Measure and Integration

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Module No. # 10

Lecture No. # 37

Fundamental Theorem of Calculus for Lebesgue Integral- II

Welcome to lecture 37 on measure and integration. If we recall in the previous lecture, we had started looking at the theorem called fundamental theorem of calculus for Lebesgue integrals. We will continue looking at the same in this lecture also and try to complete the arguments today.

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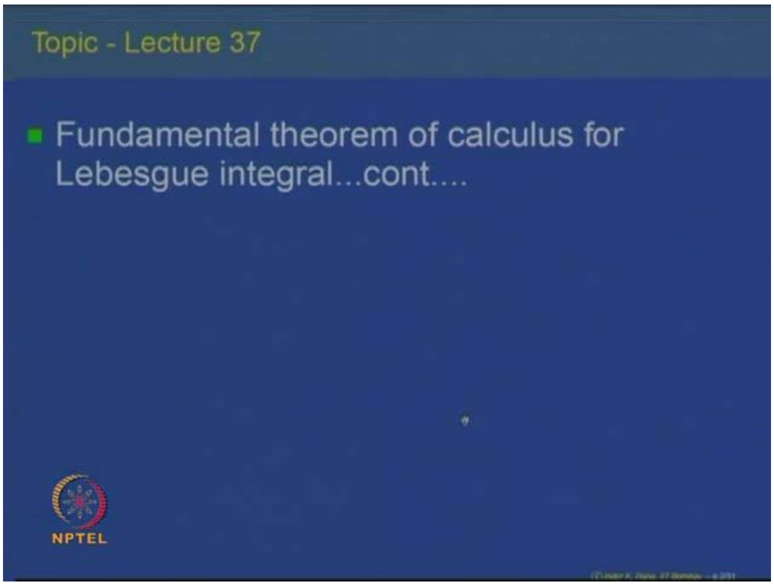

Today's topic is going to be looking at fundamental theorem of calculus for Lebesgue integrals.

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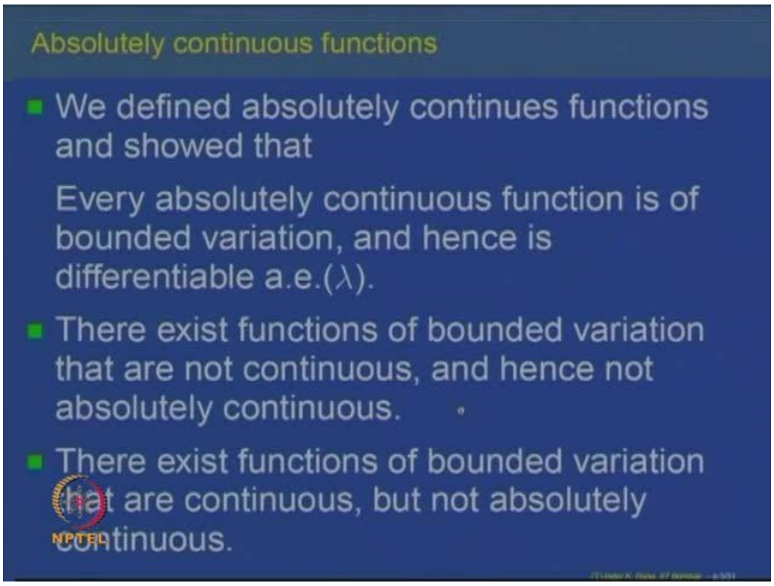

If you recall, we had defined in the previous lecture, what are absolutely continuous functions and showed that every absolutely continuous function is a function of bounded variation. Hence, it is a difference of two monotone functions and as a consequence it becomes differentiable almost everywhere.

Of course, there exist functions of bounded variation that are not continuous and hence not absolutely continuous, we had looked at such things also and there exist functions of bounded variation that are continuous but not absolutely continuous.

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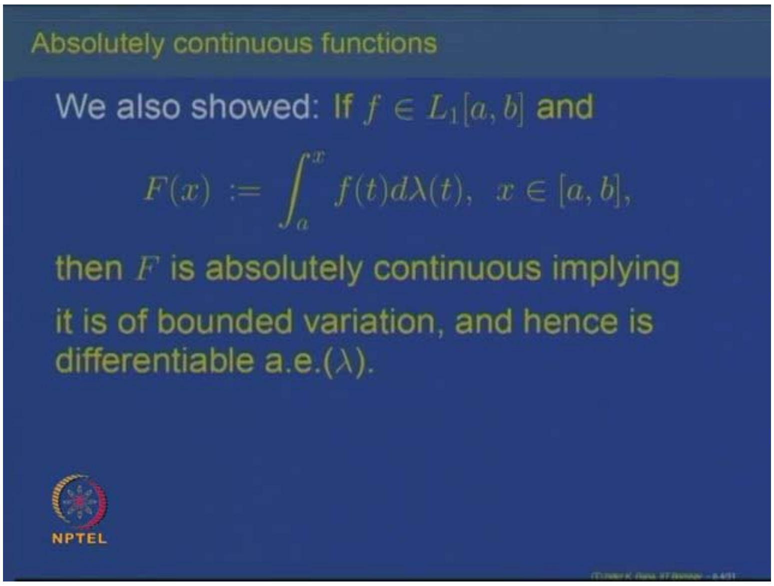

In the previous lecture, we had also looked at the notion of indefinite integral of integrable functions. So we define, what is called F of x as the indefinite integral a to x, f t d lambda t, where f is a Lebesgue integrable function. For such functions, we prove that these functions are absolutely continuous. Indefinite integral of Lebesgue integrable functions are examples of absolutely continuous functions.

Essentially fundamental theorem of calculus says, that these are the only ways of getting absolutely continuous functions and once it is absolutely continuous, it also becomes a function of bounded variation and hence becomes differentiable almost everywhere.

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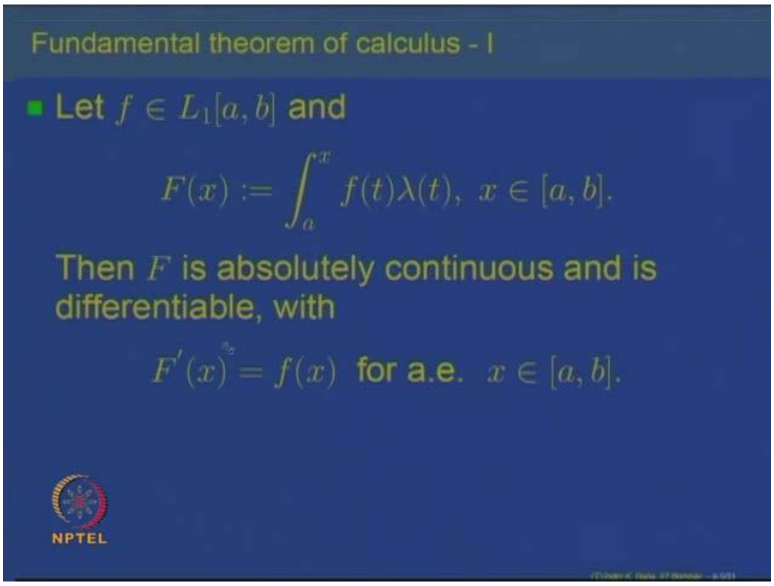

The main aim of fundamental theorem of calculus part 1 is to identify the derivative of this function (Refer Slide Time: 02:19) If f is L1 of a b and F of x is the indefinite integral of the function f of t d lambda t, then we would like to show that the derivative this function f which is differentiable almost everywhere, its derivative is equal to little f of x for almost all x belonging to a b.

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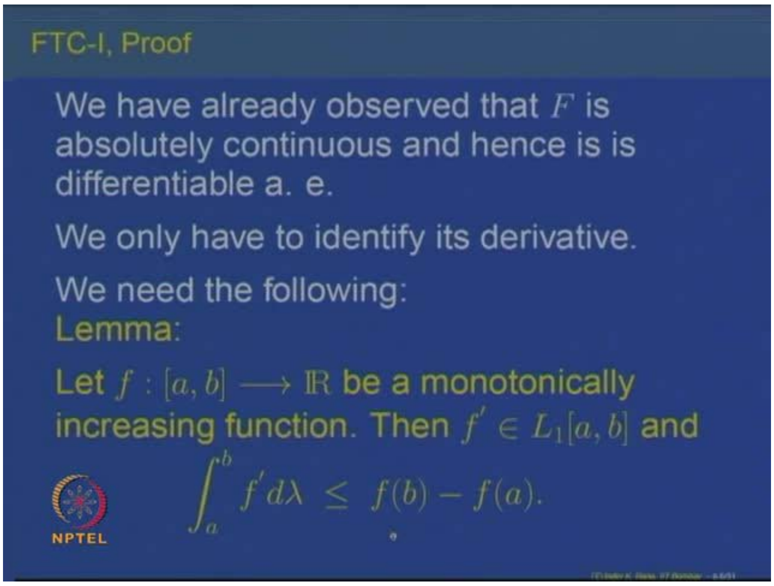

We have already observed that this function is absolutely continuous; the indefinite integral is absolutely continuous and hence differentiable almost everywhere. To prove the fundamental theorem of calculus we have only to identify its derivative.

For that we need a Lemma which essentially says, that for a function - integrable function- if its function is differentiable, if it says that if integrable function which is monotonically increasing then of course, it is differentiable almost everywhere by Lebesgue theorem.

The Lemma says, that the derivative becomes integrable and the integral of the derivative for this monotone function is less than or equal to f of b minus f of a.

