Measure and Integration

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Module No. # 10

Lecture No. # 39

Modes of Convergence

Welcome to lecture 39 on Measure and Integration. Today, we will be looking at a special topic which is called various modes of convergence for measurable functions. So, the topic for today's discussion is modes of convergence.

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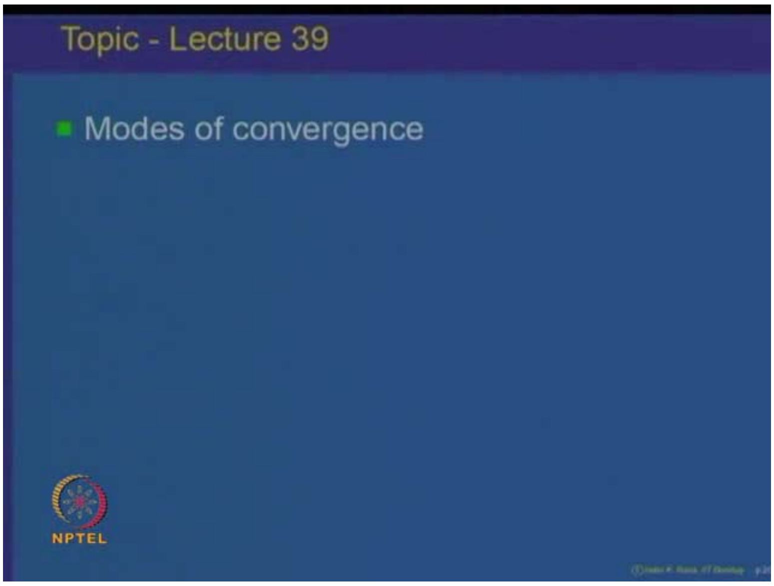

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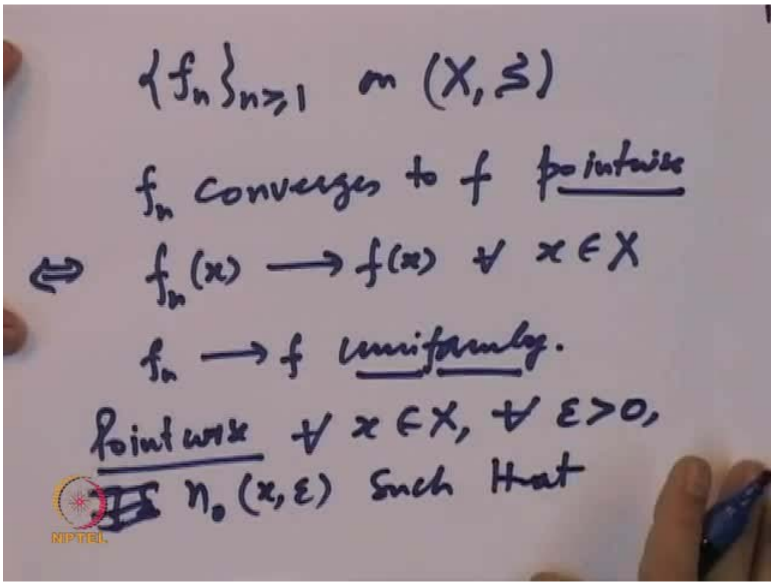

Let us recall some of the ways of convergence that you might have already looked into earlier courses. So, let us take a sequence of functions f n, it is a sequence of functions on measurable space X S. Now, saying that f n converges to f point wise, it means that f n x is same as saying that f n x converges to f of x; these are numbers for every x belonging to X, this is converging point wise. You might have already come across something called f n converges to f uniformly, so what is the difference between this point wise convergence and uniform convergence? Let us just write down in terms of our definition of epsilon delta.

Point wise means for every x, for every epsilon bigger than 0 there is n naught which will depend upon the point x and epsilon such that f n x minus f of x is less than epsilon for every n bigger than n naught. So that essentially means that the numbers f n x converges to the number f of x as n goes to infinity, so the given epsilon is bigger than 0. There is a stage after which f n x is close to f of x of course, this stage may depend upon the point x and the number epsilon.

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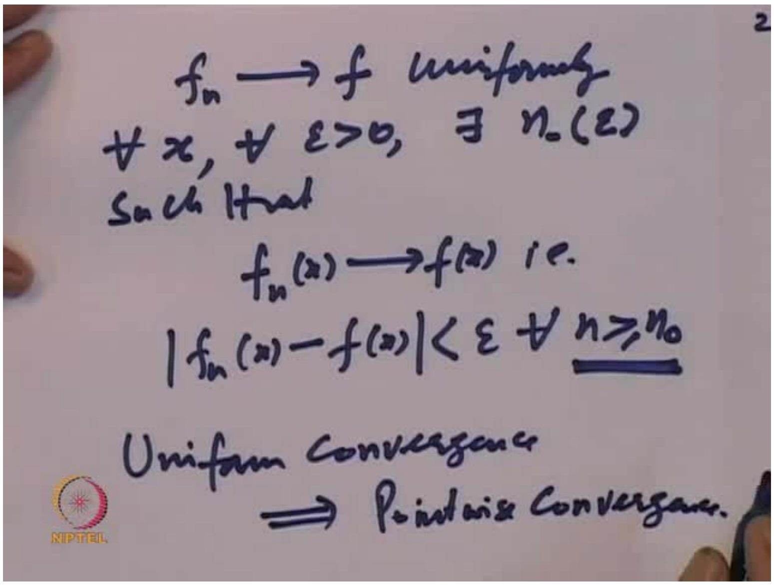

Let us look at what is point wise convergence, by saying that f n converges to f uniformly; that means for every x, for every epsilon is bigger than 0 there is the stage n naught which does not depend on x which depends only on epsilon such that f n x converges to f of x; that is, mod f n x minus f of x is less than epsilon for every n bigger than n naught and the same stage works for every x. You must have already seen that the uniform convergence implies point wise convergence and the converse need not be true.

So the uniform convergence implies point wise convergence; the converse need not be true that you must have seen in your earlier courses in analysis, but we are going to look at today is the functions which are defined not only on measure spaces; that is, actually they are defined on measure spaces not only on measurable spaces. So, we are going to look at a sequence f n of measurable functions on a measure space X S mu. We had already looked at some concept called convergence almost everywhere.

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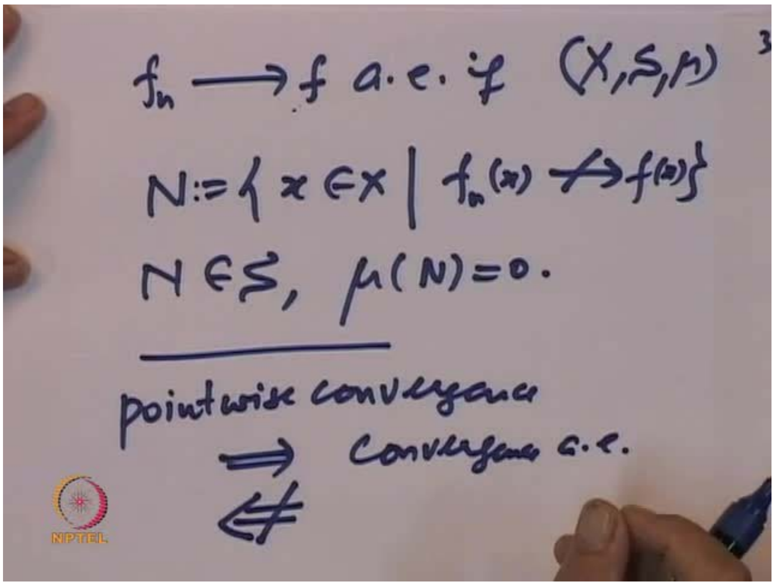

Let us define formally again that f n converges to f almost everywhere if everything is defined on a measure space X S mu; these are functions defined on the measure space X S mu. We say f n converges to f almost everywhere, if the set N all x belonging to X such that f n x does not converge to f of x. If that is a set N then, we want to N should belong to sigma algebra S and mu of N should be equal to 0, so that is convergence almost everywhere.

Obviously, point wise convergence implies convergence almost everywhere. Obvious examples one can construct to show that this other way implication need not hold; convergence almost everywhere need not imply point wise convergence.

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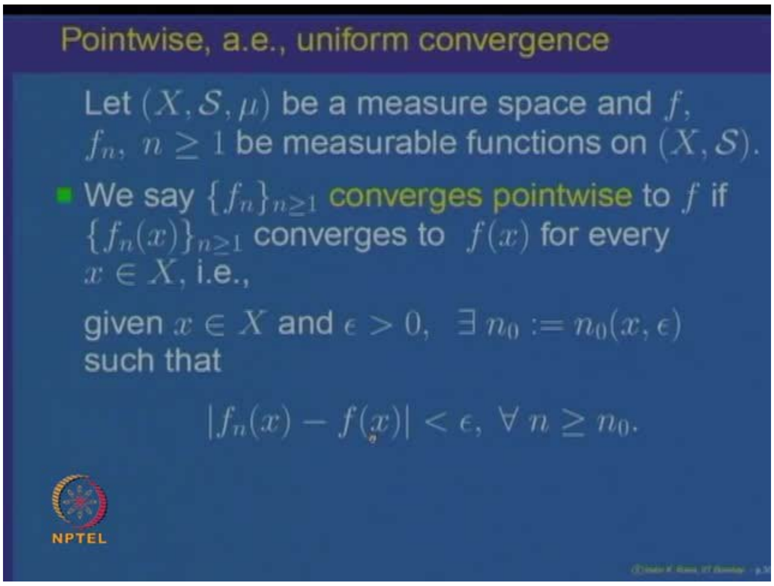

Now, let us try to look at a relation between convergences almost everywhere. Just recall once again what we are saying? We are saying f n convergence to f point wise, if the sequence of numbers f n x converges to f of x. That is same as saying that for every epsilon bigger than 0, there exists n naught such that it may depend upon the point x. The number epsilon is such that absolute value of f n x minus f of x is less than epsilon for every n bigger than n naught.

