Measure and Integration

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Module No. #10

Lecture No. #38

Absolutely Continuous Measures

Welcome to lecture 38 on Measure and Integration. Today, we will start looking at the notion of absolute continuity of measures. We started this concept in the previous lecture but, we will do it in detail in today's lecture. So, we will start with looking at what is called one measure being absolutely continuous with respect to other measure and then, we will prove an important theorem called Radon–Nikodym theorem for absolutely continuous measures.

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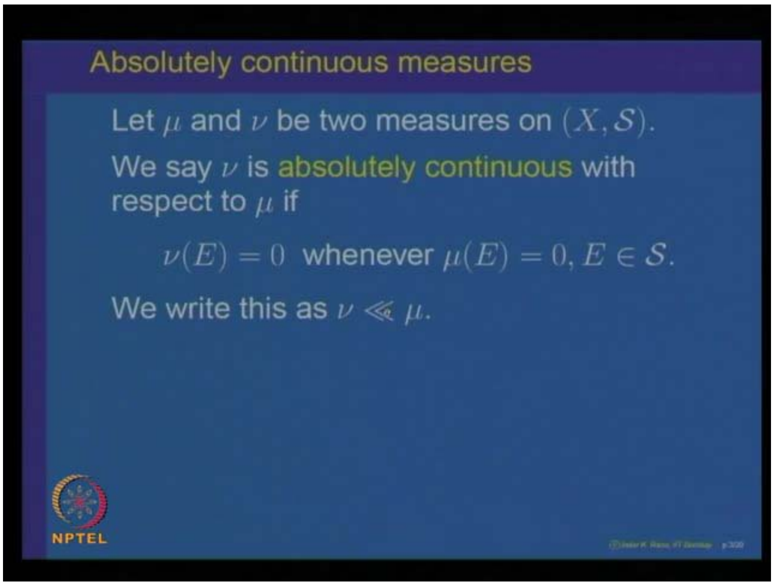

Let us start recalling what an absolutely continuous measure is. So, if two measures mu and nu given on a measure space - measurable space X, S we say nu is absolutely continuous with respect to the measure mu, if for any set E in the sigma algebra S, mu of E equal to 0 implies nu of E equal to 0. That means, if a set E has got mu measure 0 then, it should imply that nu measure of the set E is also equal to 0. So in that case, we write this relation by the symbol that nu with this special symbol which is less than twice printed, so it is called absolutely continuous with respect to nu and this is denoted by this symbol.

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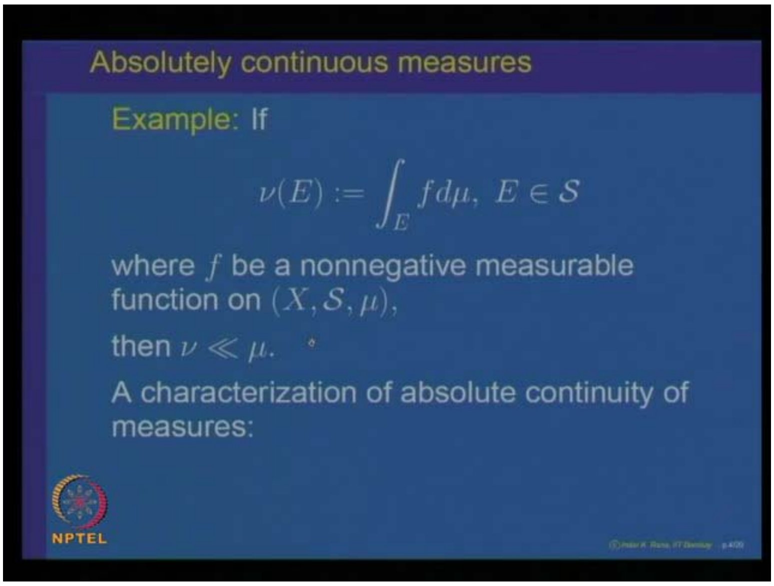

We looked at an example of absolutely continuous measures. We said, let us take a function f which is non-negative measurable on a measure space X S mu and let us integrate this function over a set E in the sigma algebra. So, this integral f is fix, mu is fix, E is varying and this is a non-negative number, which we denoted by nu of E and we had shown, when we defined the integral for non-negative functions that nu of E is a measure and it has the special property that if mu of E is equal to 0 then nu of E is also equal to 0.

So, this measure nu of E which is defined via the integral of a non-negative function f over a set E implies that this measure nu is absolutely continuous with respect to mu. Let us give a characterization of absolutely continuous measures in terms of what is called epsilon delta definitions which look similar to absolute continuity of functions.

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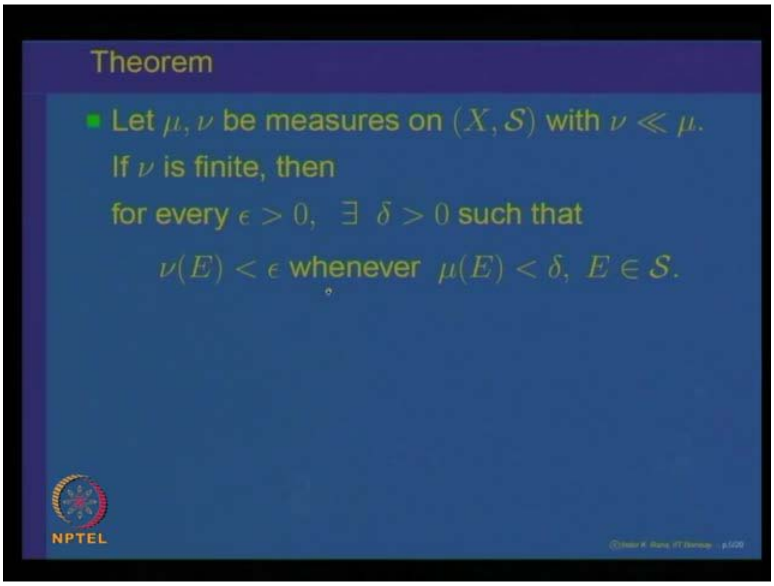

So, we want to prove the following namely, if mu and nu are two measures such that nu is absolutely continuous with respect to mu then, the following holds that if nu is finite then, for every epsilon bigger than 0 1 can find a delta bigger than 0 such that whenever nu of E is less than epsilon whenever mu of E is less than delta.

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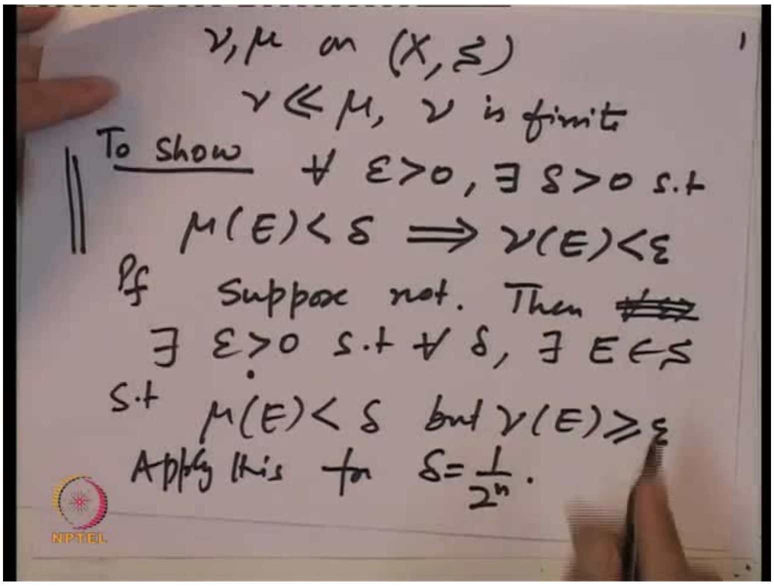

That means, given an epsilon you can find a delta such that, whenever for a set the measure mu of E is less than delta, that should imply nu of E is less than epsilon; so let us prove this result. So, we have given that nu and mu are measures on the measurable space X S and nu is absolutely continuous with respect to mu and nu is finite. To show, for every epsilon bigger than 0, there is delta bigger than 0 such that mu of E less than delta should imply nu of E is less than epsilon, so the proof is of this is by contradiction.

So, if suppose not, means suppose this claim is not true that will mean what? Claim is for every epsilon there is a delta, so that means - there is an epsilon such that then for every epsilon bigger than no - that means, then there exists epsilon bigger than 0 such that for every delta, there exists a set E belonging to the sigma algebra with the property, whenever mu of E is less than delta but, nu of E is bigger than or equal to epsilon.

So, if we assume that the required claim is not true that will imply that there is a number epsilon bigger than 0 such that for every delta you can choose a set E of course, this E will depend on delta such that mu of E is less than delta but, nu of E is bigger than or equal to epsilon. So, let us apply this result for so apply this for delta equal to 1 over 2 to the power n, so that will give us the following when I do that.

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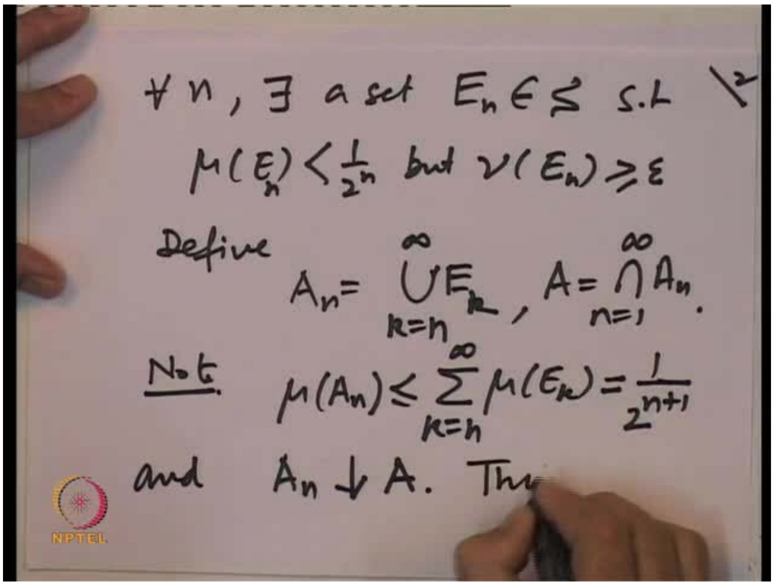

So, for every n there exist a set E n belonging to S such that mu of E n is less than 1 over 2 to the power n but, nu of E n is bigger than or equal to epsilon. So, this is what we get if we assume our result is not true.

Now, let us define a set A n to be equal to union of E n, E k from k equal to n to infinity and A to be equal to intersection of A n, n equal to 1 to infinity.