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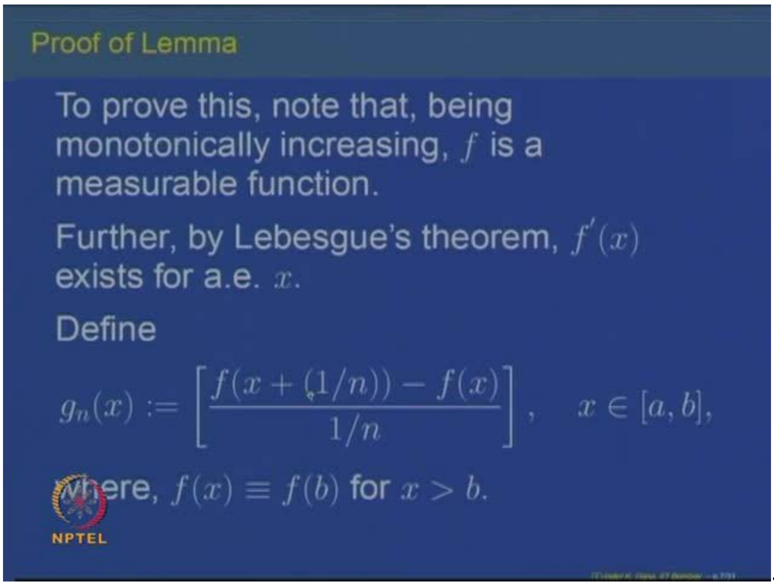

This is Lemma that we require; so we will prove this Lemma first. To prove this Lemma, we first note that being monotonically increasing this function is a measurable function. The reason for that is every monotonically increasing function is a function which is continuous almost everywhere; that is a theorem normally proved in basic courses in real analysis. If a function is monotone, it is continuous almost everywhere and hence measurable.

So, that is one way of looking at a monotone function and proving it is measurable or there is a direct way, you can use also the definition and look at the inverse image of an interval, then it is also going to be an interval for a monotonically increasing functions or a union of 2 intervals at the most; that is another way of looking at the measurability of monotone functions.

So, every monotonically increasing function is a measurable function and by Lebesgue's theorem. In fact, it becomes differentiable almost everywhere and the derivative let us noted by f dash of x. Let us define g n of x to be equal to f of x plus 1 over n, minus f of x divided by 1 over n whenever, x belongs to a b and let us, this is not f of x, let us extend f outside the interval a b to be as f of b. So, for x bigger than b we will treat f of x as f of b.

For example, when this is b, b plus 1 over n will be treated as f of b. So g n is a function which is the increment of f at the point x by an increment in x by 1 by n. So clearly because the function f is differentiable, this function g n x converges to the derivative of the function f whenever the derivative exist.

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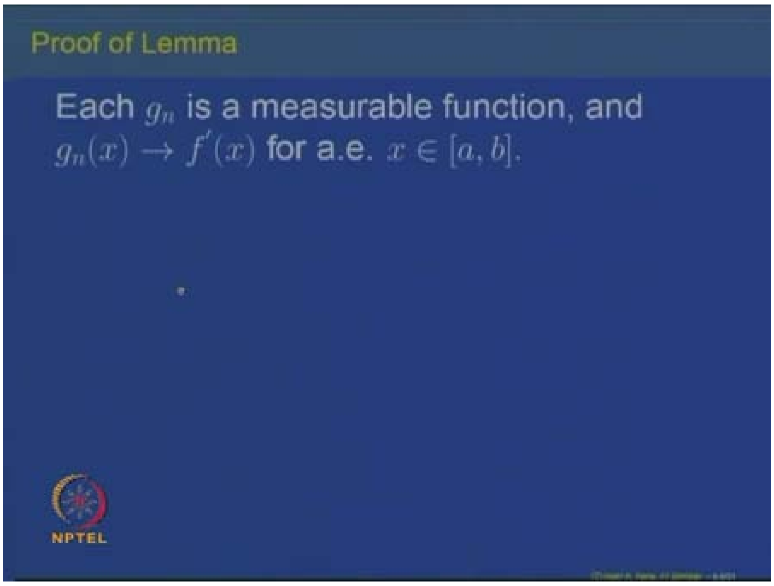

The first claim is that each of this g n is a measurable function and g n x converges to f dash of x for almost all x.

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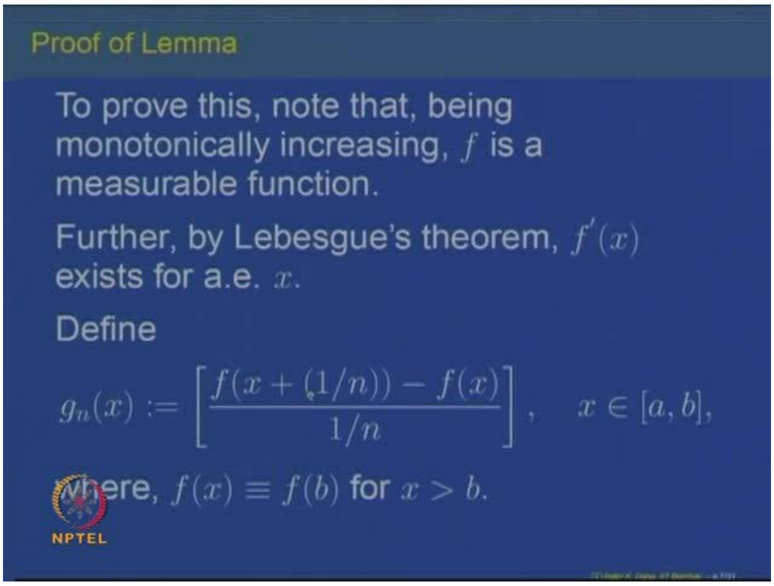

That g n is a measurable function that is obvious because f is measurable; so f of x plus 1 over n, this function is translate of the measurable function is measurable. So, difference of measurable functions and divided by a constant that is a measurable function.

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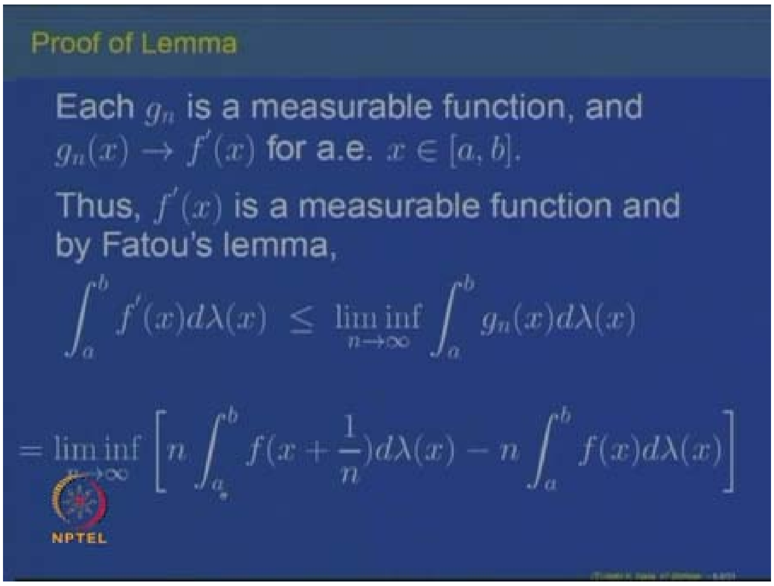

So g n x is a measurable function converging almost everywhere to the derivative of the function f dash of x.

Let us compute by applying Fatou's lemma to this, by Fatou's lemma we know that - first of all note that – g n x also is a non negative function because f is a monotonically increasing function, so the numerator f of x plus 1over n minus f of x divided by 1 over n that is a non negative function.

So, g n is a sequence of non negative measurable functions converging almost everywhere to f dash of x by Fatou's lemma. First of all, f dash is a measurable function because it is limited measurable functions. By Fatou's lemma, the integral of f dash of x which is the limit of g n's will be less than or equal to limit inferior of g n over a to b.

Fatou's lemma says, that if a sequence of non negative functions converges to a function then, the integral of the limit inferior is less than or equal to limit inferior of the integrals. So, this is the direct consequence of Fatou's lemma applied to the function in g n.

Next, let us compute this quantity and put the value of g n; g n is equal to f of x plus 1 over n minus f of x. So this integral splits into 2 integrals, n times integral a to b, f of x plus 1 over n d lambda minus n times integral a to b f x d lambda x.

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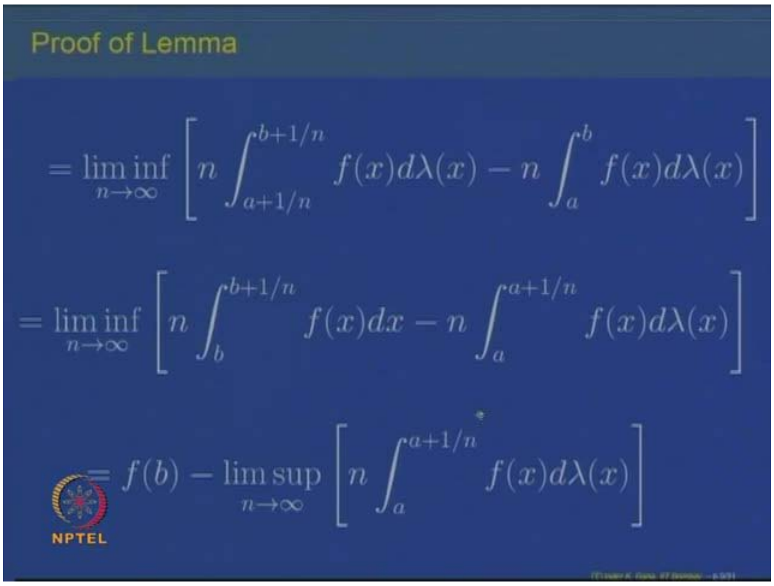

Let us use the fact that Lebesgue's measure is translation invariant; so this integral a to b f of x plus 1 over n can be written as integral a plus 1 over n to b plus 1 over n of f x d lambda x. Here, we are using the property that the Lebesgue measure is translation in variant. The right hand side is limit inferior, n going to infinity of n times integral a plus 1 over n to b plus 1 over n and f x d lambda x minus the integral a to b.

Now using the fact, that the Lebesgue integral is additive over the limits of integration. This difference is nothing but a to a plus 1 over n and then 1 over b a; so this integral a to plus 1 over a to b can be written as a to a plus 1 over n and the second integral being a plus 1 over n to b. These two difference will give us the difference namely, this is nothing but b to b plus 1 over n f x dx minus n times a to a plus 1 over n f x dx…

Here we are using the property that the integral is additive over the limits of integration. Now we use the fact, that f is a monotonically increasing function, so the first integral is b to b plus 1 over n. We had defined f of x to be equal to b whenever x exceeds b plus 1 over n. So the first integral is nothing but f of b divided by 1 over n and cancels. So first integral is just f of b, so limit inferior of minus n integral but, recall the fact that limit inferior of minus of a sequence is equal to minus of the limit superior. This becomes minus limit superior n integral a to a plus 1 over n f x of dx.

