide Time: 35:33). It will belong to one of the intervals in the $I_i$s. Let us call that as some interval $I_2$.

$b_1$ belongs to $I_2$; that means $a_2$ must start here and $b_2$ will either be somewhere here or it will be on the right side. If it is on the right side of it, that means what? Let us say I is on the right side; here is $b_2$; instead of here, let us say $b_2$ is here (Refer Slide Time: 36:04). Then, the length of the interval I which is equal to b minus a is less than or equal to $b_2$ minus $a_1$ which is less than or equal to $b_2$ minus $a_2$ plus $b_1$ minus $a_1$. So, $b_2$ minus $a_1$ is less than or equal to $b_2$ minus $a_2$ plus $b_1$ (we are adding something bigger) and then $a_1$.

(Refer Slide Time: 36:52)

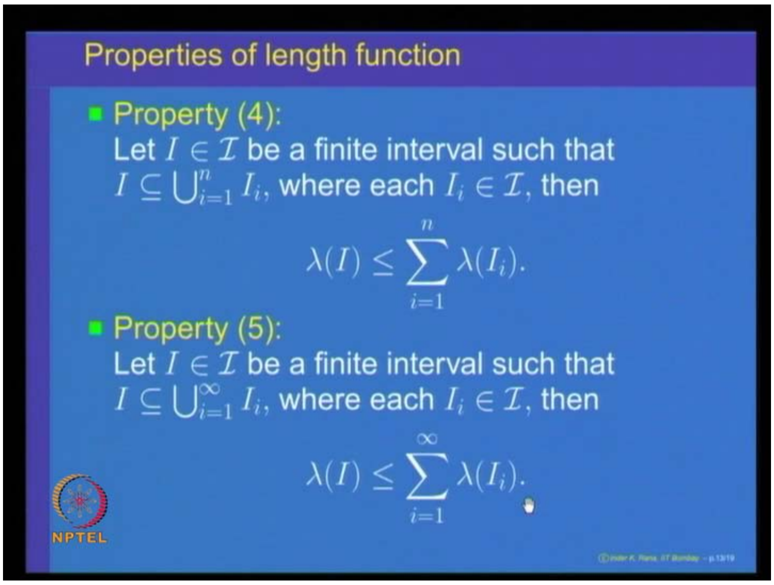



In that case, length of I will be less than or equal to length of $I_1$ plus length of $I_2$; that is anyway less than or equal to summation length of $I_i$s, i equal to 1 to n. If you go on repeating this process, what does that mean? What is the other possibility? $b_1$ is inside; that means here is a and here is b (Refer Slide Time: 37:21). If it is not outside, then it must be inside; that means here is $a_1$; here was our $b_1$; here is a and somewhere here is $b_2$; it is not on the right side; it is on the left side (Refer Slide Time: 37:34).

Once again, $b_2$ belongs… and then we can proceed in the same way. At some stage we will be through; if not, then we will have $a_1$ is less than equal to a is less than or equal to $a_2$ less than or equal to $b_1$ less than or equal to $b_2$ less than or equal to so on less than or equal to $a_n$ less than or equal to b less than or equal to $b_n$. What we are saying is either will be through at some finite stage or we can rearrange eventually after n stages the end points in that way.

