Measure and Integration

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Module No. # 01

Lecture No. # 04

Monotone Class

Welcome to lecture 4 on Measure and Integration. I recall we have been looking at classes of subsets of a set X with various properties. We started with the collection called semi-algebra of subsets of a set X. Then, we looked at what is called the algebra of subsets of a set X. Today, we will start with looking at some more classes of subsets of set X.

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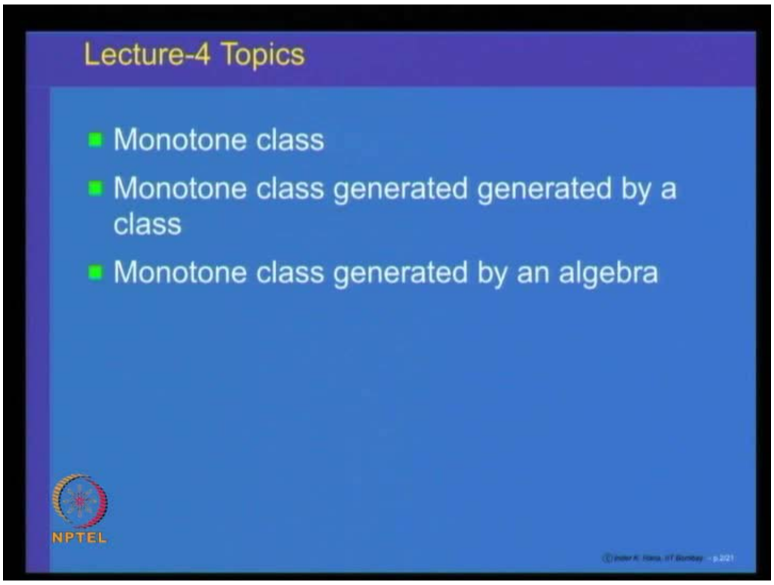

We will start with what is called a monotone class and then we will look at the monotone class generated by a collection of subsets of a set X. Then, go over to describe what is the monotone class generated by algebra. That is an important relation, which we will be using again and again.

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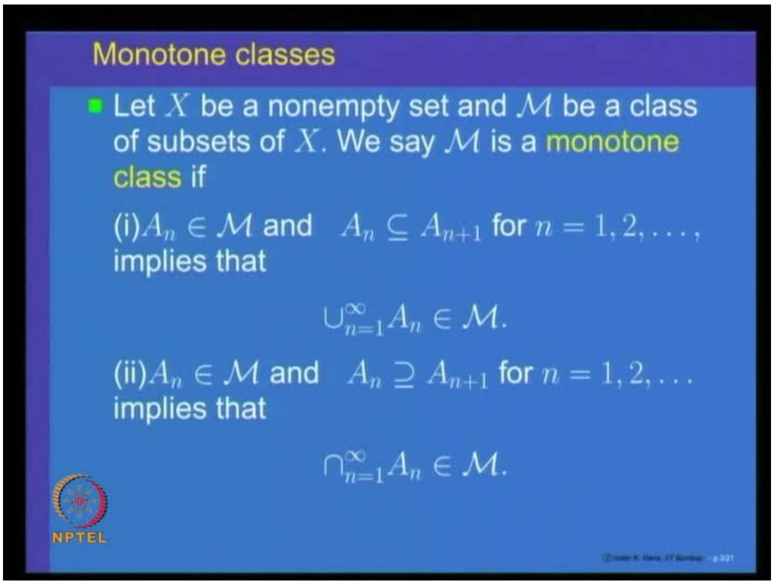

Let us start with describing what a monotone class is. A monotone class is a collection of subsets of a set X. Let us denote that collection of subsets by M. So, M is a collection of subsets of X and we say it is a monotone class if it has the following two properties: One - whenever there is a sequence of sets A n belonging to the collection M such that the sequence is increasing; that means, for every n A n is a subset of A n plus 1, then we demand that the union of these sets A n's also belong to M. So, the first property is that the collection M of subsets of set X is closed under unions of increasing sequences. The second property that we expect from this collection is that whenever a sequence A n is in M and A n is decreasing; that means A n is a subset of A n plus 1 for every n, then the intersection of the sequence of sets A n should also be an element of M.

Let us recall once again what a monotone class is. A monotone class is a collection of subsets of a set X with the two properties: one - for every sequence of sets A n in M such that A n is increasing, their union also belongs to M. Secondly, whenever A n is a sequence of sets in M such that A n is decreasing, then the intersection of the sets is also in M. That is why the name monotone comes. That means this class M of subsets of M is closed under monotone sequences. Whenever a sequence A n is increasing in M, their union belongs to M. Whenever a sequence A n is decreasing in M, then their intersection also belongs to M. So, such a collection of subsets of a set X is called a monotone class.

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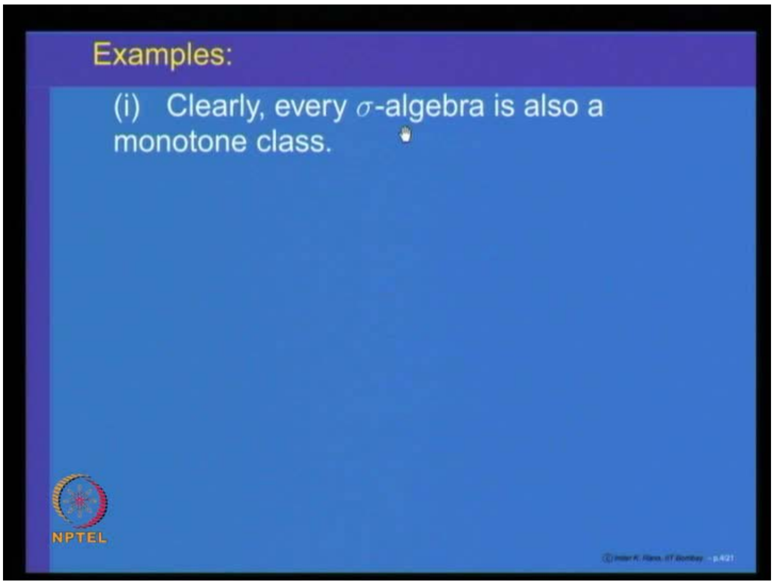

Let us look at some examples of such collections. Firstly, let us observe that every sigma-algebra is also a monotone class. Why is that true? Because a sigma-algebra is a collection of subsets of X, which is closed under any countable unions. Because it is closed under countable unions, it will also be closed under increasing unions. So, first property will be true. Secondly, if a sequence A n is decreasing in a sigma-algebra, then look at the complements of that sequence; that sequence of complements of the sets will be an increasing sequence of sets. Because it is a sigma-algebra, it is also closed under complements. So, A n complements will belong to it. So, union of A n complements belong to it. That means the intersections of A n's complement belong to it. That means the intersections of A n's belong to it.

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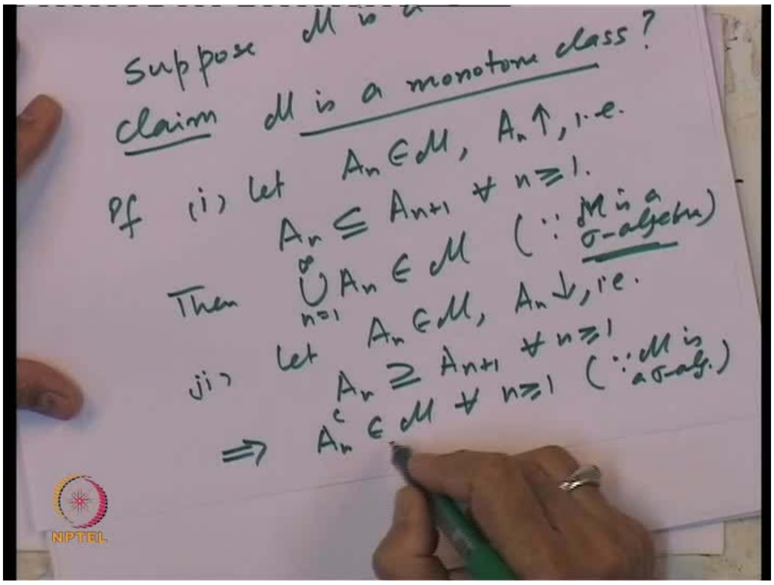

Let us look at this property, how do we write it? Let us look at… Suppose M is a sigma-algebra, claim M is a monotone class. To prove this, how do we go ahead? One - let A n belong to M and A n's increase. That is, A n is a subset of A n plus 1 for every n bigger than or equal to 1. Then, union of A n's n equal to 1 to infinity belong to M because M is a sigma-algebra. Because it is a sigma-algebra, it is closed under all unions and hence, such types also. Secondly, let us take – let A n belong to M and A n's decrease. That is, A n includes A n plus 1 for every n bigger than or equal to 1. So, in that case, that implies - because A n's belong to M and M is… So, A n complements belong to M for every n bigger than or equal to 1 because M is a sigma-algebra. Because it is a sigma-algebra, it is closed under complements.

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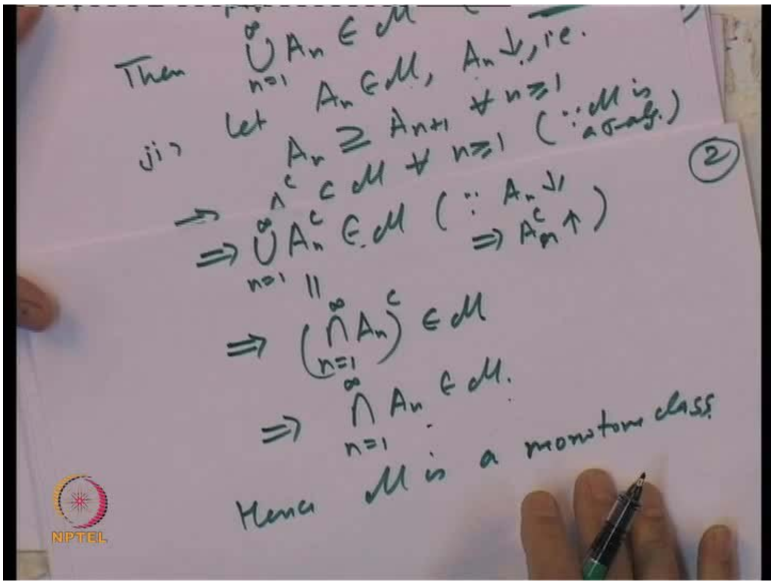

Once that is true, that implies that A n complement union n equal to 1 to infinity belongs to M because A n decreasing implies a sequence of A n complements will be increasing. Just now we saw that whenever a sequence is increasing, their unions belong to M. However, that implies but, what is the set? that is intersection of n equal to 1 to infinity A n complement. That is by de Morgan's law. So, this belongs to M.

Now, M is a sigma-algebra. So, that implies that intersection n equal to 1 to infinity A n belongs to M. We have shown that whenever sequence A n is in M and A n's are decreasing, that implies the intersection also belongs to M. Hence, M is a monotone class.

The first proposition or first observation (Refer Slide Time: 07:15) is that every sigma-algebra is also a monotone class.