Now use a fact that f is a monotone function, so integral of f of x from a to a plus 1 over n will be less than or equal to f of a, for the whole interval length of interval is 1 over n. So this is less than or equal to is a monotonically increasing function.

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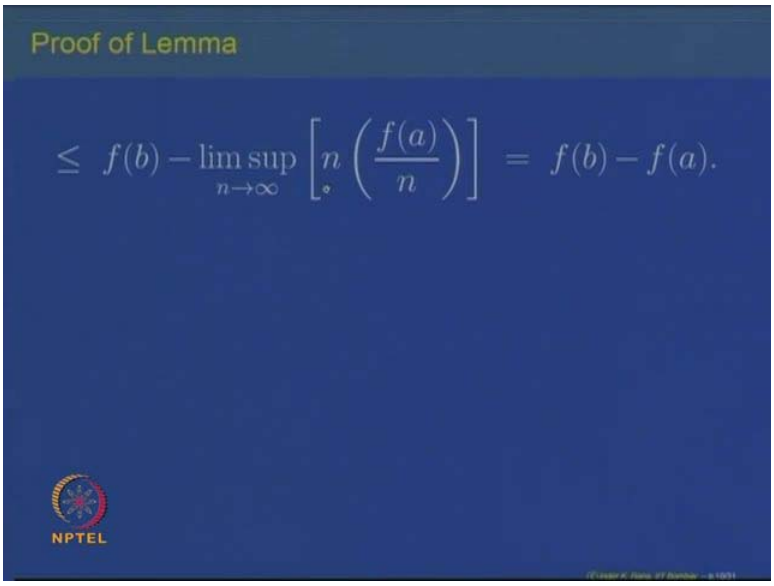

We are putting the value f of a, so it is f of a divided by n into n and that is independent of superior; so that is just f of b minus f of a. Here, we have used the fact that, f is a monotonically increasing function; that proves the Lemma namely, if f is monotonically increasing then its derivative function is integrable and the integral of the derivative function. So integral of f dash a to b f x dx is less than or equal to f of b minus f of a.

This Lemma we will be using in the proof of the fundamental theorem of calculus part 1.

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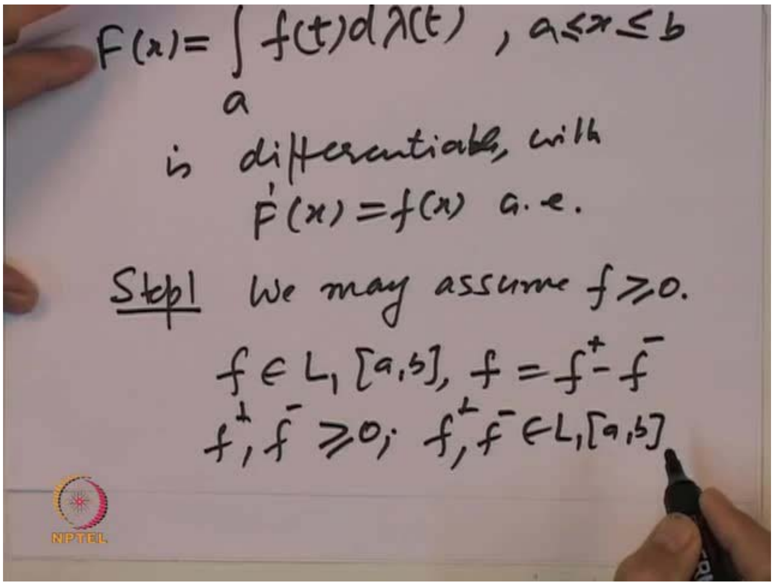

Let us prove the theorem, fundamental theorem of calculus part 1. We want to prove that the integral a to x f of t d lambda t, this function F of x for x between a and b. We want to show that this is differentiable; of course we know already that it is differentiable with derivative f dash of x equal to f of x for of almost everywhere. So this is what we want to prove (Refer Slide Time: 11:34).

As a first step, we are saying that let us assume that the function step 1 is that we may assume f is bigger than or equal to 0. The reason for that is because f belongs to f L1 of a b. So, f can be written as the positive part minus the negative part of the function and recall that f plus, f minus both are nonnegative functions and f plus and f minus both belong to L1 of a b.

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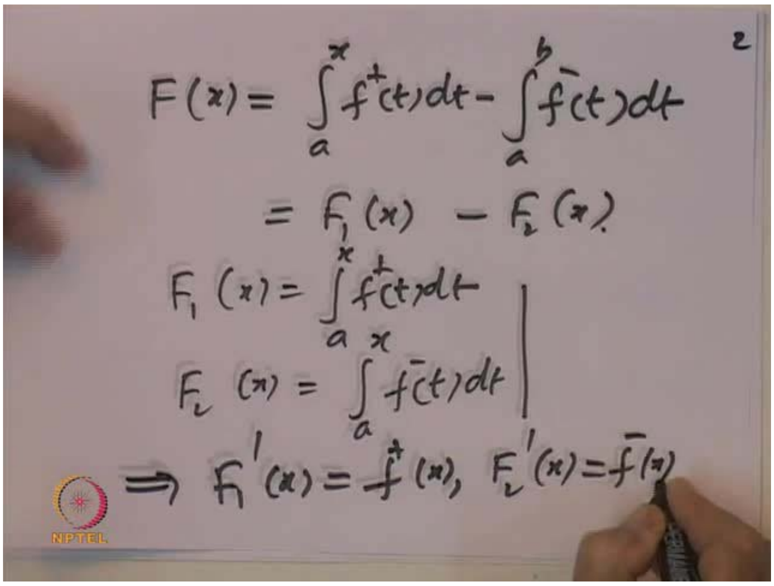

As a consequence of this, the function F of x can be written as integral a to x, f plus t dt minus a to b, f minus t dt. If you call this function as F 1 of x and the second function as F 2 of x then both F 1 and F 2, so F 1 is the indefinite integral of f plus t dt and f2 of x is indefinite integral a to x of f minus t dt.

The important thing to note in this is that both f plus and f minus are non negative integrable functions. In case, we have already proved the theorem for non negative integral functions, this will imply that F 1 dash of x is equal to f plus of x and F 2 dash of x is equal to f minus of x.

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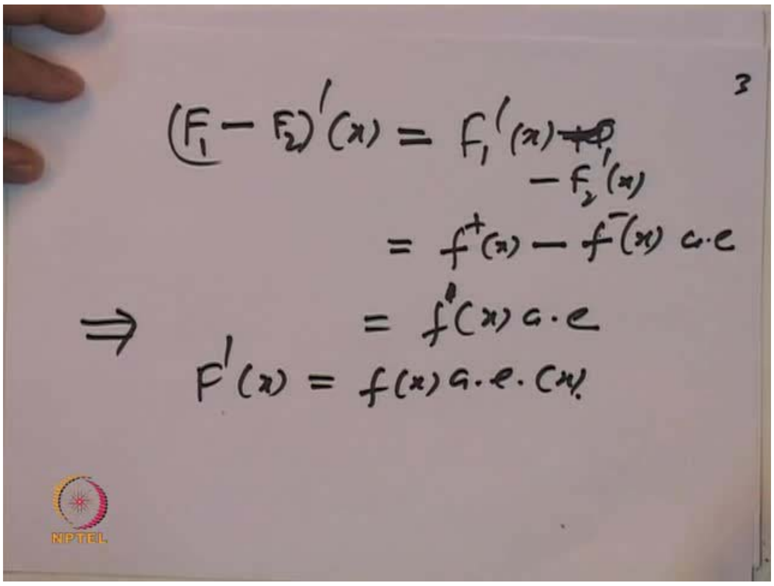

Hence as a consequence of this we will have that F 1 minus F 2 dash of x is equal to F 1 x minus F 2 dash of x and this by the step 1, if it is already true for non negative integrable functions. This is f plus of x almost everywhere and that is f minus of x almost everywhere and that is equal to f of x almost everywhere. So, that will prove that f dash of x, so implying f dash of x is equal to f of x almost everywhere of x.

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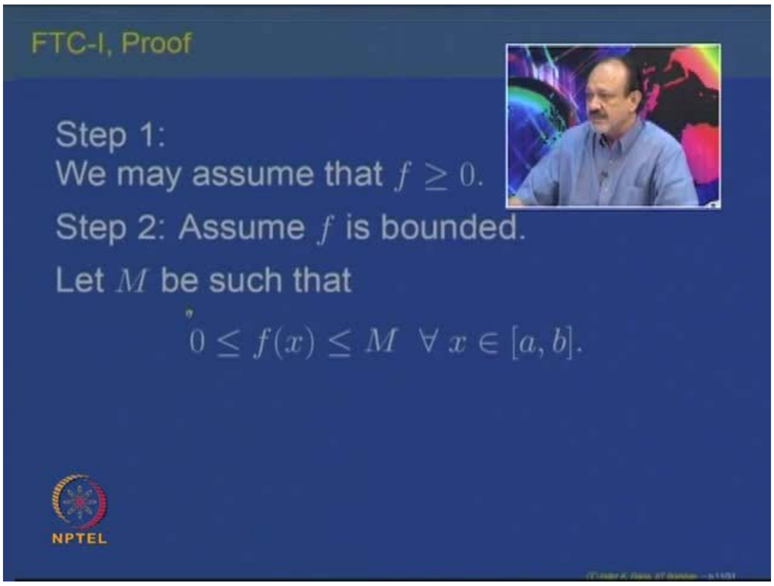

If we can prove the theorem for non negative integrable functions then, we will be through; so step 1 says let us assume f is non negative.

Second step says, let us assume that the function is a bounded function and let us prove the result when f is a bounded function. We will assume for the time being that f is a bounded function. Let us assume, M is the constant such that f of x is less than or equal to M bigger than or equal to 0 for every x belonging to a b.

