Measure and Integration

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Module No. # 10

Lecture No. # 40

Convergence in Measure

Welcome to lecture 40 on Measure and Integration. In the past two lectures, we had been looking at the various modes of convergence for measurable functions.

We will continue the study in today's lecture also. We will recall some of the properties of convergence in measure that we had proved last time. Then we will go on to what is called convergence in the pth mean.

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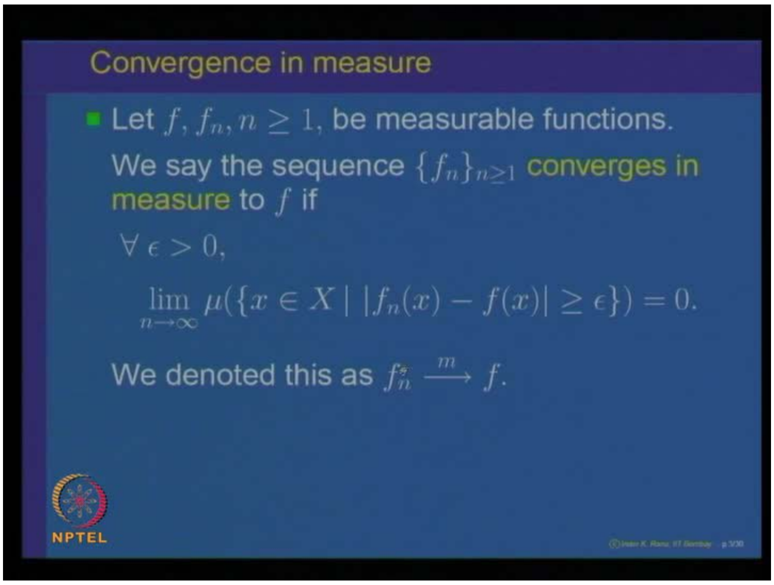

So, let us just recall what we have been doing; f and f ns are measurable functions on a measure space; then, we said that f n converges to f in measure, if for every epsilon bigger than 0, the measure of the set all x, where f n x minus f x is bigger than or equal to epsilon - that measure of that set goes to 0 as n goes to infinity for every (( )). So, in some sense, the measure of the set where f n is away from f by a distance epsilon, that goes to 0 for every epsilon bigger than 0.

This was called convergence in measure and we denoted this by saying that f n with an arrow and the symbol m above f. So, f n converges to f in measure. So, it is denoted by this symbol.

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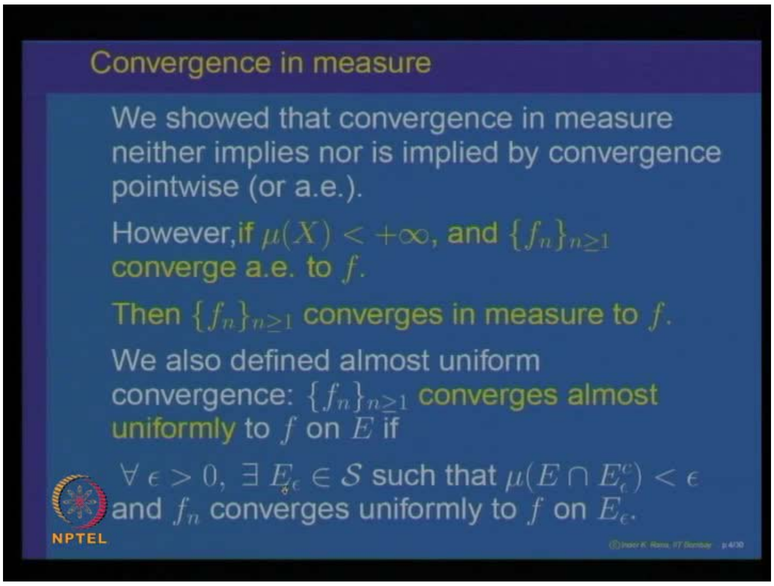

In the previous lecture, we showed that the convergence in measure neither implies nor is implied by convergence point almost everywhere. That means if f n converges to f in measure, then it need not imply that f n converges pointwise or almost everywhere. Conversely, if a sequence converges pointwise or almost everywhere, that need not imply that it converges in measure.

However, we proved that if the underlying measure space is finite and f n converges to f almost everywhere, then it also converges in measure. So, that means convergence in measure is implied by convergence almost everywhere, if the underlying measure space is finite.

After that, we looked at what is called almost uniform convergence for functions. So, we said a sequence f n converges almost uniformly to f on a set E, if we can find for every epsilon, a subset E epsilon such that the measure of the set E intersection E epsilon complement is finite and f n converges uniformly to f on E. That means except for a small set of small measure, f n converges to f uniformly. So, this is called almost uniform convergence.

Note: this is different from saying that the convergence is uniform almost everywhere. Saying that f n converges to f uniformly almost everywhere will mean that, except for a null set, f n converges to f; almost uniform convergence is saying that outside a set of measure 0, for every epsilon, there is a set E epsilon of such that outside that set the convergence is uniform.

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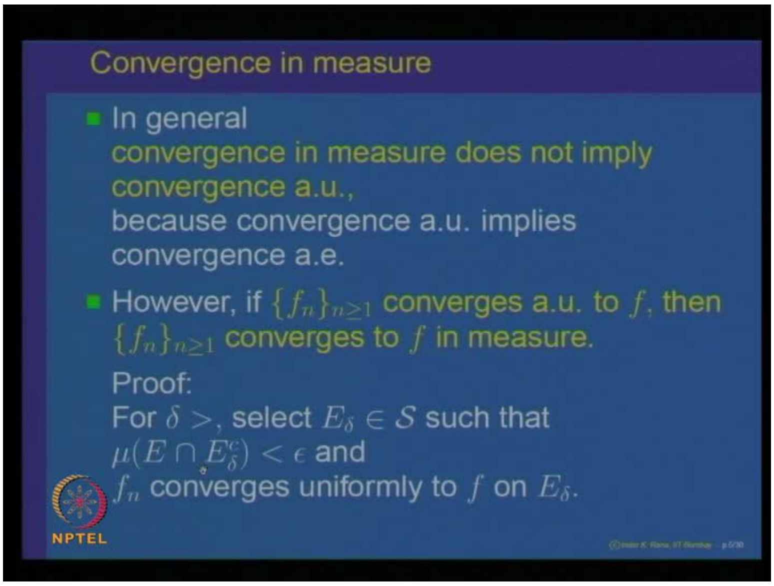

In general, we showed that convergence in measure does not imply convergence almost everywhere because if convergence almost everywhere almost uniformly implies convergence almost everywhere, we want to conclude that convergence in measure does not imply convergence almost uniform because almost uniform convergence implies almost everywhere convergence.

So, if convergence in measure implies almost uniform convergence, then it will also imply a convergence almost everywhere which is not true, in general. So, convergence in measure does not imply convergence almost uniform.

However, if f n convergence almost uniformly to f, then it also converges in measure; so, the converse is always true; namely, if f n converges almost uniformly to f when f n converges to f in measure so that the proof of this is quite simple because f n converges to f almost uniformly. So, that means for every delta bigger than 0, we can select a set E delta such that outside the measure of the set, outside E delta is small f n in convergence uniformly to f on E delta.

So, that is by the property that f n converges almost uniformly to f; so, for every delta, we can find a subset. So, let us say we are analyzing this on the set E. Then measure of the set outside E delta is small and f n converges uniformly to E delta, but what does that mean?

That means because it is converging uniformly to f on E delta; that means for every n integer N bigger than 0, f n must come close to f for every x in E delta.

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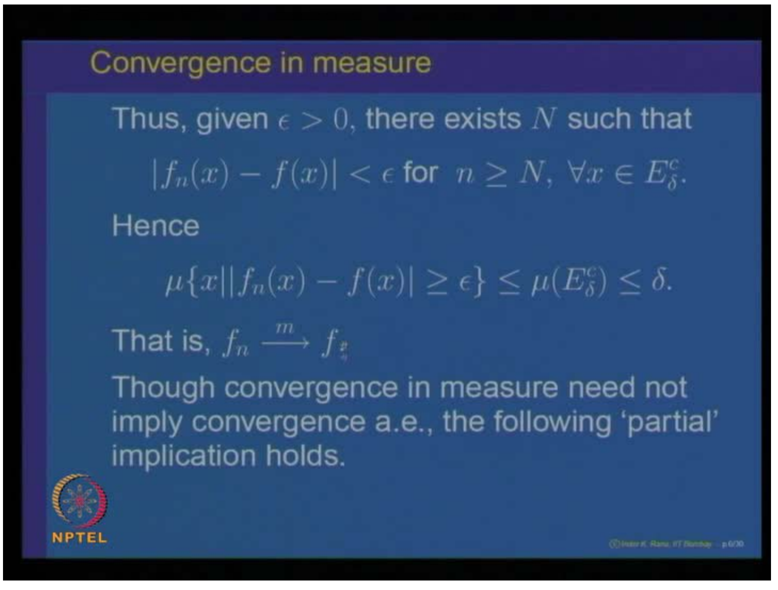

So, that means for every epsilon bigger than 0, there exists a set n, there exist a natural number n say that the distance f n minus f x is less than epsilon, for all n bigger than N and for every x belonging to E delta complement.