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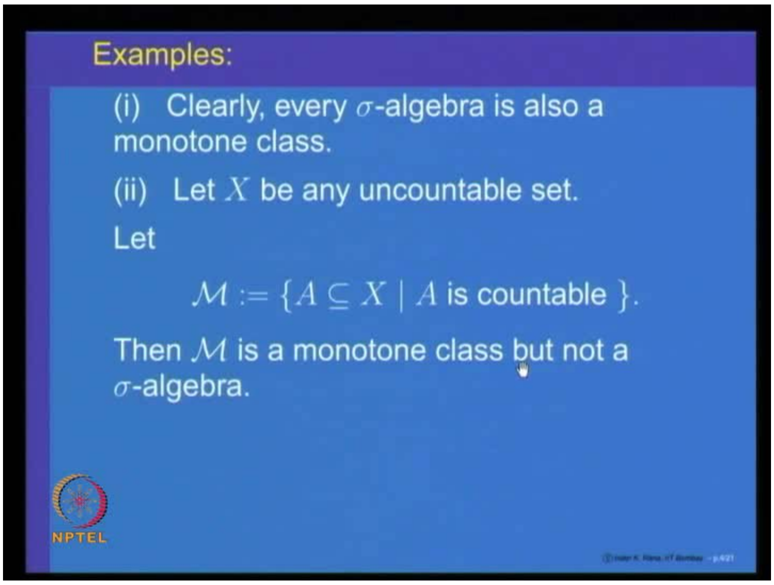

Let us go to some more properties and more examples. Let X be any uncountable set. Let us look at the collection of all subsets A of X such that A is a countable set. The claim is that this collection M is a monotone class, but it is not a sigma-algebra.

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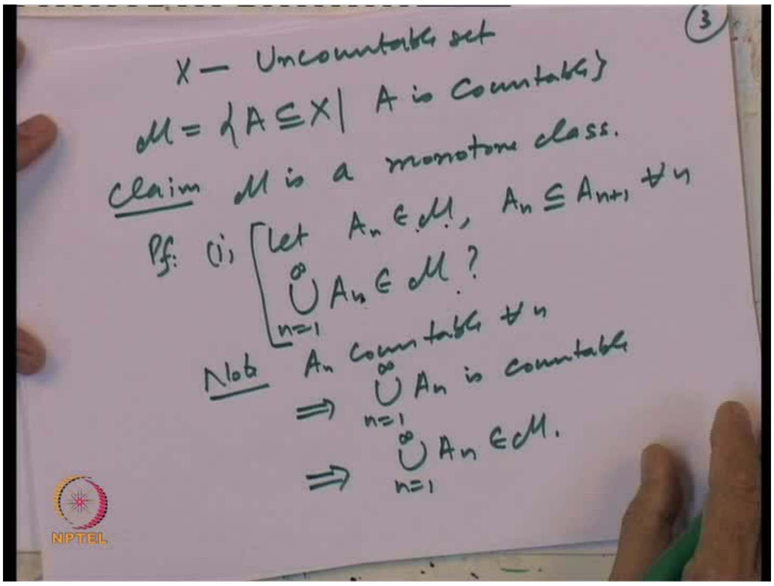

Let us look at M. X is uncountable and we are looking at the collection M of all those subsets of X such that A is countable. So, claim M is a monotone class.

Let us see how you prove it. First property - let us take a sequence A n belonging to M, A n's increasing; A n subset of A n plus 1 for every n. So, what we have to prove? We have to check that union of A n's n equal to 1 to infinity also belongs to M. That is what we have to check. However, let us note - to check that the union belongs to them, we have to show that it is a countable set. Now, each A n is given to be an element of M. That means, each A n is countable for every n; implies union A n is also countable.

Here we are using a fact that the countable union of countable sets is again a countable set. Hence, this implies that union A n's n equal to 1 to infinity belongs to M. The first property we have checked is that if A n's belong to M and A n's is an increasing sequence, then the union A n belongs to M.

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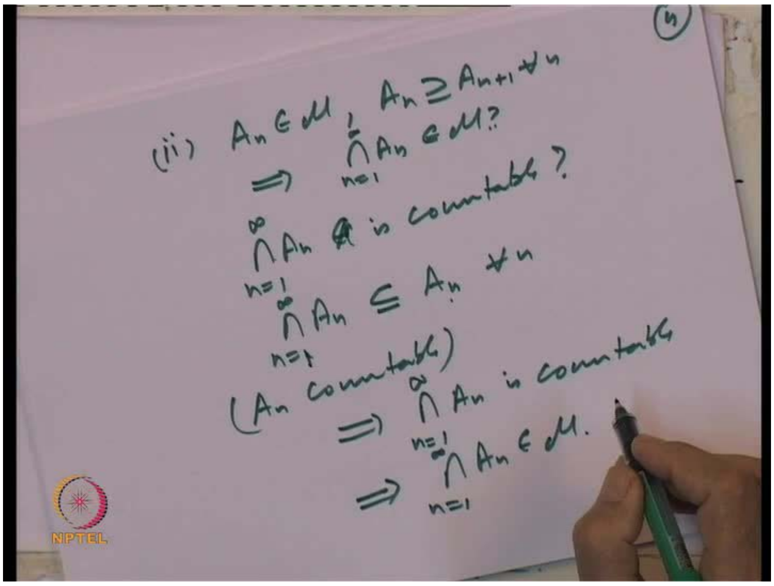

Let us check the second property, namely that A n belonging to M, A n's decreasing for every n should imply that the intersection A n's belongs to M. So, what we have to check? We have to check that intersections A n n equal to 1 to infinity belongs to M. That means, is countable. So, that is what we have to check. However, let us observe that intersection A n's; this is a subset of A n for every n because it is intersection. Each A n is countable; A n countable implies… This is a subset of it. So, intersection A n is countable. Hence, implies that intersection A n belongs to M.

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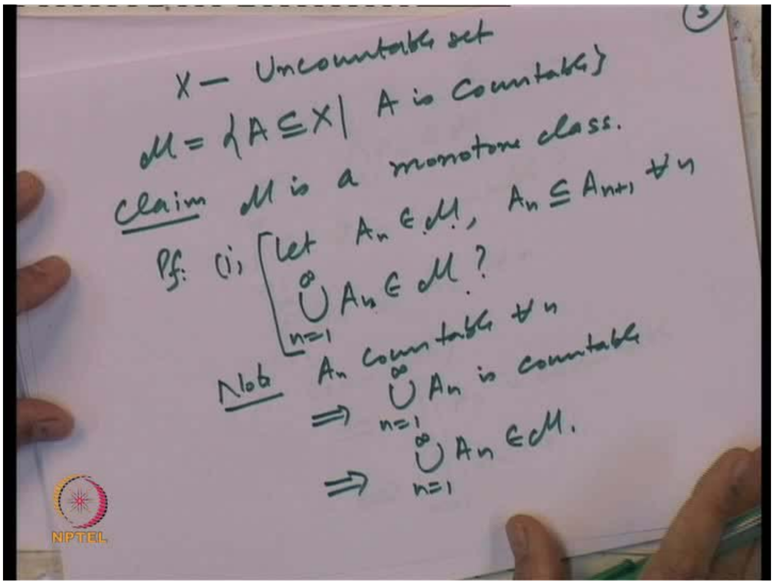

We have shown that if X is a countable or uncountable set does not matter. Actually, so far what we have not used the fact - it is an uncountable set if M is the collection of all countable subsets of a set X, then that forms a monotone class.

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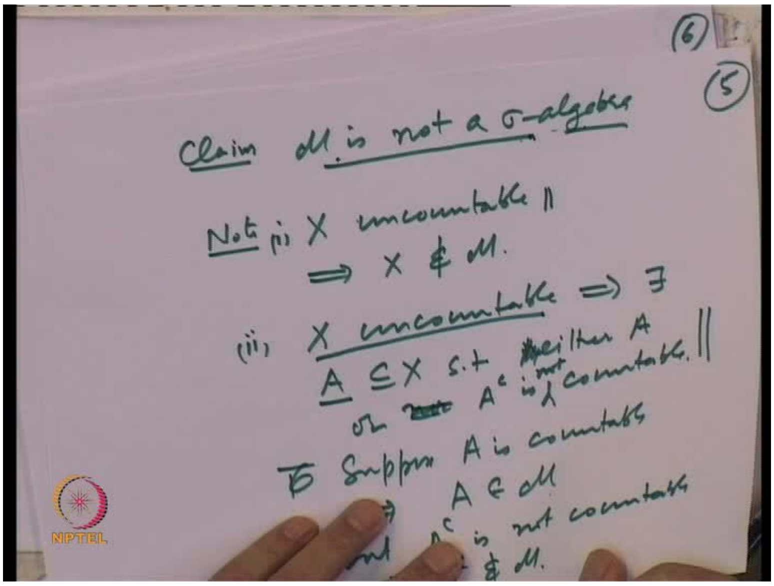

Why it is not a… Claim - finally, we want to prove that M is not a sigma-algebra. Let us observe a few things here. Note - here we will be using X uncountable. This implies first of all X does not belong to M. So, the very first property of a collection being a sigma-algebra, namely the whole space belong to it is violated because X is not countable, but it is uncountable.

Another way of looking at this is the following. So, this is one observation (Refer Slide Time: 11:59). Secondly, X uncountable implies there exists a subset A in X such that neither A nor A complement is countable. That is obvious because if X is an uncountable set, then the claim that there exists… This is not actually required and this may not be true.

Let us take a subset A of X such that either A or A complement is not countable. That is possible because if for every set A and A complement are countable, then X will be a countable set. So, let us choose a set A such that either A or A complement is not countable. Then, suppose A is countable, that will imply that A belongs to M, but A complement is not countable. That implies A complement does not belong to M.

When X is uncountable, you can have in fact for every set, which is countable A will belong to M, but A complement will not belong to M. So, this collection M is also not going to be closed under complements. So, X does not belong to it and that it is also not going to be closed under complements. That is because X is uncountable. So, when X is uncountable, collection M of all countable subsets of it is a monotone class, but it is not a sigma-algebra. So, every sigma-algebra is a monotone class, but the converse need not be true.

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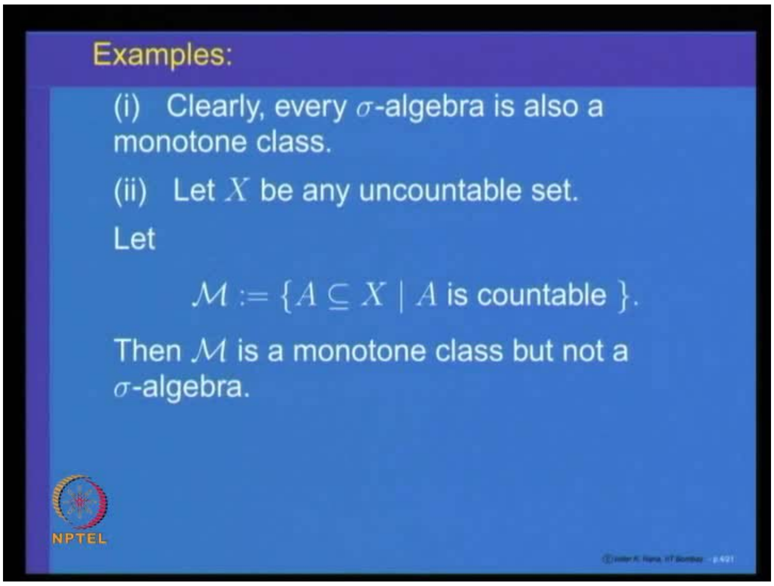

This is what we have shown just now. If M is a monotone class, every monotone class is a sigma-algebra, but there exists examples of monotone classes, which are not sigma-algebras.