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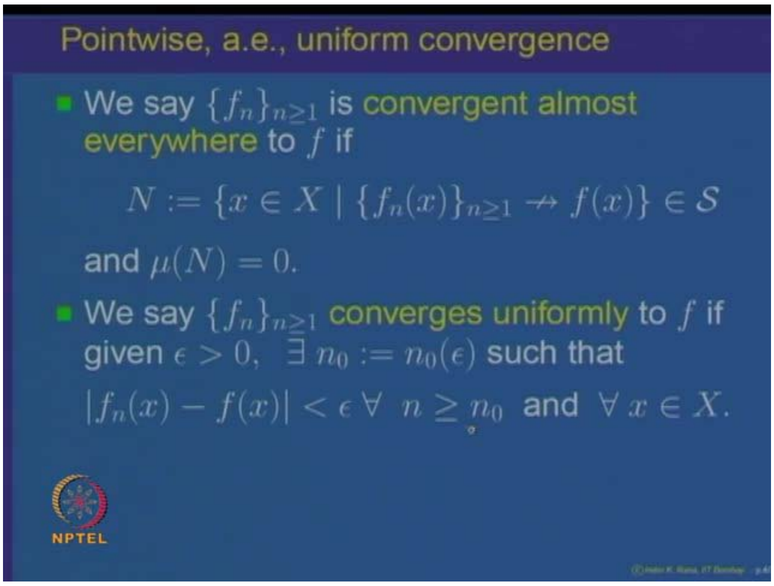

So, the numbers f n x converging - the sequence f n x is converging to f of x. By saying f n converges almost everywhere to f means that the set of points where f n x does not converge to f of x that set is a set of measure 0, so that is convergence almost everywhere. Saying that the f n converges uniformly to f that means, for every epsilon there is a stage n naught which depends only on epsilon n naught on the point x such that for every x the distance between f n x and f of x is less than epsilon for every n bigger than or equal to n naught. That stages works for every x, so that is important thing for uniform convergence.

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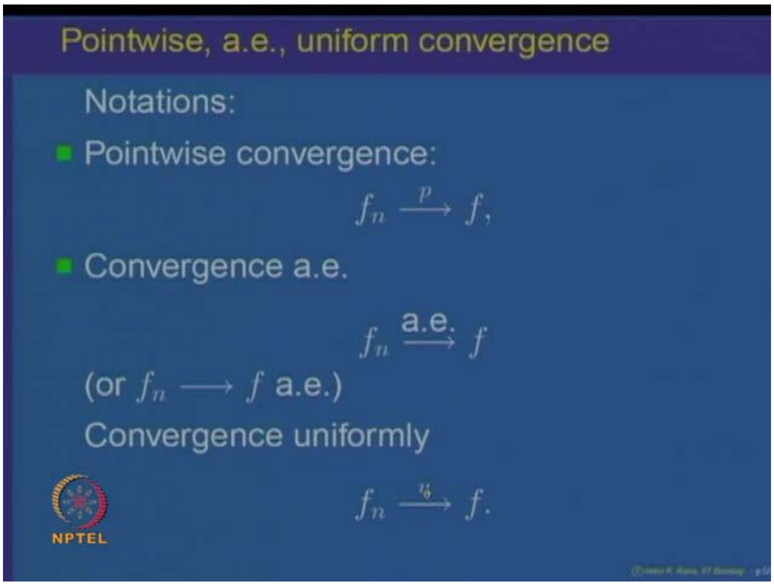

Instead of writing in English all the time whenever f n converges to a point wise, we will write f n arrow with the p above indicating it is a point wise converges, f n converges to f point wise. Convergence almost everywhere will be indicated by f n an arrow pointing right side to f with a symbol a dot e dot saying almost everywhere to f, but this is also written as f n converges to f, f n right arrow f, a e indicating and this converges is almost everywhere.

Similarly for uniform convergence, we will denote it by symbols f n arrow by pointing towards the right and u above the arrow indicating that it is uniform convergence.

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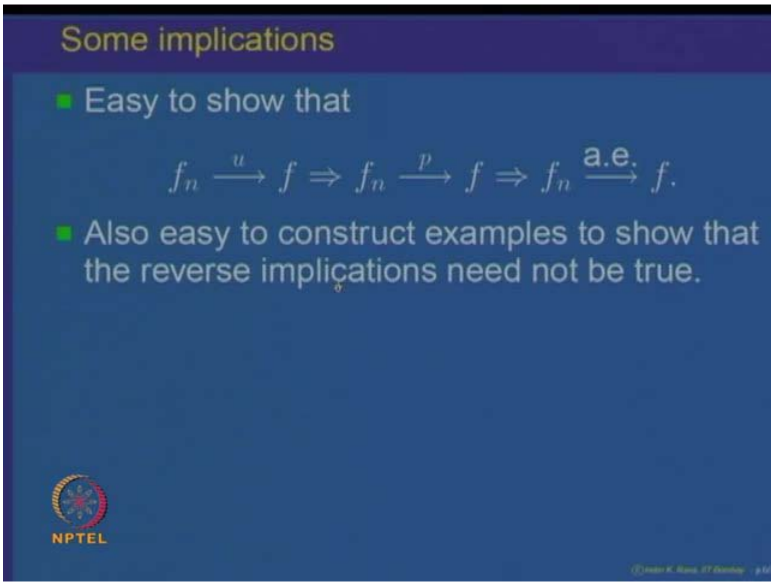

So as I pointed out earlier that f n converges to f, uniformly implies that f n converges to f point wise and that implies f n converges to f almost everywhere. None of the backward implications is true, so one can easily construct counter examples. We show that convergence almost everywhere need not imply convergence point wise and point wise convergence need not imply convergence which is uniform.

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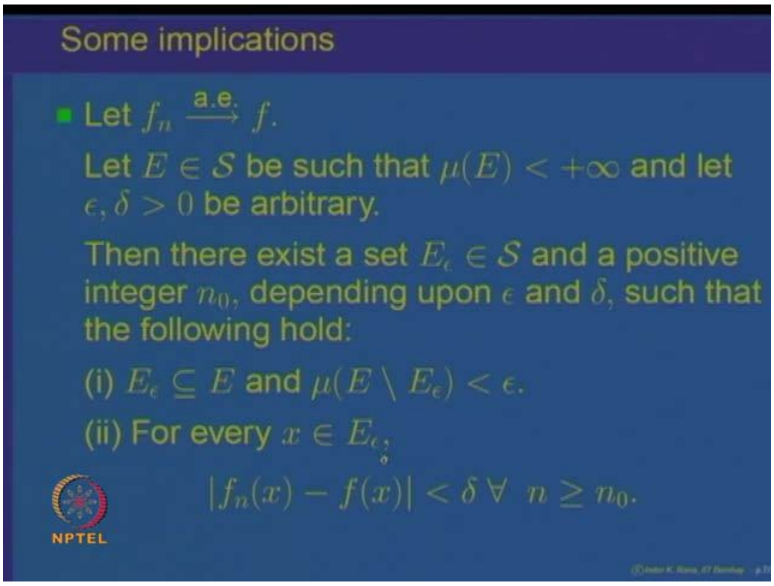

However, we would like to still analyze whether some conclusions can be drawn from the point wise convergence or almost everywhere convergence in terms of uniform convergence. To analyze, let us assume that f n converges to f almost everywhere. Let us take a set E which is in the sigma algebra S such that the measure of this sigma set E is finite and let us be given with two numbers epsilon and delta arbitrary.

Then, we would like to show that there exist a set E lower epsilon - a measurable set E lower epsilon of S and a positive integer n naught which will depend upon epsilon and delta such that with the following properties namely, that E epsilon is a subset of E and the difference E minus E epsilon is small. For every x belonging to E epsilon, on E epsilon f n x minus f of x is less than delta for every n bigger than or equal to n naught. That means, for every x the same stage n naught works, so essentially saying that the difference between f n and f stays less than delta for all n bigger than n naught and the stage will not depend on epsilon.