So, let us note the following set A n, so mu of A n is less than or equal to sigma mu of E k; k equal to n to infinity and mu of E k is less than 1 over 2 to the power k and summation n equal to 1 over 2 to the power n, so this is going to be equal to 1 over 2 to the power n plus 1. So, mu of A n because mu of E k will be less than 1 over 2 to the power k. So, this says mu of A n is less than or equal to 1 over 2 to the power n and these A n's; A n is union of n to infinity, so as n increases these A n's are going to be smaller and smaller, so A n is a decreasing sequence and it decreases to the intersection namely A.

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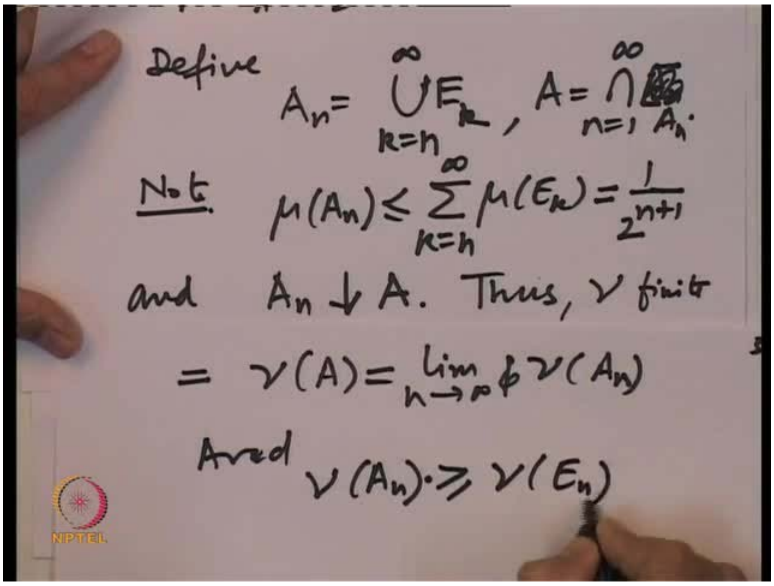

Thus, nu being finite A n decreasing to A will imply, so this will imply that nu of A is equal to limit n going to infinity nu of the set A n but, nu of A n; A n is the intersection and oh sorry intersection of E sorry A n sorry that is ok A n. So A n is union of E k and nu of A n is summation, so it is bigger than or equal to nu of E n because A n is union of E n's, so A n includes E n. So nu of A n going to be bigger than or equal to nu of E n which is bigger than or equal to epsilon.

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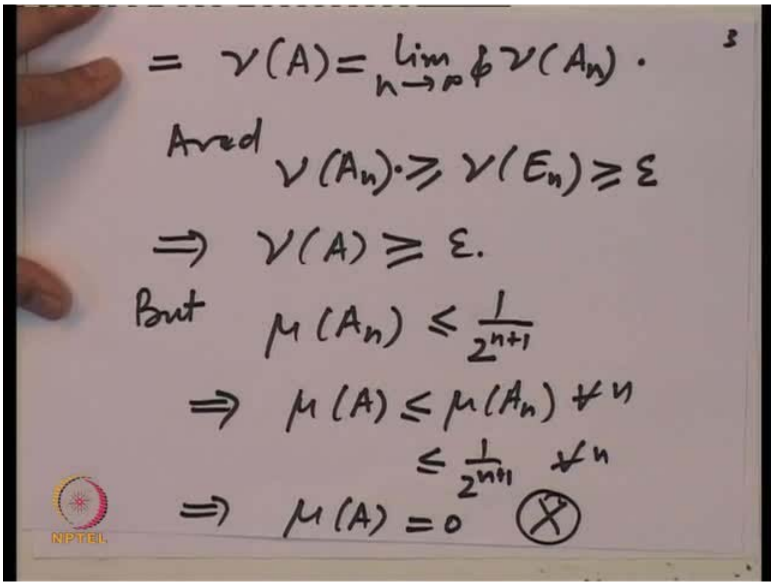

So, this along with this implies that nu of A is bigger than or equal to epsilon but, we just now observed that mu of A n was less than or equal to 1 over 2 to the power n plus 1, so what does that imply? That implies that mu of the set A because A is equal to intersection of this, so mu of A is less than or equal to mu of A n for every n, so less than or equal to 1 over 2 to the power n plus 1 for every n, so that implies that mu of A is equal to 0.

So, what we have done is assuming that the required condition does not hold we have shown that there is a set A, such that mu of A is 0 but, nu of A is not equal to 0; it is bigger than or equal to Epsilon which is a contradiction; this is the contradiction. So that implies, what we assumed is not true and hence, we have the required claim namely, that if nu is absolutely continuous with respect to a measure mu and nu is finite then, for every epsilon bigger than 0 there is a delta bigger than 0 such that mu of A less than delta implies, nu of the set A has to be less than or equal to epsilon. This is what we have proved.

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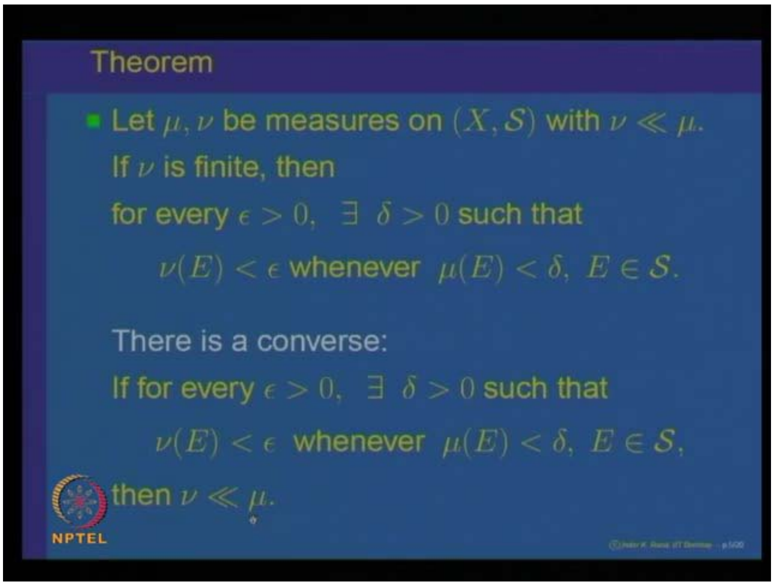

So, we have shown that if nu is finite then, the condition that nu is absolutely continuous with respect to mu implies this required claim. Let us prove actually, the converse of this statement is also true namely for every epsilon bigger than 0 if there exists a delta such that nu of E is less than epsilon whenever mu of E is less than delta then nu is absolutely continuous with respect to mu.

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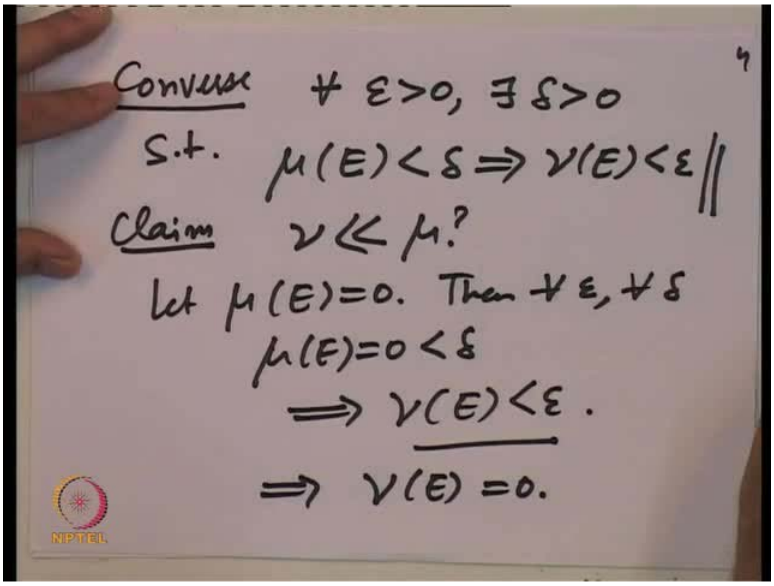

Let us prove the converse. So, converse says the following that we have the property that for every epsilon bigger than 0 there is a delta bigger than 0 such that mu of a set E less than delta implies, mu of E is less than epsilon and we want to claim that this property means that nu absolutely continuous with respect to mu.

Let us take suppose, mu of E be equal to 0. If mu of E is equal to 0 then for every epsilon - whatever delta we take - for every delta mu of E is equal to 0 is less than delta. So, that will imply by the given property that nu of E is less than epsilon. So nu of E is less than epsilon for every epsilon, so that implies nu of E is equal to 0. So, saying that for every epsilon there is a delta such that mu of E is less than delta implies the nu of E is less than epsilon implies, that the measure nu is absolutely continuous with respect to mu. So the converse is easier and it is true even if nu is not finite. So, we do not use anywhere the fact that nu is finite.

One way, we require nu to be finite that means, absolute continuity and nu finite implies this condition that for every epsilon, there is a delta such that mu of E less than delta implies nu of E less than epsilon. For the converse, we do not need this property. This is one characterization of absolutely continuous measures.

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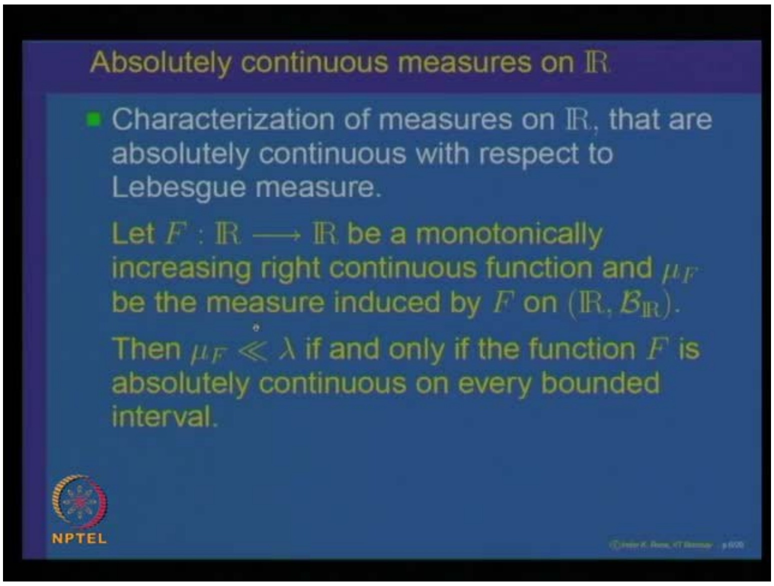

Next, let us recall that - we had defined - we had characterized all measures on the real line on the Borel sigma algebra. We had shown that if say given a measure mu on the collection of all borel measurable sets in the real line there exists a monotonically increasing right continuous function such that for any interval a comma b mu of a b is given by f of b minus f of a. So, that gave us the characterization of all countably additive set functions on the sigma algebra or Borel subsets of real line. Using that we would like to characterize what are all absolutely continuous measures on the real line with respect to the Lebesgue measure.