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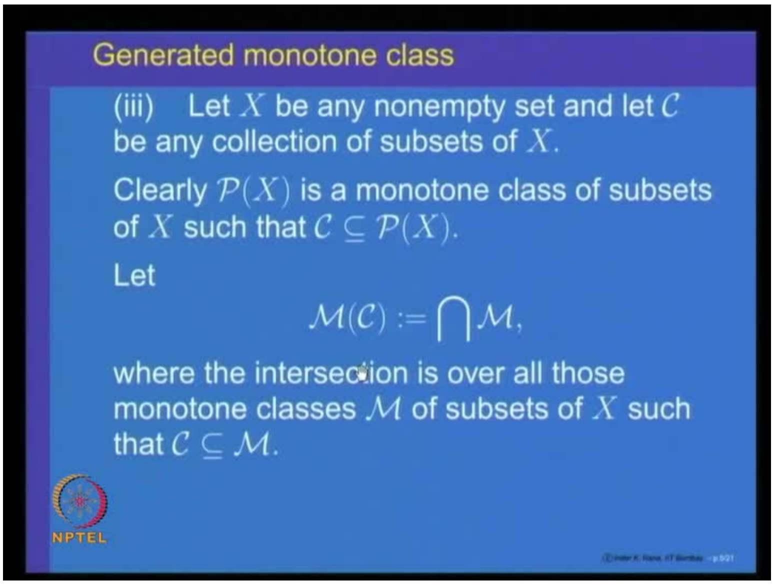

Let us look at the next scenario. Let us start with a collection C of subsets of a set X; C is any collection; it may or may not be a monotone class. We would like to find a monotone class of subsets of X, which includes C and is smallest. First of all, let us observe that all subsets of the set X is a monotone class of subsets of X. C is because it is a subset, is sub collection. So, C is sub collection. Given any collection of subsets of a set X C, it is always included in a collection, namely the power set of X, which is a monotone class. So, given a collection, there always exists a monotone class of subsets of X including it.

However, this is too large, we want to have the smallest monotone class including C. Whether such a thing exists or not. The proof is something similar to what we have shown is algebra generated by a class; the sigma-algebra generated by a class. So, let us look at M of C. This is a notation for the intersection of all monotone classes M of subsets of X, which include C. Look at the collection of all monotone classes of subsets of X, which includes C. Take their intersection and call this as M of C. So, what we want to prove is that M of C is a monotone class, it includes C and it is the smallest.

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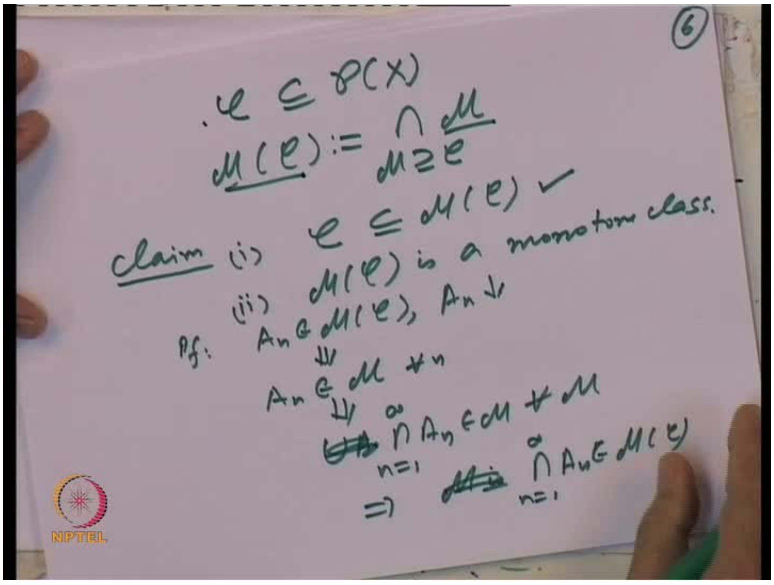

Let us look at a proof of that. C is any collection of subset of set X. M of C is the intersection of all monotone classes M; M including C. Claim 1 - C is inside M of C. That is obvious because C is inside every collection M and M of C is the intersection. So, this property is obvious.

Second - claim that M of C is a monotone class. The proof goes on the same lines as that of algebra and sigma-algebra. To prove this, let us look at A n's belong to M of C, A n's decreasing. However, this implies that each A n also belongs to M for every collection M, which include C for every n. That implies that union of… This is decreasing (Refer Slide Time: 17:28). So, we want to show that intersection A n's; n equal to 1 to infinity belong to M for every M. Hence, that implies that intersection A n's belong to M of C. So, essentially saying that if A n is a sequence in M of C, which is decreasing, then this is also a sequence, which is decreasing in each M. Hence, the intersection belongs to each M and belongs to the intersection M of C. Similar proof works for the union.

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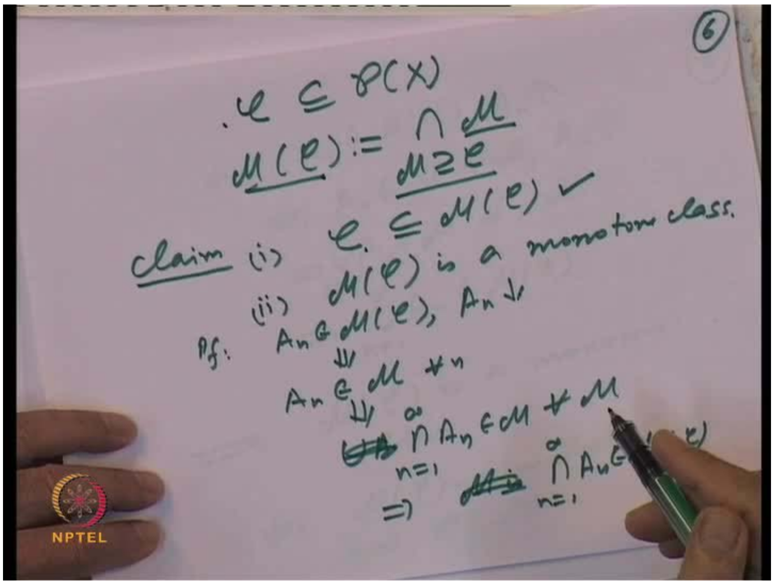

Let us look at second part. A n’s belong to M of C and A n’s increasing. That implies A n’s belong to M for every M and A n is increasing. That means A n’s union because each M is a monotone class. So, the union belongs to M for every M and that implies that the union A n’s n equal to 1 to infinity belongs to M of C. Hence, M of C is a monotone class. So, it is a monotone class that includes the collection C (Refer Slide Time: 19:01).

Third - M of C is smallest such that C is M of C; smallest monotone class. That is obvious because it is the intersection (Refer Slide Time: 19:20) of all monotone classes. So, obviously it is going to be the smallest.

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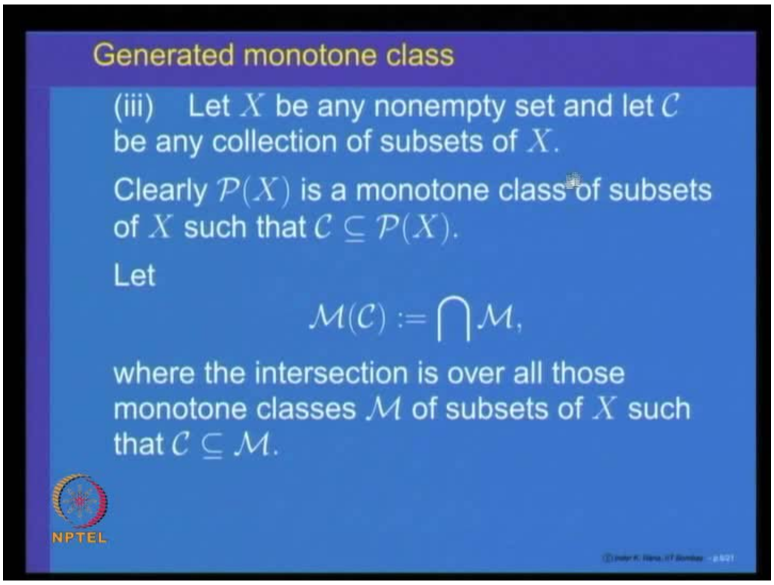

What we have shown is that given a collection C of subsets of a set X, there exists a monotone class M of C, which includes C and which is the smallest. So, this monotone class is called…

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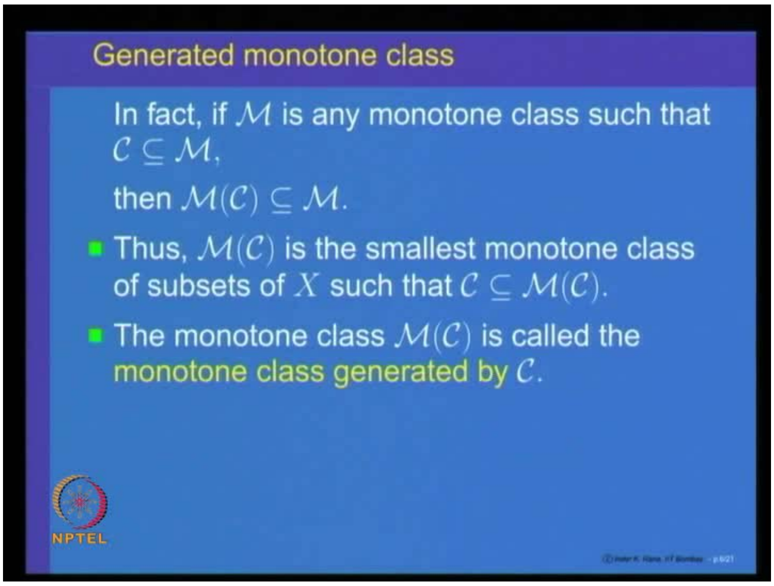

This is what we have proved. Thus, M of C is the smallest monotone class of subsets of X such that C is inside M of C. This collection is called the monotone class generated by C. Every collection C has got the smallest monotone class in which includes it. So, that is called the monotone class generated by it.

So, given a collection C of subsets of a set X, we are able to generate an algebra, a monotone class and a sigma-algebra out of it. The next question that we want to analyze is - what is the relation between these collections?

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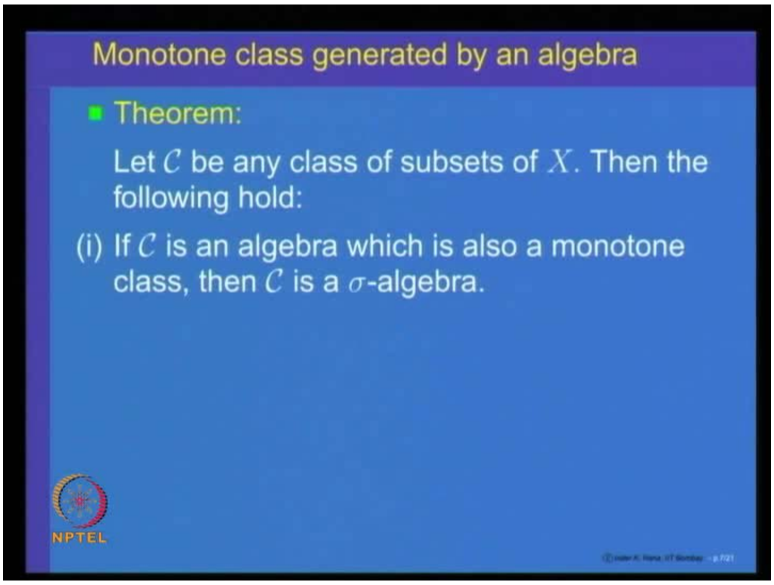

We want to prove a theorem, which relates these concepts. First of all, let us start with any collection of subsets of a set X. Then, the first observation is - if C is an algebra, which is also a monotone class, then C is sigma-algebra.