Then, that means if we look at the set up points where this is bigger than equal to epsilon, that will be contained in E delta and the measure of E delta is less than epsilon. So, that will prove that for every delta, we have got a stage n bigger than N naught such that x is f n x minus f x bigger than epsilon is less than delta; that means f n converges to f in measure. So, we have shown that almost uniform convergence implies convergence in measure. so these are the basic properties of relations between convergence in measure with almost uniform convergence, convergence pointwise, and so on

Now, we just now said that convergence in measure need not imply convergence almost everywhere, in general.

However, one can prove a partial result in this direction namely, if f n converges to f in measure, then there is a subsequence which converges almost everywhere. This result is quite useful sometimes when you are analyzing sequences of measurable functions which converge in measure.

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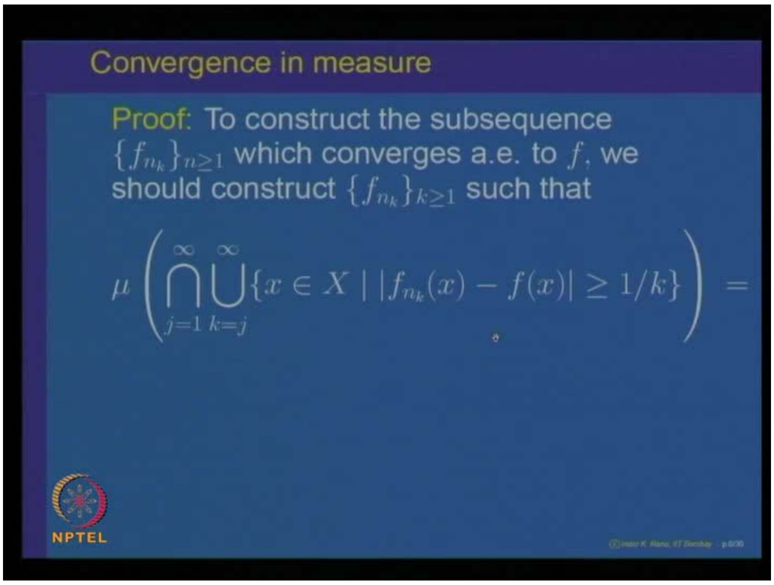

So, this theorem is called Riesz theorem. So, let us prove Riesz theorem. So, he says, let f n be a sequence of measurable functions converging in measure to a measurable function f; then there exist a subsequence f n k such that f n k converges to f everywhere, to almost everywhere; actually we should be saying to f n almost.

So, every sequence which converges in measure is a subsequence which converges almost everywhere.

So, let us see a proof of this. So, to prove this, we have what we are looking at we are looking at how to construct a subsequence f n k which converges almost everywhere to f. So, that means we have to find f n k such that…

So, we reformulate the problem as follows: Look at the set of points. So, f n k we want with the property that wherever f n k x minus f of x is bigger than 1 over k, the set of these points for k equal to union of such sets for k equal to j to infinity intersection over j equal to 1 into infinity - that must be equal to 0. So, we want a sequence f n k such that this measure of the set is equal to 0. Why is that rough?

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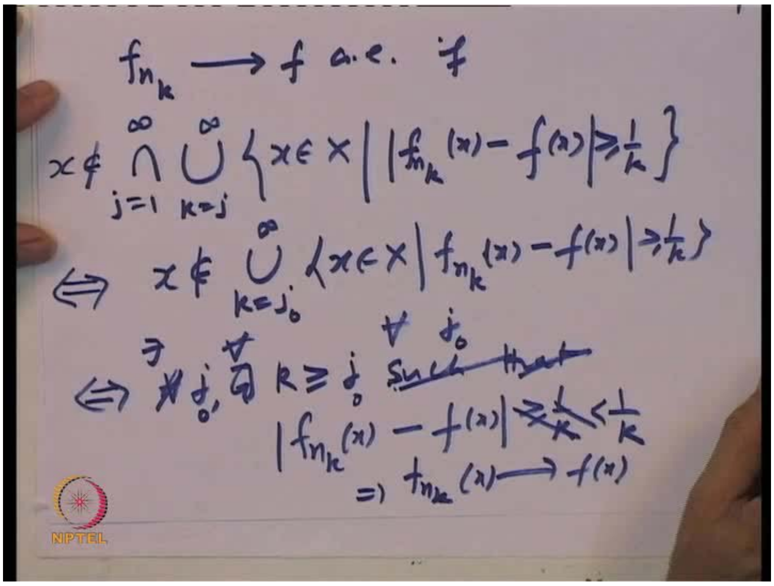

We are saying that f n k will converge to f almost everywhere if the following: if we look at the set intersection j equal to 1 to infinity union over k equal to j to infinity, the set of points x belonging to x such that mod of f n k x minus f of x bigger than 1 over k -This set has got measure 0. Why is that? Because if you take an element x belonging to this, what will that imply? That if and only if x belongs to; this is belonging to intersection; so, x belongs to union k over j to infinity, x belonging to x such that f n k x minus f of x is bigger than or equal to 1 over k, for every j - That must happen for every j, because it belongs to intersection. But what does that mean?

That is if and only if this belongs to j - that means for every j there exists k equal to j, k equal to j to infinity; so, that means for every j, there exist k bigger than or equal to j such that, saying that x belongs to this union means x will belong to at least one of them; that means x belongs to for this set, for some k bigger than or equal to j. so there is a k such that bigger than or equal to j such that x belongs to this such that that means f n k x minus f of x is bigger than or equal to k.

For such an x, this must hold. It is bigger than or equal to k; so, that means it does not converge. So, if we show that this set has got measure 0 (Refer Slide Time: 10:50), that means if x does not belong, then it must converge. So, what is the meaning of saying? If you want to say that if x does not belong to the set - that will mean what? It does not belong to the intersection; that means it does not that means there exist so x belongs to this does not belong to intersection; that means at least it does not belong to one of them. So, there exist some j naught such that x does not belong to this.

If x does not belong to this - for j some j naught, it does not belong to union. So, that means there exist A j naught such that from j naught onwards, x does not belong to this (Refer Slide Time: 11:31); that means x cannot belong to any one of them; that means for every k bigger than or equal to j, there is A j naught such that for every k bigger than or equal to j; bigger than or equal to does not hold; equal to j; that means this must happen for less than 1 over k.

So, saying x does not belong to this set will mean there is a j naught such that every k bigger than or equal to j naught; the distance is less than k; that will imply that f n k x converges to f of x. So, what do we have? What we are saying is essentially that this is the set of points (Refer Slide Time: 12:18) where f and k does not converge to f n. You want that set to have measure 0.

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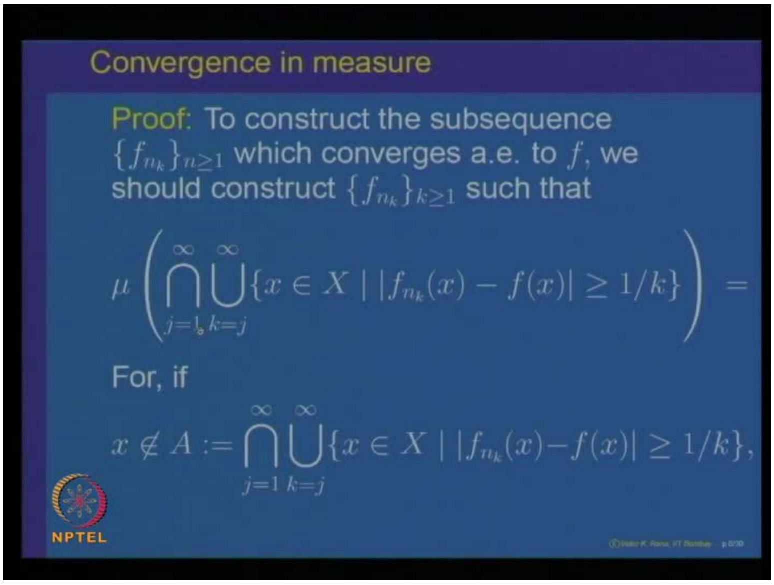

So, once again let me revise. Saying that f n k does not converge to f x is same as saying x does not belong to this set. So, if x does not belong, so for every x does not belong to intersection over j of some sets, that means there is a set, there is A j naught such that x does not belong to j naught. So, it belongs to this.