So, the theorem states the following namely, let F be a monotonically increasing right continuous functions and mu have be the measure induced by this monotonically increasing right continuous function F on B R. Then, the claim is that this measure mu F is absolutely continuous with respect to the Lebesgue measure lambda if and only if, the function F is absolutely continuous on every bounded interval. So, we will like to prove this result.

This will give us that what are all measures which are absolutely continuous with respect to the Lebesgue measure. It ties up with the property of the function F being absolutely continuous. So, this relates the two concepts: absolutely continuous measures and absolutely continuous functions on the real line.

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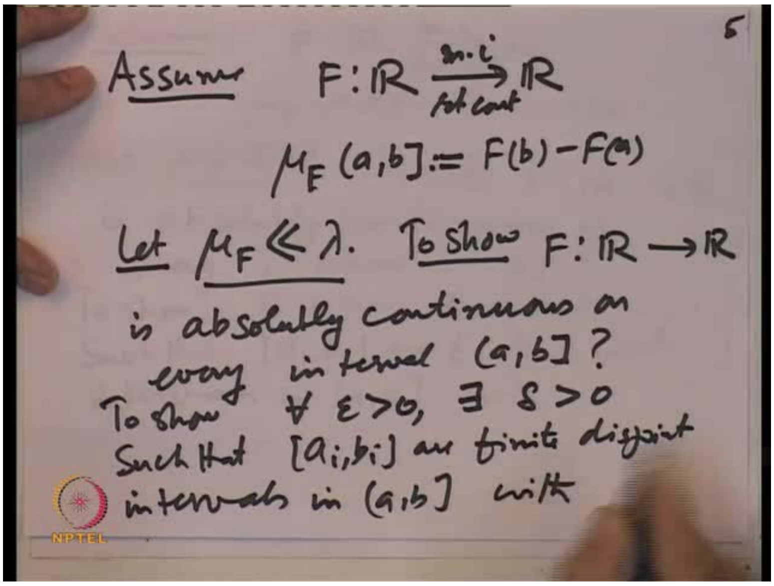

Let us look at a proof of this. Let us assume that F is monotonically increasing and right continuous and mu F is the measure on the borel sets which is given by for the left open right closed interval a, b it is defined as - if you recall - we defined it as F b minus F of a and then, we showed that this mu F is the measure which we called as the measure induced by the function F.

So this is given, let mu F be absolutely continuous with respect to the Lebesgue measure. To show that the function F is absolutely continuous on every interval say a to b, what we have to do?

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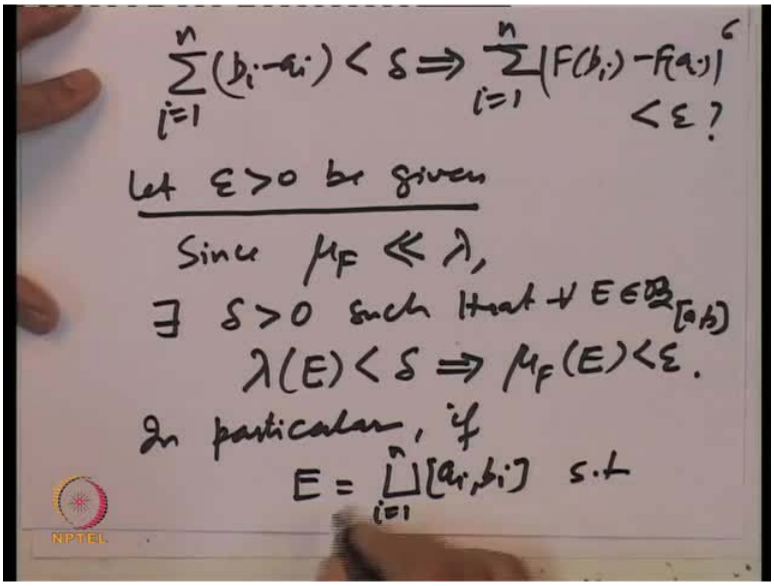

We have to show that for every epsilon bigger than 0 there is a delta, bigger than 0 such that if I take a sequence of intervals say a i, b i are finite disjoint intervals in the interval a b with the property that sigma of b i minus a i is less than delta and that should imply that sigma i equal to 1 to n F of b i minus F of a i is less than epsilon, so this is what we have to show.

So let us fix, let epsilon greater than 0 be given. Let us start with an epsilon bigger than 0 be given. Now, since mu F is absolutely continuous with respect to lambda there exists delta bigger than 0 such that lambda of a set E less than delta will imply that mu F of E is less than epsilon, so this is by the property. We just now proved that absolute continuity is equivalent to this property and mu of F on the interval a b is finite, such that for every E inside belonging to the borel sigma algebra of the interval a b, this is true.

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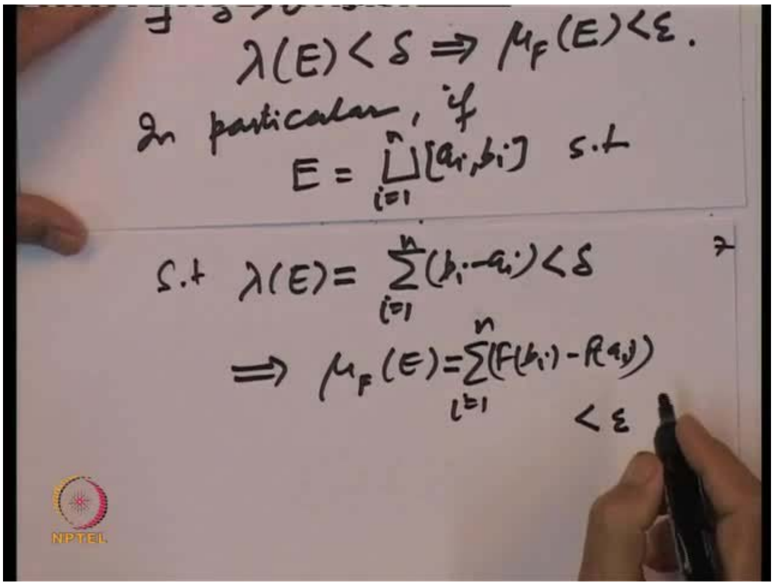

So in particular, if E is a finite disjoint union of intervals a i b i inside, such that lambda of E which is equal to sigma b i minus a i is less than delta.

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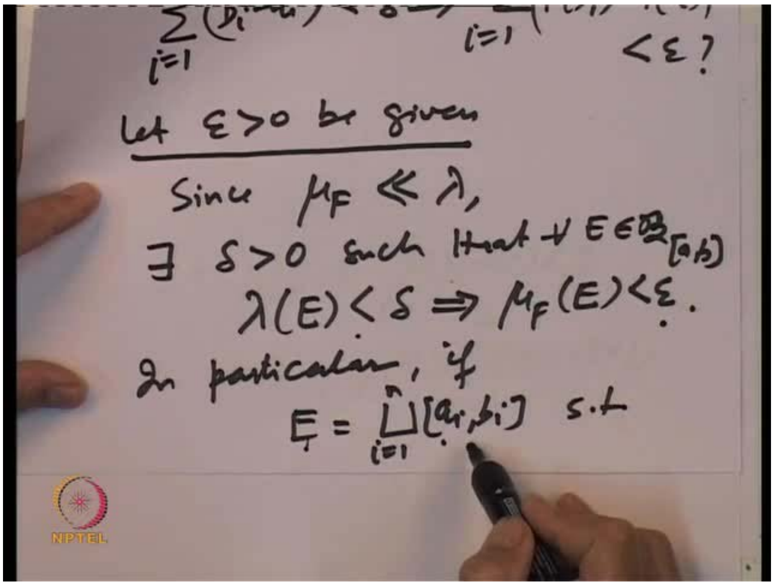

So that will imply that mu of F of this set E; mu of F of this set E is nothing but, F of b i minus F of a i, summation i equal to 1 to n is less than epsilon that is because, we have just now given epsilon, we have chosen delta with that property.

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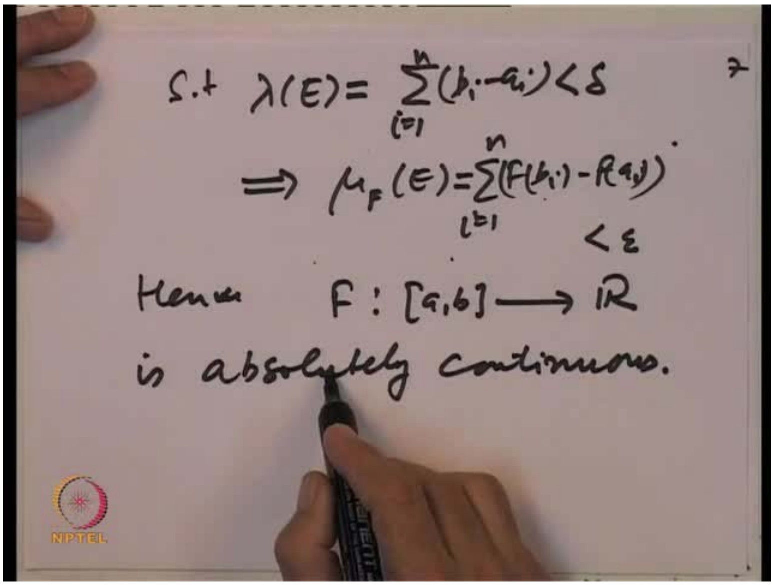

So, in particular we applying this for a set E which is a finite disjoint union of intervals a i b i inside the interval a, b. So, this will imply mu F and by definition mu F of E - which is the finite disjoint union of intervals - is nothing but, sigma F of b i minus. Hence, this proves that F on a b to R is absolutely continuous. So, one way we have proved that if F of mu of F the measure induced is absolutely continuous with respect to the Lebesgue measure. Then, the corresponding function F monotonically increasing right continuous function F which is inducing that measure is also absolutely continuous.

Let us look at the converse of this statement that we are given that the function F - capital F - which is monotonically increasing and right continuous is also absolutely continuous then, we want to show that the corresponding measure mu F is absolutely continuous with respect to the Lebesgue measure. So, let us prove that fact.