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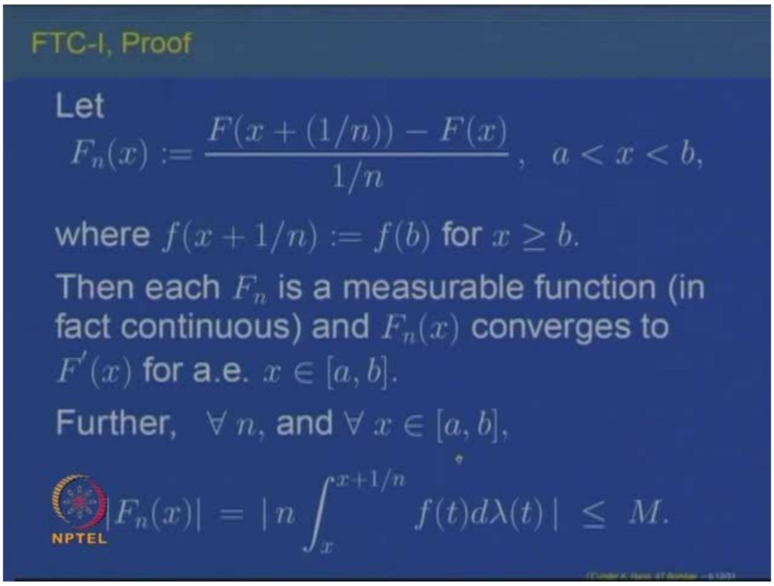

In that case, let us define F n of x to be equal to F of x plus 1 over n minus F of x divided by 1 over n, whenever x belongs to a b, where as usual f of x plus 1 over n is defined as equal to f of b for x bigger than or equal to b. Then each of F n is a measurable function in fact each F n is an absolutely continuous function and hence is also measurable.

F n x converges to F dash of x for almost all x belonging to a b, because it is function is differentiable almost everywhere being absolutely continuous; so the derivative exist almost everywhere and right hand side is nothing but, converging to the derivative of the function F of x.

Since, we have assumed that f is a bounded measurable function, f is bounded; so this capital F also is a bounded measurable function, because capital Fn of x is less than or equal to n times, this 1 over n gives you n times f of x plus 1 over n is the integral form, x to x plus 1 over n of f of t d lambda t.

This difference numerator is nothing but the integral from x to x plus 1 over n f t d lambda t. This absolute value of the integral is less than or equal to integral of the absolute value and hence this absolute value is less than or equal to n times absolute value of f which is less than M into the length of the interval which is 1 over n, so that cancels. This is less than or equal to M, so F n is a sequence of measurable functions and each of them is bounded by a constant M and we are over a finite space a b.

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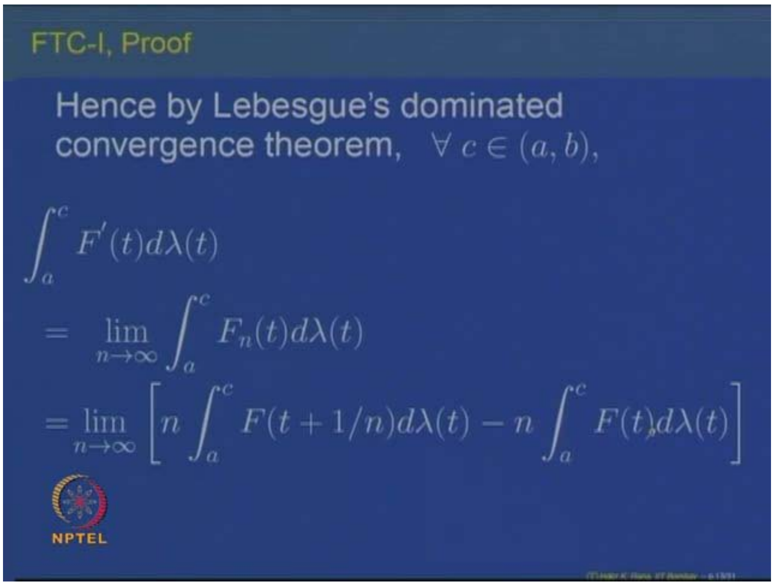

Lebesgue's bounded convergence theorem or also the dominated convergence theorem says that, for every c belonging to a b, if we apply it over the interval a to c then, integral over a to c F prime t d lambda t is equal to limit of the functions F n t d lambda t. Here, we are applying bounded convergence theorem or dominated convergence theorem; that gives you because F n converges to F dash and all of them are integrable functions. So, integral a to c of F prime t d lambda t is equal to limit n going to infinity a to c, F n t d lambda of t.

Now, we will write down the values of F n. What is F n? F n is equal to n times F of t plus 1 over n minus f of t divided by 1 over n. This integral a to c is nothing but, n times integral a to c of F of t plus 1 over n d lambda t minus the second term that is n times a to c, F of t d lambda t.

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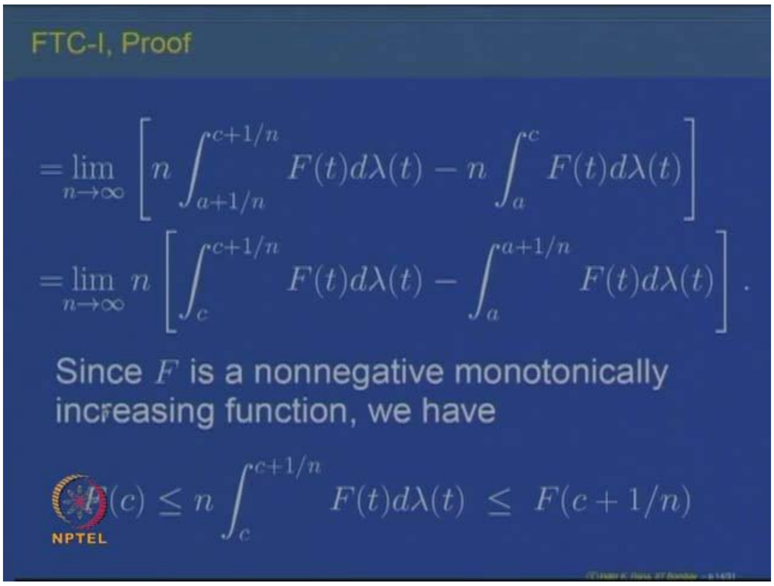

Now, we will use the fact that this integral is with respect to lambda which is translation invariant. The first integral which is F of t plus 1 over n can be written as integral of F of t d lambda t from a plus 1 over n to c plus 1 over n. The first integral is transformed to n times integral a plus 1 over n to c plus 1 over n, F of t d lambda t and of course, the second integral remains as n times a to c F of t d lambda t. This is using the fact that the Lebesgue's measure is translation invariant.

Now, once again as we have seen earlier Lebesgue's integral being additive over the limits of integration. So, this integral from a plus 1 over n to c plus 1 over n minus the integral, a to c gives us the integral of f of t from c to 1 plus 1 over n to integral from a to a plus 1 over n of F t d lambda t. Now, we use the fact that f is a non negative monotonically increasing function and why is this capital F non negative monotonically increasing function? It is because of the function small f.

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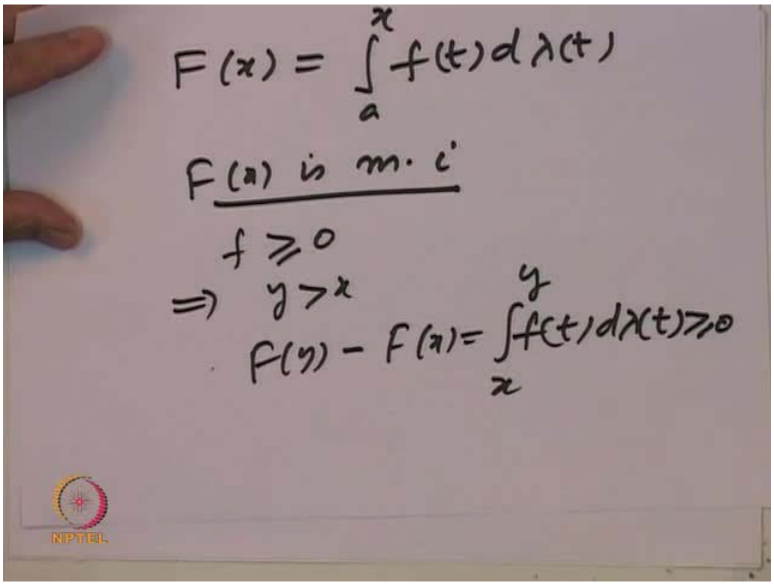

The reason for that is this function F of x which is defined as a to x f t d lambda t then, F of x is monotonically increasing and the reason is because f is non negative. So that implies that for y bigger than x, F of y which is minus F of x will be equal to integral x to y of f t d lambda t, which is bigger than or equal to 0. So, that implies that the function F t is monotonically increasing.

Since the function f is of course, a non negative - you are integrating a non negative function - and it is monotonically increasing. The integral c to c plus 1 over n, F t d lambda t will be bigger than or equal to the value at the lower end point, that is f of c into the length of the interval 1 over n into n, so that gives you F of c.

The first integral is n times c to c plus 1 over n is bigger than or equal to F of c and of course, it is less than or equal to the value at the upper end point into the length of the interval. So, that gives you F at c plus 1 over n. That is the first integral is between F of c and F of c plus 1 over n.