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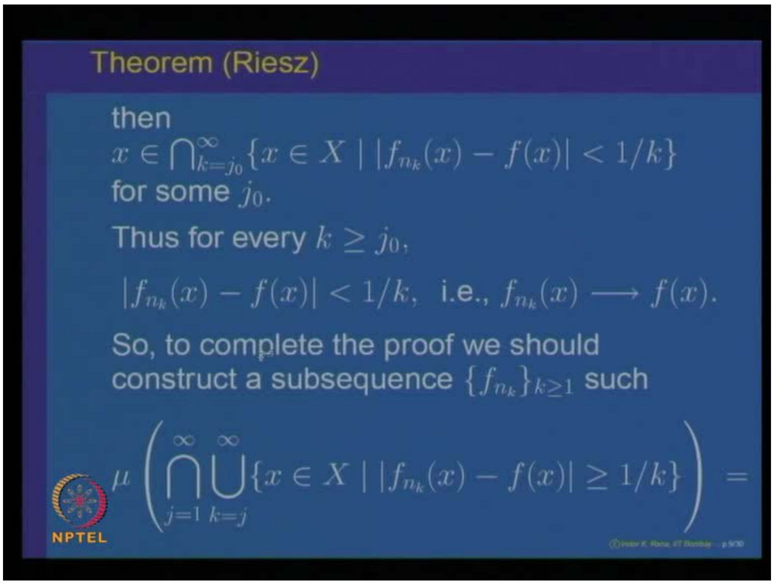

For some it should be union of k; so, that is a union here. So, that means for every k bigger than j naught, this is less than k. So, we have to only find a subsequence. So, we have to complete the proof. We have to construct a subsequence so that measure of this set is equal to 0.

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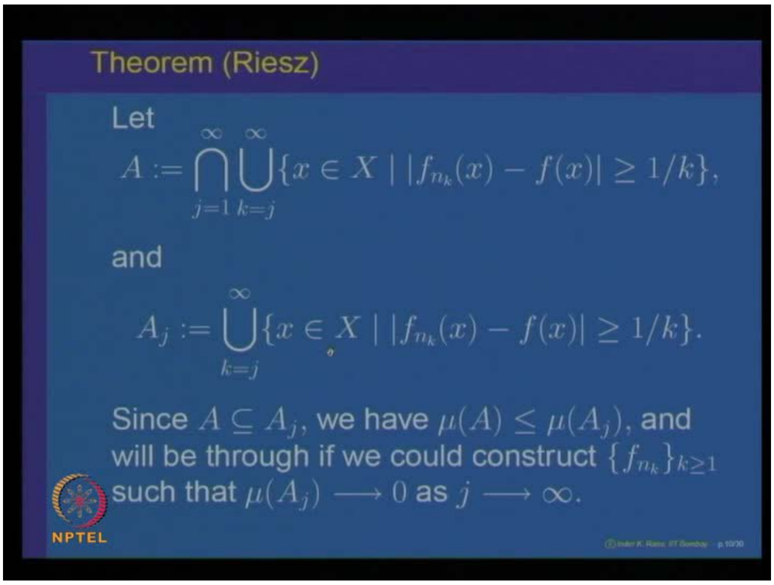

Let us write this set as A. So, A is the set of points where intersection over j equal to 1 to infinity union over k equal to j to infinity, where f n k x minus f x is bigger than or equal to 1 over k. Let us call the inner portion which is a union set as A j. So, what we want to prove is that mu of intersection A j is equal to mu of… So, we want to prove that mu of A is 0, but this set A is contained in A j because this intersection is smaller. So, A is subset of A j.

So, showing that mu of the set A is 0, A is contained in A j, that means mu of A is less than or equal to mu of A j. So, we will be through if we can prove that mu of A js converge to 0 as j goes to infinity. So, we have to construct a subsequence f n k with this property that mu of A js of these sets must go to 0 as j goes to infinity. So, this will be true.

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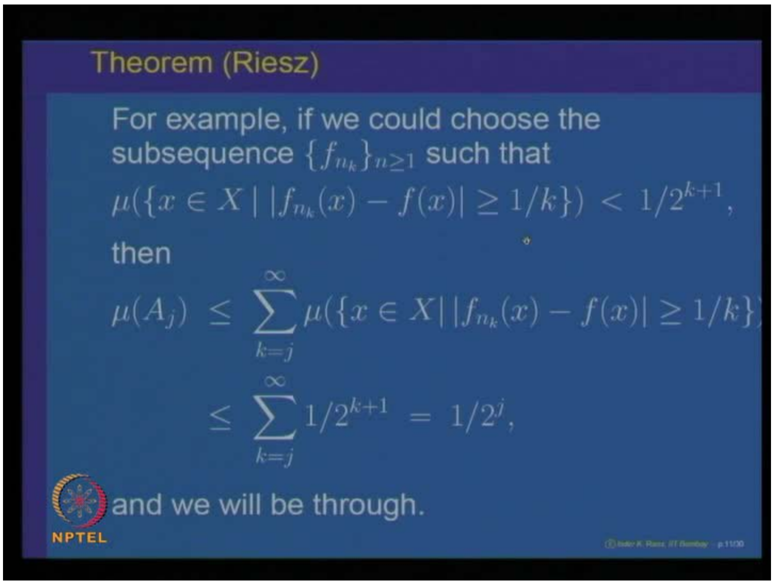

We want mu subsequence that mu of A js go to 0. That will be true if you could find, say for example, a sequence f n k such that the measure of the set where f n k minus f x is bigger than or equal to k, say it is less than 1 over 2 to the power k plus 1.

Suppose we can choose our subsequence in that way; then what will happen? Then, mu of A j which is nothing but union of from k equal to j to infinity of this set; that is by countable sub additive property mu of A j will be less than or equal to summation k equal to j to infinity mu of these sets and each mu of these set is less than 1 over 2 to the power k plus 1. So, that is summation j equal to infinity. So, this is a geometric series with common ratio less than half. So, that will give you 1 over 2 to the power j.

So, we will get mu of A j less than 1 over 2 to the power j. So, as j goes to infinity, we will get mu of A j equal to 0 and hence we will be through.

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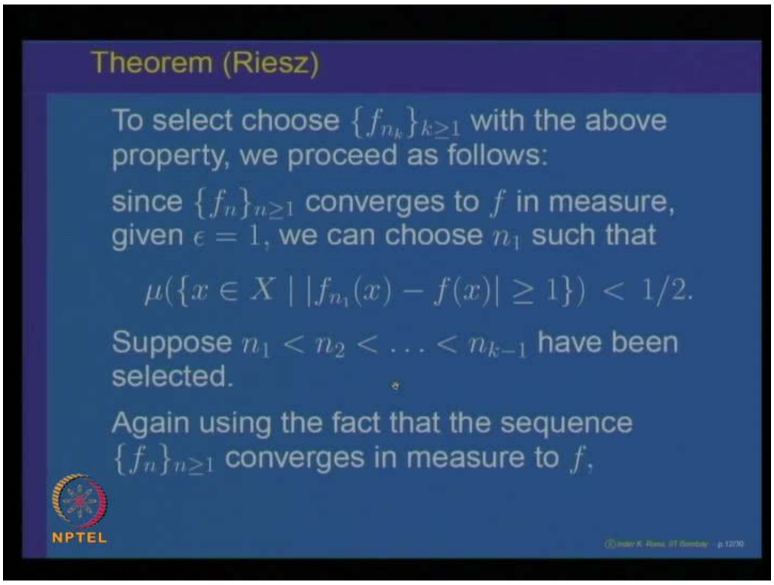

So, we have to only find a subsequence f n k with this property and that is done by using the fact that f n convergence to f in measure. Because it converges in measure, by the property of convergence in measure, let us start with epsilon equal to 1. Then convergence in measure says that mu of set of those points where f n minus f x is bigger than 1 will be less than half for every… that goes to 0. So, that means after some stage, the difference of the measure. So, measure of the set where f n minus f x is bigger than or equal to 1 will be small after some stage. So, that stage we called as n 1.

So, using the fact the convergence in measure, we can choose n 1 such that measure of the set where f n 1 minus f x bigger than 1 is less than half.

Now, you proceed inductively. Supposing we have selected n 1 less than n 2 up to n k minus 1 have been selected with those required property, then by fact that is sequence f n is converging in a measure for epsilon equal to 1 over 2 to the power k plus 1.

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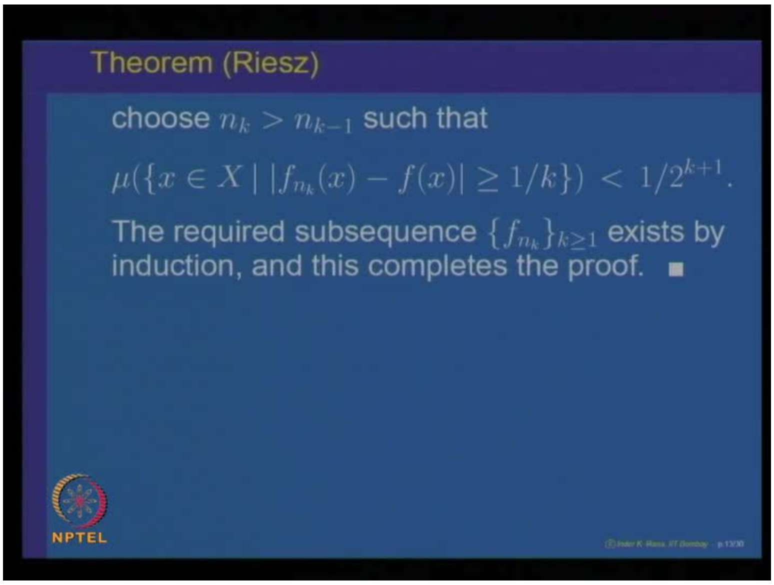

So, let n k a stage bigger than n k minus 1 such that the measure of the set f n k minus f x bigger than 1 over k is less than 1 over 2 the power k plus 1. So, by induction, this existence is instance of a sequence f n k with this property is complete. Hence, we will have that this subsequence converges almost everywhere to f. So, this is an important result which is used sometimes to analyze sequences which converge in measure. So, this is called Riesz theorem.