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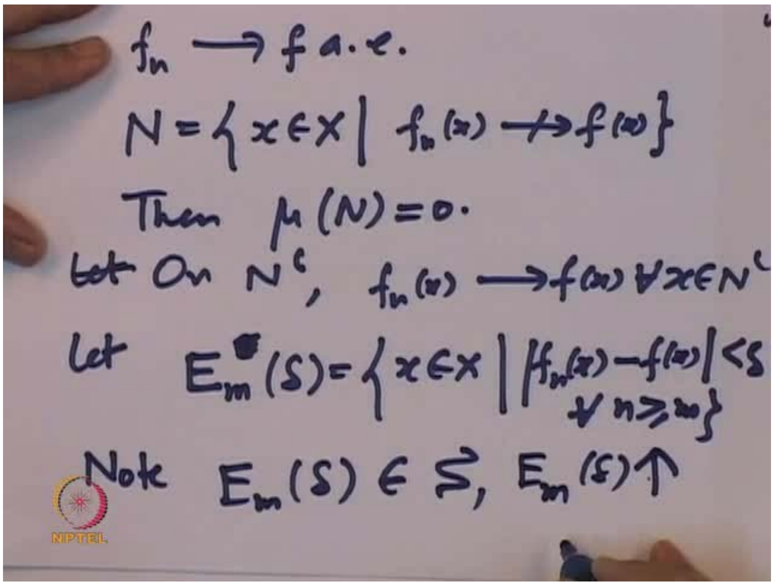

So let us prove this result. To prove this, we are given that f n converges to f almost everywhere. That means, if we define the set N to be the set of all x belonging to X such that f n x does not converge to f of x then, this set the mu of the set N is equal to 0, so that is what is given to us. Now, let us look at the set that means, on N complement on the complement of this set f n x converges to f of x for every x belonging to N complement.

Let us write the set E say m, we are also given delta, so let us write E m delta to be the set of all points x belonging to X such that f n x such that the difference f n x minus f of x f n x minus the difference f n of x is less than delta for every n bigger than or equal to m. Let us look at this set E m delta, this set depends on m and on delta, so the difference between f n and f is less than delta for every n bigger than or equal to m.

Let us note that these sets E m delta they belong to the sigma algebra. If the difference between f n and f m f n and f x is less than delta for all n bigger than m then, that is also going to be true for all n bigger than n plus one.

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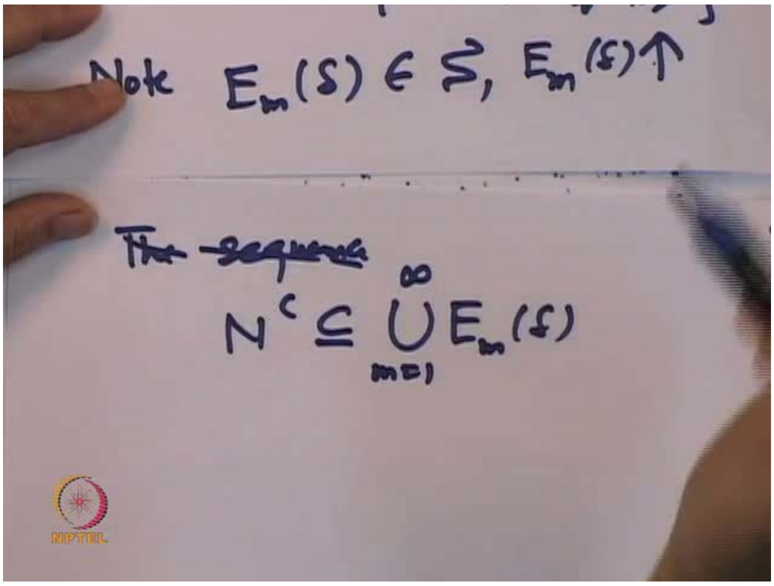

That means, the E m delta is an increasing sequence. So, E m delta is a subset of E m plus one delta further, not only it is increasing because we are given that f n x converges to f of x for every x, So that means the sequence is increasing and on n complement was that know the difference is so the sequence is increasing sequence and we can say that n complement is contained in union of E m delta m equal to 1 to infinity, because on n complement it converges. So for every x, we can find some stage after which it will be less than for a given delta.

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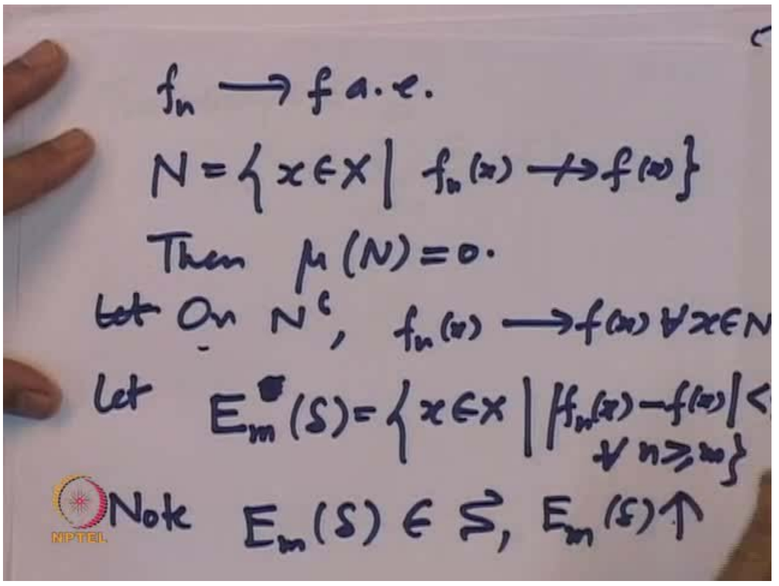

So, the fact that on n complement f n x converges to f of x that implies that the set n complement is a subset of the union of E m delta.

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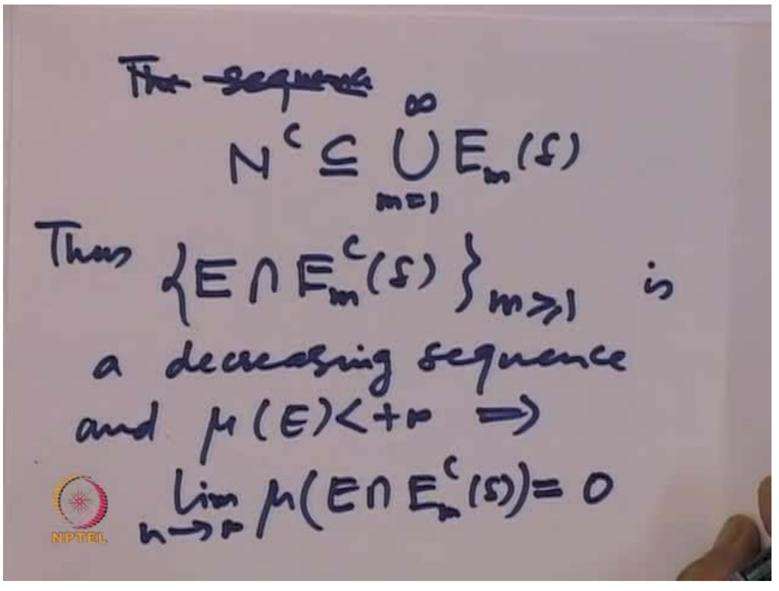

Thus, if you look at the sequence E intersection E m complement delta then, this is a decreasing sequence; it is a decreasing sequence of sets and mu of E being finite implies that limit n going to infinity mu of E intersection E m complement delta. It must be equal to this, because E m was increasing sequence and the complements will be decreasing sequence, so it must converge to intersection of all of them and that is contained in the complement the e intersection and complement that is equal to 0, because this sequence of sets decreases to E intersection n complement, so that is equal to 0.

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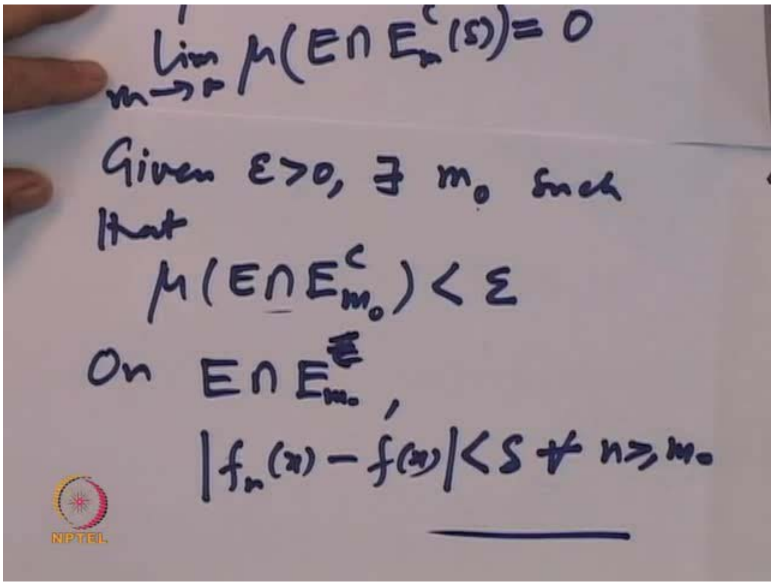

So what does that imply? Saying that this converges to 0 that implies that given there is the stage. Let us find some E m m naught such that mu of E intersection E complement m is m naught is less than epsilon. On E intersection E m naught complement, we know that f n x minus f of x is less than delta for every n greater than or equal to m naught.

So with the given epsilon delta and mu of E finite, we have found a stage m naught and a set E m naught such that E intersection is nothing but E minus the set that is less than epsilon and on the set E intersection the difference between f n and f is less than delta (Refer Slide Time: 16:05).

So that proves the required claim namely, if f n converges to f almost everywhere and we are given a set E of positive finite measure - mu of E is finite then, for every epsilon and delta we can find a subset E epsilon inside E such that the measure of on E epsilon f n x minus f x is less than delta for every n greater than the stage n naught and the measure of the difference E minus E epsilon is small.