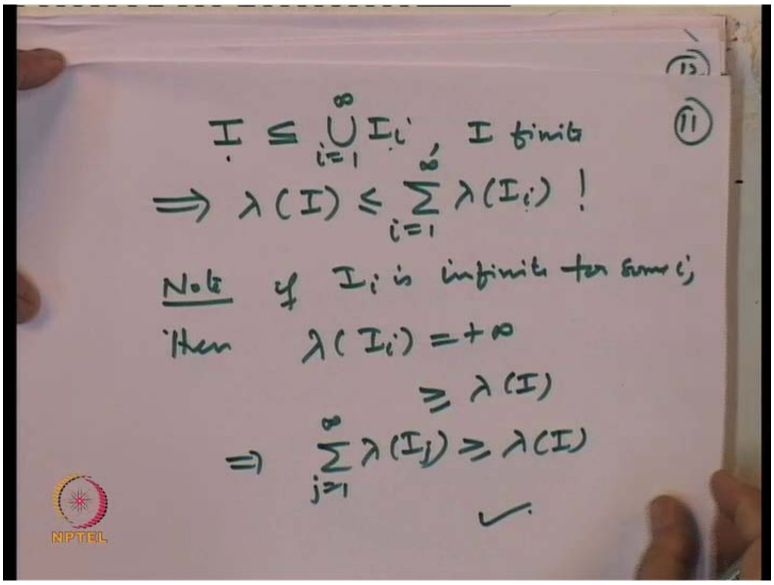
In that case again, lambda of I which is equal to b minus a, here is a and here is b (Refer Slide Time: 38:23), is less than or equal to same idea $b_n$ minus $a_1$; go on adding and subtracting; it is less than or equal to $b_n$ minus $a_n$ plus $b_{n \text{ minus } 1}$ minus $a_{n \text{ minus } 1}$ and so on plus $b_1$ minus $a_1$; that is equal to sigma lambda of $I_j$, j equal to 1 to n. Whenever we are in a finite stage, the end points can be rearranged nicely and we get this property; the length function is having the property that whenever a interval I is covered by a finite union of intervals, then the length of I is less than or equal to summation length of $I_i$s (Refer Slide Time: 39:18).

(Refer Slide Time: 39:19)



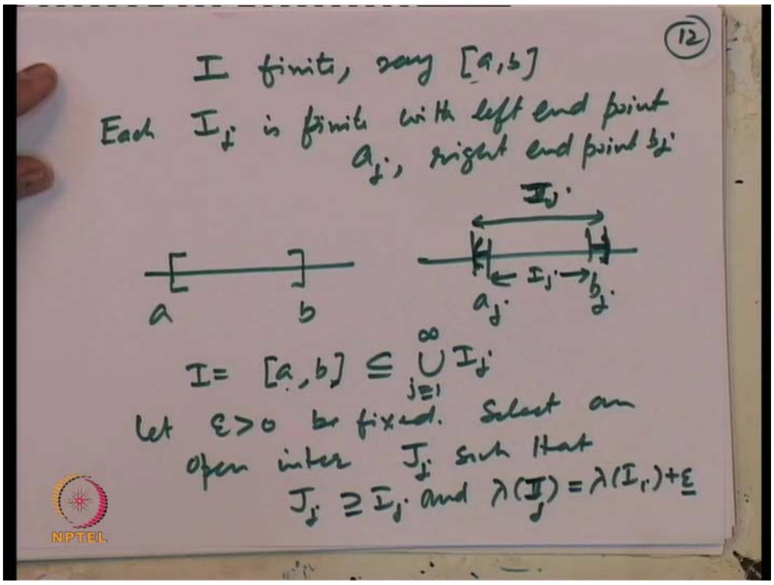

Let us look at an extension of this property. Supposing I is a finite interval such that I is covered by a union of intervals $I_i$s 1 to infinity, that means the interval I is covered by a countable union of intervals $I_i$s; then again, the claim is length of I is less than or equal to summation length of $I_i$s. Let us prove this property. Keep in mind that here we are assuming our interval I is a finite interval.

(Refer Slide Time: 39:57)




Interval I is contained in union of intervals $I_i$s, i equal to 1 to infinity; these are intervals; I is finite. This implies length of I is less than or equal to summation length of $I_i$s, i equal to 1 to infinity; this is what we want to prove. Obvious case: if any one of the terms on this side – lambda of $I_i$ – is infinite, then we are through. Note: if $I_i$ is infinite for some I, then what will happen? Length of $I_i$ will be equal to plus infinity which is bigger than or equal to length of I, whatever it may be – whether finite or infinite. It implies sigma length of $I_j$, j equal to 1 to infinity is also bigger than or equal to lambda because one of them is infinite; that case is obvious. Let us assume that not only is I finite but all the intervals $I_i$s are also finite; we want to check this property (Refer Slide Time: 41:16).