So, we have looked at the various properties of so till now we looked at various properties of convergence: convergence almost everywhere, convergence pointwise, convergence almost uniform, convergence uniform and convergence in measure, and relations between such modes of convergence,

There is another mode of convergence which arises when the functions f ns are in L p spaces. Recall, we have defined L p spaces the pth power integrable functions. So, one would like to know that when sequences converge in L p, does this convergence have any relation with convergence in measure - pointwise convergence, or convergence almost uniformly? So, we will analyze these things next.

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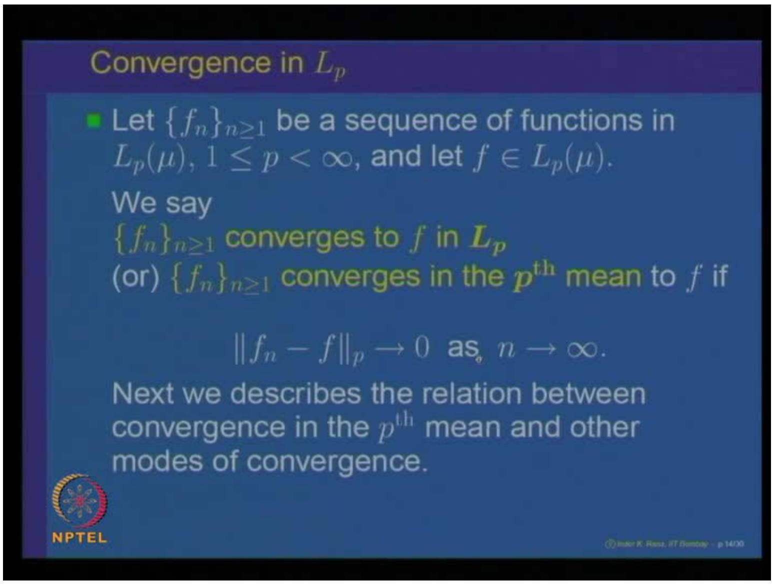

Let us recall what is the meaning of saying that have a sequence converges in the pth mean or L p. So, saying that a sequence f n, there p is bigger than 1 less than infinity and f belongs to L p.

So, let us take functions f ns and f in L p. So, saying that f n converges to f in L p or sometimes one also writes this as f n converges in the pth mean to f, if the pth norm of f n minus f goes to 0, as n goes to infinity. This called convergence in the pth mean.

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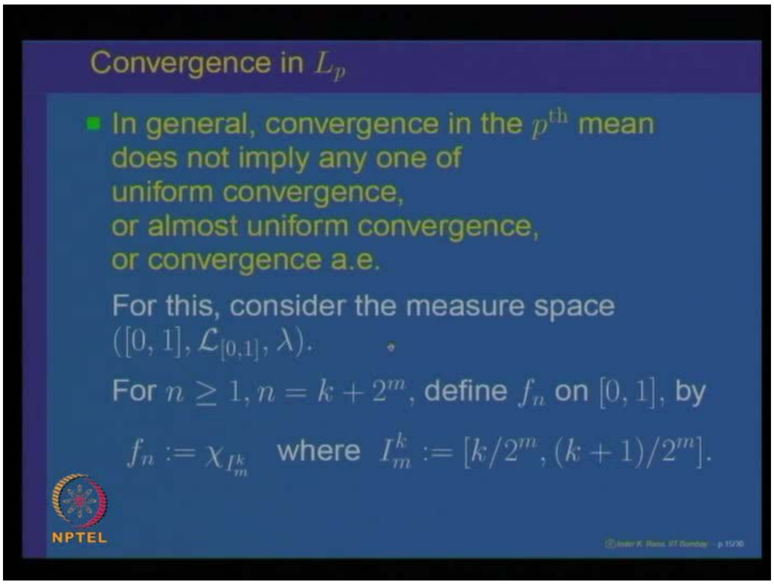

So we would like to find out the relations between the pth means the and other modes of convergence

So the first property is that in general convergence in the pth mean does not imply uniform convergence it and it need not imply almost uniform convergence or convergence almost everywhere. So convergence in pth mean need not imply any one of uniform convergence, almost uniform convergence or convergence almost everywhere

So, given example of measure space and a sequence of functions in L p such that f n converges to f in L p, but f n does not converge uniformly or almost uniformly or almost everywhere. So, we will give an example of a sequence which converges in L p, but does not converge almost everywhere. So, that example itself will imply that pth mean cannot imply uniform convergence because uniform implies converges almost everywhere.

Similarly, same example will be sufficient to say that pth mean does not imply almost uniform convergence because uniform convergence implies almost uniform convergence.

So, we want to construct a sequence of functions in L p such that the sequence converges in L p, but does not converge almost everywhere. So, for that, let us look at the measure space - the interval 0 1, Lebesgue measurable sets in 0 1, and lambda - the length function.

let us write, if you recall - we had constructed a sequence of measurable functions on this measure space by looking at dividing the interval into equal parts by using the binary points, and that was an example which showed that convergence in measure does not imply convergence almost everywhere.

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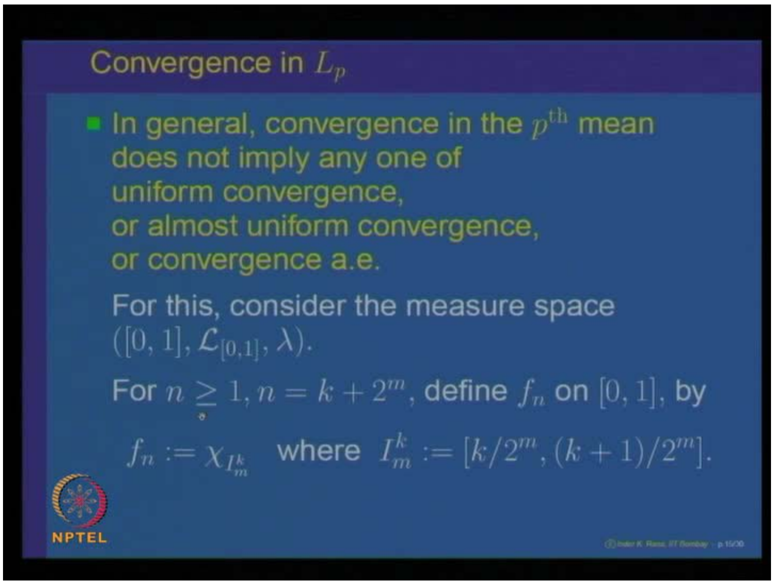

So, that is the sequence we are looking at again. So, let us recall that sequence: f n for every n bigger than or equal to 1; let n be represented as k plus 2 to the power m where this k is between 1 and 2 to the power m. So, we showed that, for every n can be written uniquely in the form k plus 2 to the power m where m is a positive integer and k is between 1 and 2 to the power m.

So, whenever m n is represented this way, we define f n to be the indicator function of the interval k by 2 to the power m and k plus 1 by 2 to the power m. So, look at the k th interval of length 1 over 2 to the power m and define f n to be the indicator function of this interval.

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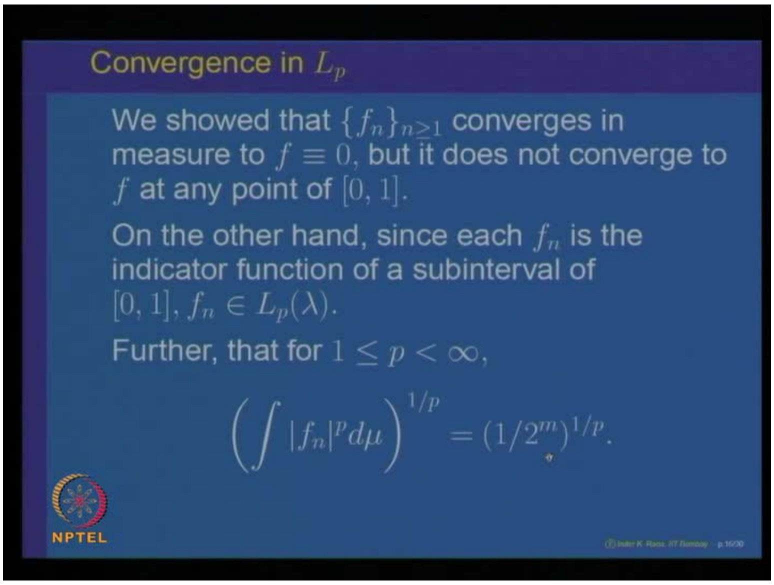

So, we showed earlier that this sequence of measurable functions does not converge almost everywhere and this converges in measure to the function identically equal to 0. So, this sequence of measurable functions converges in measure to f identically 0, but it does not converge to f at any point in 0 1. So, this does not converge almost everywhere or at any point, actually.