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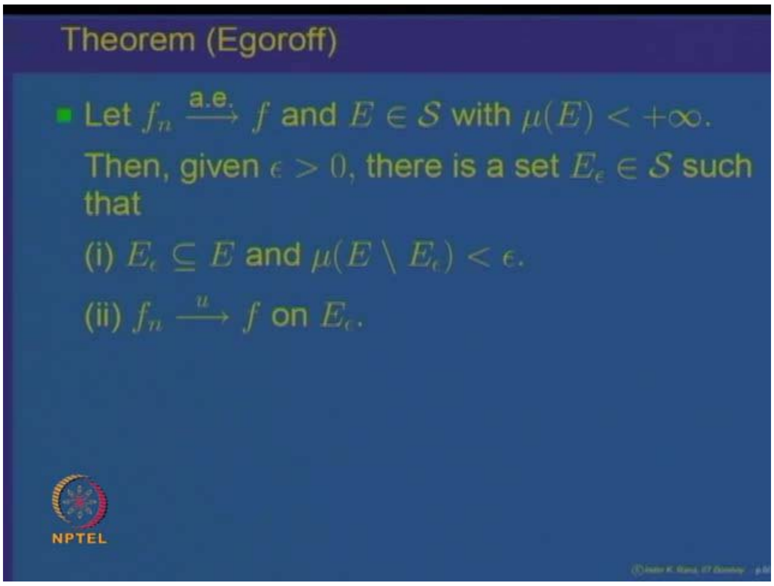

So, this is consequence of the f n converges to f almost everywhere and mu of E has got finite measure. As a consequence of this, we prove theorem called Egoroff's theorem; it says that if f n converges to f almost everywhere and E is a set of finite measure then for given epsilon, there is a set E epsilon such that measure of the set E minus E epsilon is less than epsilon and f n converges to f uniformly on E epsilon.

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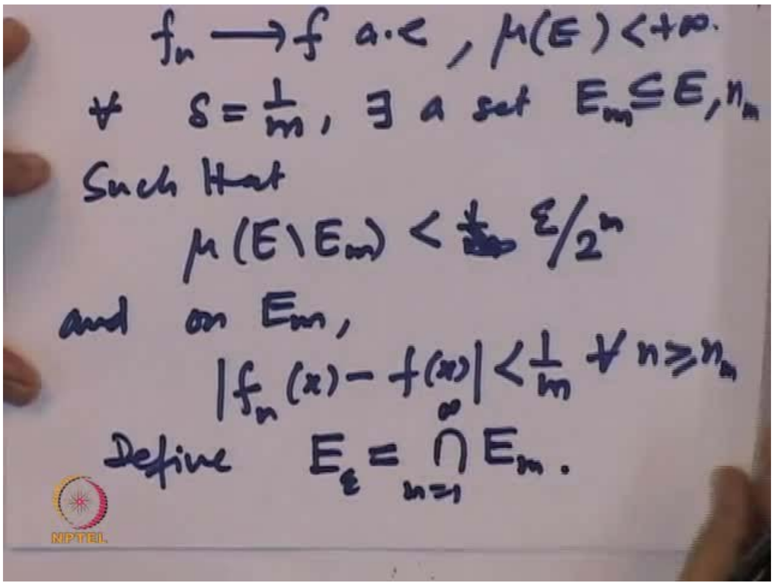

We are given that f n converges to f almost everywhere and mu of E is given to be infinite. So by the result, it just now proved for every delta equal to 1 over n; let us apply the previous result for epsilon equal to delta equal to 1 over n there exists a set E n contained in E such that mu of E minus E n is less than 1 over n. On E n on the set let us write E m here just for the sake of clarity, because we are going to apply it for every n. For every E m there is a set and a stage n m such that f n of x minus f of x is less than 1 over m for every n greater than that stage n m.

By the result, we have proved that whenever f n converges to f almost everywhere and E is a set of finite measure then with delta equal to 1 over n, sorry, this is less than there is a set E m with that this is less than epsilon; for every epsilon, let us do it for epsilon two to the power m (Refer Slide Time: 19:17).

So, we are applying the previous result with delta equal to 1 over m by replacing epsilon by epsilon divided by 2 to the power m. We will say that is set, so that the difference between E and E m is less than epsilon by 2 to the power m and on the set E m the difference f n minus f x is less than 1 over m for every n bigger than or equal to E m.

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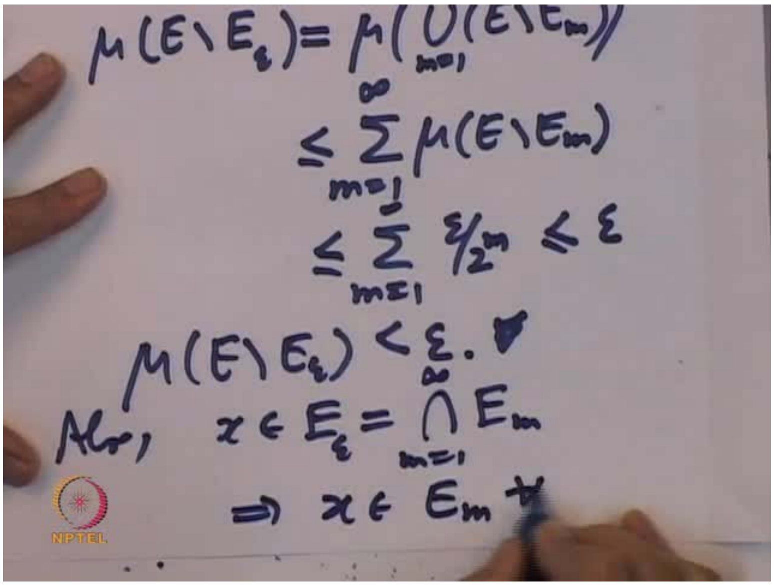

Let us define the set E epsilon equal to intersection of E m, m equal to 1 to infinity. With that let us compute that this is the required set. So mu of E minus E epsilon is equal to mu of union of E minus E m, m equal to 1 to infinity, because E epsilon is intersection, so minus that will make it union, which is less than or equal to sigma m equal to 1 to infinity mu of measure of E minus E m which is less than epsilon by 2 m; so which is less than or equal to m equal to 1 to infinity of epsilon by 2 to the power m which is less than or equal to epsilon.

So, we get that measure of the set E minus E epsilon is less than epsilon. Also if we look at x belongs to E epsilon that is equal to intersection of E m, m equal to 1 to infinity.

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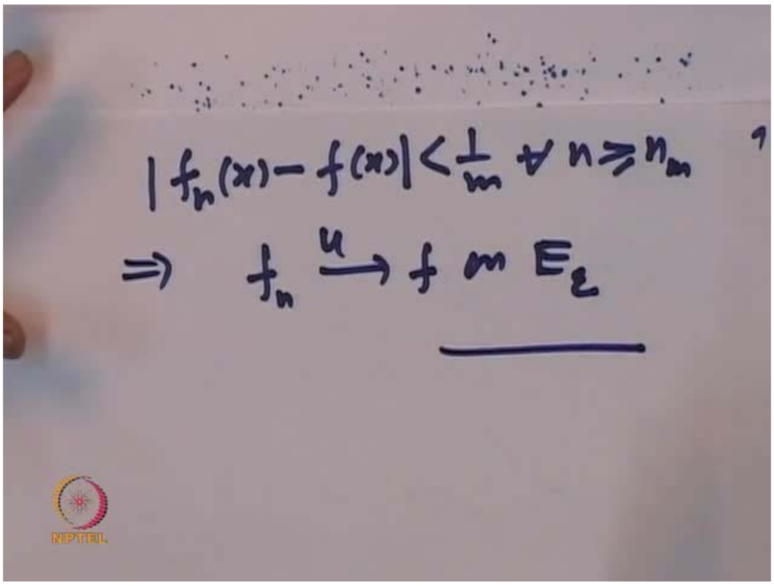

What does that imply; that means, because it belongs to intersection it implies that x belongs to E m for every m; mod of f n x minus f of x will be because it belongs to E m for every m it is less than 1 over m for every n bigger than or equal to n m. So that means for every m we can find a stage after which the difference f n x minus f x is less than 1 over m for every x; so that implies that f n converges to f uniformly on E epsilon that converges to E epsilon uniformly.

So, this proves Egoroff's theorem namely, which says that if f n converges to f almost everywhere then with given any set E of finite measure, we can find a part of it. So that on the part of that set E f n converges to f uniformly and the measure of the remaining that is E minus E epsilon is small is less than epsilon. So for every epsilon we can find a set E epsilon contained in E such that mu of E minus E epsilon is less than epsilon and on E epsilon f n converges f uniformly, so this is what is called Egoroff's theorem.