(Refer Slide Time: 41:19)

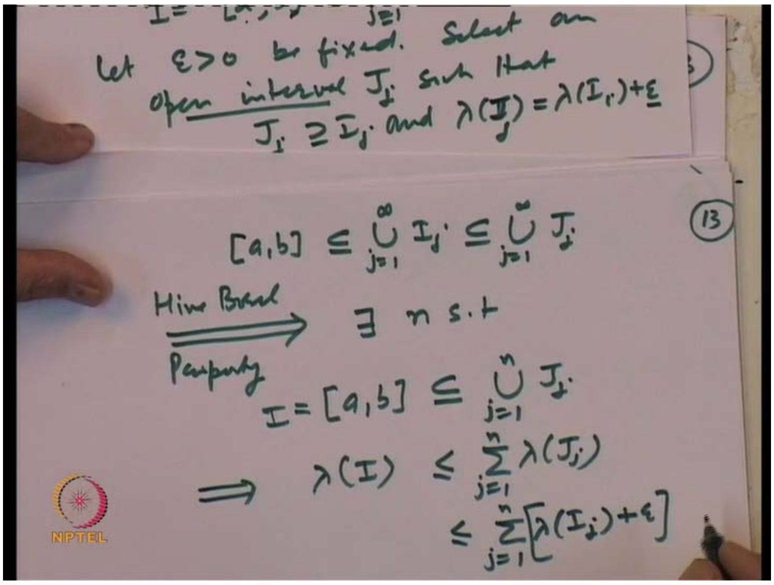



What we want to check is the following. I is finite with end points a and b' we can assume it is a closed interval because the length of I is not going to change. Each $I_j$ is finite with left end point $a_j$ and right end point $b_j$. We are not saying that we are assuming these $I_j$s are open or closed or anything' we are just naming the end points. We are saying I looks like this – a and b; each $I_j$ is $a_j$, $b_j$ (Refer Slide Time: 42:17). We are not saying that these end points are included. We are given that I which is a comma b is contained in union of $I_j$, j equal to 1 to infinity.

If this was finite, then we already know how to manipulate that; that we have already done earlier in the previous case. The idea is: from that infinite union, bring it to a finite union. Here is a closed bounded interval contained in an infinite union and we want to say this is going to be contained in a finite union. Somewhere, the compactness property of the interval a to b is going to be used, but for that we need the intervals to be open.

Let us make these intervals $I_j$s open but, of course, the lengths will change. Let epsilon greater than 0 be fixed. Select an open interval, say, we call it as $J_j$ such that this $J_j$ includes our interval $I_j$ and does not change the length much. Length of this $J_j$ is equal to say length of $I_i$ plus epsilon; so, slightly increase. What we are saying in this picture is take an interval from here to here the open interval from here to here (Refer Slide Time: 44:05); call that as $J_j$. Each $I_j$ which was from $a_j$ to here ($b_j$) is enclosed in an open interval slightly bigger but the length portion that you had is at the most equal to epsilon.