Let us just prove that this sequence of measurable functions is in L p because there is an indicator function of an interval. So, the function takes the value 1 on this interval and interval is finite. So, obviously, it implies this is an L p integrable function. So, indicator functions of a sub interval; so, f n belongs to L p. So, we have got a sequence of functions in L p.

Let us look at the what is the L p norm of f n because it is the indicator function of an interval of a length 1 over 2 to the power m. So, it is precisely equal to 1 over 2 to the power m; the function takes the value 1 on the interval of length 1 over 2 to the power m; so, its L p norm is 1 over 2 to the power raise to the power 1 over p.

So, this implies that the norm of f n minus f which is identically 0 goes to 0 as n goes to infinity because as n goes to infinity, m also goes to infinity.

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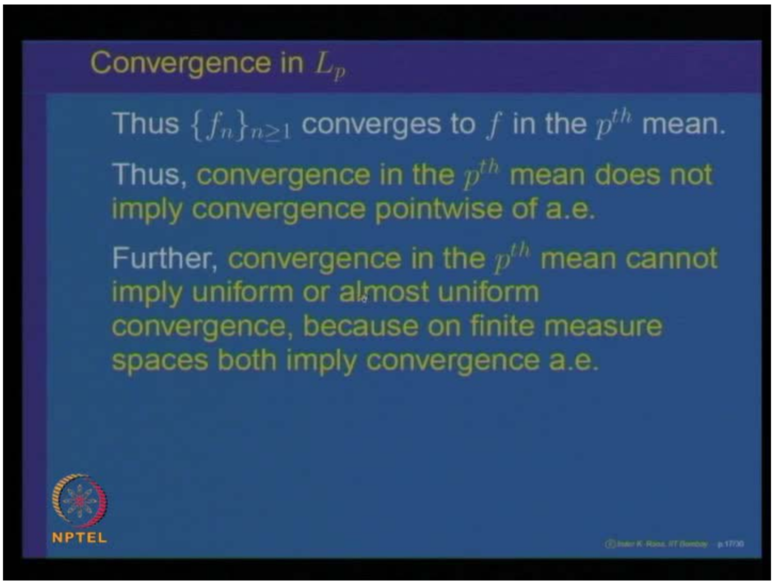

So, this is the sequence of functions in L p which converges to the function identically 0, but the sequence does not converge almost everywhere.

So, that proves our claim namely, convergence in the pth norm does not imply convergence uniform or convergence almost uniform or convergence pointwise. So, convergence in the pth mean does not imply this; so, as we indicated, this cannot imply uniform or almost uniform because both of them imply convergence almost everywhere.

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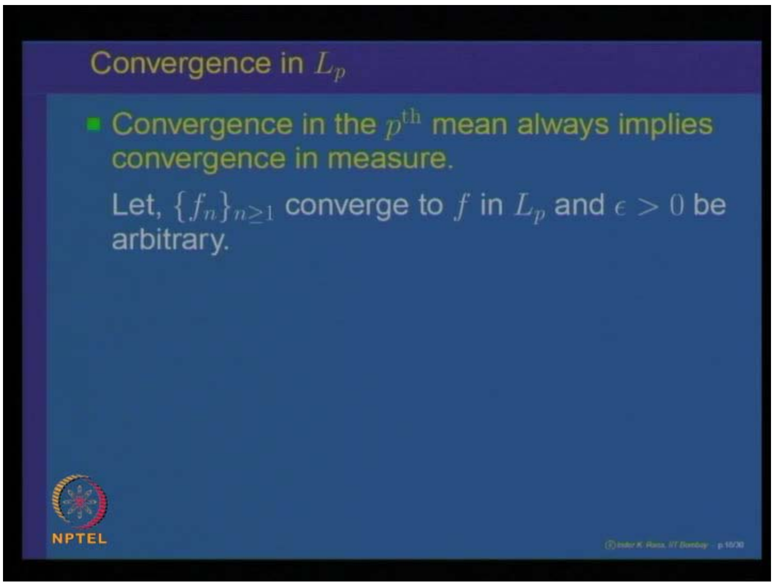

Next, let us look at fact that convergence in the pth mean always implies convergence in measure. So, let us take a sequence f n which converges to f in L p and let epsilon greater than 0 be arbitrary. So, what we have to show? So, let us just look at what we have to show - f n converges f in L p.

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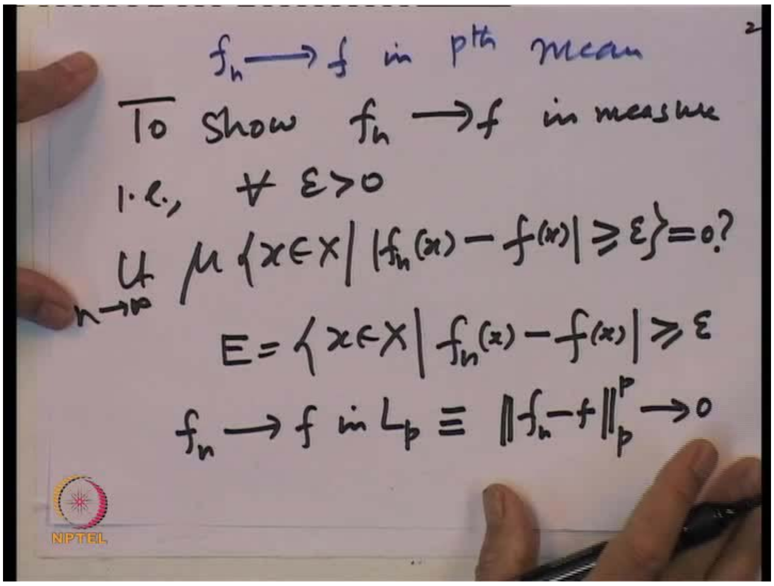

We have given that f n converges to f in pth mean. To show - f n converges to f in measure; so, what does it mean? That is for every epsilon bigger than 0, we have to look at the measure of the set x belonging to x such that f n of x minus f of x bigger than or equal to epsilon, limit n going to infinity; that must be equal to 0. So, this is what we have to show.

So, let us call E to be the set x belonging to x where f n x minus f of x is bigger than or equal to epsilon. Now, we know that f n goes to f in L p; that is equivalent to saying that f n minus f norm of that; here the norm goes to 0. Let us take the power p also. what will that mean is the following.

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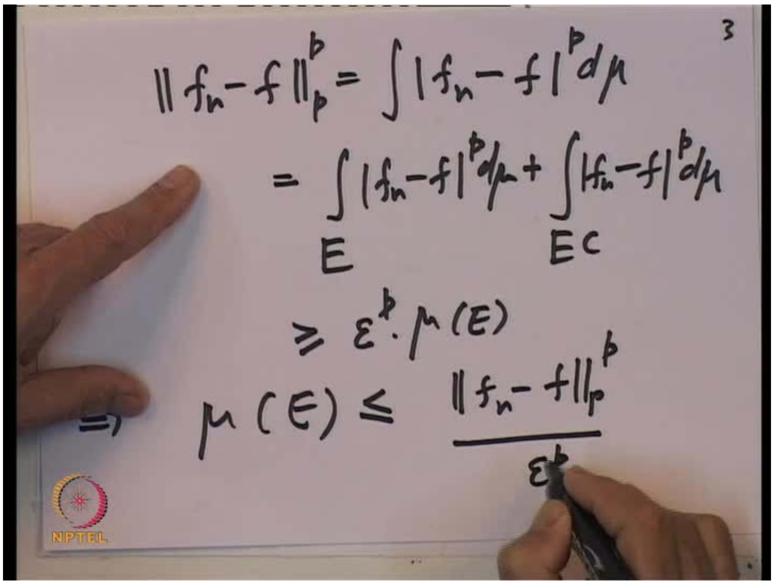

So, saying that in L p, let us look at the norm f n minus f p b; this is equal to integral mod f n minus f to the power p d mu. So, this goes to 0, but this is equal to integral over E of the same thing - f n minus f to the power p d mu plus integral over E complement of the same thing; so, mod f n minus f to the power p d mu.

Now, on the set E, f n is bigger than or equal to epsilon; so, on the set E, this difference is bigger than epsilon. So, it is bigger than or equal to epsilon to the power p into mu of the set E plus integral over E complement, but this is a non-negative function; so, even if I drop this term, it will still remain bigger than or equal to mu of E. So, that implies that mu of E is less than or equal to mod mu of E is norm of f n minus f to p divided by epsilon point to power p.