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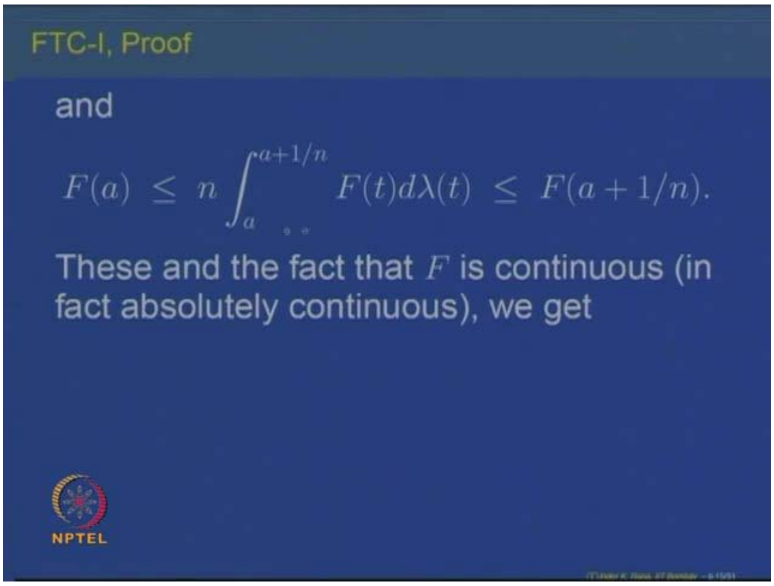

Similarly, the second integral is between F of a and F of a plus 1 over n. Now, these two inequalities along with the fact that the function capital F is absolutely continuous and hence continuous gives that as n goes to infinity F of a plus 1 over n is going to go to F of a. The second integral will go to F of a, which is equal to 0 and the first integral will go to F of c because it is sandwiched between F of c and F of c plus 1 over n.

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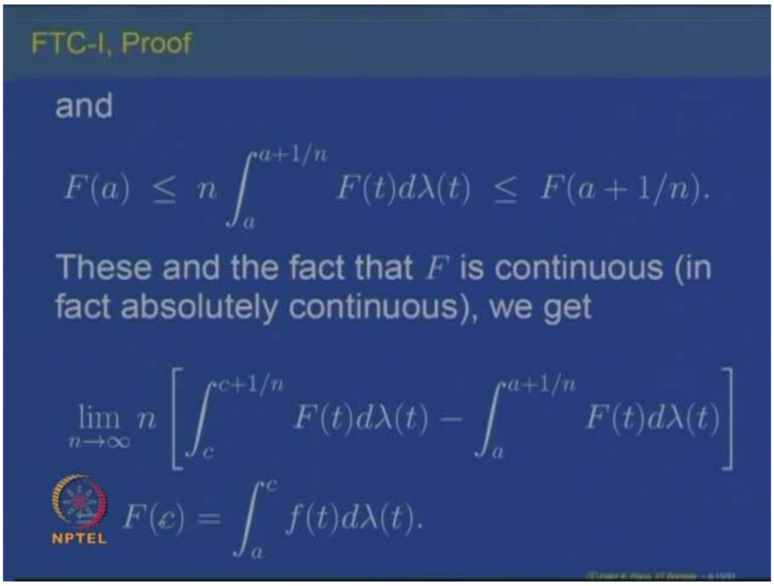

That implies that, the limit n going to infinity of n times the integral c to c plus 1 over n F t d lambda t minus a to a plus 1 over n F t d lambda t is equal to F of c which is equal to nothing but a to c of ft d lambda t.

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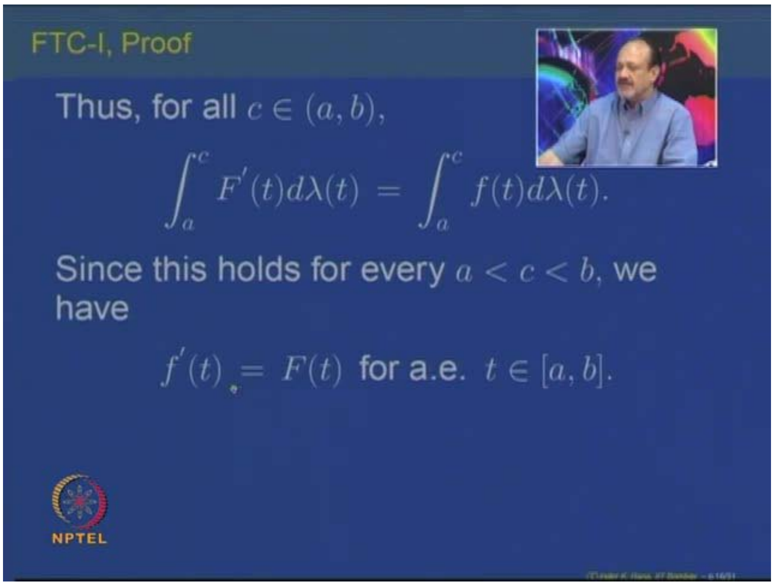

What we have shown is that the integral of the derivative of capital F. So integral of the derivative of capital F over the interval a to c is equal to integral over a to c of small f to t d lambda t. This holds for all values of c between a and b and that implies because these two are equal, so that implies that small f of t must be equal, so this is a mistake, we should be writing at f dash of t, capital F dash of t must be equal to small f of t (Refer Slide Time: 22:40).

What we are saying is integrals of two functions – non negative functions - are equal over all intervals of the type a to c where c belongs to a to b, that means the two functions must be equal almost everywhere. So f prime of t must be equal to F of t, for almost all t. That proves the fact that, the derivative of the function f of capital F of t is nothing but, small f of t. Here the two written wrongly (Refer Slide Time: 23:12) it should be capital F dash of t is equal to small f of t.

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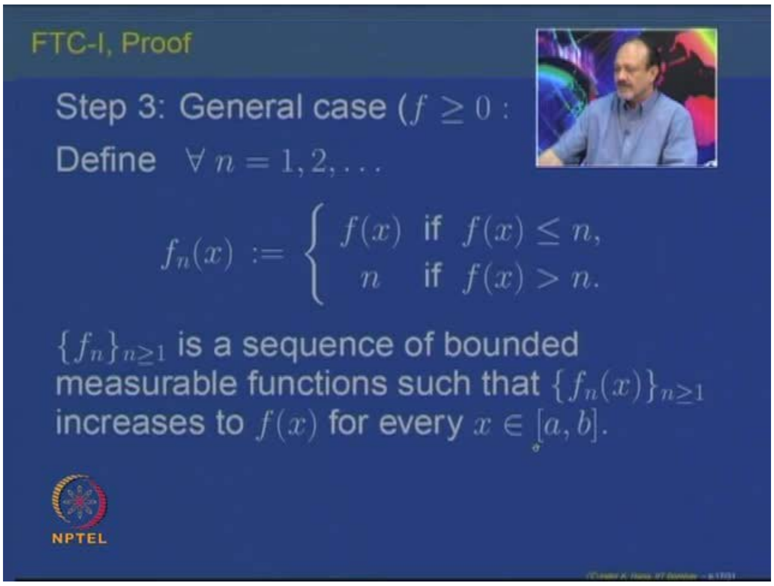

So that proves the fundamental theorem of calculus for bounded non negative integral functions. For the general case, when f is general non negative function one uses what is called the truncation of the function f.

So I will just give you outline, so what one does is look at the function f on x which is f of x, if f of x is less than or equal to n and is equal to n if f of x is bigger than n. So whenever the graph of the function goes beyond n you cut it and put it equal to n; so if this f of n is called the truncation of the function f beyond n.

So each f n is a non negative integrable function and f n converge to the function f for every x belonging to a b.

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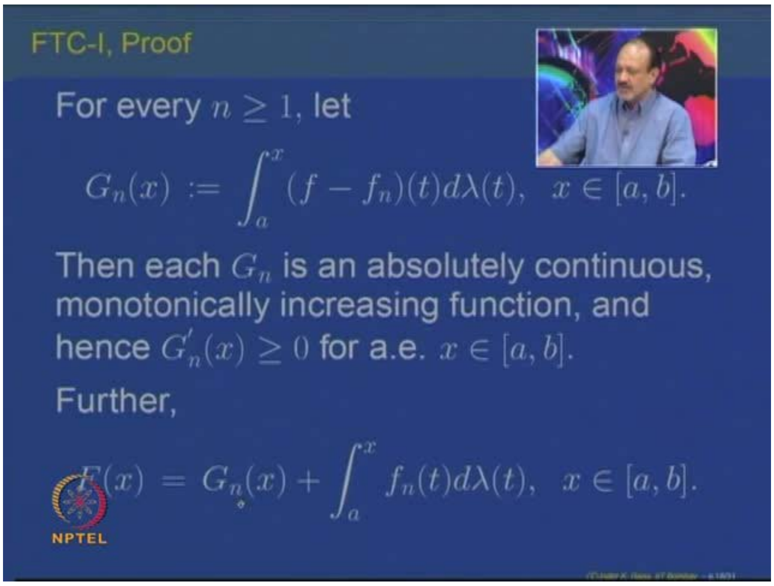

So one defines G n of x to be equal to a to x, f minus f n, t d lambda t then G n is a sequence of functions which is absolutely continuous, because difference of it is indefinite integral of an integrable function. It is monotonically increasing because the function f minus f n is nonnegative; so G n is increasing sequence of functions and its derivative G n of x non negative for every x, because it is a monotonically increasing function. So its derivative must be non negative and it exists because it is absolutely continuous function.

So let us look at the function F of x which is G n of x minus a to x f n t d lambda t that is the indefinite integral of a to x, f of x dx from here, because this integral G n of x is a to x f minus f n d lambda t. So that is integral a to x f d lambda t minus integral of f n, so if you take it on the other side we get F of x is equal to G n x plus a to x f n t d lambda t.

From here, because f dash exist for almost everywhere so by the earlier case, we have got that f n is a non negative bounded measurable function; so applied, when we apply earlier case to this indefinite integral its derivative exist and is equal to small f n.