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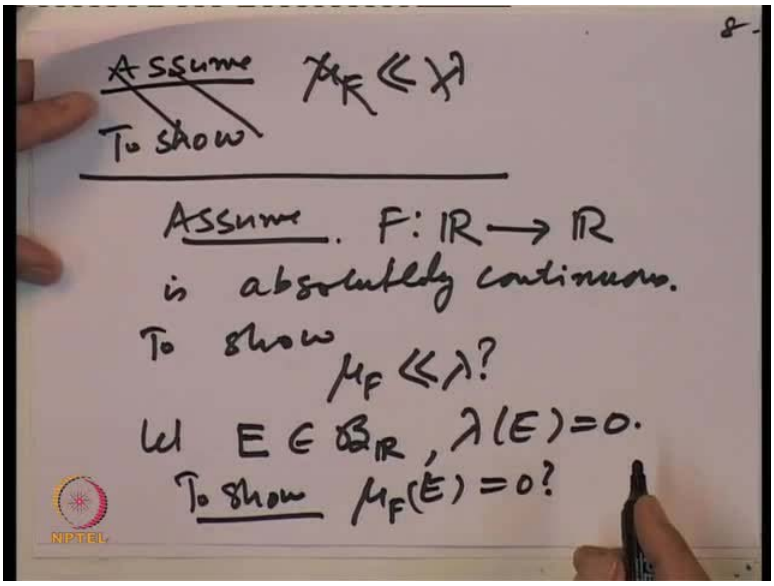

Let us prove, assume that mu F is absolutely continuous with respect to lambda. we have to show the other way round sorry this is not, this is just now we have shown. So, assume that F is absolutely continuous. To show, we have to show that mu F is absolutely continuous with respect to lambda. So this is what we have to show for the other way round proof. Let us assume, let E belong to B R and lambda of E equal to 0.

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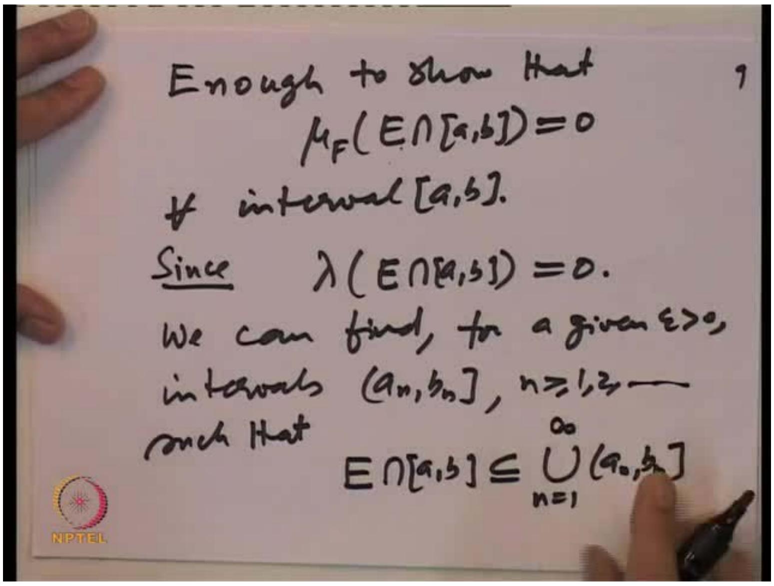

So, to show mu F of E is equal to 0, this is what we have to show mathematically. Now let us observe first of all by showing that mu F of E is equal to 0 enough to show that mu

F of E intersection every interval a to b is equal to 0, for every interval a b then, we can split E into every interval. So, countable additivity will give us that this is 0 for every E.

Let us assume that we have got enough to show that this is equal to 0. Now, since lambda of E is 0; lambda intersection a b is also equal to 0 that means, the set E intersection a b is a set of Lebesgue measure 0. So, by the properties of sets of Lebesgue measure 0, we can find so let us or a given epsilon, we can find intervals say a n b n. We can choose them to be left open right closed n bigger than or equal to 1 2 and so on, such that this set E which is a null set intersection a, b is covered by these intervals a n b n and the total length of this and sigma b n minus a n, n equal to 1 to infinity is less than epsilon.

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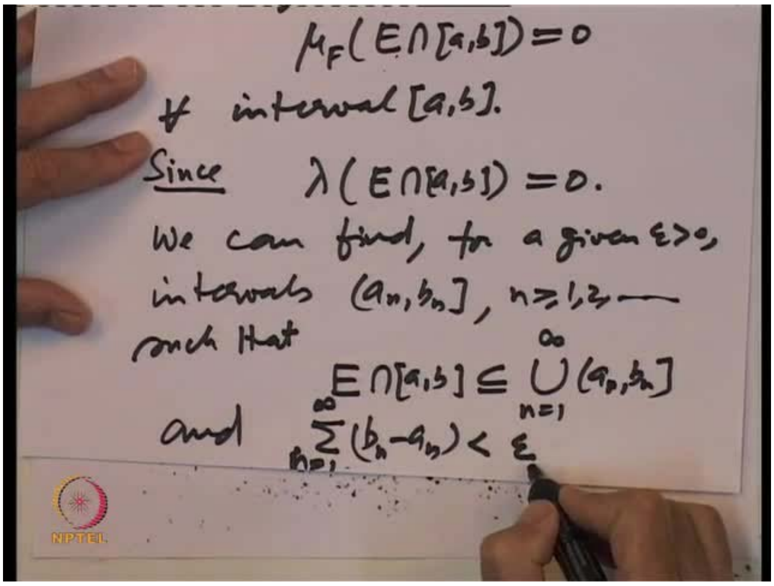

So, the total length of this is small. This is by the property that E is a null set, so E intersection a b is a null set. So, by the definition of after measure if you like, you can find a sequence of left open right closed intervals which cover this set and the total length of this recovering intervals is small is less than epsilon.

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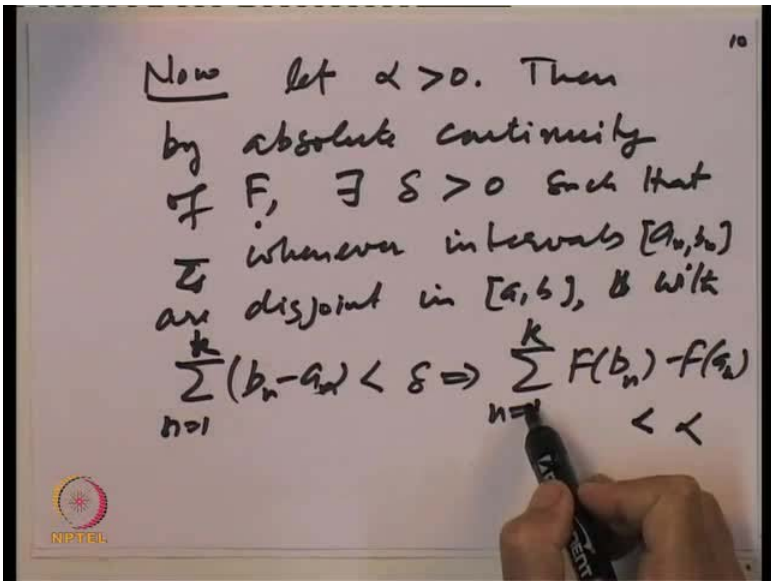

Now, what we want to show? Our aim is to show that the function F is absolutely continuous. So, let us take any number say alpha; let alpha bigger than 0 be given then, it is enough to show that mu of that thing is equal to. So, what is to be shown? We are given F is absolutely continuous then by the absolute continuity of F there exist some delta bigger than 0 such that whenever intervals a n b n are disjoint in a b with sigma b n minus a n some n equal to some 1 to k finite number of disjoint intervals a b less than delta will imply that F of b n minus F of a n, n equal to 1 to k is less than alpha.

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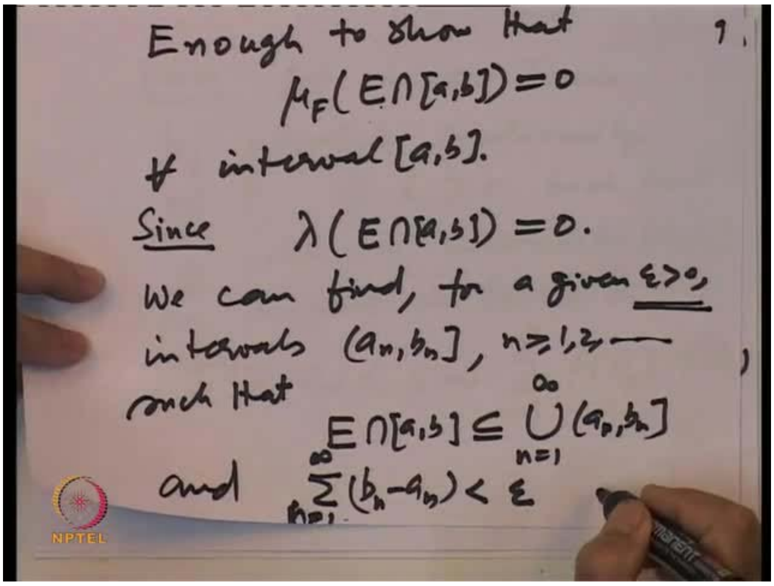

So, this is by the absolute continuity of the function F which is given to us. Now, we start with given any alpha bigger than 0, we find a delta with this property. Now for this delta, we apply our earlier things that lambda of this was a null set, so for any given epsilon; we will use this when epsilon is equal to delta because, when epsilon is equal to delta we will have the summation of this intervals less than delta and that will imply the corresponding.

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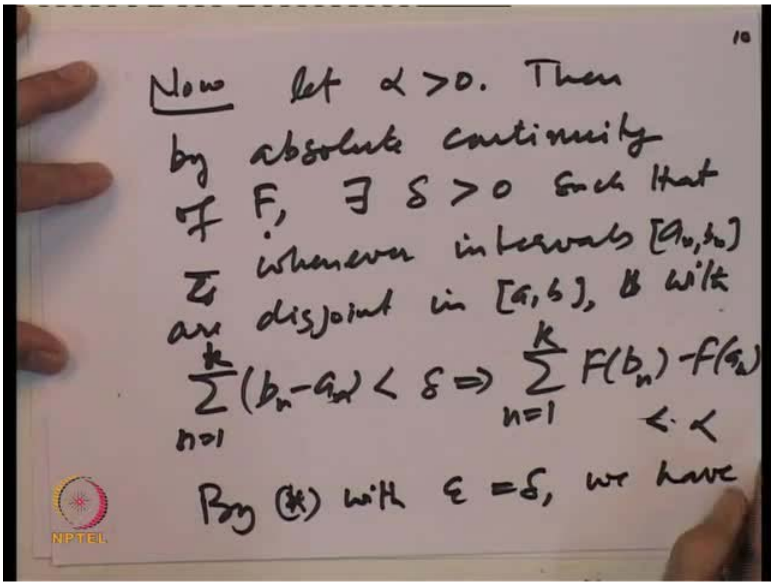

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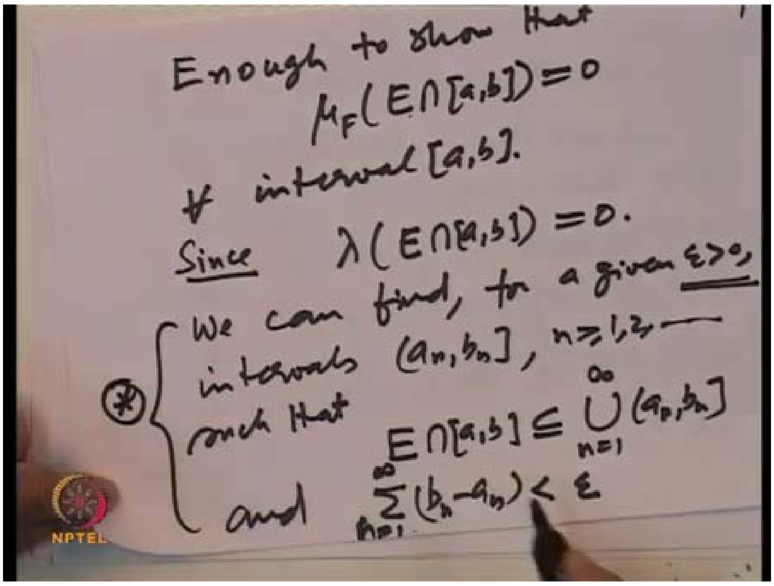

Let us call this thing as star.