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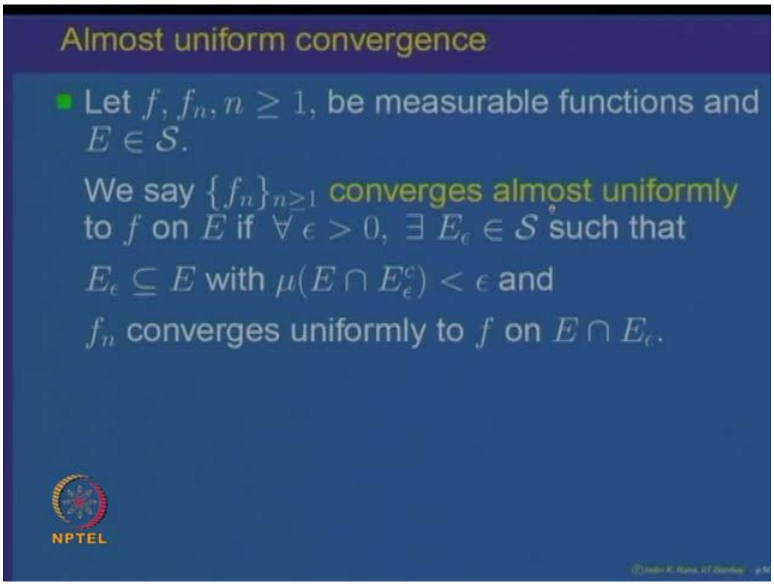

So as a particular case of Egoroff's theorem; let us define and make new definition for measurable functions f n and f, and a set E in the sigma algebra. One says, f n converges almost uniformly to f on E if for every epsilon there is a set E epsilon belonging to the sigma algebra such that the measure of the difference E minus E epsilon is small. On E epsilon f converges a uniformly, such convergence is called almost uniform convergence essentially, saying it is uniform except on a set of measure small that is what almost uniform is to be understood.

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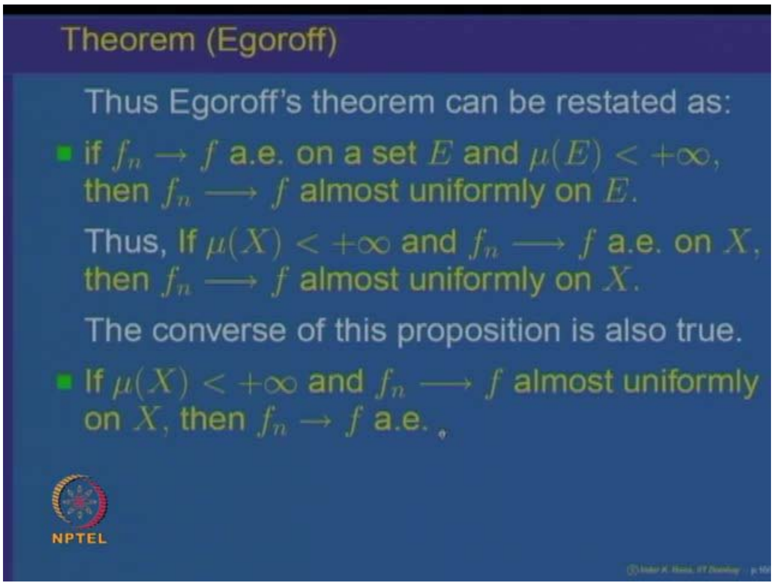

So, Egoroff's theorem can be restated as if f n converges to f almost everywhere on a set E of finite measure then, f n converges to f almost uniformly on E. So almost everywhere convergence implies almost uniform convergence on every set of finite positive measure. In particular case, when mu of x is finite this will imply f n converges to f almost everywhere on the set x, so this will imply that f n converges to f almost uniformly on x.

On finite measure spaces almost everywhere convergence implies convergence which is almost uniform. In fact converse of this proposition is also true when we have got the measure space finite. So, the converse says that if mu of x is finite and f n converges to f almost uniformly on x then f n converges to f almost everywhere.

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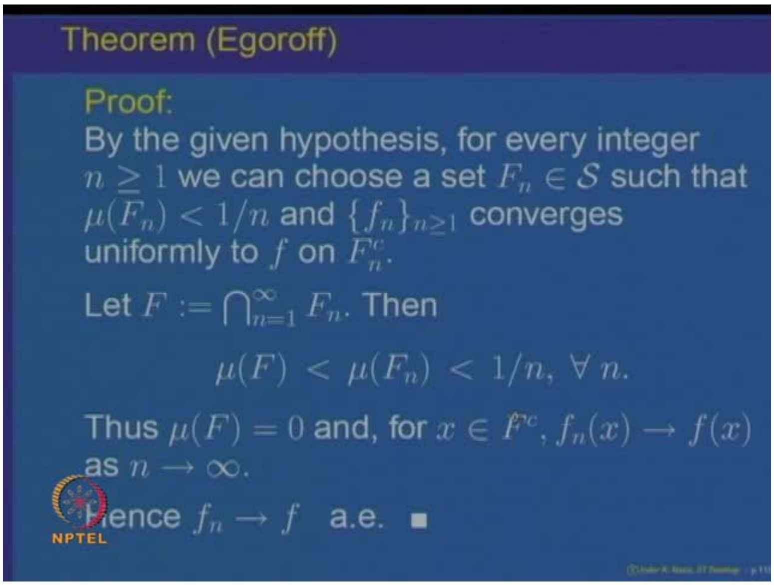

So, the proof of that is almost obvious because by the given hypothesis for every integer n, we can find a set f n in S in the sigma algebra such that the measure of the set f n is small and f n converges to f uniformly on the complement of it. Let us define the set F which is intersection of all this F n's then measure of the set f is less than or equal to measure of each f n which is less than 1 over n.

The set f is a set of measure 0. Outside this if f if a point x belongs to f complement then that mean it belongs to some f n complement for some n and on that convergence is uniform, so f n converges to f in particular. That will imply that on f complement f n x converges to f of x, so that proves the converse part of the Egoroff's theorem for finite measure spaces.

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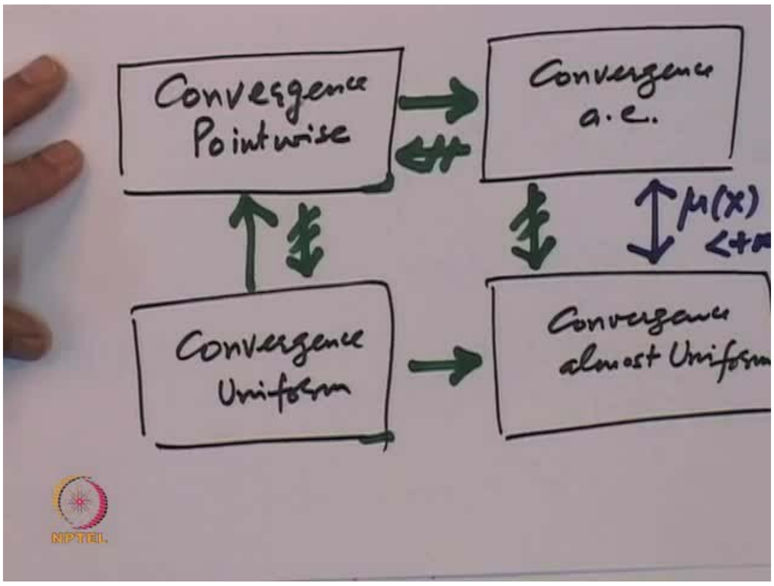

Let us draw a picture and try to see what this implication mean. We have got convergences; one is convergence point wise, so this is point wise convergence. We have got convergence almost everywhere, convergence which is uniform and convergence which is almost uniform.

We know that the point wise convergence implies convergence which is almost everywhere. Convergence uniform implies which is point wise convergence. So convergence uniform implies convergence point wise and point wise implies almost everywhere. The other way around inequality is this is not true and this implication is also not true in general (Refer Slide Time: 26:15).

We just now showed that convergence almost everywhere obviously does not imply this convergence. Convergence uniform obviously implies convergence almost uniform, because almost uniform means outside a set of measure small, so this is uniform implies that convergence also almost uniform (Refer Slide Time: 27:20).

So, almost everywhere convergence or point wise convergence cannot imply this because it does not imply this (Refer Slide Time: 28:00). This implication convergence almost everywhere obviously does not imply in general this, but what we can say is that when it is finite almost this implies this when mu of X is finite and of course, this also implies the other way round when this is finite.

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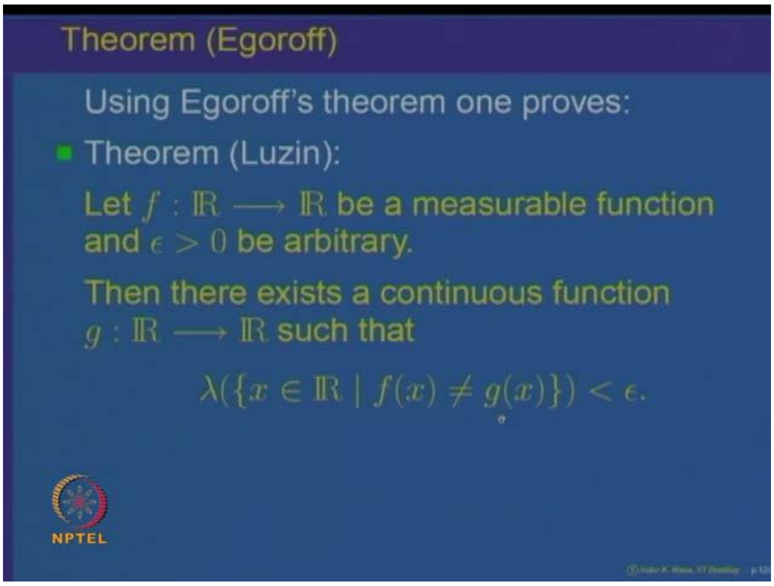

So this is what one way, almost everywhere implies convergence almost uniform that is Egoroff's theorem and the converse always hold, so these two are same if underlying measure spaces is a finite measure space. These are the implications that we can say as true.