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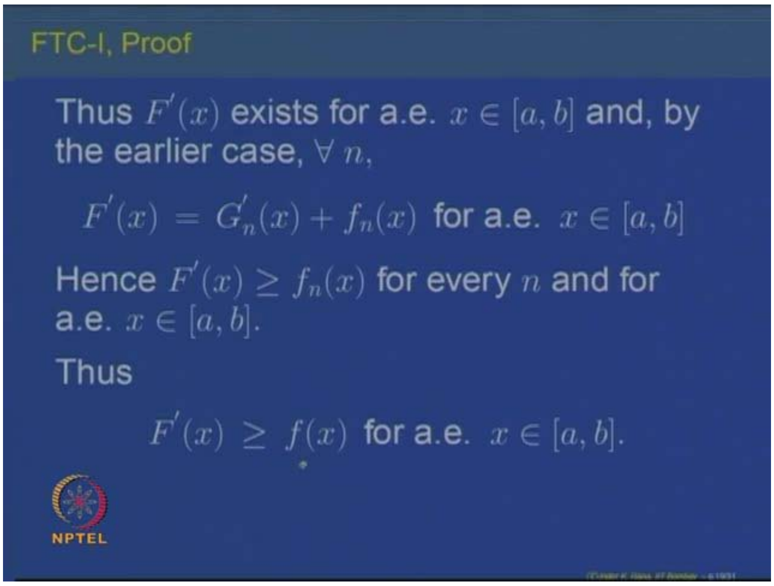

So that gives us that for almost all likes F dash of x is equal to G n dash of x plus f n of x for almost all x belonging to x. G n is non negative so that implies that F dash of x must be bigger than or equal to F of x for every n and for almost all x belonging to x.

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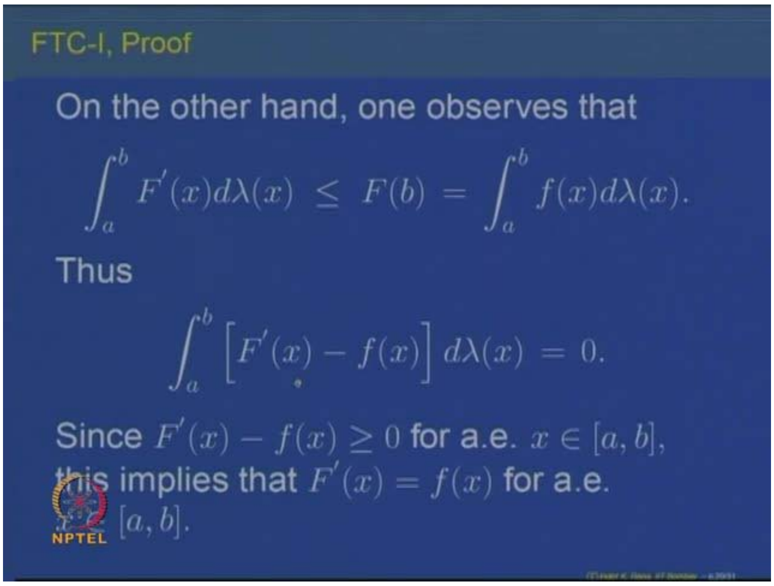

This is happening for all n, so in the limit that gives us that F dash of x is bigger than or equal to f of x for all x belonging to a b but, on the other hand we know that integral of F dash of x is less than or equal to F b minus F of a which is nothing but, a to x f x d lambda x. So that implies, the integral of F dash of x minus small f of x is equal to 0 over the interval a to b.

This is a non negative function whose integral is 0 that means this function F dash of x must be equal to f of x for almost all x belonging to a b. So that completes the proof of fundamental theorem of calculus part 1 namely, the indefinite integral of a function from a to b f x.

If I take indefinite integral a to x f t d lambda t for integrable function then this indefinite integral is absolutely continuous. Hence derivative exists and not only that the derivative F dash of x is equal to small f of x for almost all x belonging to a to b.

So, this is fundamental theorem calculus part 1 which corresponds to the fundamental theorem of calculus part 1 of Riemann integrable functions, namely if you take a function f which is continuous on a interval a b then the indefinite integral is differentiable everywhere and the derivative is equal to small f.

In this case for Lebesgue integrable functions, we get that the indefinite integral is an absolutely continuous function and it is differentiable almost everywhere and derivative is equal to the integrant f of x for almost all x. If you recall the fundamental theorem of calculus also had the second part for Riemann integrable functions namely if you integrate the derivative, then you get back the values of the function.

So a corresponding result is true for Lebesgue integrable functions also, we would like to prove that. So, the claim is that if f is absolutely continuous, so is derivative exists almost everywhere and the claim is that if we integrate the derivative, it is integrable function and its integral is nothing but, the values of the original function.

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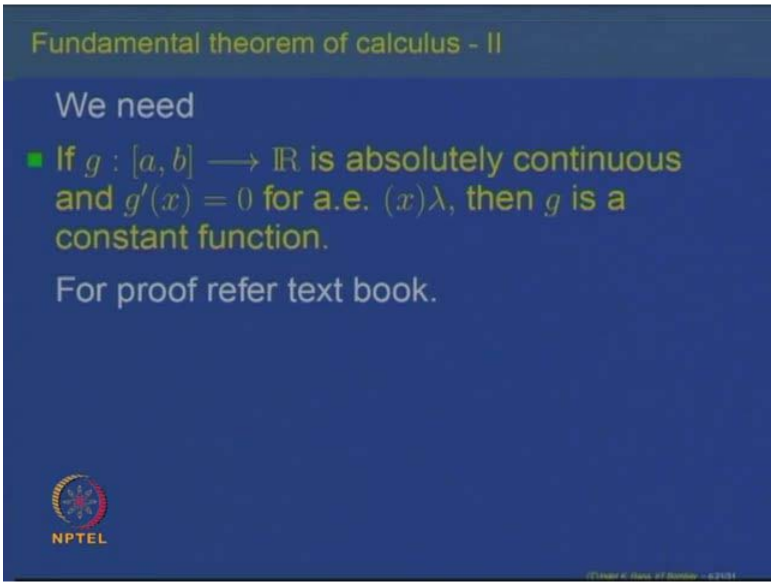

So to prove that we need a theorem, need a Lemma which will not prove which says that if g is an absolutely continuous function and its derivative is equal to 0 almost everywhere, then g is a constant function.

If you recall that is also required, similar result is required for functions which are Riemann integrable. In the fundamental theorem of calculus part 2, namely if g is a function whose derivative is equal to 0 everywhere then, the function is a constant function.

So this is parallel of that result for absolutely continuous functions (Refer Slide Time: 29:20). If for an absolutely continuous function the derivative is equal to 0 almost everywhere and the function is a constant function. So we will assume this result because a proof requires this is slightly technical proof and requires the use of what is called vitali's covering theorem, which we have not covered in this part of the lectures.

Those interested can look up the proof in the text book which is referred earlier. So, the result Lemma is that if a function is absolutely continuous and its derivative is 0 almost everywhere, then the function is a constant function.

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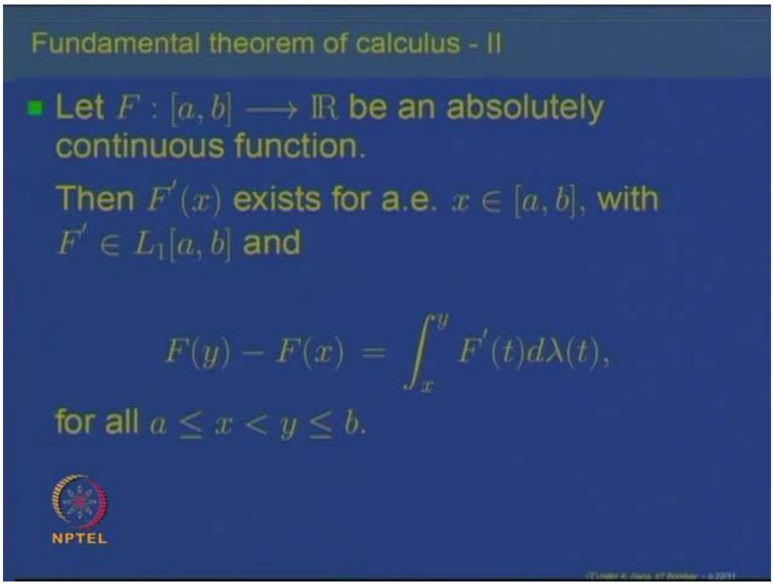

So using that we will prove fundamental theorem of calculus part 2 which says that if F is absolutely continuous function then of course, we know that the derivative exists almost everywhere. The claim is that the derivative is integrable function and the integral of the derivative from x to y is equal to F y minus F of x for all points x less than y between a and b. So that will complete the statement and analysis of fundamental theorem of calculus.

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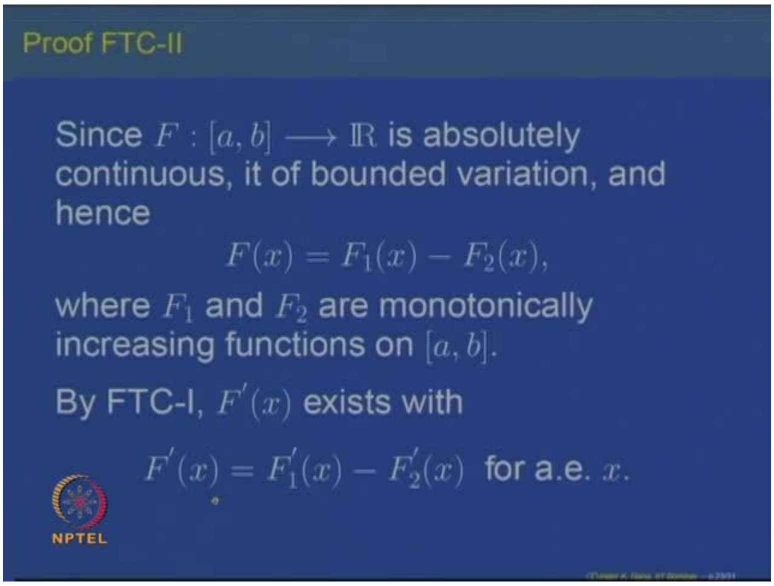

So to prove this, let the function capital F which is given to be absolutely continuous and every absolutely continuous function is of bounded variation. Hence, we can write F as a difference of two monotone functions, monotonically increasing functions, say F 1 and F 2 where, both F 1and F 2 are monotonically increasing functions. Of course, by fundamental theorem of calculus we know that F 1 dash exist, derivative F dash exist by fundamental theorem of calculus and it being a difference of two monotone functions which are also differential almost everywhere by Lebesgue's theorem.