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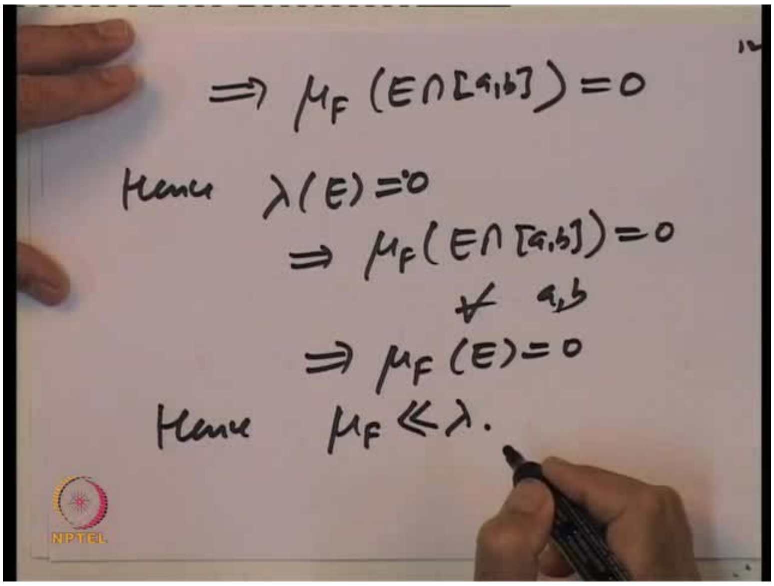

So by star with epsilon is equal to delta, so we will have that sigma of F b n minus F of a n, n equal to 1 to k is less than alpha for every k, so this happens for every k. So, that implies that sigma of n equal to 1 to infinity F of b n minus F of a n is also less than alpha because, this is happening for every k but, that implies nu of E intersection a b which was nu F of this which was less than or equal to because, E intersection a b was contained in the union of intervals a n b n. So, it is less than or equal to sigma mu F of a n b n, n equal to 1 to infinity and this is equal to sigma n equal to 1 to infinity of F of b n minus F of a n which is less than alpha.

What we have shown is that for every alpha, nu F of E intersection a b is less than or equal to alpha. so that implies so this- This happens because this is happening for every alpha, so this implies that nu F of E intersection a b is equal to 0 because, this is happening for every alpha, so let alpha go to 0. So hence, we have shown that lambda of E equal to 0 implies mu F of E intersection a b is equal to 0, for every a b. That implies - because this is happening for every a b so this is implies - that mu F is of E is equal to 0. So hence, mu F is absolutely continuous with respect to lambda.

So, this proves the other way round theorem. This completes the proof of the theorem that mu F is absolutely continuous with respect to lambda if and only if the function F is absolutely continuous on every bounded interval. So, this completely characterizes and ties up the notion of absolutely continuous measures on the real line with respect to Lebesgue measure with absolutely continuous functions on the real line.

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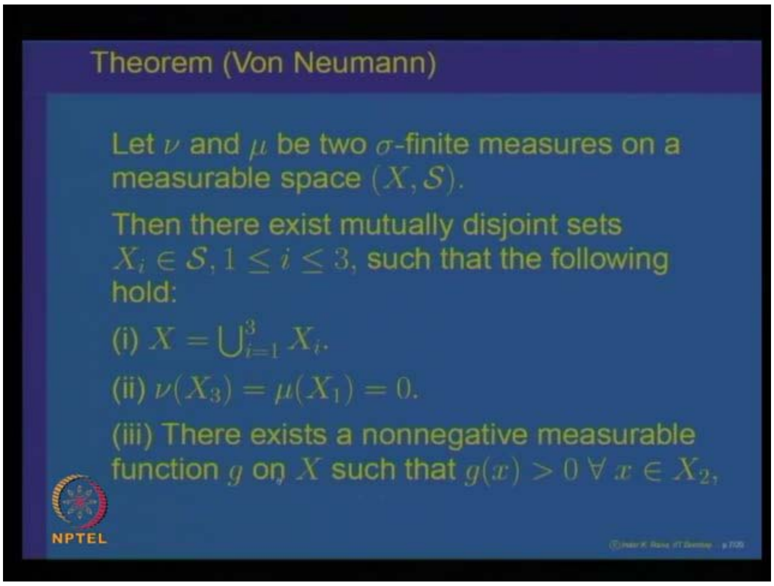

Next, we want to prove a theorem called Von Neumann theorem, which is a very nice theorem and it uses very nicely, very intelligently. The fact that the dual of the l 2 of measure spaces itself that we called as the Riesz representation theorem namely, that if t is a continuous linear functional on l 2 of a measure space then, it is essentially given by the inner product. So this is used very effectively, we will not give a proof of this theorem. Those who are interested can read the text book but, we will see how this theorem is use to prove some results about measures.

Let us first state this theorem called for Von Neumann theorem. The theorem says, let us take two measures mu and nu which are sigma finite measures on measurable space X S then, it says there exists mutually disjoint sets X i a measurable sets such that the following properties hold, first of all this X 1, X 2, X 3 give a partition of the space X.

So, X is partitioned into three parts; X 1 union X 2 union X 3 on X 3 and X 1, nu of X 3 is equal to 0 that means, nu for any subset in X 3 the measure of a set is equal to 0. That means, nu at the most gives values to subsets on X 1 and X 2. On the other hand, mu of X 1 is equal to 0 that means, mu gives values to subsets possibly in X 2 and X 3. So, nu of X 3 is equal to 0, mu of X 1 is equal to 0.

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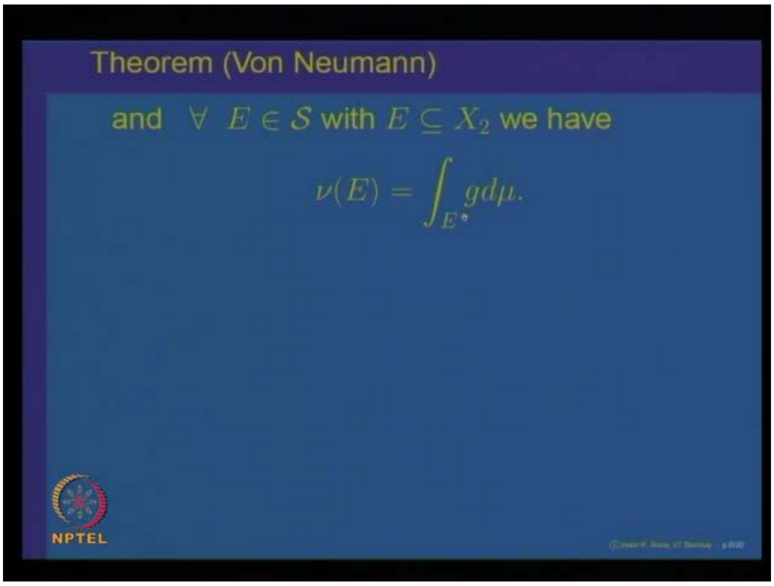

On the set X 2, there is a function g which is a non-negative measurable function such that for every subset E of measurable subset E of X 2, nu of E can be written as integral of the function g over the set E.

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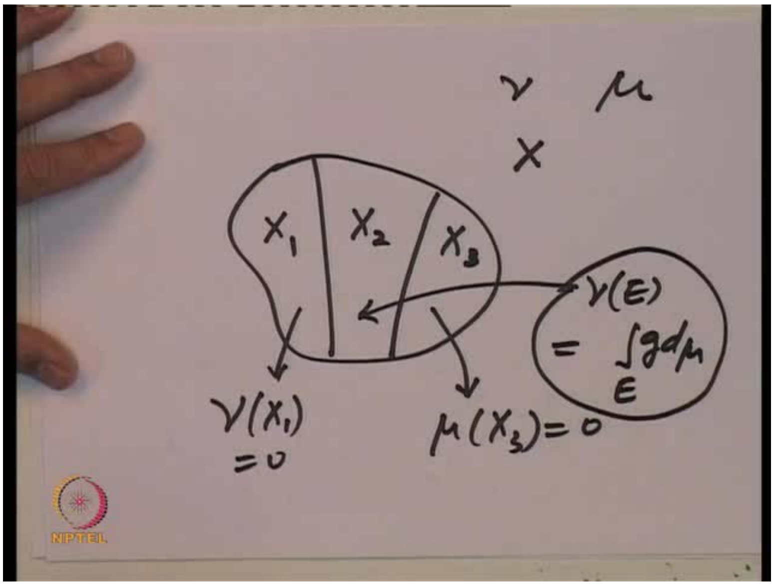

So in some sense it describes, let us take this is my set X and we have got two measures nu and mu. It says we can decompose into three parts X 1 X 2 and X 3. On this part, nu of X 1 is 0, on this part mu of X 3 is equal to 0. In this part nu is 0; in this part mu is 0 and on this part X 2, this nu of any set E; it can be written as integral over E of g d mu, so this property holds on this set.

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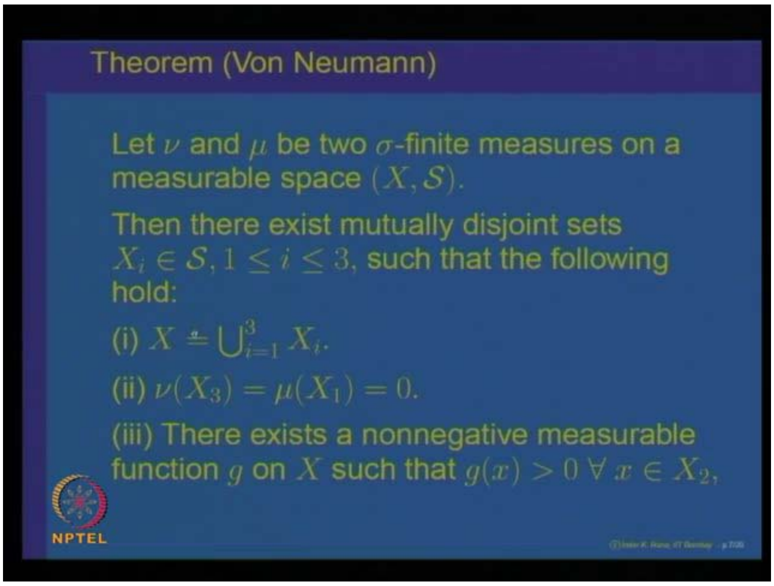

So this is a decomposition of Von Neumann. Let us just recall, the Von Neumann decomposition theorem says that given two measures nu and mu which are sigma finite.