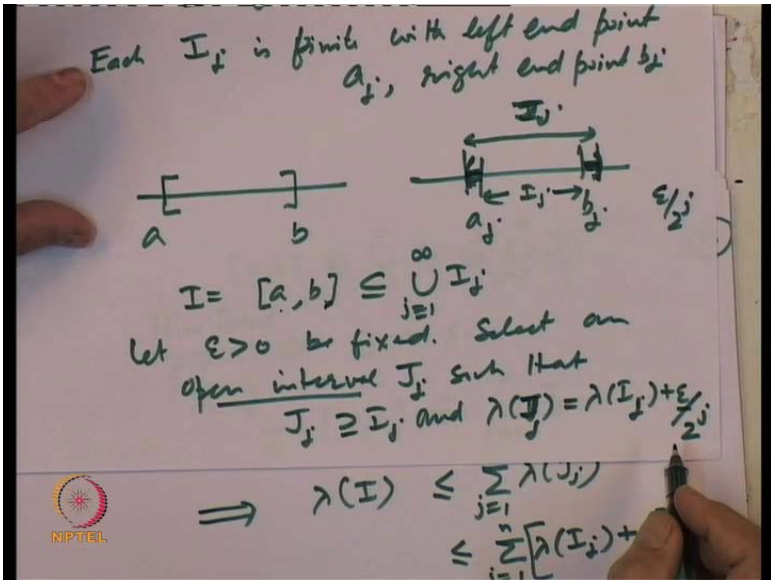
(Refer Slide Time: 44:33)




Now, what happens is the following. a, b is contained in the union of $I_j$s; each $I_j$ is contained in the union of $J_j$s; each $J_j$ is an open interval; we had taken an open interval (Refer Slide Time: 44:55). We have got an open cover of the closed bounded interval a, b. Heine–Borel property of the real line which says that whenever a closed bounded interval is covered by a collection of open intervals implies there exist some n such that a finite number of them will cover it; so a, b will be contained in union of j equal to 1 to n $J_j$s; a finite number of them will cover it.

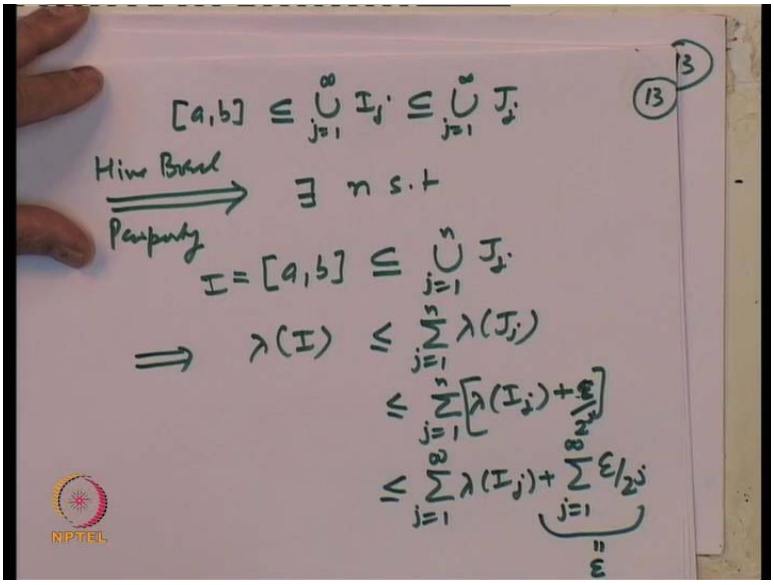
This implies by our earlier case that length of I, this was my interval I (Refer Slide Time: 44:46), is less than or equal to sigma length of $J_j$s, 1 to n. Each one of them is less than or equal to sigma j equal to 1 to n length of $I_j$ plus epsilon. Now, we want to separate out this summation and let it go to infinity ((.)) to infinity but the problem will come because of the summation epsilon added n times. That summation will tend to become very very large; we do not want that to happen. What we do is we revise our construction.