Next, let us point out the fact that Egoroff's theorem says that almost everywhere convergence on finite measure spaces gives rise to almost uniform convergence. This can be use to prove a important theorem called Luzin's theorem, which says that if f is a real valued function on reals which is measurable and epsilon greater than 0 is arbitrary then, there exists a continuous function g such that where f differs from g is a set of measure small.

So, what it says that if every measurable function is almost continuous then, you can look at it this way. We will not prove this result. Basically, the idea is on every interval one tries to apply Egoroff's theorem and then try to patch it up. Those who are interested should look at the text book, refer for this result by saying that Egoroff's theorem has an application namely every measureable - real valued measurable function on reals is almost equal to a continuous function; that means, the set of we can find for given any epsilon one can find a continuous function such that the difference f x not equal to g x, it is the measure of that set which is small.

So, these are the relationships between point wise convergence, convergence almost everywhere, uniform convergence and almost uniform convergence, but keep in mind almost uniform convergence is not uniform almost everywhere, so these two are different things. Almost uniform means that except outside a set of measure small the convergence is uniform, but if you say uniform almost everywhere that means, outside a set of measure 0 the convergence is uniform. So almost uniform is not same as uniform almost everywhere, so keep that in mind.

Next we would like to discover another important mode of convergence called convergence in measure. This mode of convergence is very useful in the theory of probability analyzing what are called convergence of random variables. We will not be go to the probabilistic aspect of this; we will just look at the measure thyroidic aspect of what is convergence in measure is.

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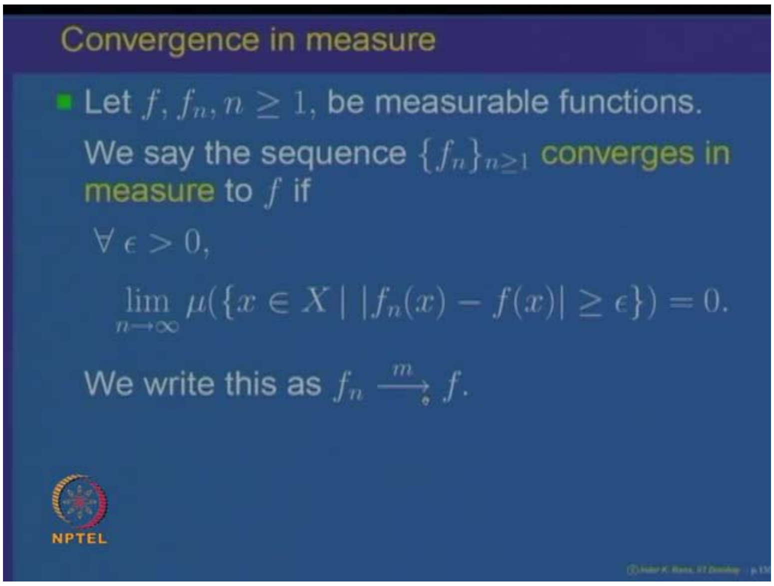

So, a sequence of measurable functions f n is set to converge in measure to a measurable function f on a measure space excess mu. If for every epsilon, look at the set, where f n minus f x is bigger than or equal to epsilon, this is the kind of set where f n is not going to; the difference remains bigger than epsilon.

So, collect all such points and if we look at the measure of this set, if the measure of this set goes to 0 and it becomes smaller and smaller as n goes to infinity, then we say f n converges to f in measure. So saying f n convergence to f in measure is for every epsilon, this is important for every epsilon bigger than 0.

Look at the measure of the set where f n minus f is bigger than or equal to epsilon, measure of that set limit n going to infinity should be equal to 0; this is called convergence in measure for function. This will write as f n with an arrow f and above the arrow, we will put the symbol m indicating that f n converges to f in measure.

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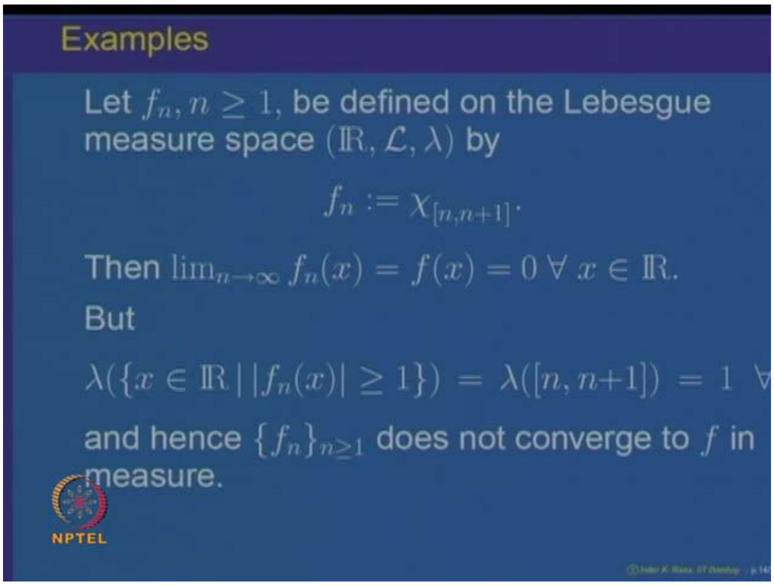

Let us look at some examples to understand convergence in measure, these are the almost everywhere convergence.

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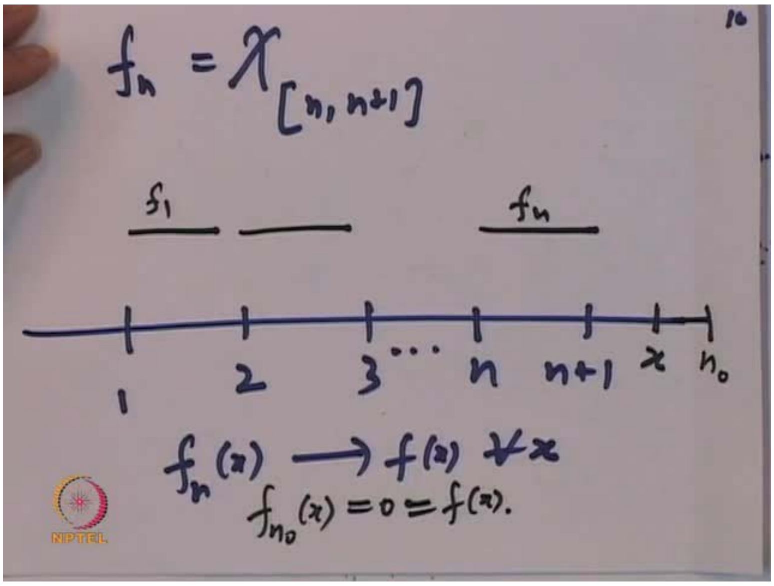

Let us look at the sequence of functions f n which is defined on the Lebesgue measure space R that is real line Lebesgue measurable sets and the length function. Let us take the function f n to be equal to the indicator function of n to n plus 1. So, what we are doing is f n it is the indicator function of the interval n to n plus 1, if this is a line, this is 1, 2, 3, n and n+1.

What is f 1? f 1 is the indicator function of 1 to 2, f 1 is nothing but the function, so this is f 1. What is f 2? f 2 is the indicator function from 2 to 3, so this is f 2, this is f 3 and so on and this is f n (Refer Slide Time: 32:45). So what is f n? f n is the function, it nothing but the constant function one on the interval n to n plus 1, this graph is shifting as you go.

Clearly, f n x converges to f x for every x and that is obvious because, if we are given a point x somewhere, here is a point x then, after some stage we can find n which is bigger than this. So we can find something - some n naught which is bigger than this, then f n naught of x will be equal to 0 (Refer Slide Time: 33:50).

That means for every x, we can find a stage n naught so that this is equal to 0; that will happen for every n bigger than that; that means, the f n x converges to f of x which is identically equal to 0 (Refer Slide Time: 34:15). So, the sequence of functions f n x point wise converges to the function f of x which is identically 0, so f n converges to f identically 0 point wise.

Let us look at the measure of the set, where f n minus f is bigger than or equal to say 1, because f n gives the value 1 or 0 only. The set where f n x differs from f of x is precisely from the interval n to n plus 1. Lebesgue measure of the set where f n x minus f of x is bigger than or equal to epsilon equal to 1 is equal to 1 and that never goes to 0. This is the sequence of measurable functions which converges point wise, but it does not converge measure to the function f identically equal to 0.

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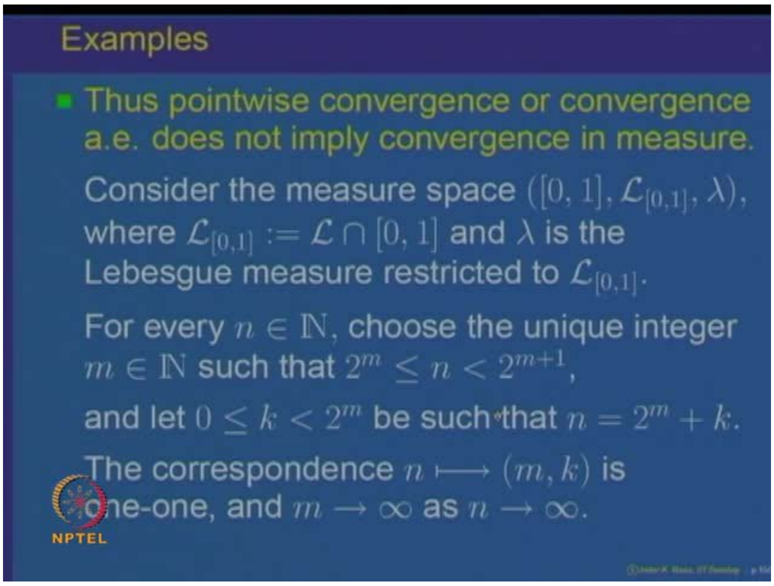

So, point wise convergence need not imply convergence in measure. Point wise convergence - convergence almost everywhere in particular also does not imply convergence in measure.