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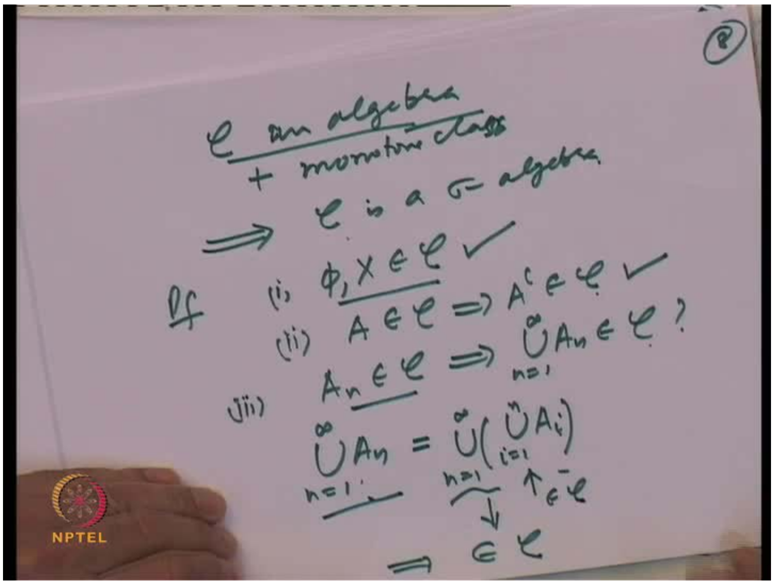

Let us first prove this fact that if C is an algebra plus monotone class, implies C is a sigma-algebra. So, proof; what we have to prove? To prove C is a sigma-algebra wait a proof first - empty set, the whole space belong to C. That is true because C is an algebra. So, this property is true. Second - we should show that if a set A belongs to C, implies A complement belongs to C. That is again obvious because the collection C is a algebra. So, these two properties are true. The third property is only property to be checked that if A n belong to C, then it should imply that union of A n's n equal to 1 to infinity also belongs to C. So, C is closed under countable unions. That is what we have to prove.

What we are given is - C is a monotone class. A monotone class is a collection, which is closed under only increasing and decreasing. So, let us look at union of A n's. Can we represent this as a union of increasing sets? The one possibility is - let us take union of A i's i equal to 1 to n then this collection will be an increasing sequence as n increases. Their union n equal to 1 to infinity will be a union of increasing sequence of sets, which are namely union A i's.

Now, observe that each one of these sets union A i i equal to 1 to n; that is a finite union of elements in the algebra, A n's belong to C and that is an algebra. So, that means each one of them belong to C. We have written the union A n's as union of sets in C and this is an increasing sequence (Refer Slide Time: 23:04). C is a monotone class. So, that implies that this right hand side set belongs to C. Note: We have represented any union as an union of increasing sequence of sets and each set here is a finite union of elements of the algebra C. Hence, this belongs to it. So, this becomes an increasing union of sets. Hence, this belongs to C because it is a monotone class.

Every A n belonging to C implies the union 1 to infinity also belong… So, C is in fact closed under countable unions. So, it becomes a sigma-algebra. This proves the first property, namely if C is an algebra, which is also a monotone class, then it is a sigma-algebra.

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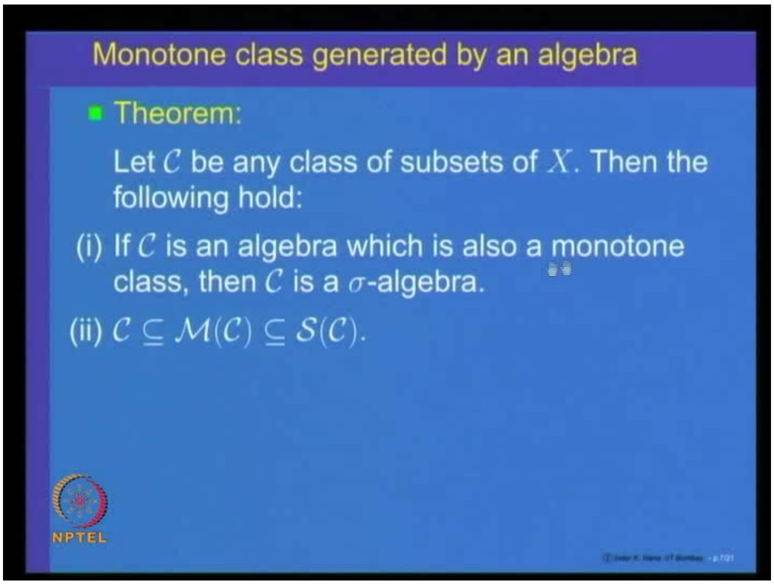

Let us look at the next property that if C is any collection, then C is contained in M of C. That is obvious because M of C is the smallest monotone class including C. So, this is obvious. Now, C is also subset of S of C because S of C is the sigma-algebra generated by it. So, C is inside S of C. Just now, we proved S of C is also a monotone class. So, S of C is a monotone class including C. Hence, the smallest one must come inside. So, that will prove this C is inside M of C is inside S of C. That means given any collection of subsets of X, it is always included in the monotone class generated by it. The monotone class generated by it is also inside the sigma-algebra generated by it.

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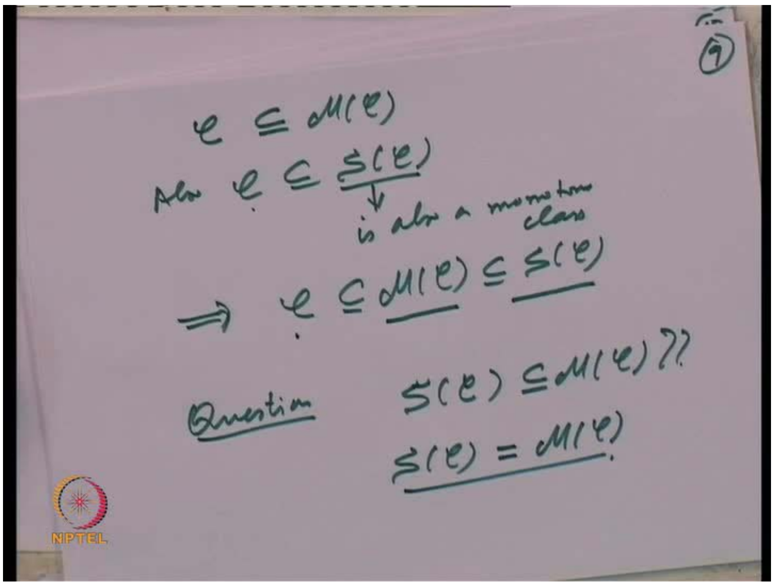

Let me repeat these arguments once again. First of all, C is contained in M of C. That is because M of C is the smallest monotone class including C. Also, C is included in S of C because S of C is the smallest sigma-algebra of subsets of X, which include C. Just now we proved this, which is also a monotone class. So, this is a monotone class including C. That implies the smallest one must come inside it and the smallest one is M of C that comes inside S of C.

What we have shown is - for every collection C of subset of a set X, the monotone class generated by it is a subset of the sigma-algebra generated by it. We want to analyze the question - When can we say S of C is also a subset of M of C? When is this true? That is same as saying that when is S of C, the sigma-algebra generated by a collection? Can I say it is equal to M of C?

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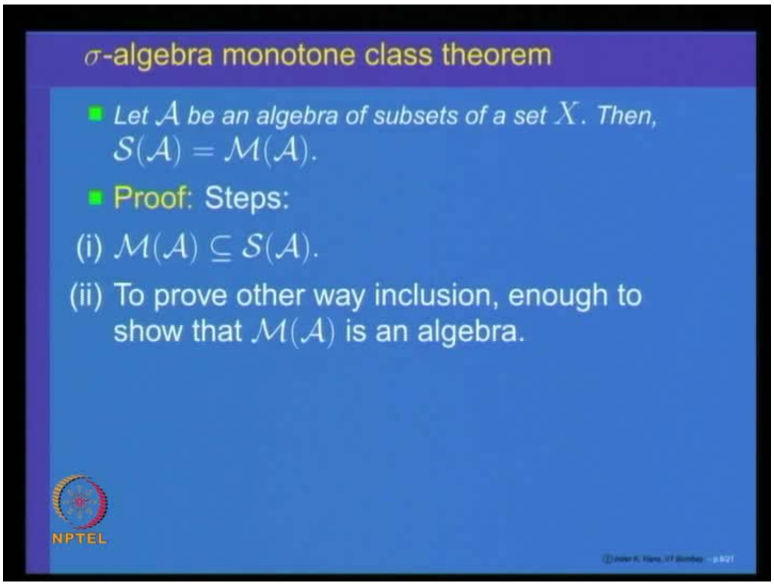

The answer is given by the next theorem, which says that if C is an algebra, then this is true. So, this is an important theorem called the sigma-algebra monotone class theorem. It says - if A is an algebra of subsets of a set X, then the sigma-algebra generated by it is same as the monotone class generated by it.

We have already observed the first part that M of A is a subset of S of A. For that, one need not have even A as algebra for any collection M of C is contained in S of C. So, in particular, if A is an algebra, then M of A is a subset of S of A.

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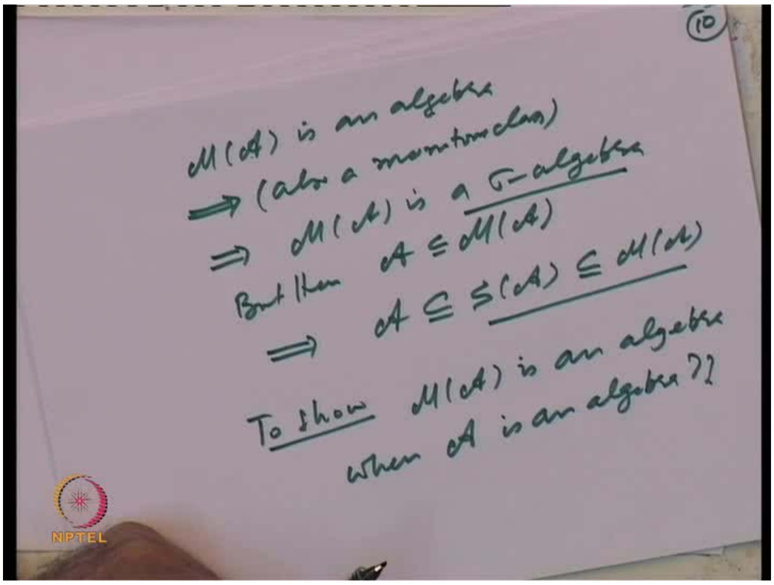

We have to prove the second, the other way around inclusion, namely M of A includes S of A when A is an algebra. To prove the other way around inclusion, let us observe that it is enough to prove that M of A is an algebra; why it is enough to prove this? Let us observe enough to show M of A is an algebra will imply - Because M of A is an algebra, it is also a monotone class. Just now we proved that every algebra, which is a monotone class will imply M of A is a sigma-algebra because M of A is a monotone class. If we are able to show it is an algebra, then it will be also a sigma-algebra. However, then we have A is inside M of A.

Now, if M of A is a sigma-algebra, that will imply the smallest one must come inside it. So, S of A will be inside M of A. So, that will prove S of A is a subset of M of A and will be through. We have to only prove to show that M of A is an algebra when A is an algebra. This is what we have to prove.