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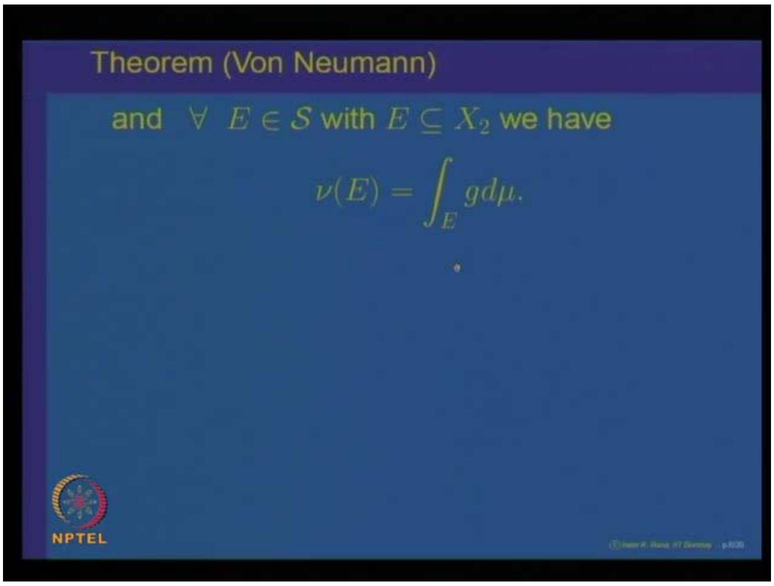

Of course, we can decompose X into three parts; X is equal to X 1 union X 2 union X 3; nu of X 3 is equal to 0; mu of X 1 is equal to 0. On the middle part X 2, the measure nu can be represented in terms of the measure mu by the property that nu of E is equal to integral g d mu.

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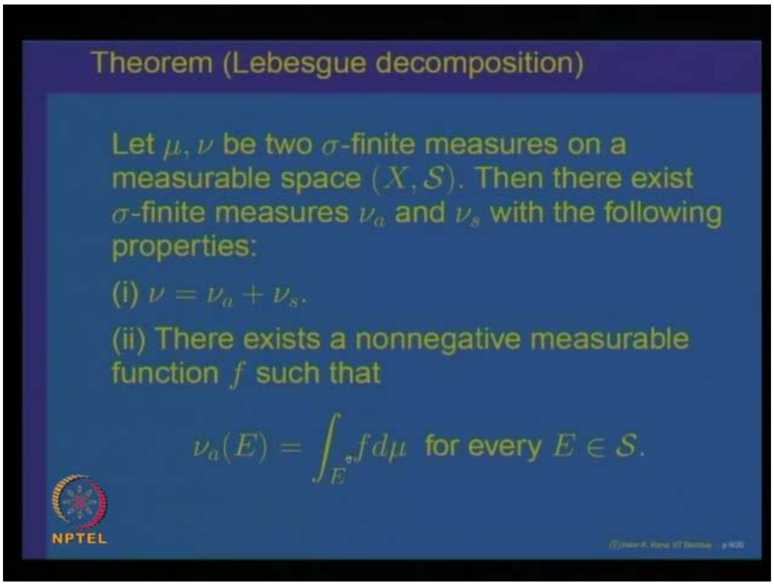

This is called Von Neumann's theorem and it is very useful theorem we will see soon. Using this theorem, we prove what is called the Lebesgue decomposition theorem. So, Lebesgue decomposition theorem says the following that suppose, mu and nu are two sigma finite measures on a measurable space X S. Then it implies that there exists a sigma finite measures nu a and nu s with the following properties namely, this measure nu can be written as a sum of two measures nu lower a and plus nu lower s.

So, the measure nu can be written as the sum of two measures nu of a plus nu of s. What are the properties of these two measures? The first property says that measure nu of a is representable in terms of the measure mu via integrals. So it says, nu of a for any subset E is integral of a non-negative measurable function f over the set E.

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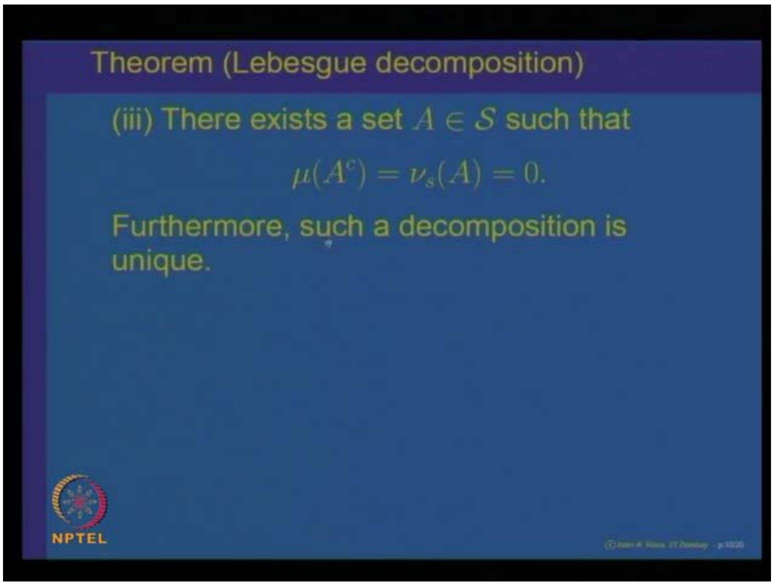

So it says, there exists a unique non-negative measurable function f. Such that to compute nu of a of any set E, we just integrate f over the set E. The second part says, it describes what is the measure nu s? It says there is a set A such that mu of A compliment is 0 and nu s of A is equal to 0 that means, mu and nu s are setting on disjoint sets, mu of A compliment is 0 that means, mu sits on A and nu s of A is equal to 0, so nu s it is on A compliment essentially. So, this is what is called Lebesgue decomposition theorem. So, we want to show that how it arises as an application of Von Neumann's theorem, this is also part of the consequence that such decomposition is unique.

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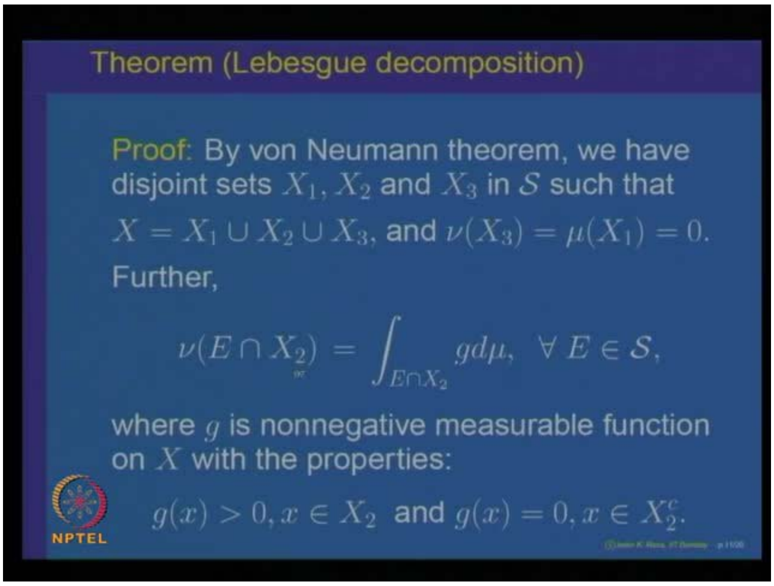

Let us look at by Von Neumann theorem. Given the measures nu and mu, we have disjoint sets X 1, X 2 and X 3, such that the following property holds. We recall that X is equal to X 1 union X 2 union X 3 and nu of X 3 is equal to 0 and mu of X 1 is equal to 0. So, mu does not give any mass; does not give any measure to the set X 1 and nu does not give any measure to the set X 3. On X 2, we recall that for every set E, a measurable set if we look at E intersection X 2 then, we can compute this measure as an integral of a non-negative measurable function g over the set E intersection X 2.

Moreover recall this is a non-negative, g is a non-negative measurable function and g is equal to 0 on the compliment of the set E 2 because the measure nu on X 3 is equal to 0, so this is the Von Neumann's theorem.

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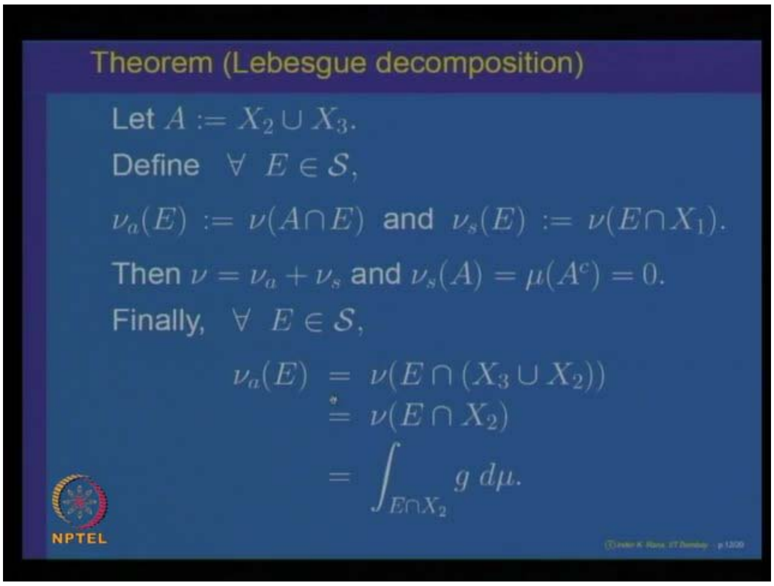

Now it is easy to define, what should be our measures nu a and so. On the set X 2, we should define nu a to be equal to nu on X 2, so let us put A equal to X 2 union X 3. On this set for every set E belonging to S nu of a of E is defined as nu of A intersection E, on the compliment of this part, so nu s of E that is going to be E intersection X 1 because X 1 is the compliment of A that is A compliment.

The measure nu as nu of A intersection E restrict nu to E and restrict nu to X 1 to get the measure nu s. Now obviously, these two are nu s and nu a are the measures that is obvious from the definition. Also it is clear from the definition that the measure nu is nothing but nu a, because nu a is on A and nu s is on E compliment. So, nu is nothing but nu a plus nu s and the measure nu s on A compliment is equal to 0, so this satisfies the required properties. Only we have to check that nu of a is given by the integral and that is obvious because nu a of E is nu of E intersection X 2 union X 3 by the definition, because A is X 2 union X 3 and on X 2 it is given by E intersection X 2 integral g d mu because nu of X 3 is equal to 0.