So, the inequality that we get for this is actually an important inequality in the theory of probability and this is called Chebyshev's inequality; very simple inequality, but you see, it gives the powerful outcome. So, that is following because f n converges to f n L p, so this goes to 0 (Refer Slide Time: 28:00); that means that mu of E equal to 0.

So, here f n is bigger than f by distance epsilon mu of that it depends on epsilon; so that goes to 0 for every epsilon. So, that will prove that, if f n converges to f in the pth mean, then it converges in the measure.

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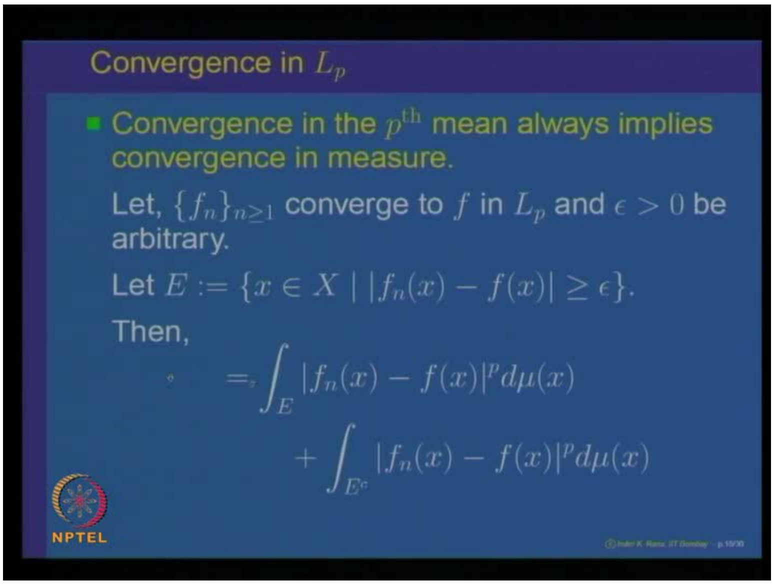

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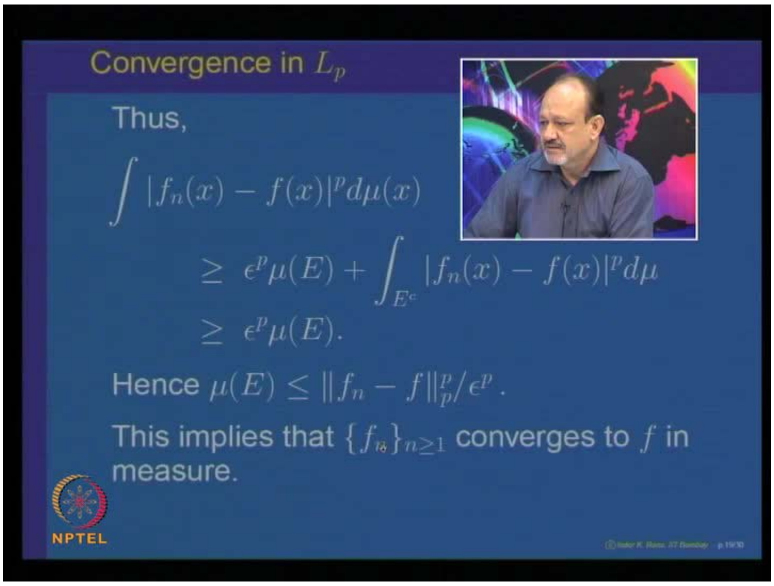

So, as we looked just now, the proof is simple look at. The set where f n minus f x is bigger than or equal to epsilon, the integral mod of f n minus f n, norm of f n minus f to the power p can be separated into two parts: One part where the function f n minus f is bigger than epsilon is this and remaining part will drop. So, inequality still stays; so, mu of E is less than or equal to this, which goes to 0.

So, that proves that convergence in the pth mean implies convergence in measure.

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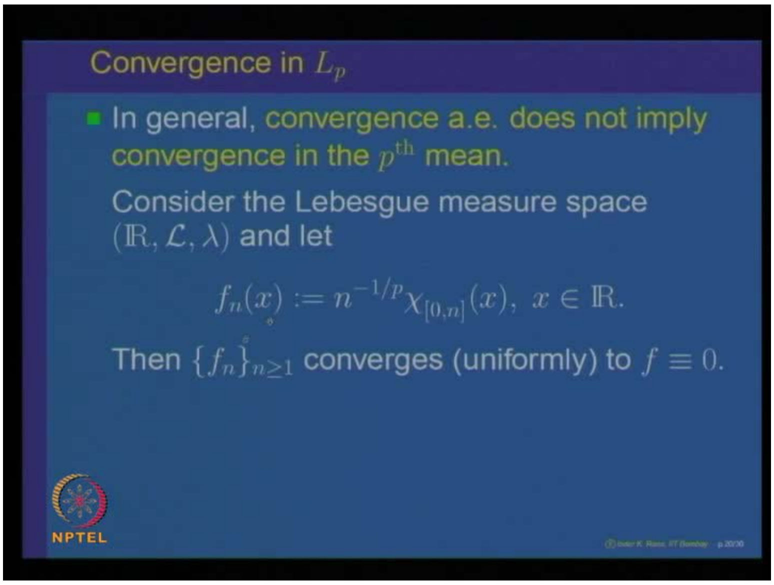

However, in general, the convergence almost everywhere does not imply convergence in the pth mean.

So, for that, let us look at a simple example. Look at the Lebesgue measure space R Lebesgue measurable sets lambda. So, that is the Lebesgue measure, the Lebesgue measure space and look at the function f n which is defined as n raise to power minus 1 by p into the indicator function of the interval 0 to n.

So, first of all, let us observe that this function belongs to L p because integral of f n raise to power p is just n to the power minus 1 to the power p. So, that is 1 over n and into the measure of the interval 0 and that is n. So, each f n is a L p function; its norm is equal to L p norm is 1 and this convergence uniformly to f. That is obvious because the values of the function that is taken on a larger and larger interval is 1 over n; so, becoming smaller and smaller. So, we can always find for every x, we can find a stage after which the distance will be less than 0. So, that means f n converges. So, it is easy to check. Let us just verify that f n converges to f uniformly.

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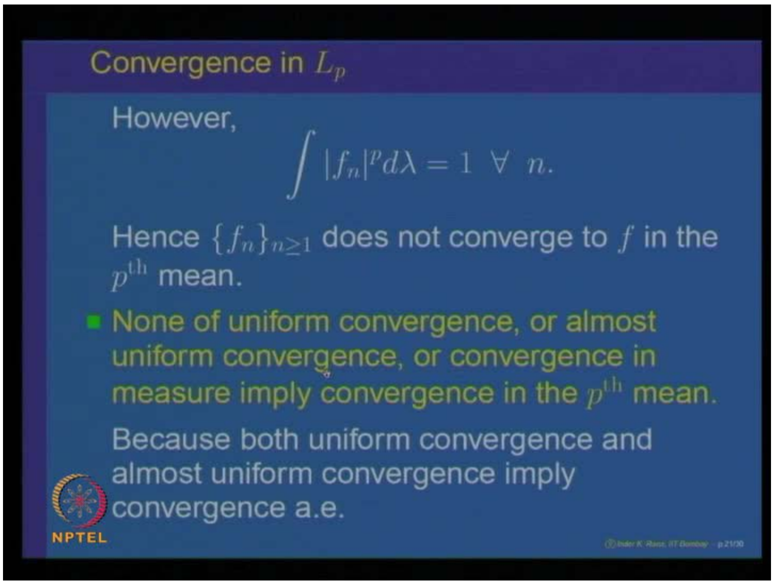

However, we just now said that the L p norm of each f E is 1; that does not converge to f in the pth norm. So, what we are saying is convergence almost everywhere does not imply convergence in the pth mean. This is convergence almost everywhere.

Here is another observation that none of uniform convergence or almost uniform convergence or convergence in measure imply convergence in the pth mean - that means none of these; so, that means in general, uniform convergence does not apply convergence in the pth mean or convergence almost uniform does not imply, and similarly, convergence in measure need not imply convergence in the pth mean.

Because obviously both uniform convergences are almost uniform convergence, apply convergence almost everywhere. So, if this were true, then we will have a contradiction; so, this is true.