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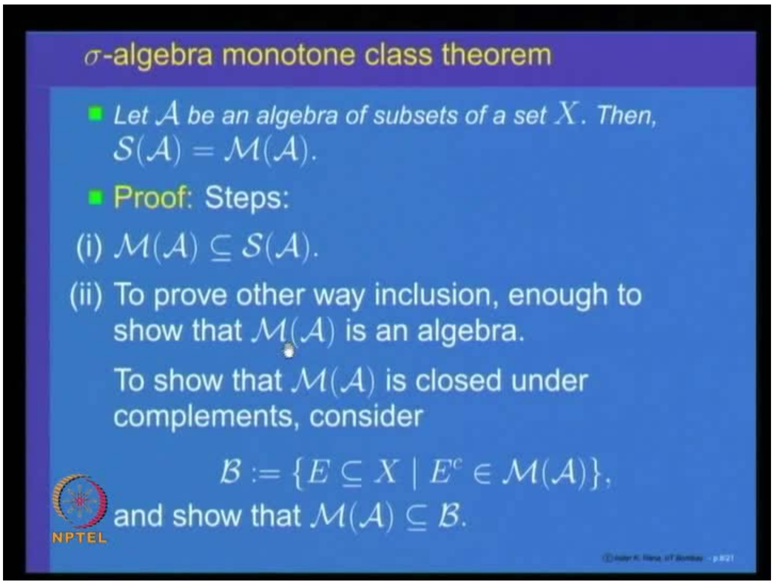

Let us start looking at a proof of this. First of all, to prove that M of A is an algebra, we should show that it is closed under complements. So, let us try to prove it is closed under complements. That means what? To show it is closed under complements, I have to show that for every subset in M of A, its complement is also in M of A. So, this is a technique, which we are going to use very often.

Let us collect together all the sets B, which have the property that whenever B is the collection of all those subsets, say that E complement belongs to M of A. To prove M of A is closed under complements, what we have to show is that M of A is a subset of B. So, we have to show that M of A is a subset of B.

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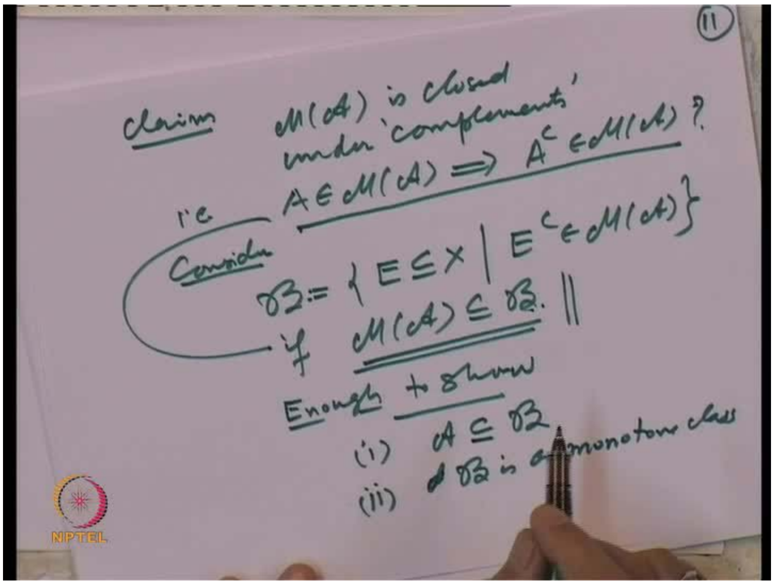

Let us try to prove that. Claim - let me repeat; we want to show that M of A is closed under the operation of complements. That is, A belonging to M of A should imply A complement belongs to M of A.

To show that, let us consider the collection B of all those subsets E contained in X such that E complement belongs to M of A. To prove that this will be true (Refer Slide Time: 30:28), the required claim will be true. This will be true if we can show M of A is contained in B. So, that is what we want to prove. We want to show because for every set A belonging to M of A, it will belong to B. That means complement will belong to M of A. So, this is what we have to show.

Now, let us observe. We are trying to show that M of A is inside a collection B. What is M of A? M of A is the smallest monotone class including A. Suppose we are able show that A is inside B and B is a monotone class, then this claim (Refer Slide Time: 31:19) will be true. So, to prove this claim, it is enough to show that one - A is inside B. Secondly, B is a monotone class because once B is a monotone class including A, the smallest one will come inside. So, let us try to prove these two facts.

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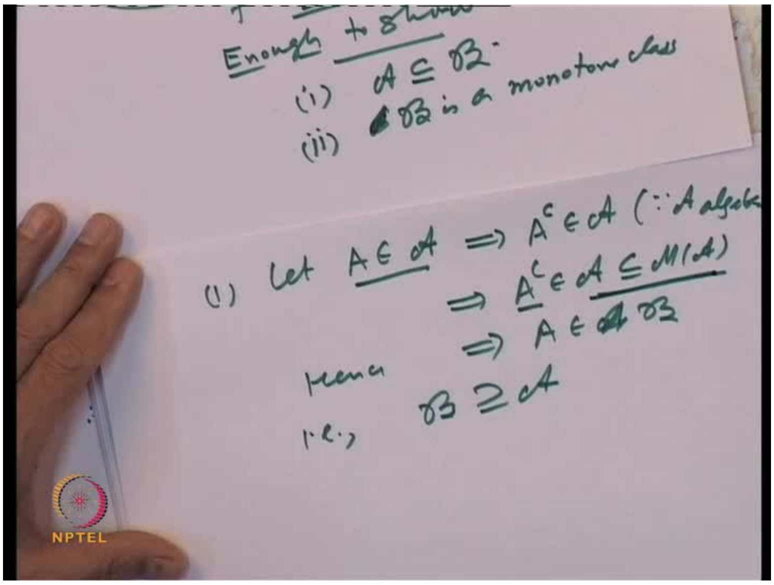

Proof of one that A is a subset of B. Let set A belong to A. That implies A is an algebra. That implies A complement belongs to A because A is algebra. Note that it implies A complement belongs to A, which is inside M of A because A is always inside M of A. What we have shown that if A belongs to A, then its complement belongs to M of A. Hence, that is same as saying that A belongs to the collection B. That is, we have proved that B includes A. So, first property is true.

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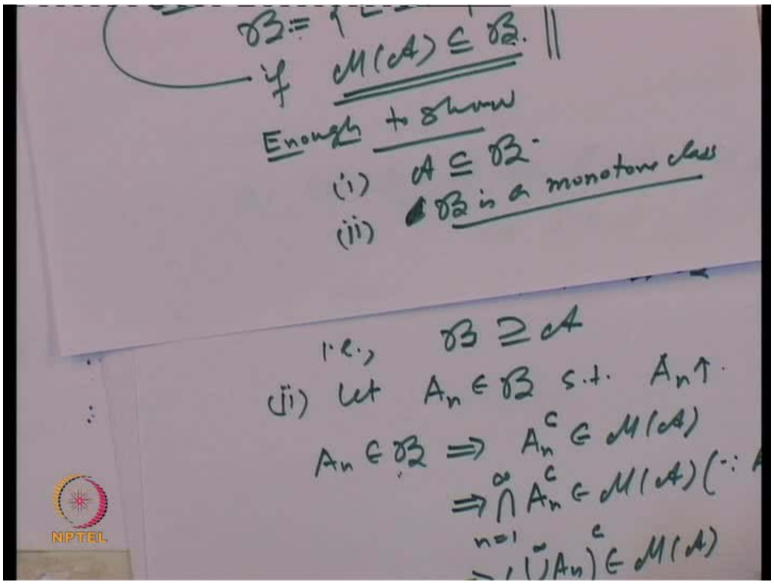

Let us look at the second property. What is the second property we want to prove? The second property we want to prove is that B is a monotone class. Let us take a collection A n, a sequence belonging to B such that it is decreasing or increasing. Let us say A n is increasing, but A n belonging to B means what? A n complements belong to M of A. That is the definition of the class B. So, saying that we have got a set A n means that A n complements belong to A.

Now, M of A is a monotone class. That implies that A n complements intersection will belong to M of A provided you can say A n complements are decreasing. That is true because A n's are increasing because A n complements are decreasing and M of A is a monotone class. So, that means this intersection belongs to it. That means union of A n's n equal to 1 to infinity; complement of this belongs to M of A.

Whenever a sequence A n belongs to B and A n's are increasing, the complement of the union belongs to it. So, that means union of A n's n equal to 1 to infinity belongs to B. So, the collection B is closed under increasing unions. Finally, let us prove that it is also closed under decreasing sequences.

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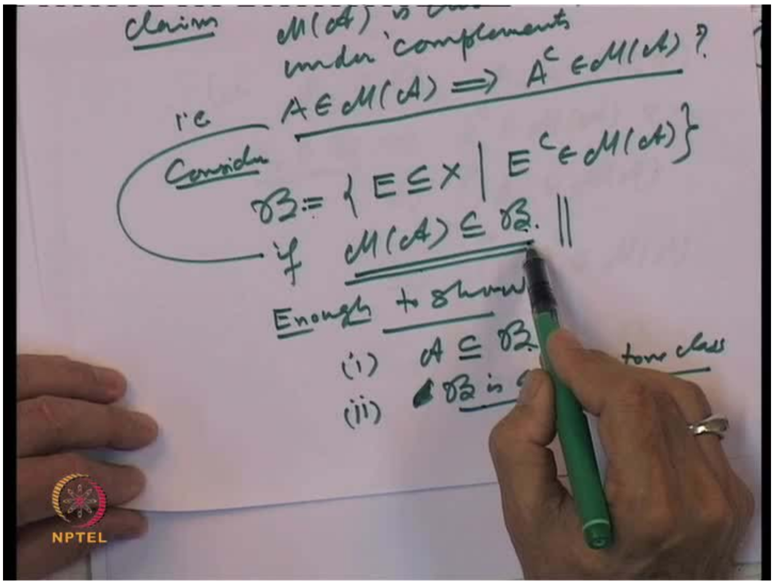

Let us take a sequence of sets, which is decreasing. Let A n belong to B and A n's decrease. We want to show that the intersection of A n's belong to it. However, A n's belong to B implies that - just now, we observed by definition that A n complements belong to M of A. By definition, A n belong to B means A n complements belong to A for every n. A n complements is a sequence because A n's are decreasing; that is same as A n complements are increasing. So, that implies union of A n complements belong to M of A because M of A is a monotone class. That implies that if the intersection of A n's n equal to 1 to infinity complement belongs to M of A. So, whenever A n's belong to it and take the intersection of A n's, their complement belong to it. That means intersection of A n n equal to 1 to infinity belongs to B.

What we have shown is the collection of B is closed under increasing union is closed under decreasing intersections. That means B is a monotone class (Refer Slide Time: 36:17). A is inside B; B is a monotone class. That will prove that M of A is a subset of B because this is a monotone class including A. So, it must include the smallest one.

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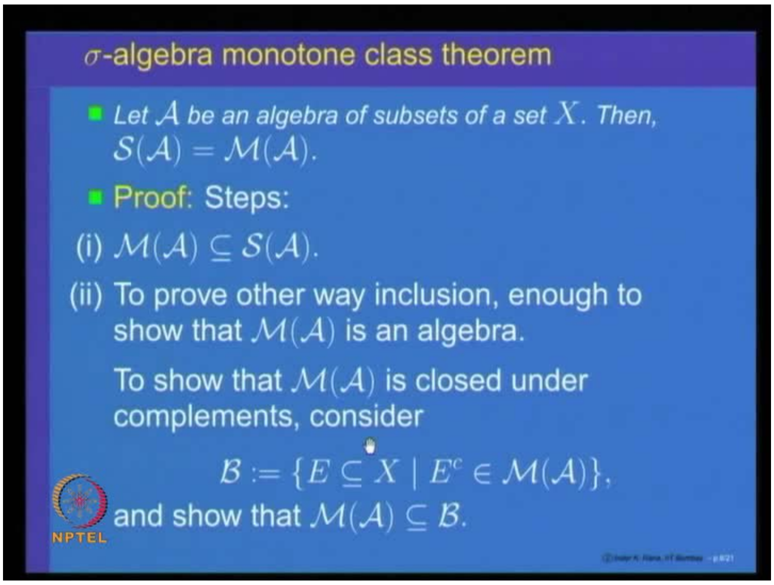

That proves the first step of our claim, namely that the collection M of A is closed under complements. We wanted to show that it is an algebra.