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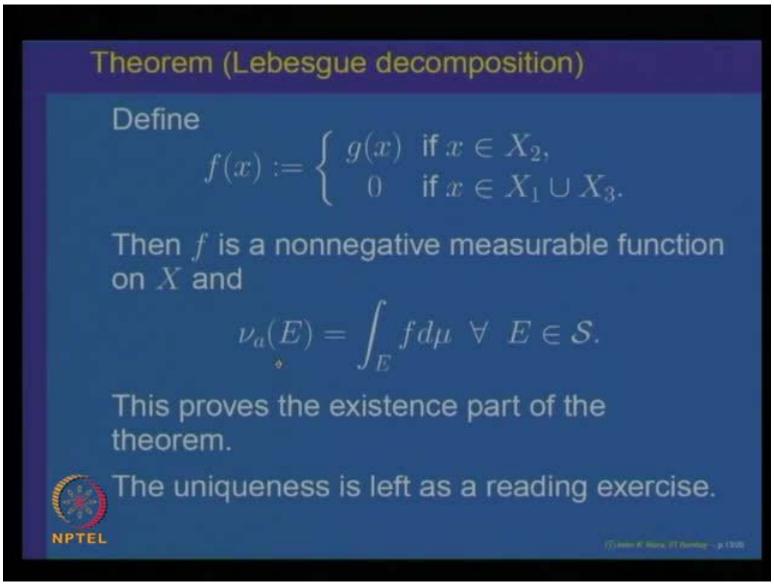

So, nu of a is given by the integral. So that proves the theorem, except for the fact that at present g is defined only on X 2; this is no not a issue, we can extend it to the whole by putting it equal to 0. Then, it has the required property that f is defined is a non-negative measurable function defined on the whole space X and nu a of E is given by the integral. So that proves Lebesgue decomposition theorem and the uniqueness is only a manipulation of the measures, which we will leave it as a reading exercise for the reader to verify. So, we will assume the uniqueness part of it.

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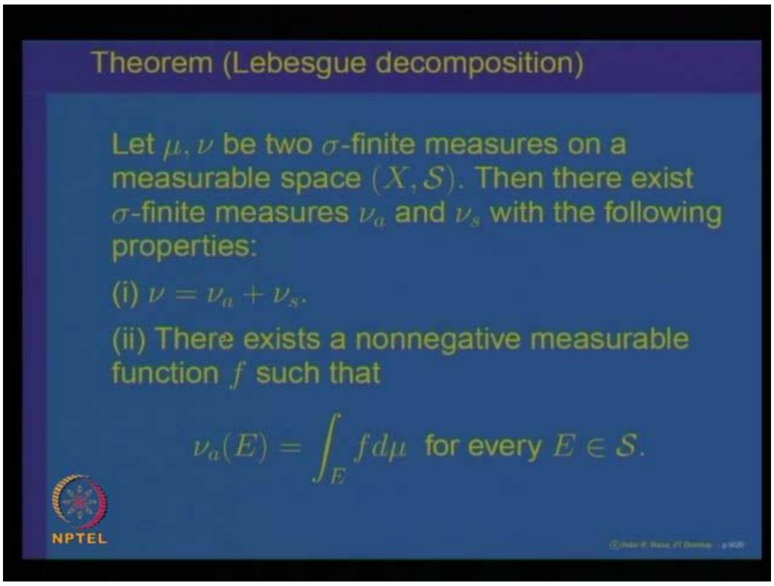

So Lebesgue decomposition, let me go back and state understand what is Lebesgue decomposition theorem? It says that given two sigma finite measures on a measure space. One of the measures say let us take nu can be written as a sum of two measures nu a and nu s, where nu a is given by integration and nu s sits on a part which is compliment to the part of nu of a.

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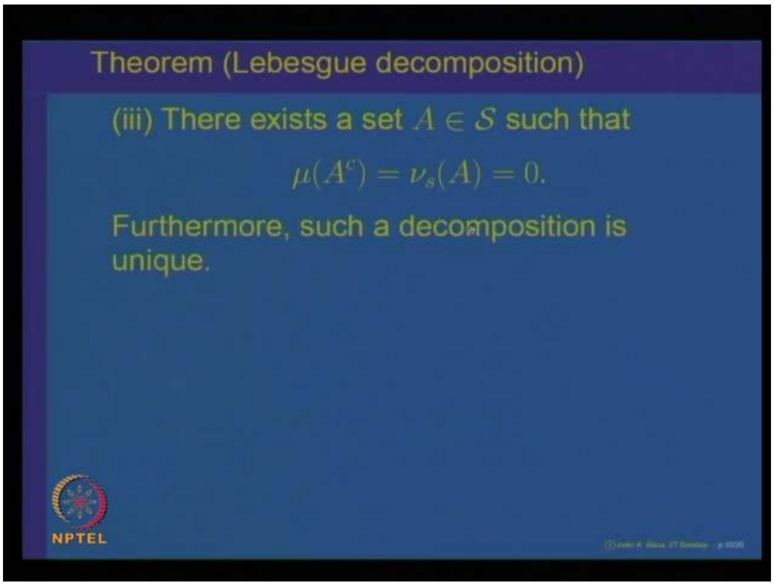

This is nu of a is given as a integral and mu of A complement is same as nu s of A is equal to 0. So this theorem is called Lebesgue decomposition theorem.

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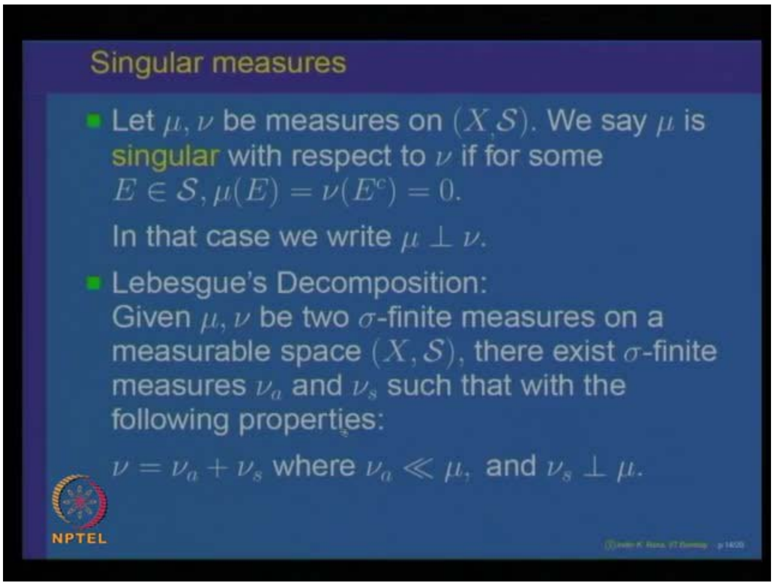

This is the measure nu s that we have defined just now- Let us make it as a definition, so two measures mu and nu are said to be singular with respect to each other or we say mu is singular with respect to nu. If there is a set such that mu of the set E is 0 and nu of the E complement is equal to 0. That essentially says, we can decompose the space X into two parts E and E complement and mu sits on one part and nu sits on the other part.

So such measures are called singular and the obvious that if mu is singular with respect to nu then nu is singular with respect to mu, it is a commutative relation of singularity while, absolute continuity was not. So, nu absolutely continuous with respect to mu need not imply mu is absolutely continuous, while the singularity is true that namely this singular is also written as mu perpendicular. So this is also read as singularity, is also said mu is orthogonal or mu is perpendicular to nu and written as mu perpendicular to nu.

So Lebesgue decomposition theorem, can be stated in terms of singularity that given two measures which are sigma finite on a measure space X S, there exist sigma finite measures nu a and nu s. Such that the following properties hold namely, nu is decomposed into two parts nu a plus nu s, where nu of a is absolutely continuous with respect to mu and nu s is orthogonal with respect to mu.

So, this is the decomposition that essentially it says that the measure nu has got absolutely continuous part and absolute continuous part says, you can obtain nu of a via integration. Singular part says that this is the other part which is completely orthogonal to mu, so the sets – disjoint sets - you cannot do anything there, they are disjoint sets.

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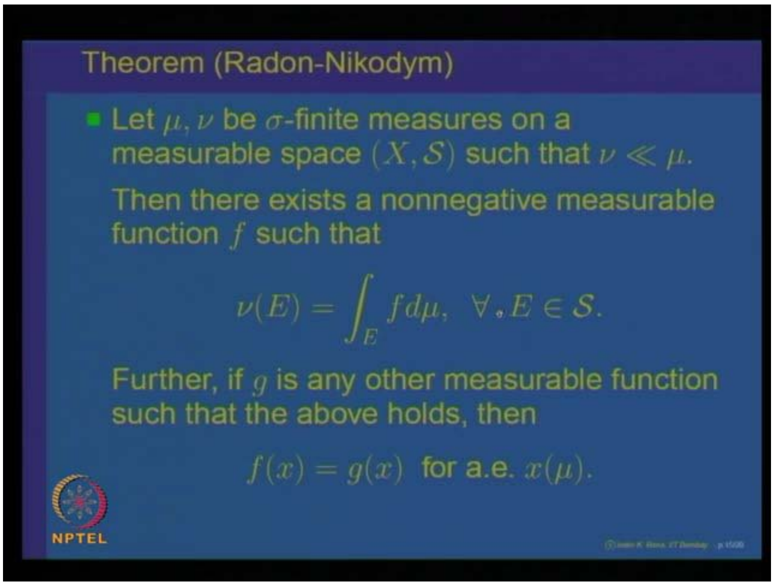

They sit on what one says they sports are disjoint essentially, so this is what is called Lebesgue decomposition theorem. As a consequence of this, we will get what is called the Radon–Nikodym theorem, which characterizes absolutely continuous measures. It says that if two measures mu and nu which are sigma finite on a measurable space X S, such that nu is absolutely continuous with respect to mu. Then, there is a non-negative measurable function f such that nu of E is given via integration over of f over the set E. So, that completely characterizes absolutely continuous measures.

Recall if we define nu of E by this then, we have already shown that was the beginning of our analysis saying that any measure nu defined in terms of integral with respect to mu is absolutely continuous. This is the converse part of it namely if nu is absolutely continuous with respect to mu, then there must be a non-negative measurable function f such that nu of E is equal to integral of f over E with respect to the measure mu.

Proof is obvious from the Lebesgue decomposition theorem and the uniqueness part of it, because we know that nu is absolutely continuous with respect to mu, so when we apply Lebesgue decomposition theorem to the measures nu and mu; nu will be decomposed into two parts the absolutely continuous part and the singular part but, there is no singular part, there is only absolutely continuous part and the absolutely continuous part, we have already seen in Lebesgue decomposition theorem is given via integrals.