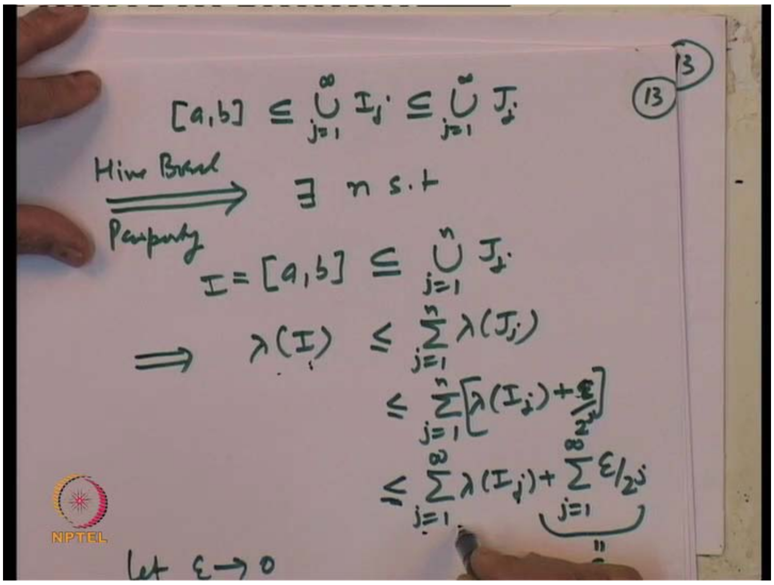
(Refer Slide Time: 46:37)




For a given epsilon, select an open interval $J_j$ says that this holds. So, instead of epsilon for the interval $I_j$, let us divide it by 2 to the power j. Instead of having this extra length to be equal to same length as epsilon for every interval $I_j$, for $I_j$ we want this extra length to be equal to epsilon by 2 to the power j (Refer Slide Time: 47:04).

(Refer Slide Time: 47:07)

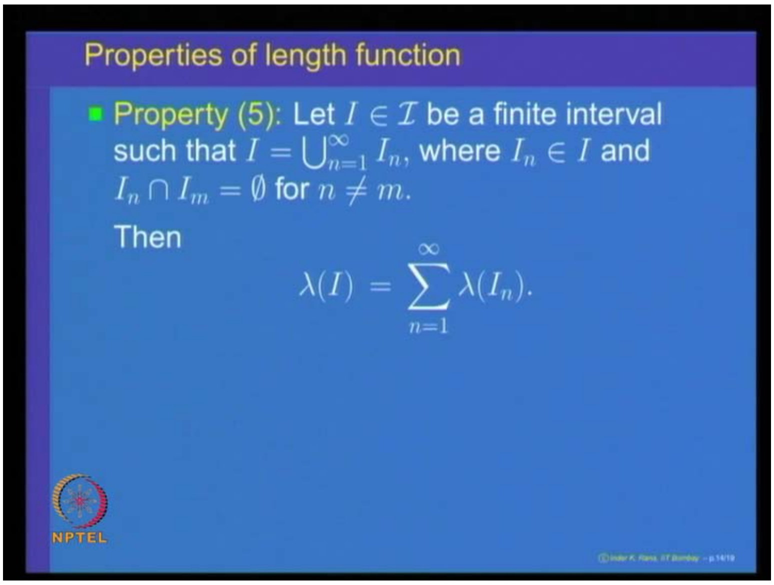





Once we do that, we are in a better shape because now this ((.)) will be 2 to the power j. That means it is less than or equal to summation j equal to I can put it 1 to infinity because this is less than or equal to lambda of $I_j$ plus summation epsilon by 2 to the power j, j equal to 1 to infinity. Now, this series is convergent because it is a geometric series with common ratio 1 by 2, which is less than 1. This term is equal to epsilon (Refer Slide Time: 47:42).