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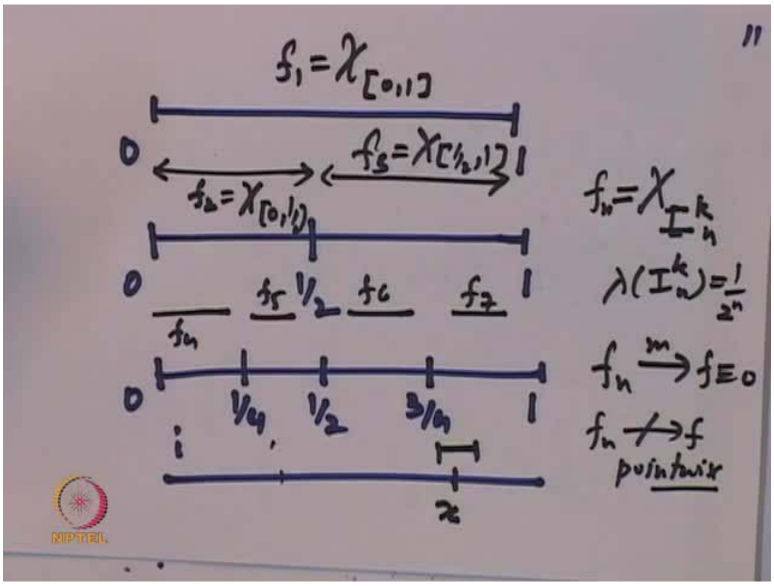

Let us look at another example which is easy to describe pictorially. So, what we are going to look at is, look at the interval 0, 1 divided into two equal parts that is 0, half and 1. In the next stage divided into four equal parts 1, 2; that is 0, 1 by 4, half, 3 by 4, 1 and so on.

So define f 1 to be the indicator function of the whole interval 0 to 1; define f 2 to be the indicator function of the interval 0 to half, so this is f 2 which is the indicator function of 0 to half. This is f 3 which is equal to indicator function of half to 1, so that is f 3. Similarly, this is f 4, this is f 5, this is f 6, f 7 and so on. So each f n is nothing but the indicator function of interval with n points which are obtained by dividing the interval into equal parts at every stage (Refer Slide Time: 36:20).

So, it is a indicator function of a interval I k n, where the length of I k n is equal to 1 over 2, it is going to depend on which stage? It is something like interval of length 1 over 2 to the power n or something like this, because it is going to be shrinking (Refer Slide Time: 37:12).

If this is how the sequence f n is obtained then, it is clear that the set on which f n is 1 is becoming smaller and smaller. So, guess is this f n converges to f identically 0 in measure, but this sequence f n does not converge to f point wise. For that the reason is given any point x we can always find some f n like this, where the value of the function f n is going to be equal to 1, because of that what is happening is f 1 is the indicator function of the whole interval, f 2 it is this (Refer Slide Time: 38:10).

So, the set over which the function is non 0 is taking the value 1, it is fluctuating but it is moving, at that most stage it becomes 0. Actually at every point x there will be some f n which will be decreasing the value 1, so this will not converge almost everywhere or point wise. This will be an example of a sequence which converges in measure but not point wise.

One can formally write this as follows. To write this more rigorously, let us look at this. For every n, first of all choose unique integer m such that n lies between 2 to the power m, n 2 to the power m plus 1 that is obvious. That for any n you can find an integer with this property (Refer Slide Time: 39:00).

Next look at an integer k, for any integer k between 0 and 2 to the power m this number n can be written as 2 to the power m plus k, because n to n plus 1 can be divided into intervals of length 1 over 2 to the power m. Once, you do that you can find a k such that this n is equal to 2 to the power m plus k.

So, what we are saying for every natural number n? It can be expressed uniquely as in the form. To every n, you can associate a pair namely, small m comma k where m is such that n lies between 2 to the power m less than or equal to and less than 2 to the power m plus 1 and k is between 0 and 2 to the power m. This correspondence n with this pairs is unique. Obviously, as n goes to infinity, this m will also go to infinity and conversely as m goes to infinity, n goes to infinity (Refer Slide Time: 40:06).

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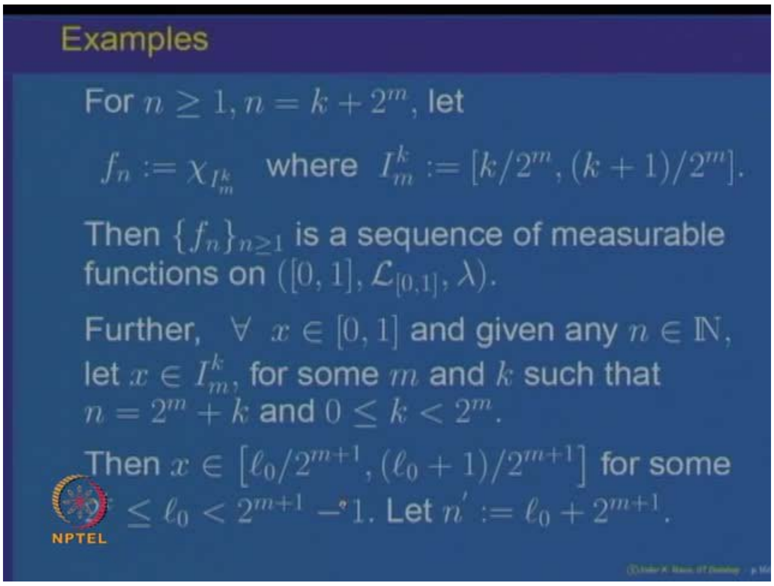

How the function f n is defined? So define f n for any n, represent it uniquely as k plus 2 to the power m and take f n to be the indicator function of I k m. This is the interval starting at k by 2 to the power m and going to k plus 1 2 to the power m; that means, it is a interval of length 1 over 2 to the power m. f n is 1 on this interval and you see that this interval is shifting but never becoming smaller and smaller, and never vanishing.

So, f n is the sequence of some measurable functions because of the indicator functions of intervals. For every x we can find a stage; for any n, if n is equal to this we can find a stage, so that x will also belong to an interval of the starting with l 0 with base as 2 to the power m plus 1. We mean that for every n there is a stage and dash such that f n n dash of f n dash at the point x is also equal to 1 (Refer Slide Time: 41:35).

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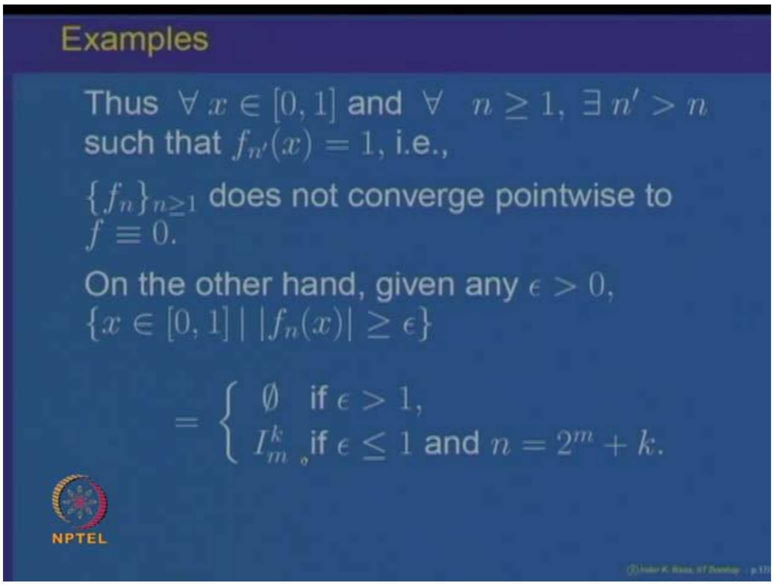

So, this will prove that the sequence f n does not converge to the function identically 0 point wise. On the other hand, it is quite obvious that the set of points where this is bigger than epsilon for every epsilon bigger than 1, it is empty set and less than or equal to 1, it is the interval which is becoming smaller and smaller.

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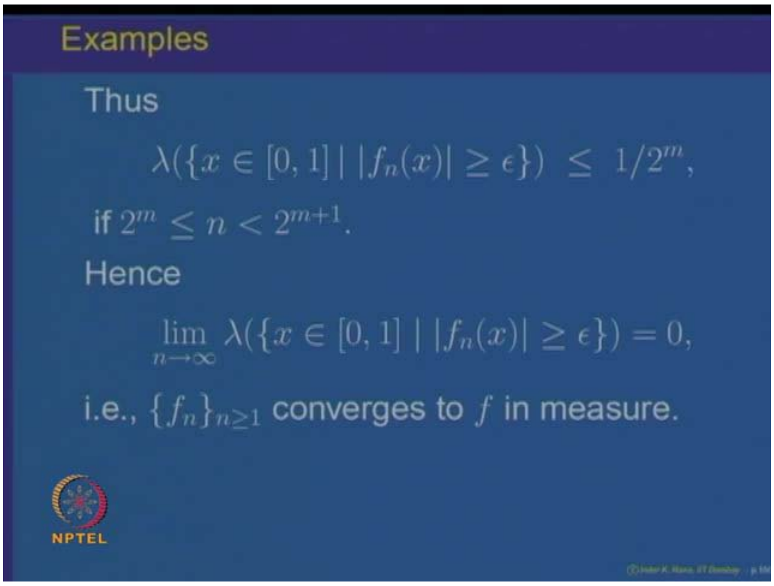

So, it says that this is a sequence of functions that will go to 0. This is a sequence of functions which converges in measure, but not point wise. What we have shown is that the convergence in measure is neither implied by point wise convergence or convergence almost everywhere and neither implies convergence in measure. So, convergence in measure and point wise convergence are neither implied nor implied by each other. This is the concept of convergence in measure which is quite different from point wise convergence.