So, this is a direct application of Lebesgue decomposition theorem. This function f is also unique in the sense that if there are two functions, then f must be equal to g almost everywhere. That happens because, if there is another function g with the same property, then integral of f over every set E is equal to integral of g over every set E that implies that f must be equal to g almost everywhere that we have seen earlier.

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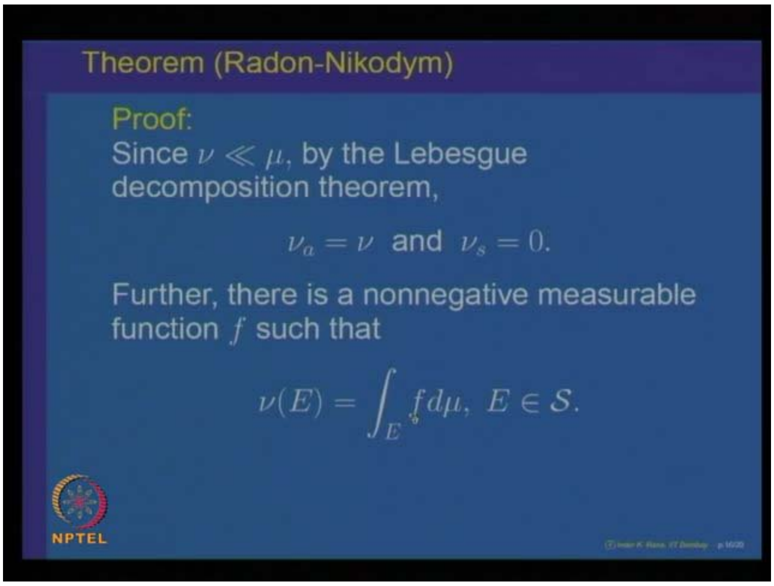

So Radon–Nikodym theorem, the proof is an application of Lebesgue decomposition theorem and Lebesgue decomposition theorem is an application of Von Neumann's theorem. Once again saying that nu is absolutely continuous with respect to mu by Lebesgue decomposition theorem, nu of a must be equal to nu, because that is absolutely continuous and nu of s is equal to 0.

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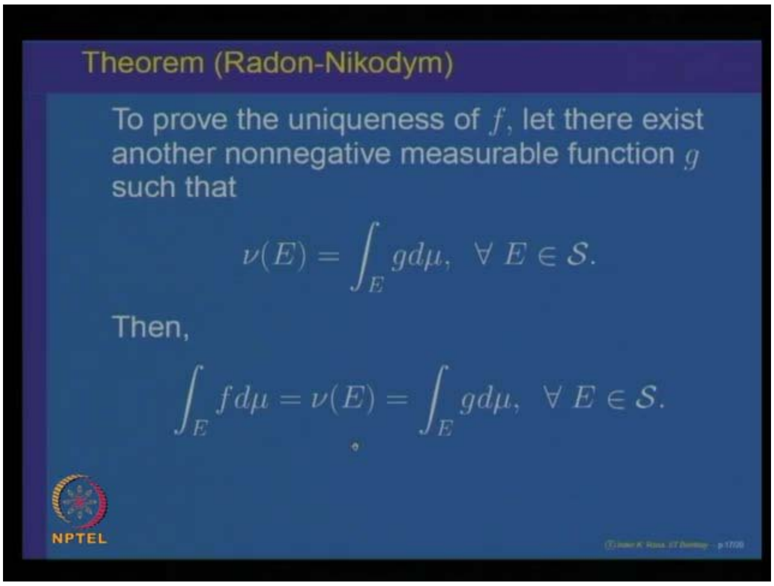

So that says, nu of E is given by integral over f by the Lebesgue decomposition theorem. As such we said, uniqueness is obvious because integral f is equal to integral g must imply that f is equal to g almost everywhere. Radon–Nikodym theorem is one of the most important and subtle theorems of our subject, because just from the existence of same null sets mu of E equal to 0 implies nu of E equal to 0, this simple property about null set says that nu must be obtainable from mu via integration.

It is really a deep and amazing theorem of our subject. Let me also point out there are many proofs available of this theorem. We have given a proof which is via Von Neumann's theorem and Von Neumann's theorem uses the fact that the dual of l 2 is l 2. There is another purely measure theoretic proof of this but, that goes into the realm of what is called signed measures. So, one looks at signed measures and then, one decomposes a signed measure. There is something called a Hahn decomposition theorem, that every signed measure is a difference of two measures and then from there one deduces Lebesgue decomposition theorem and then comes to Radon–Nikodym theorem.

So that is another route possible for proving this theorem. So, both the proofs are available in the text and we have outlined here only one proof which is more function theoretic using that dual of l 2 is l 2.

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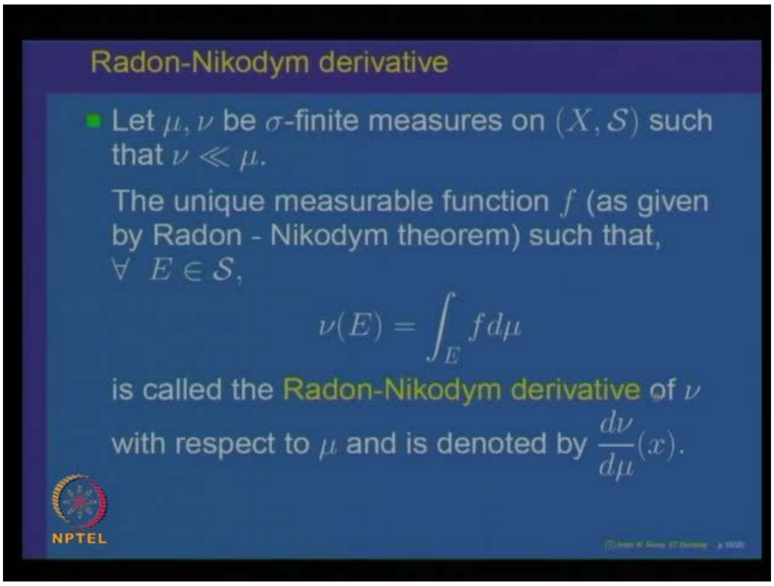

This gives the notion of absolute continuity also gives rise to the notion of what is called the derivative. Let us define that whenever two measures mu and nu are sigma finite and nu is absolutely continuous with respect to mu, so Radon–Nikodym theorem says that nu of E must be given by a function - a unique function - f this unique function f is called the Radon–Nikodym derivative of the measure nu with respect to mu and is denoted by d nu by d mu of x.

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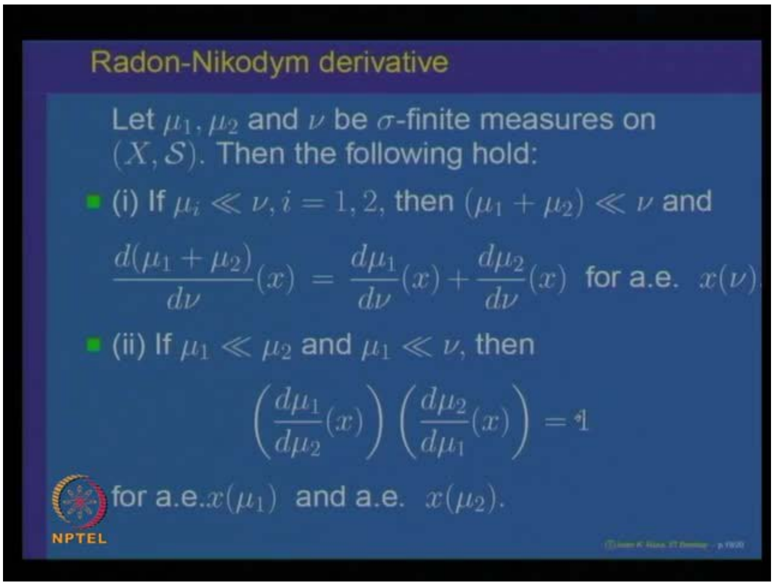

So here are some simple properties of this Radon–Nikodym derivative which are very much similar to the derivative of functions namely, if mu 1 mu 2 and nu are sigma finite measures then, and if mu i each mu 1 and mu 1 is absolutely continuous with respect to mu then, mu 1 plus mu 2 is also absolutely continuous and the Radon–Nikodym derivative of the sum is equal to sum of the Radon–Nikodym derivatives of course, almost everywhere.

So very much similar to the derivative of the sum is equal to sum of the derivatives for functions. Similarly, if mu 1 is absolutely continuous with respect to mu 2 sorry this is a mistake here and mu 2 is absolutely continuous with respect to mu 1. That means, both are absolutely continuous with respect to each other then the product of the derivative is equal to 1. So that is the second property, so here it should be mu 2 absolutely continuous with respect to mu.

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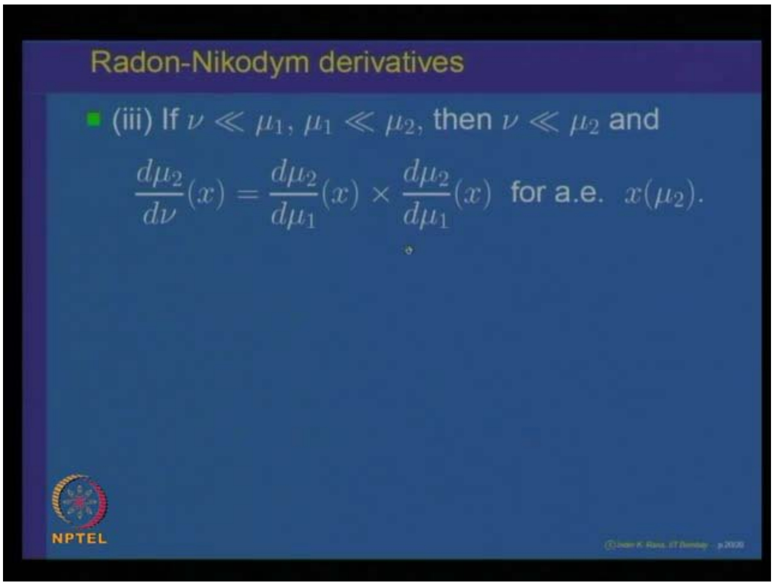

Finally, if nu is absolutely continuous with respect to mu 1 and mu 1 is absolutely continuous with respect to mu 2. Then there is a kind of associatively property namely, this implies that nu must be absolutely continuous with respect to mu 2 and the derivative, so d mu 2 over d nu is computable as d mu 2 over d mu 1 multiplied by the derivative d mu 2 over this should be d mu here this should be d mu. So, it is like a chain rule for the derivative functions.

So with that idea of Radon–Nikodym - This set of ideas namely, absolutely continuous functions and absolutely continuous measures, is complete and in the remaining next lecture, we will look at some special properties of sequences of measurable functions and the various ways they can converge to a function f. Thank you.