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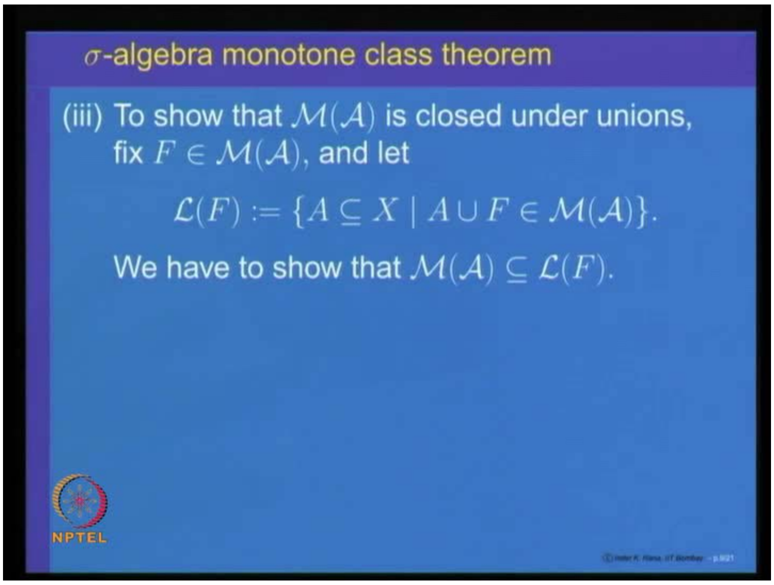

What is the next step? Next step should be to show that M of A is closed under unions. That means whenever two sets E and F belong to M of A, their union must belong to M of A.

Let us fix one of them. Let us fix the set F in M of A and let us look at the collection L of F such that it is the collection of all those sets say that A union F belongs to M of A. So, what we have to prove? In this, to prove that M of A is closed under unions, we have to prove that M of A is a subset of L of F. So, once again the required property that M of A is closed under unions; we are translating into a property of a collection of subsets.

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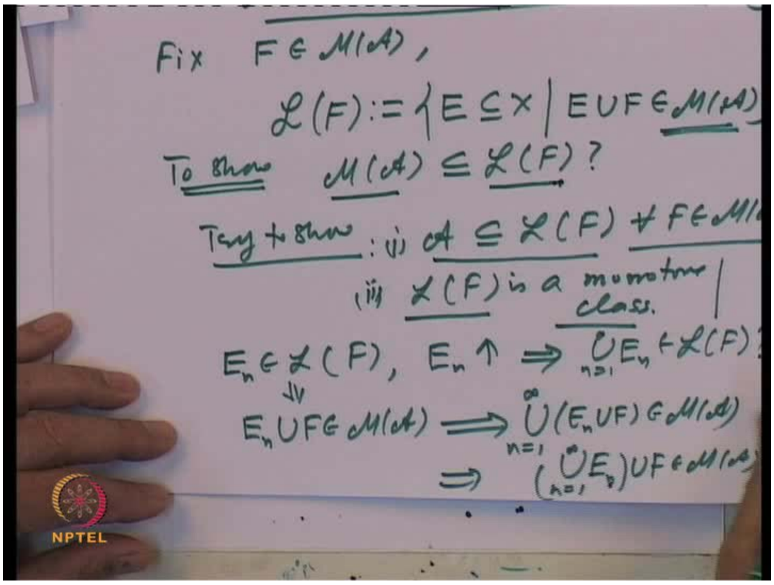

Let us try to show that M of A is contained in L of F. That is the first we should try to show. To show that the collection M of A is closed under unions; this is what we want to show. Let us fix a set F belonging to M of A and consider the collection; let us call it as L of F. What is this collection? It is the collection of all those subsets in X such that E union F belongs to M of A.

Saying that M of A is closed under unions, we should show that M of A is a subset of L of F. That is what we should show. Once again we want to show that M of A is a subset of L of F. M of A is a monotone class generated by A and we want to show that it comes in some other collection L of F. That means we should try to show that A is inside this collection and this collection L of F is a monotone class.

We should try to show that A is inside L of F for every F belonging to M of A. Second - L of F is a monotone class. Let us observe second one, which is quite obvious. So, let us observe that L of F is a monotone class. For that, what we have to show? Let us take a sequence E n belonging to L of F; E n's increasing. However, that will mean if E n's are in L of F, that will imply that E n union F belongs to M of A. That is by definition of L of F. M of A is a monotone class; E n's are increasing. So, E n union F is also increasing. So, implies that union of E n union F belongs to M of A because E n's are increasing, E n union F is increasing and belong… That means it is same as saying that union of E n's union F belongs to M of A. What does that mean? That means union of E n s belong to the class L of F.

Whenever E n’s belong to L of F; E n is increasing; this implies that union E n’s belong to L of F. So, this is what we have just now proved. A similar proof will work for decreasing also. Saying that L of F is a monotone class is a straight forward argument because M of A is a monotone class.

Let us try to check that A is inside L of F for every F belonging to M of A. So, we want to check namely the first property. This is the property we want to check that this collection algebra (Refer Slide Time: 41:25) is inside L of F for every F belonging to M of A.

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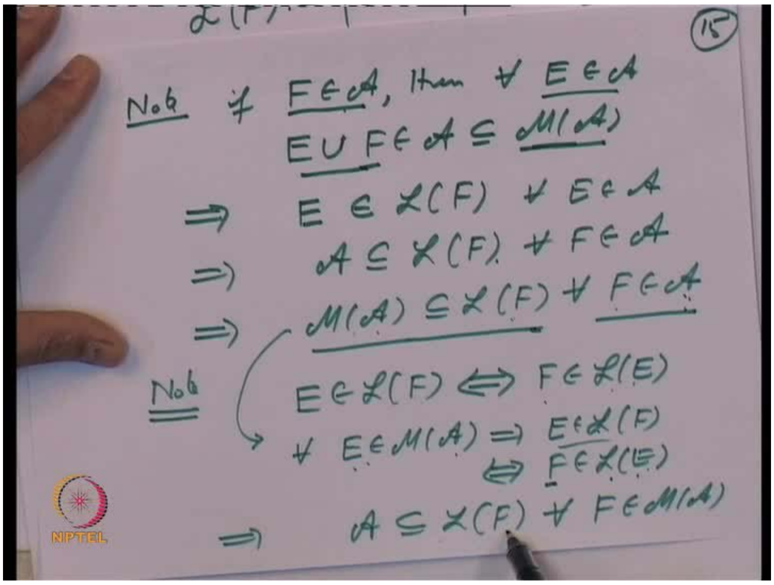

For the time being, we want to check this property for every F in M of A. Let us note that if F belongs to A, then for every E belonging to A, E union F belongs to A because A is the algebra. If E and F are two sets in A, A is algebra that will mean that the union belongs to algebra. That is included in M of A. What does this imply? This means for every F in A, E union F belongs to M of A. That means that the set E belongs to the collection L of F.

Once again we are starting with a very simple observation - if a set F belongs to A and E belongs to A, then their union belongs to A. A is always inside M of A. So, that means E union F belongs to M of A for every E belonging to A. That means we have shown that A is inside L of F.

Now, A is inside L of F for every F in A implies L of F is a monotone class; just now we proved. This implies M of A is inside L of F for every F belonging to A. What we have shown is - M of A is a subset of L of F for every F belonging to A. However, we wanted to check that (Refer Slide Time: 43:23) M of A is inside F for every F belonging to M of A. We have got only for F belonging to A.

Here is a very simple observation, which helps us. Note that a set E belongs to L of F if and only if F belongs to L of E. This is an observation, which is going to be very important and very useful for us. A set E belongs to L of F means what? E union F belongs to M of A. However, if E union F belongs to M of A, that is same as F union E belongs to… That means F belongs to L of E. So, saying that E belongs to L of F is same as F belongs to L of A.

Now, let us translate this property (Refer Slide Time: 44:17). Here it says M of A is inside L of F for every F belonging to A. That means for every E belonging to M of A implies that E belongs to L of F. That is, if and only if F belongs to L of E. Here F was in the algebra. So, what we have got? For every F in the algebra, it belongs to L of E whenever E belongs to L of A. That means what? That means A is inside L of F for every F belonging to M of A. See how nicely we have turned the tables.

Earlier, we had M of A is inside L of F for every F in A (Refer Slide Time: 45:16). That means every element here E is element in L of F, but here, E belongs to L of F means F belongs to L of E. Now, F is in A. That means A is in L of F for every F belonging to A. That means once that is true…

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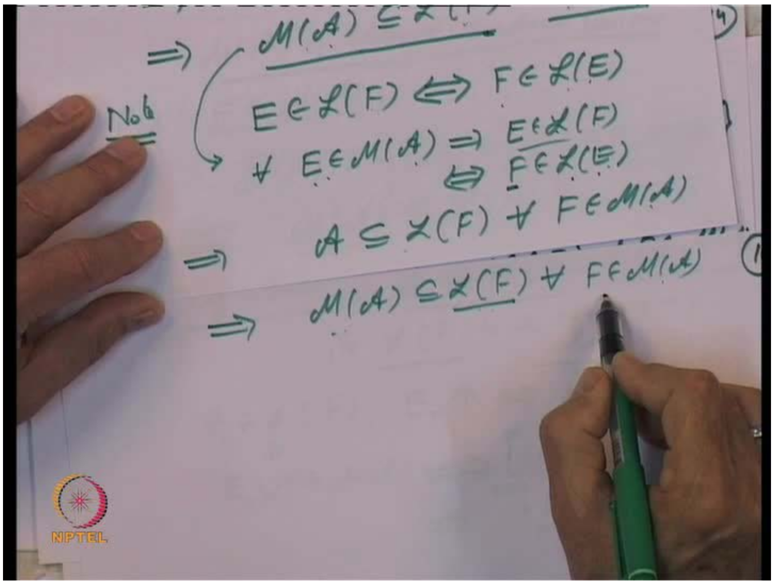

Now, A is inside L of F for every F in M of A. That implies that M of A is inside L of F for every F belonging to M of A. Because L of F is a monotone class, it includes algebra A. So, it must include the smallest one. So, M of A is inside L of F for every F belonging to M of A. So, we have proved the required thing, namely M of A is inside L of F for every F belonging to M of A. Hence, that means M of A is also closed under…

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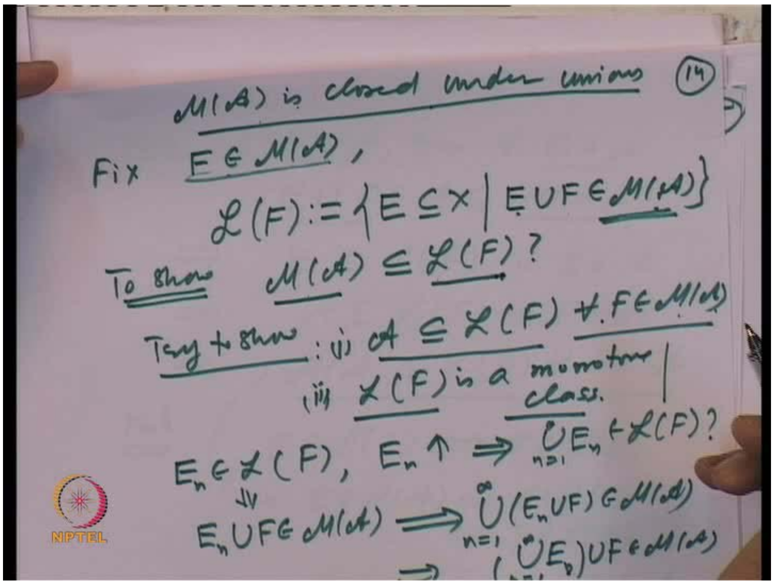