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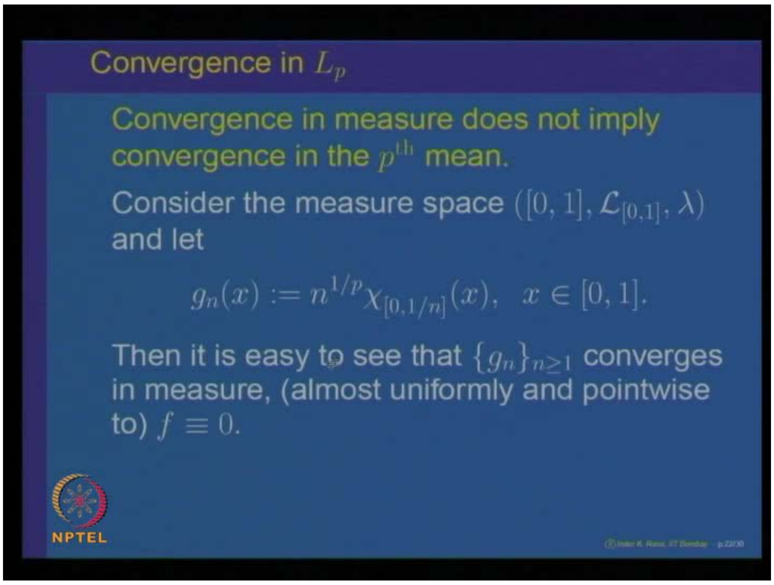

Next, we want to say that convergence in the measure need not imply convergence in the pth mean. So, for that, let us look at the example of the measure space 0 1, Lebesgue measure space. Let us define g n of x equal to n to the power 1 over p; that is something similar to the previous one; instead of minus 1 over p, it is n to the power p and into the indicator function of 0 to 1 over n.

So, here, the interval where function is nonzero is shrinking, but the value is increasing. In the previous one, the value was decreasing and the length was increasing. So, it is other way round of this.

So, look at this function - g ns all of this functions. g n are in L p because integral mod g n will be equal to the power p will be n into to indicator function; so, integral is equal to 1. So, each one of them has got an L p with integral equal to 1. It is obvious that the sequence g n converges in measure; it converges in measure; that is obvious because the set where it is going to be bigger than or equal to epsilon is going to be shrinking. It is 0 to 1 over n; so, that will go to 0. So, converges in measure and identically equal to 0, and its integrals L p norms are equal to 1.

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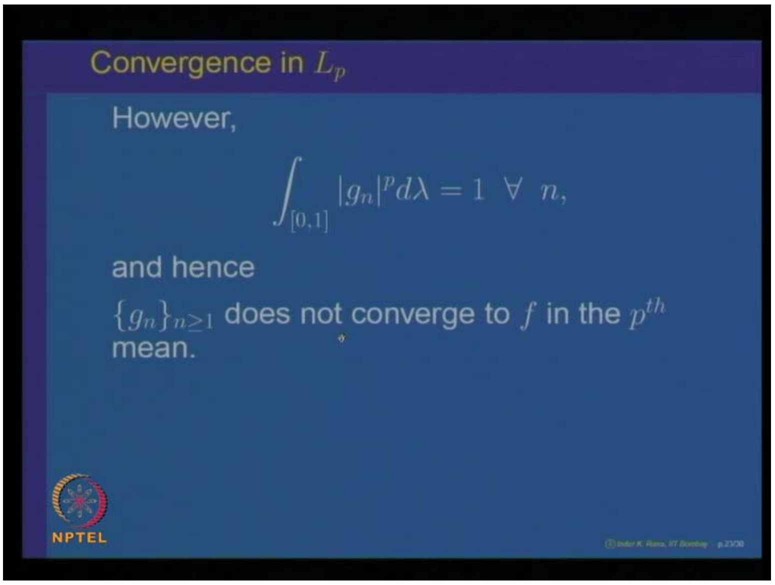

Hence, g n does not converge to f identically 0 in the pth mean. That means, we as produce a sequence of functions which is convergence in measure, but does not imply convergence in L p space.

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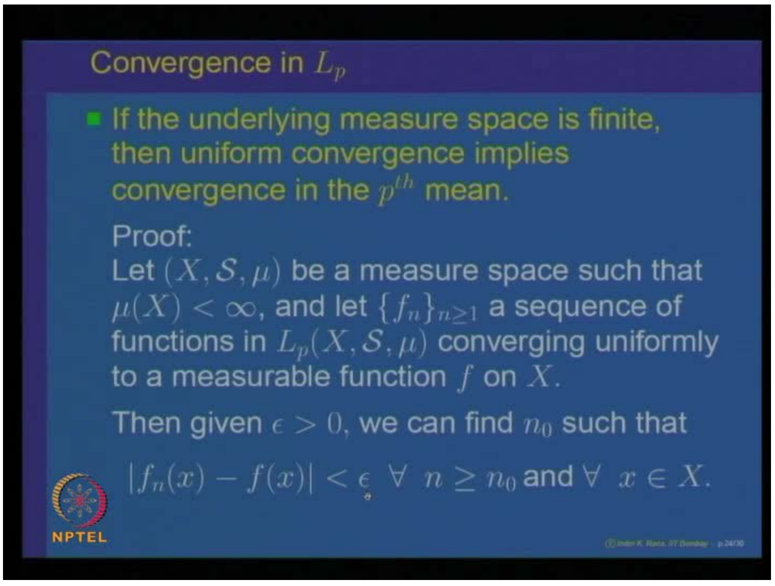

However, if the underlying measure space is finite, then the uniform convergence does imply convergence in the pth mean. So, that is quite obvious to verify.

Let us just look at look at mu of a measure space so that mu of x is finite and let f n be a sequence of functions in L p, converging uniformly to a measurable function f. We want to show that f n converges to f in L p also.

What is uniform convergence? uniform convergence means for every epsilon bigger than 0, there is a stage after which… given every epsilon bigger than 0, you can find a stage n naught such that the distance between f n and f is less than epsilon, for every n bigger than or equal to n 0 and for every x. So, uniform means for every x, the same stage satisfies the required property. So, for every x, there is a single stage n naught such that f n x minus f of x is less than epsilon.

So, now, let us look at the absolute value of the function mod f to the power p. See we have just given that f is converging uniformly and each f n is in L p, but we do not know whether f is in L p or not.

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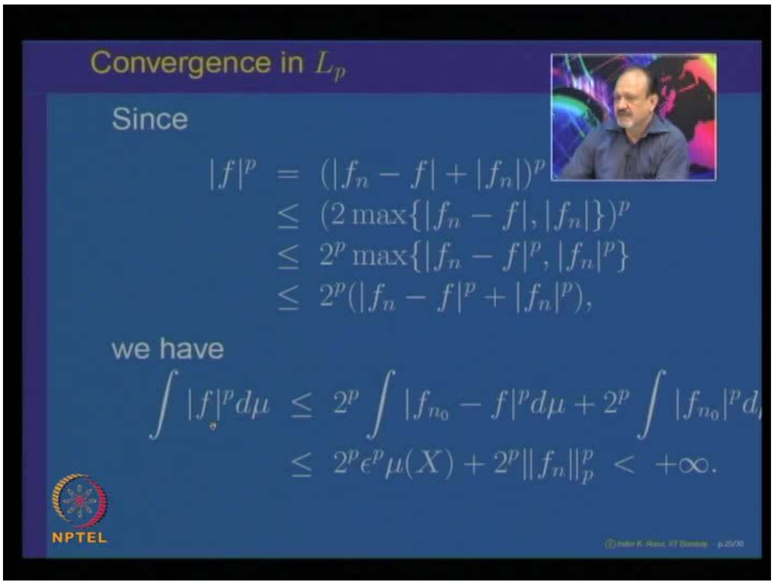

We have to first prove that this is in L p, but look at the absolute value of f to the power p. So, by triangle inequality, I can write it as mod of f n minus f plus. So, it should be always less than or equal to; by triangle inequality add and subtract f n to the power p.

Now, absolute value of a plus b is always less than or equal to 2 times the maximum of the 2 values. So, this is less than or equal to 2 times the maximum value f n minus f and absolute value of f n; of course, everything raise to power p. But that is same as less than or equal to 2 to the power p and the maximum of this to the power p. Now, 2 to the power p and this maximum will always be less than or equal to the sum. So, we can write this is less than or equal to 2 to the power p mod of f n minus f to the power p plus mod of f n to the power p.

So, what does that imply? We can integrate both sides with respect to mu. So, integral will be less than or equal to 2 to the power p into f n naught minus f to the power p d mu plus 2 to the power p of f n naught - we just now showed this is less than epsilon. So, this will be less than 2 to the power p epsilon to the power p mu of x plus 2 to the power p norm of f n to the power p; each of them being finite says that the function f is in L p.