However, when the underlying space is with finite measure one can draw some conclusions, because in the theory of probability the underlying measure space has got total mass 1; so we want to look at this concept of convergence in measure when the underlying measure space has got finite measure.

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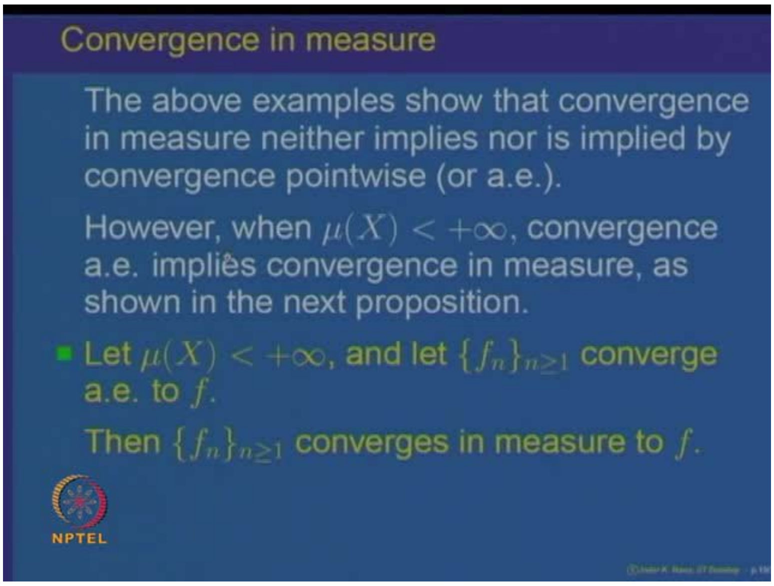

We want to prove that if mu of x is finite and f n converges to f almost everywhere then, f n converges also in measure to the set f. What we want to show is that convergence for finite measure spaces convergence almost everywhere implies convergence in a measure.

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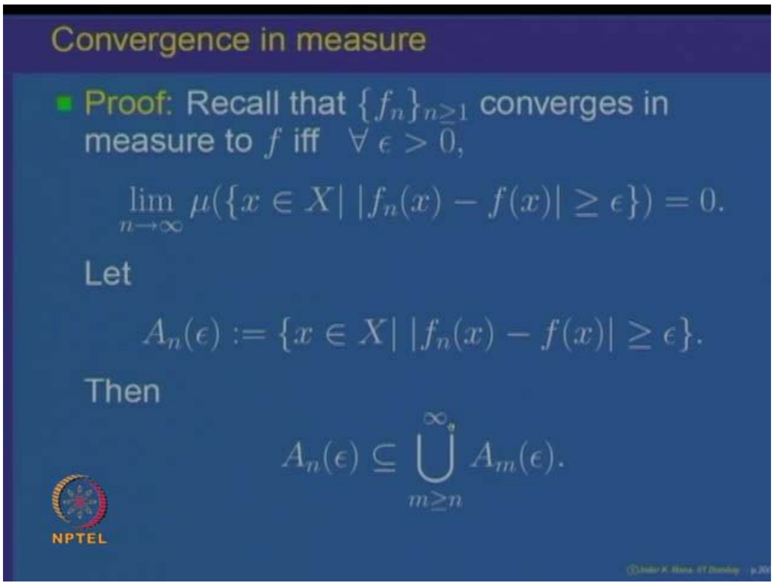

Let us look at a proof of this fact; recall that saying f n converges in measure to f is same as said; this is what we want to show? We want f n converges to f in measure, so what we have to show? We have to show that for every epsilon, if we look at the set of those points where f n minus f is bigger than or equal to epsilon then, the measure of this set goes to 0 as n goes to infinity for every epsilon.

Let us call this set x belonging to X f n x minus f of x bigger than or equal to epsilon; a set, it depends on epsilon and it depends on n, let us call this set as A n epsilon. So A n epsilon is equal to set of points where f n minus f is bigger than or equal to epsilon and we want to show that the measure of this set goes to 0 as n goes to infinity.

Now, let us observe that A n epsilon is obviously a subset of the union of the sets A m epsilons from m equal to n, because this is one of the sets in the union. So this is showing that mu of A n epsilon goes to 0 is same as enough if we can show that measure of this union goes to 0.

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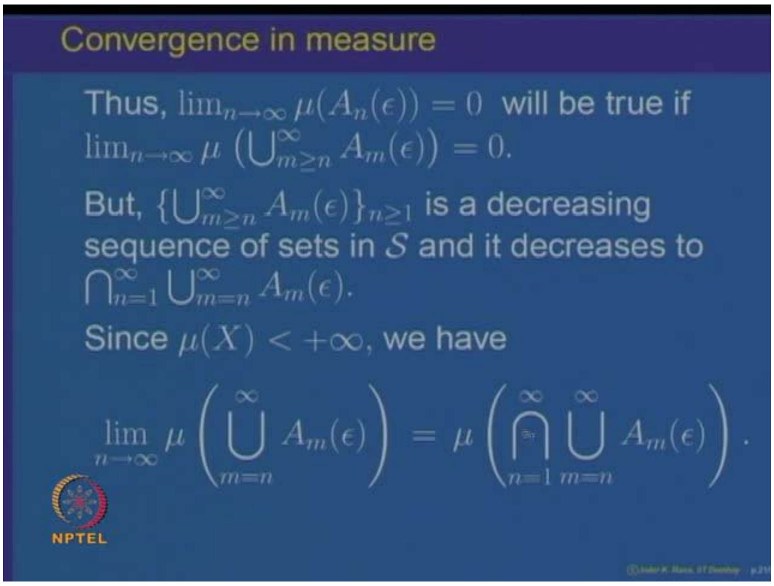

Look at this unions A m epsilon m bigger than or equal to n, this is a sequence of sets as n increases this union is going to become smaller and smaller. This is a sequence of sets which is decreasing, so it is a decreasing sequence of sets and it decreases to; where it will decrease? It will decrease to the intersection of the sets n equal to 1 to infinity of these sets; mu of the set x being finite, this is a sequence of sets which is decreasing to this set and mu of x being finite implies mu of the limit is equal to limit of the mu of the sets. So, limit n going to infinity mu of this is equal to this.

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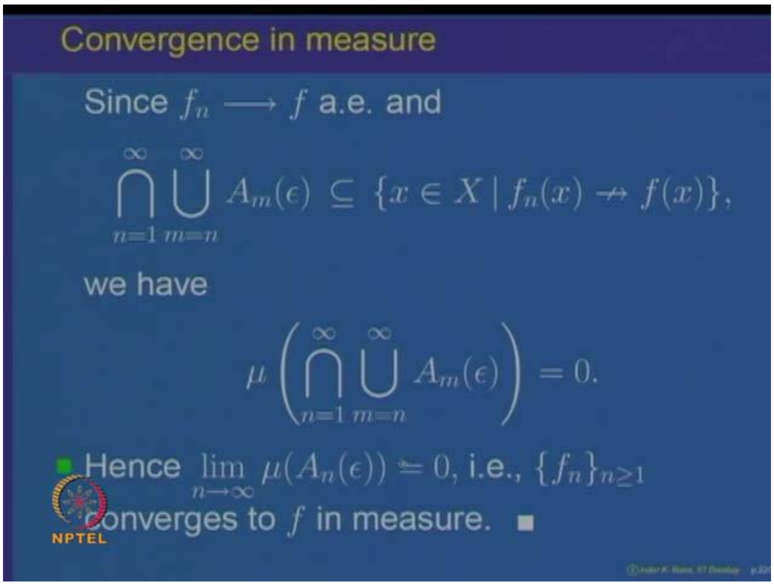

Let us observe that mu of the set which is intersection n equal to 1 to infinity, union m equal to 1 to infinity A m epsilon, what is this set? If x belongs to this set that means, for every n there is m such that x belongs to A m for some m bigger than or equal to n, which is same as the saying. So x does not converge f of x because A m epsilon is the set where it is bigger than epsilon.

What it says? It says that this is a set where f n does not converge to f of x and we are given that f n converges to f almost everywhere. So this set has got measure 0, this limit is equal to measure 0. So that proves the fact that this is of measure 0 that proves the limit of A n epsilon is 0 that means, f n converges to f n measure.

So, we had started looking at the convergence in measure and we have proved, gave examples to illustrate that neither point wise convergence implies convergence in measure nor convergence in measure implies point wise convergence. However, when the underlying space is finite convergence almost everywhere does imply convergence in measure.

We will continue this study of convergence in measure and also look at its relation with what is called convergence in mean or convergence in L P spaces. We will do that in next lecture, thank you.