Here is what we wanted to prove. So, M of A is closed under unions. That we translated into the property that M of A is inside L of F for every F in M of A. Finally, we proved that. You see again and again whenever we want to show something is true, we converted into a property of a collection of objects, show generators come inside and everything comes inside. So, that proves that M of A is an algebra; it is already a monotone class. So, it must be a sigma-algebra and S of A is a sigma-algebra. So, that will prove that the required theorem, namely…

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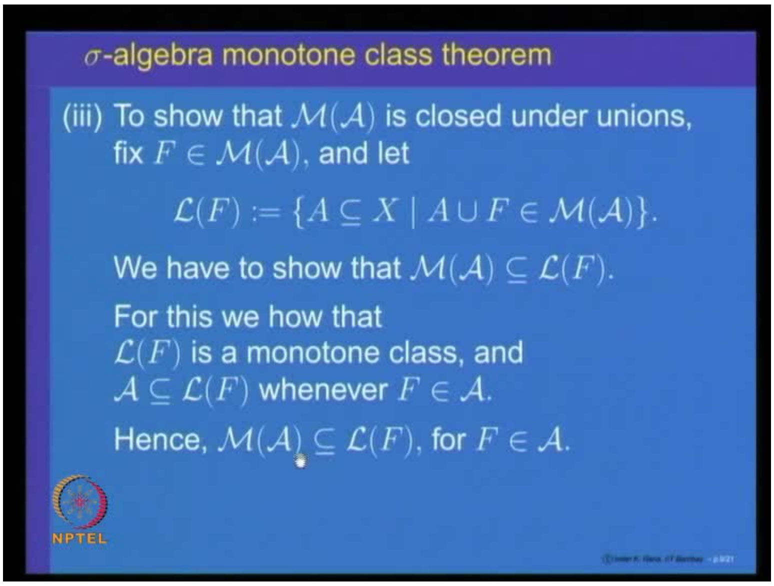

For this, let us just go through this proof again. To show that M of A is closed under unions, we fix F in M of A and look at this collection. We want to show that M of A is inside L of F. For this, we have to show that L of F is a monotone class and A is inside L of F whenever F belongs to A. So, that says M of A will come inside for F belonging to A.

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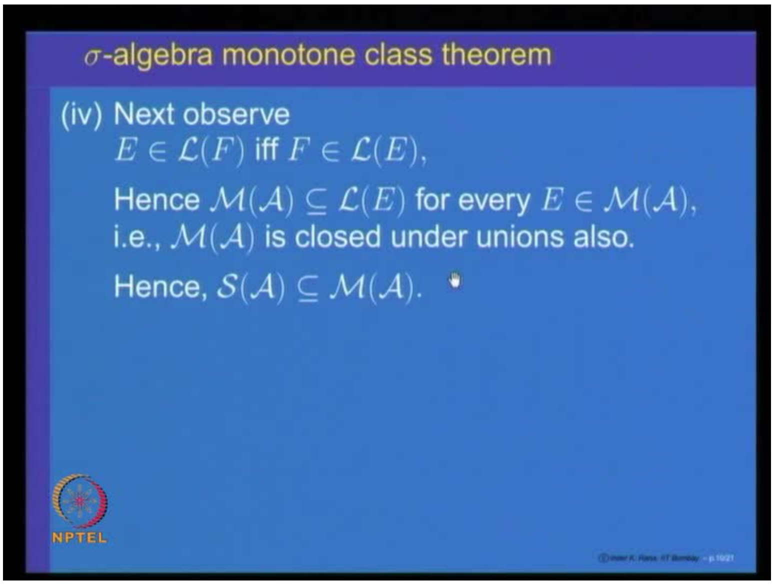

Now, reverse the roles of these two that M of A for F in A. That means E belongs to L of F. So, F belongs to L of E. Reverse the roles and that comes. That gives you the property that M of A is inside L of E. So, it is closed under unions. Hence, it is a sigma-algebra. So, it must have included the smallest one. That proves the fact that the sigma-algebra generated by algebra is same as the monotone class generated by the algebra.

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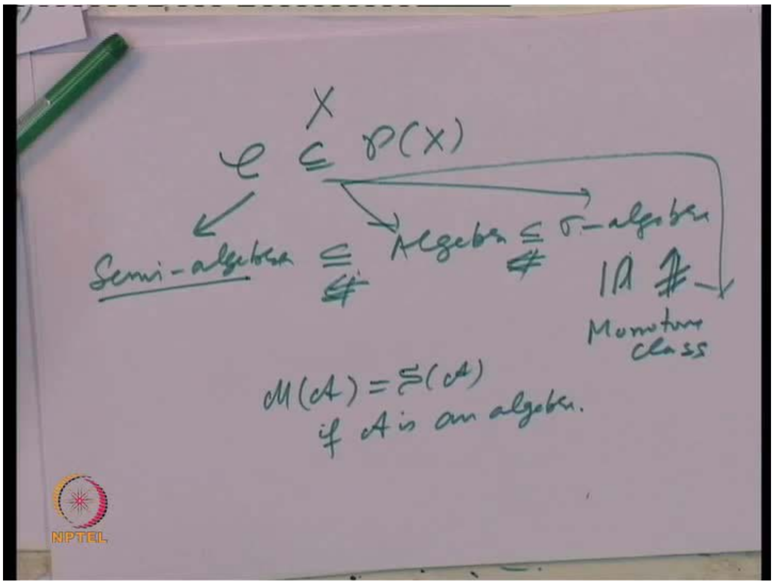

Let us recall what we have done till now. We started with a set X, looked at a collection C of subsets; C contained in P X. The first thing we all looked at is what is called a semi-algebra. Then, we looked at this collection C to be an algebra, this collection to be a sigma-algebra, and then this to be a monotone class. So, a semi-algebra. Every algebra is a semi-algebra; every sigma-algebra is also an algebra; every sigma-algebra is also a monotone class. So, this is something here. This way around implication may not be true; this way around implication may not be true and this way around implication may not be true. Finally, we proved monotone class generated by an algebra is the sigma-algebra generated by algebra if A is an algebra.

That finishes our study of collections of subsets of X with special properties. I just want to leave you with some exercises, which you should try, which are important.

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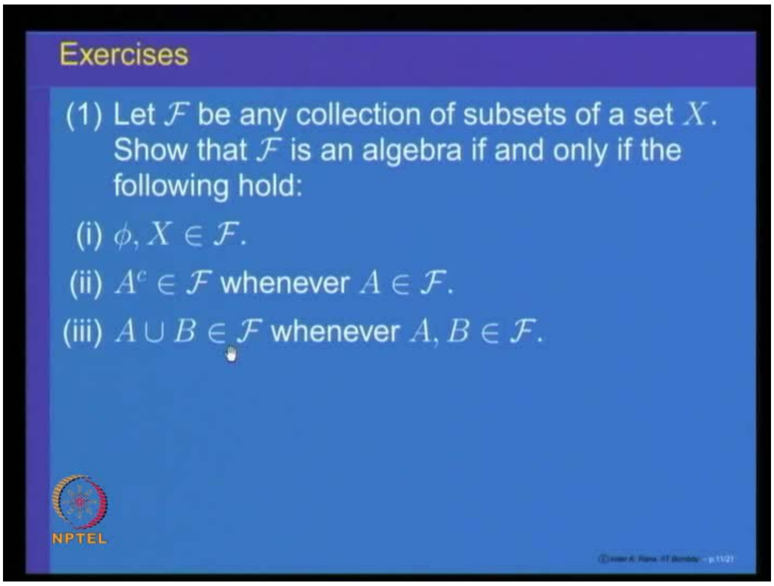

The first exercise is that whenever a collection is an algebra it is equivalent to saying that empty set in the whole space is closed under complements and it is closed under unions. That is equivalent to saying whether it is closed under complements because of this de Morgan laws.

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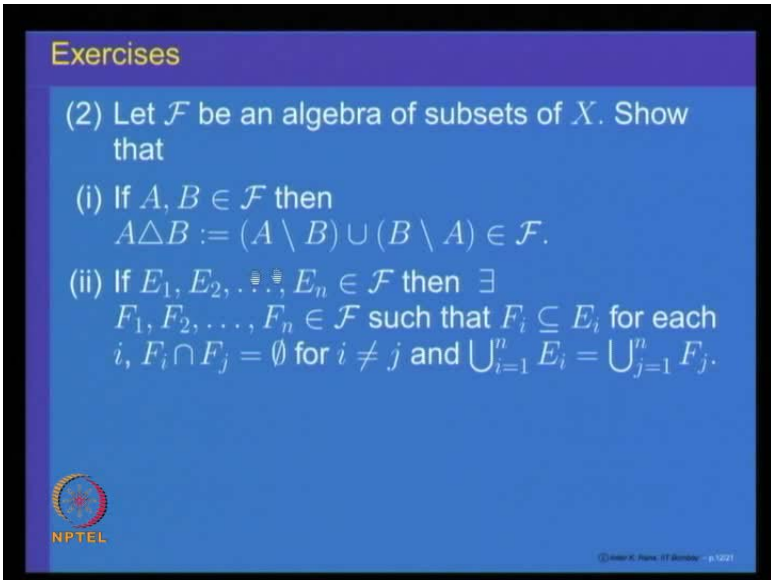

Another property that whenever something is an algebra, a collection algebra, it is also closed under symmetric references. Any finite union in an algebra can be represented as a finite disjoint union whenever you are inside the algebra. That property we have seen, but you should try to prove this exercise yourself.

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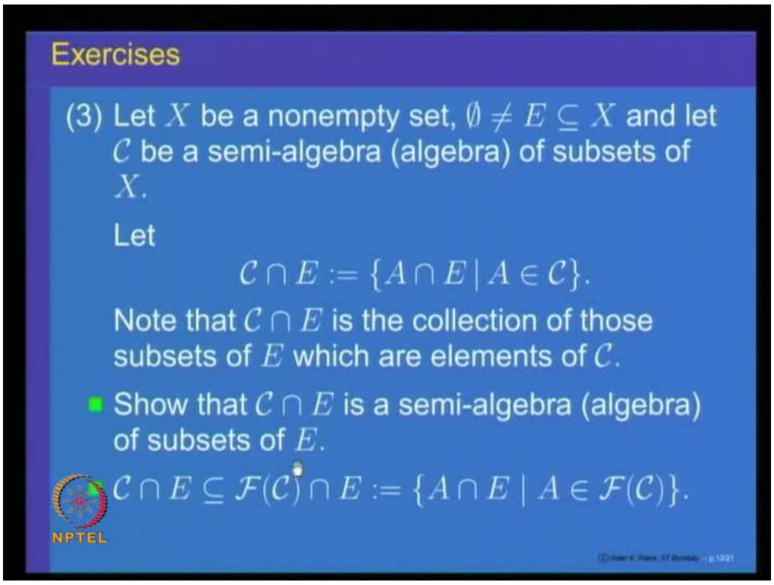

Another property about semi-algebras or sigma-algebras is that you can restrict. Take a collection C of subsets of a set X and restrict it to a set E. That means, take intersection of all sets in C with E. Then, this is C restricted to E. The property we want to prove is that if C is a semi-algebra, then C intersection E is a semi-algebra of subsets of E.