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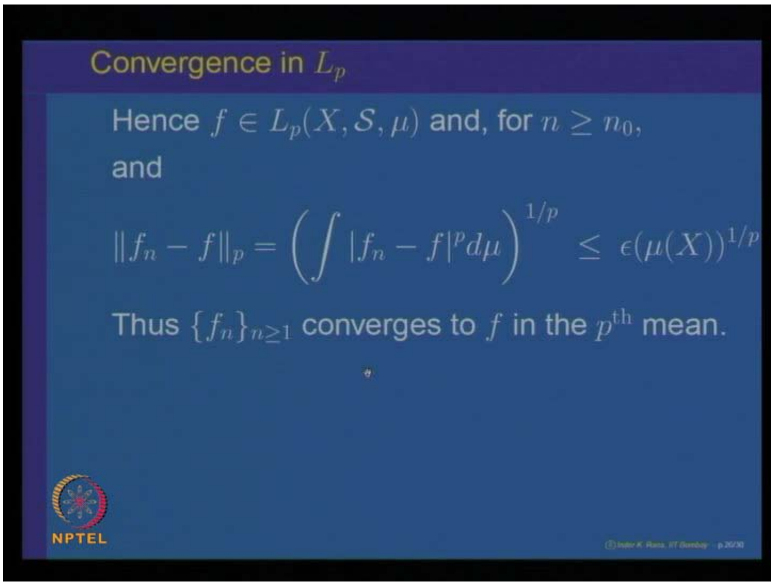

So, what we have shown is - if the underlying space is having finite measure and f ns belongs to f, f ns belongs to L p and converges uniformly to f that the limit function is also in L p; so that is what we have shown.

Now, look at the difference. So, the difference of f n minus f to the pth norm is by definition integral of mod f n minus f to the power p raise to power 1 over p; but for n bigger than n naught, this difference is less than epsilon to the epsilon to the power p raise to the power 1 over p; so that is epsilon into mu x raise to the power 1 over p. So, as epsilon goes to 0, this will go to 0. So, that proves that f n converges to f in the pth mean.

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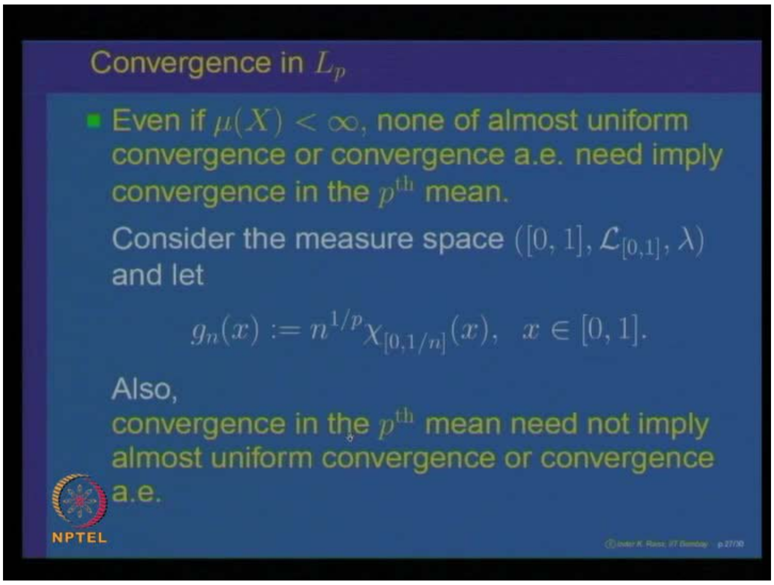

Let us also analyze what happens: The relations between L p spaces and other modes of convergence and the underlying spaces of finite measure. So, the result says, even if the underlying convergence space is finite, then none of uniform convergence or convergence almost everywhere need imply convergence in the pth mean. So, this condition is not good enough to ensure that almost uniform convergence that will imply.

We have just looked at the function g n x is equal to n to the power 1 over p. So, these are all L p functions and their L p norm equal to 1. So, they can all converge in L p, but we know that this converges in measure and hence almost uniform. Also, the convergence in the pth mean need not imply almost uniform convergence or convergence almost everywhere. Other way round, inequality - even when this is finite, the convergence in pth mean need not imply almost uniform or convergence.

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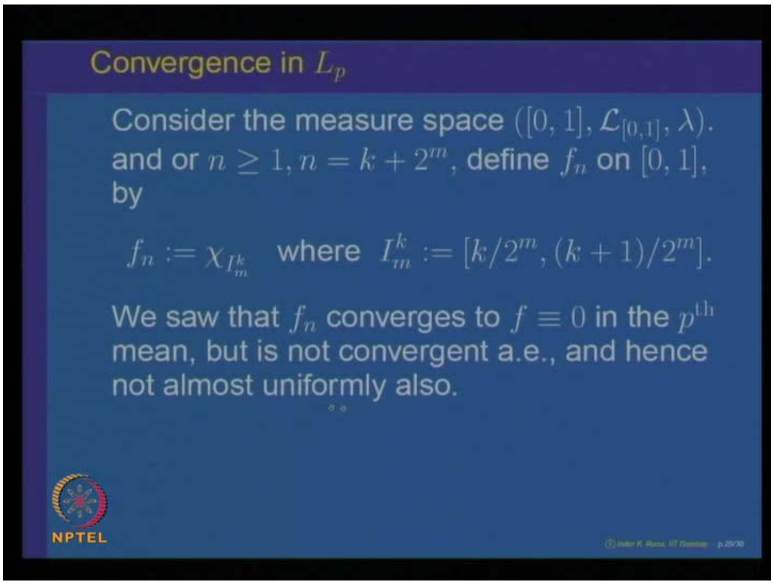

So, that earlier example of the measure space 0 on 0 1, Lebesgue measurable sets, and f n to be the indicator function of the interval I k m, where I k m is the interval of length to the power m - just now we consider this example. So, these examples of functions they converge to f uniformly. So, it converges to f identically 0 in the pth mean, but we know that does not converge almost everywhere. Hence, it can also converge almost uniformly because almost uniform convergence implies converges almost everywhere.

So, these are the various ways of looking at modes of convergence and analyzing them. Let me just look at all the implications put together in the form of a diagram. So, here is the diagram; look at this.

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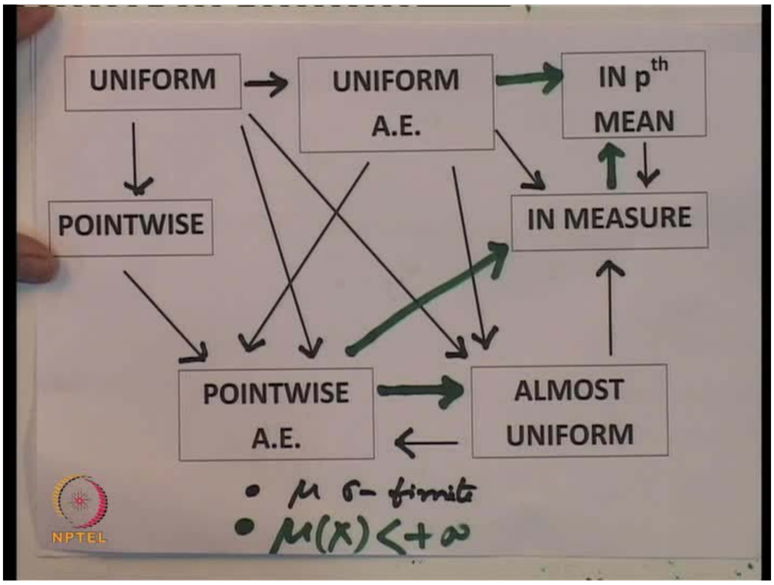

So, here, we have got the vision of uniform convergence uniform almost everywhere in the pth mean here is pointwise, pointwise almost everywhere, in measure and almost uniform. As we all know that uniform convergence is the strongest one. So, uniform we have already seen earlier; the uniform implies pointwise; of course, pointwise implies pointwise almost everywhere, and this other way round - implications need not hold; simple examples.

Uniform will imply uniform almost everywhere; actually uniform is uniform almost everywhere with the set to be empty set. So, uniform implies uniform almost everywhere. Just now we have proved that mu of x is finite; so, this green arrow indicates that implication under the condition that mu of the whole space is finite. So, uniform convergence almost everywhere and underlying space finite implies convergence in the pth mean - that we saw just now.

We have already seen uniform implies pointwise apply pointwise almost everywhere. and we also saw that pointwise converges pointwise almost everywhere; in general linearly implied convergence in measure, but when an underlying measure space is finite, pointwise will imply convergence in measure. Of course, pointwise almost everywhere need not imply almost uniform, but we showed today that if the underlying space is finite measure, then the pointwise implies almost uniform. Obviously, almost uniform we have shown implies pointwise almost everywhere.

Finally, we also showed today that almost uniform implies in measure and when the underlying space is finite, in measure implies convergence in the pth mean; we use Chebyshev's inequality. Just now we said that converges always in the pth mean always implies convergence in measure. So, this is the overall picture of various modes of convergence.

With this, we come to the end of the basic concepts of measure theory; whatever we have not done is essentially looking at various ways of necessary and sufficient conditions under which a sequence f n converges to f in L p - that is one thing we have not done, but the limited scope of lectures.

We wanted to cover in almost 40 lectures the basic concepts of measure theory and that we have covered. Another thing that we have not proved is looking at the change of variables formula for R n; that again requires a bit of work and technical things. So, we have not covered that in our discussions.

So, with that, we come to an end of this course video lecture on Measure Theory. In the next lecture, I will just try to give an overall view of the things that we have done in our course.

Thank you.