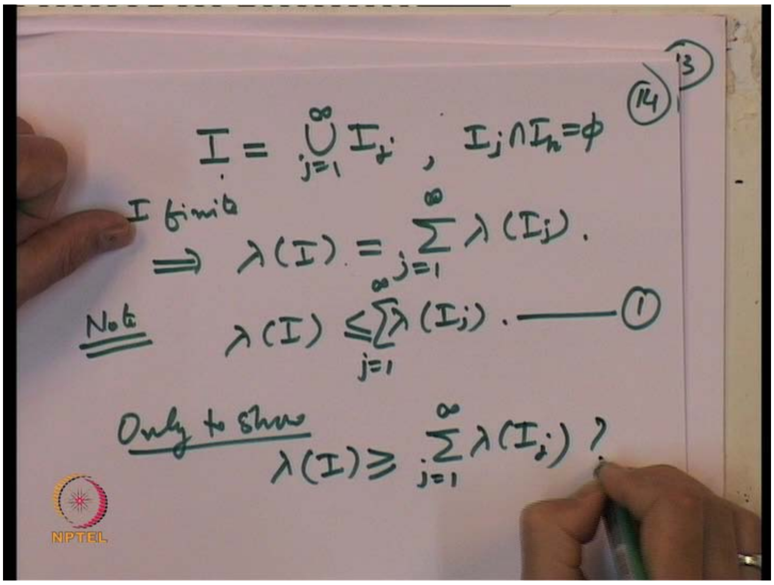
What we are saying is length of I is less than or equal to summation length of $I_j$s plus a number epsilon but epsilon was arbitrary. Let epsilon go to 0. We will get length of I is less than or equal to summation length of $I_j$s. What we are saying is that the countable property that we looked at namely length of I is less than or equal to summation length of $I_i$s whenever an interval I which is finite is covered by any countable union, then the length of I is less than or equal to length of $I_i$s. We have extended that earlier property; whenever a finite covering is there, we have extended it to a countable infinite covering but only for finite intervals. We would like to extend this to even arbitrary intervals which are not necessarily finite.

(Refer Slide Time: 48:50)



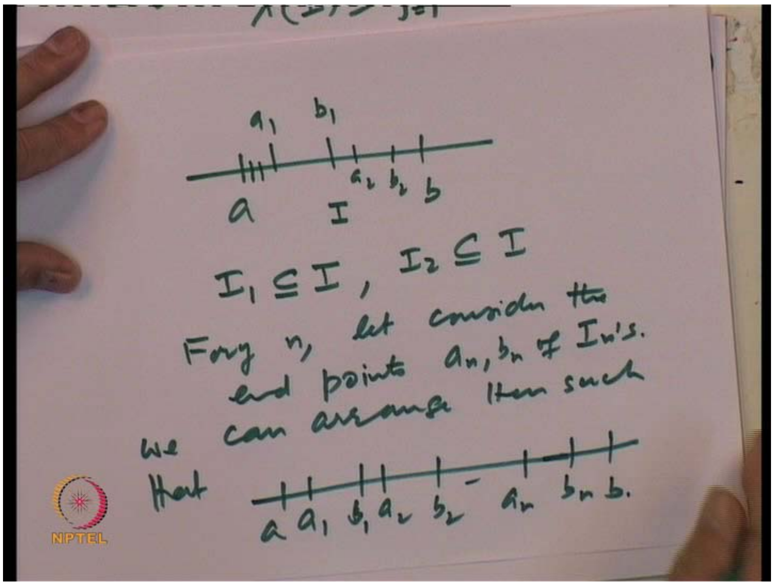

For that, we will have to do a little bit of more work. Let us look at the next property which says the following. Let I be a finite interval such that I is equal to union 1 to infinity $I_n$ where $I_n$s are pairwise disjoint. Then, at least we can conclude that the length of I is equal to summation length of $I_n$s. Whenever a finite interval is a countable union of pairwise disjoint intervals, then the length of I is equal to summation length of $I_n$s. Let us prove this property.