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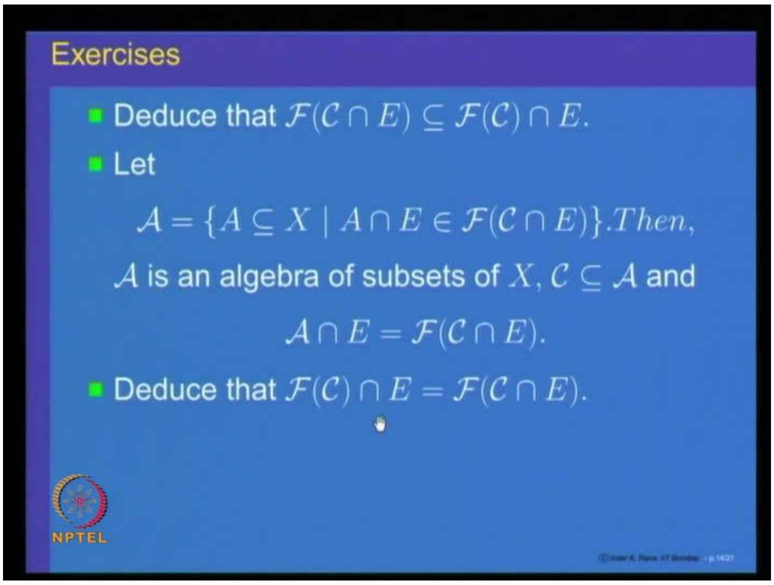

Similarly, you prove the property that the algebra generated by the restricted sets is same as the algebra generated by…; generate the algebra and restrict. So, F of C intersection E is equivalent to F of C intersection E. So, we restrict and generate. It is the same as generate and restrict.

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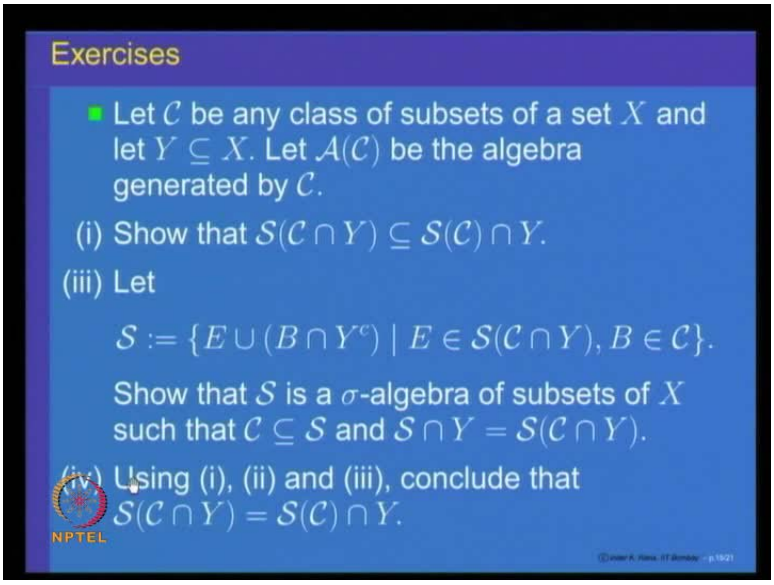

The same property is true for sigma-algebras. So, that is the property about sigma-algebras. You should try these exercises to prove yourself. The steps are outlined here for you to prove. We have already proved these things in our lectures, but I will strongly advise that you prove these things yourself.

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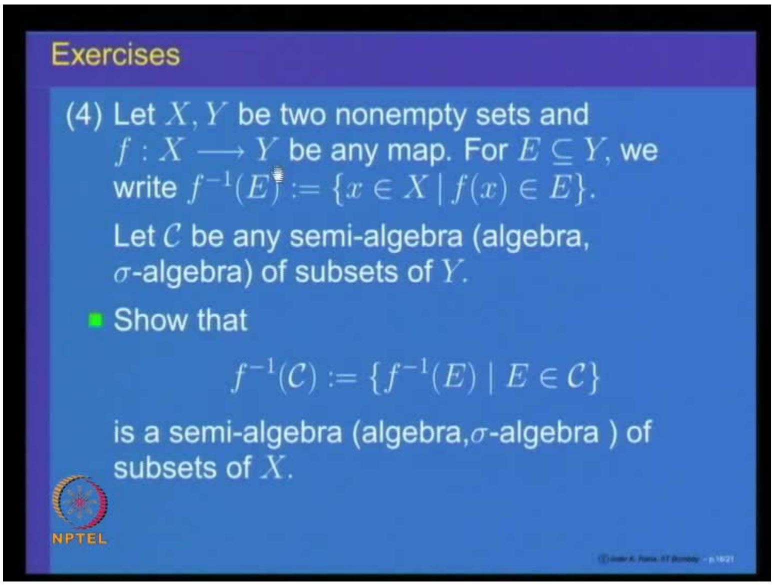

Here is another example of generating new sigma-algebras or semi-algebras. F is a function from X to Y. If you take sets in Y and take inverse images, that gives you a collection of subsets of X. So, try to show that whenever if C is a collection of subsets of Y, which is a semi-algebra or a sigma-algebra, then the pullback sets also form a semi-algebra or sigma-algebra.

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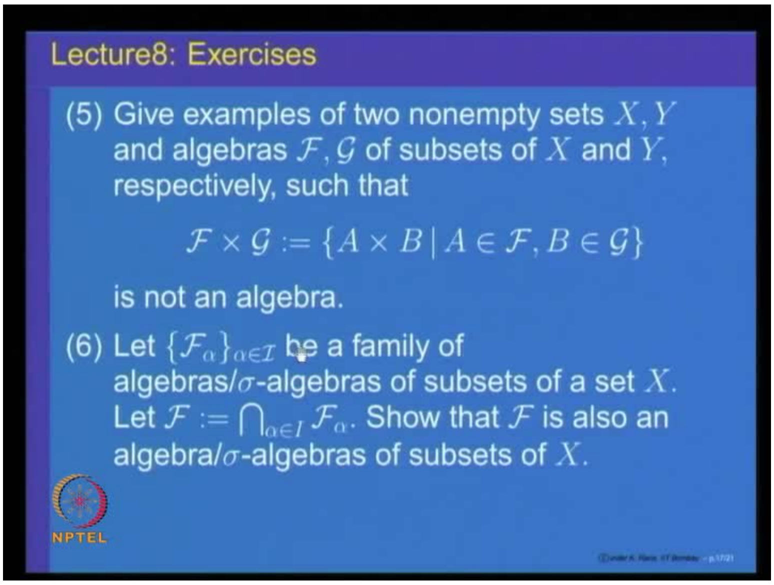

Here is an another example. Take two collection of subsets F and G of two sets X and Y. Look at the Cartesian products of these collections show that in general it is not an algebra. If you take a collection of subsets F alpha, which are all algebras or sigma-algebras, the corresponding intersection also is an algebra or a semi-algebra.

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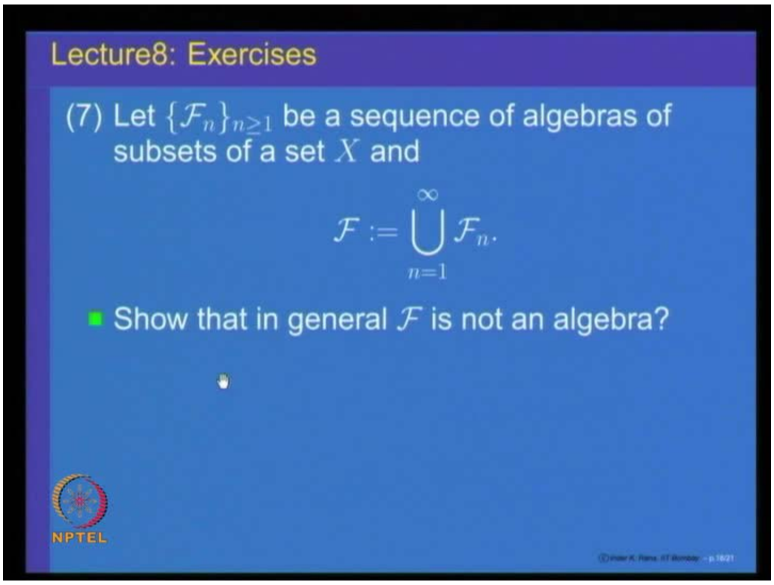

For these properties, unions may not be true. So, show that for union, this property need not be true.

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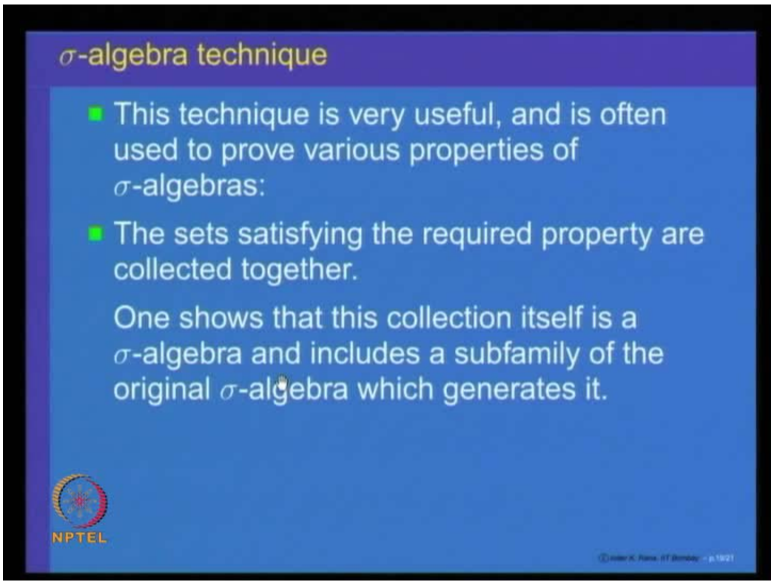

That sigma-algebra technique that something is inside, then the generated sigma-algebra comes inside that we use.

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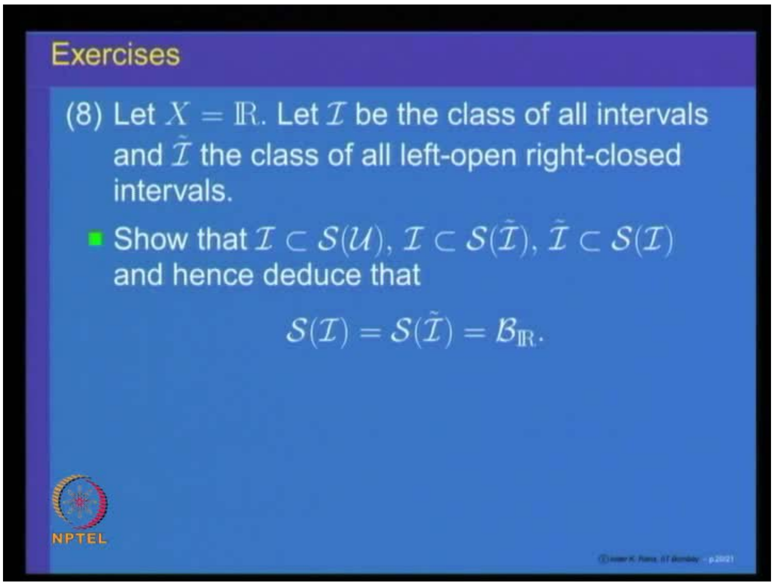

Use that to prove that if you take the collection of all intervals in I left-open and right-closed intervals, then the sigma-algebra generated by all intervals is same as the sigma-algebra generated by all left-open right-closed intervals and the Borel sigma-algebra. I would strongly advise you to try these properties to get used to the concepts of algebra, semi-algebra, sigma-algebra and monotone class. Let us stop here today. Thank you very much.