So the derivative is equal to F 1 dash x minus F 2 dash x for almost all x. So F of x can be written as F 1 dash x minus F 2 dash x for almost all x. Now, because F 1 and F 2 are monotonically increasing functions, so the derivatives are integrable functions. So that follows from the Lemma that we just now proved for the monotonically increasing functions, the derivative exist and are integrable.

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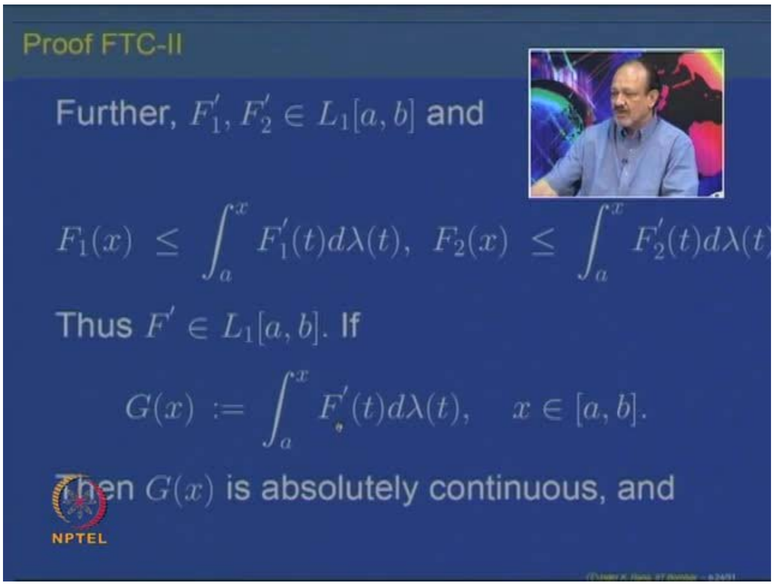

So, as a consequence we get F 1 dash is integrable, F 2 dash is integrable and being a difference of two integrable functions, the functions capital F dash is also integrable and not only that; we also have that F 1 of x is less than or equal to a to x F 1 dash of t d lambda t, that are also a part the Lemma. Similarly, F 2 of x is also less than or equal to a to x F 2 dash of t d lambda t.

As a consequence we get F dash is L 1 that we have already observed, so let us write G of x as equal to a to x F dash of t d lambda t. So, what we want to prove is that this g of x is f of x minus f of a, which is 0; so we want to prove g of x is equal to f of x. Now, let us observe that if we define G of x to be equal to this then, G of x is absolutely continuous its derivative exists and derivative is equal to F dash for almost all x.

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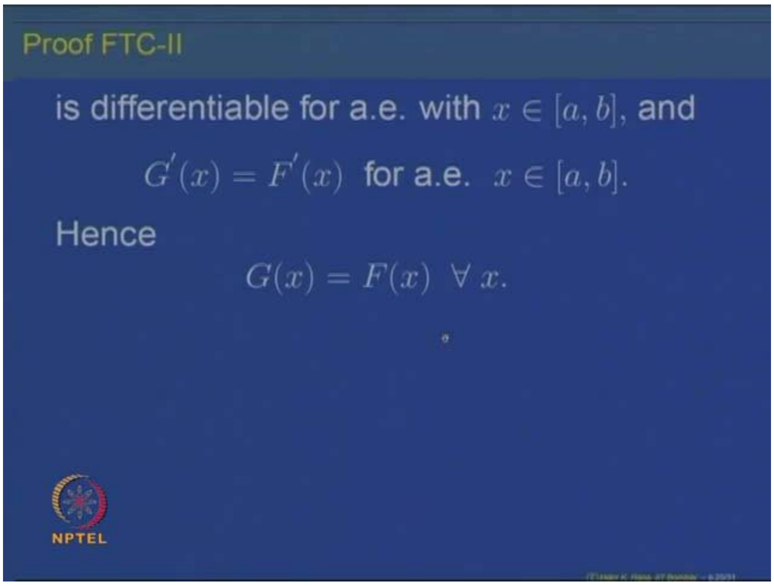

G dash is equal to F dash of x for almost all x and we know that this is a function G of x; its derivative is equal to 0 almost everywhere and G of x is absolutely continuous function. So by the Lemma just now we stated the dash G of x is absolutely continuous function whose derivative is equal to F dash of x.

So that implies that the function G of x must be equal to F of x for almost all x. So that completes the proof of fundamental theorem of calculus part 2; so fundamental theorem of calculus has 2 parts. Let me once again recall fundamental theorem of calculus.

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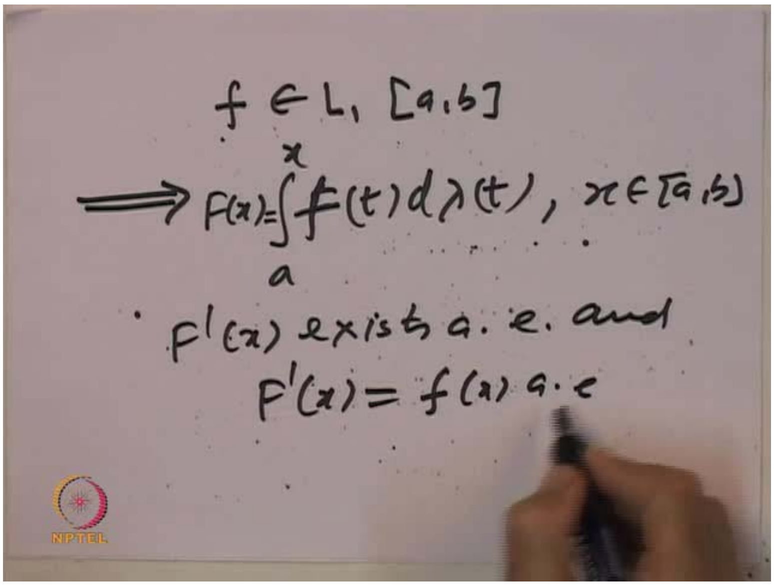

It says that if f is L1 of a b then that implies that the integral f of t d lambda t a to x, x belonging to a to b; so if you call this function as capital F of x then, F dash of x exists and F dash of x is equal to f of x. So, that is fundamental theorem of calculus part 1.

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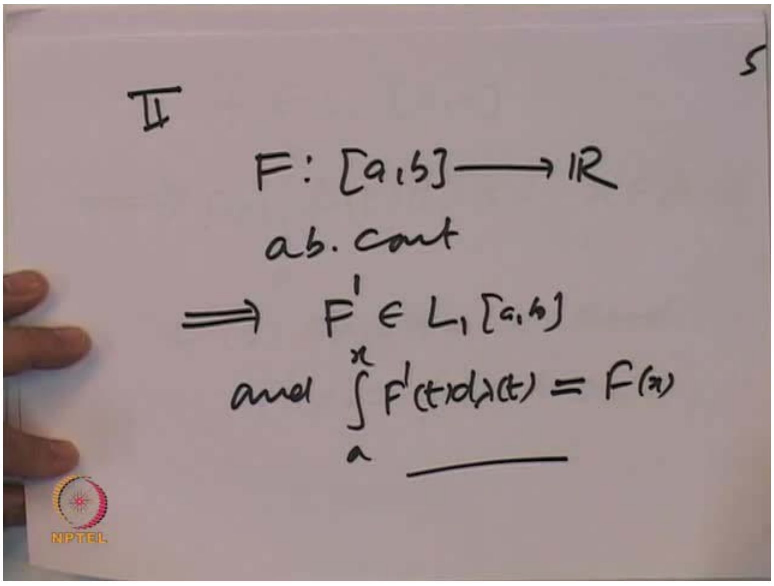

Fundamental theorem of calculus part 2 says, that if F of a to b is absolutely continuous then this implies, that the derivative exist of course, because absolutely continuous implies it is differentiable is L1 of a b and integral of F dash of t d lambda t say a to x is equal to F of x. These two put together are called fundamental theorem of calculus for Lebesgue's integrals (Refer Slide Time: 35:10).

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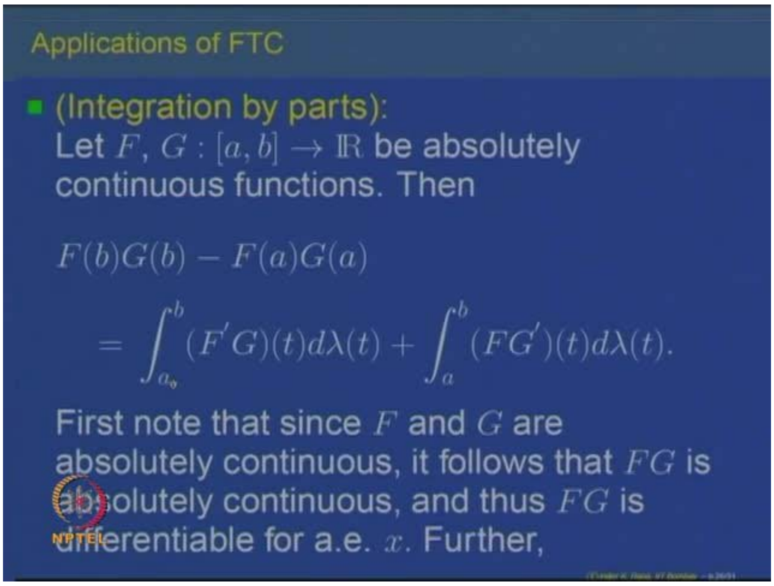

As a consequence if you recall for Riemann integrable functions, fundamental theorem of calculus gives many consequences essentially saying that the integration of the derivative is equal to the function gives consequences like integration by parts, it gives you integration by substitution, it gives you chain rule and so on.

Similarly, the Lebesgue integral fundamental theorem of integral calculus for Lebesgue integrals gives all similar results. We illustrate this by giving one result called integration by parts.

Integration by parts states the following: namely supposing F and G are two functions which are absolutely continuous functions then, the claim is that because they are absolutely continuous. So the derivatives exists, so the claim is that F of b into G of b minus F of a, G of a is equal to integral a to b of F dash G d lambda t plus integral a to b F G dash d lambda t.

Basically it is same as for Riemann integral that if F and G are absolutely continuous then we know that F dash exist. It says integral of F dash G d lambda t is nothing but, if you take it on the other side it is same as you look at the integral of F dash which is going to be F of b.