(Refer Slide Time: 49:32)



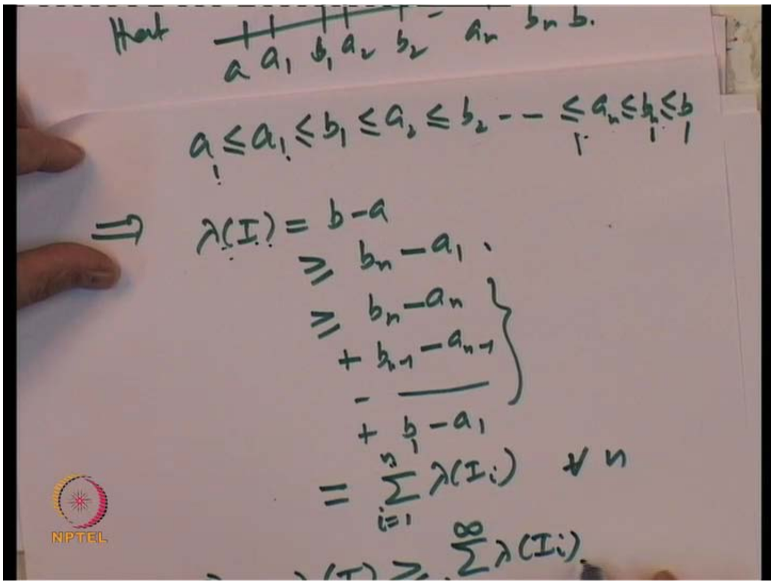
What we have got is I is equal to union of $I_j$s, j equal to 1 to infinity; $I_j$s are pairwise disjoint; I is finite implies length of I is equal to summation length of $I_j$s. Note that we have already proved, just now, that if an interval is written as this – a union of countable disjoint union, the length of I (we have just now shown) is less than or equal to length of $I_j$s added up, j equal to 1 to infinity; call it (1). Length of I is less than or equal to this (Refer Slide Time: 50:26); this we proved just now for finite intervals. We need only to show that length of I is bigger than or equal to summation j equal to 1 to infinity length of $I_j$; only this is to be shown.

(Refer Slide Time: 50:51)




Here is the interval a to b. I is finite; here is the finite interval I. $I_1$ is a subset of I; it should be somewhere inside; somewhere is $a_1$ and somewhere is $b_1$ (Refer Slide Time: 51:12). Similarly, $I_2$ is also inside I; somewhere it has to be; either it has to be $a_2$ here and $b_2$ here or it could be here somewhere and so on. For every n, let us consider the end points $a_n$, $b_n$ of $I_n$s. We can arrange them, there are only finitely many of them, such that here is a, here is $a_1$, here is $b_1$, here is $a_2$, here is $b_2$ and so on and here is $a_n$ and here is $b_n$ and here is b.

(Refer Slide Time: 52:24)




That means we can arrange them in such a way that a is less than or equal to $a_1$ less than or equal to $b_1$ which is less than or equal to $a_2$ less than or equal to $b_2$ and so on less than or equal to $a_n$ less than or equal to $b_n$ which is less than or equal to b. This implies by simple algebra that length of I is equal to b minus a. This is b and this is a and I am going to make it shorter $b_n$ and $a_1$; this is bigger than or equal to $b_n$ minus $a_1$ which is bigger than or equal to $b_n$ minus $a_n$ plus $b_{n \text{ minus } 1}$ (the next one here) minus $a_{n \text{ minus } 1}$ and so on plus $b_1$ minus $a_1$.

This put together is nothing but equal to sigma i equal to 1 to n length of $I_i$. What we are saying is for every n, the end points of the intervals $I_1$, $I_2$ up to $I_n$ can be rearranged in this fashion. Hence by looking at the ordering of this, the length of I is bigger than this (Refer Slide Time: 53:42). This happens for every n; that implies length of I is bigger than or equal to sigma I equal to 1 to infinity because this is happening for every n, I can let it go to infinity, length of I.

The other way around inequality is also proved. That means we have proved that whenever I is a finite interval which is written as a countable union of pairwise disjoint intervals, then length of I is equal to sigma length of $I_n$s (Refer Slide Time: 54:04). With that, we prove an important property of the length function for finite intervals. We will continue our study of the length function in the next lecture. Thank you.