So, it is F of F into G evaluated at b minus evaluated at a minus the integral F G dash. So, that is precisely the integration by parts for Riemann integrable functions. A proof of this is obvious from the fundamental theorem of calculus. Let me illustrate that, first note that F and G are both absolutely continuous. So it follows that the product function F G is absolutely continuous that we have already indicated and hence it is differentiable.

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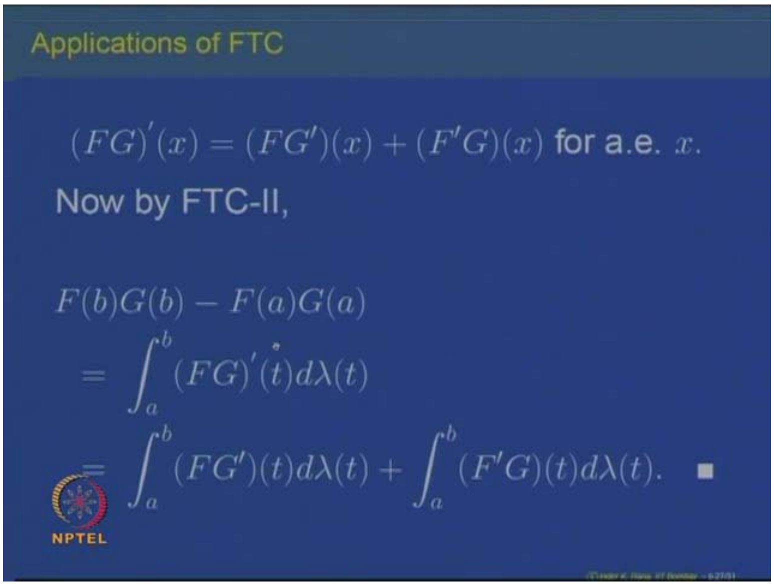

In Riemann integrable functions, it is just saying that F and G are differentiable, so product is differentiable. Here because they are absolutely continuous, so the product is absolutely continuous and hence is differentiable almost everywhere. Further by the product rule for differentiation, the derivative of the product function of course is equal to F into G dash is equal to F multiplied by G dash into x plus f dash into G into x for almost all x in the interval a to b.

This is just, where ever the function is differentiable the derivative of the product is equal to FG dash plus F dash of G and this is true, so this is equal to almost everywhere. Once that is true, now we can apply fundamental theorem of calculus part 2. We can integrate FG dash, so integrate both sides; so Integral of FG dash will give us F b minus F b into G b minus F of a into G of a because this integral will give you the value of the function at b minus the value of the function at a, the function being the product function F into G.

So the left hand side is F b G b minus F a G a and the right hand side is the integral of FG dash x plus F dash G x for almost all x, so that is the right hand side. So, that is called the integration by parts.

Similarly, other results like integration by substitution and so on can be obtained by using chain rule and other things. Those interested should refer the text book for the same.

Basically what we have tried to prove in the last 2 lectures is that for corresponding to the theorem on fundamental theorem of calculus for Riemann integrable functions, there is a theorem for Lebesgue integrable functions and it says that a function capital F from a to b defined on real values is absolutely continuous if and only if it is the integral of its derivative function.

Basically saying that because it is absolutely continuous, the derivative exist and the integral of the derivative is nothing but function can be obtained as the integral of the derivative. This is a perfect extension of the fundamental theorem of calculus for Lebesgue integrable functions.

In proving this fundamental theorem of calculus, we needed the notion of absolute continuity of functions. This absolute continuity of functions is related in another way to the properties of measures called absolutely continuous functions - absolutely continuous measures. In the coming today and in the next lecture, we will look at what are called absolutely continuous measures.

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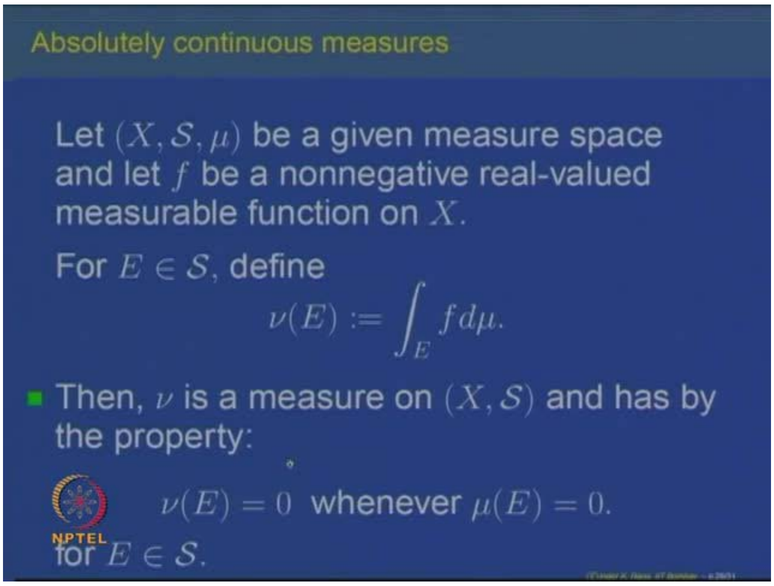

Let us define, what is absolutely continuous measure? Let X, S, mu be a given measure space and let f be a non negative real-valued measurable functions on it. Let us define, for any measurable set E in the sigma algebra S, nu of E to be integral over E of f d mu. The right hand side is a number which depends on E, so nu of E is the integral of f with respect to the measure mu over the set E.

This gives us a set function E going to nu of E and if you recall, we had shown that nu is a measure on the measurable space X, S and has a very special property namely if the set E over which you are integrating the function f has got mu measure 0 then obviously, this implies that nu of that set E is also equal to 0.

This measure nu, which is defined via integration with respect to the measure mu is related these two measures are related by the property that mu of the set E is equal to 0, whenever the measure mu of the set E is equal to 0. If you measure nu is defined via integration then the null sets of mu are also null sets of E. So, this property is known as absolutely continuity of measures.

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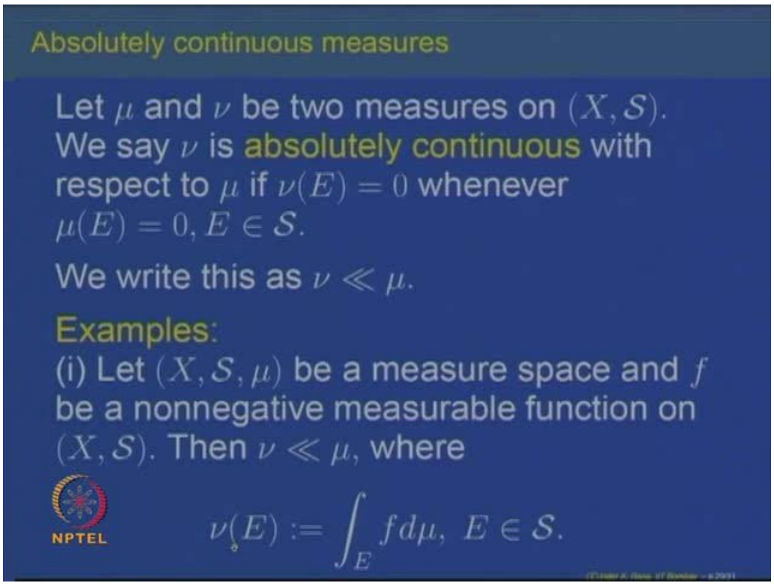

Let us write this, define two measures mu and nu are said to be absolutely continuous with respect to- so we say that the measure nu is absolutely continuous with respect to- the measure mu if nu of E is equal to 0 whenever mu of E equal to 0. So, nu is absolutely continuous with respect to mu, whenever the null sets of mu are also null sets of nu. This we write as nu with these two arrows less than, less than double sign of less signifies that nu is absolutely continuous with respect to nu. It is easier to remember mu of E equal to 0 implies nu of E equal to 0.

Let us give some examples of absolutely continuous measures. Of course, just now we said that if you take a non negative function f, which is integrable, just a non negative measurable function on the measure space X S and integrate it with respect to nu. Then integral of f over E with respect to mu gives you a measure nu which is absolutely continuous, so this measure nu is absolutely continuous (Refer Slide Time: 43:43).

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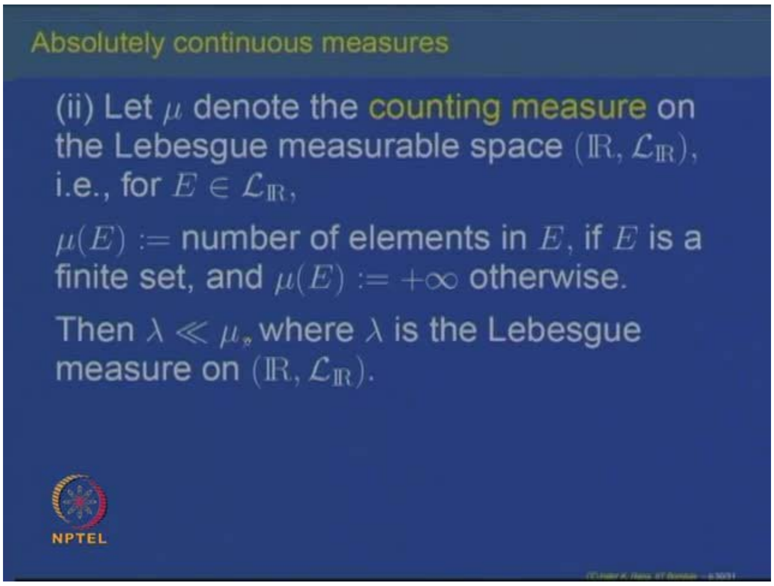

Let us look at the counting measure on the Lebesgue measure space that is a real line. What is the countering measure? For every set E in R the counting measure mu of E is defined as the number of elements in the set E. If E is a finite set and it is defined as mu of E equal to infinity if E is a set which is not finite.

For finite sets, nu of E is the number of elements in E and for an infinite set it is number we call it as plus infinity. Then we claim that lambda is absolutely continuous with respect to mu where mu is the Lebesgue measure.

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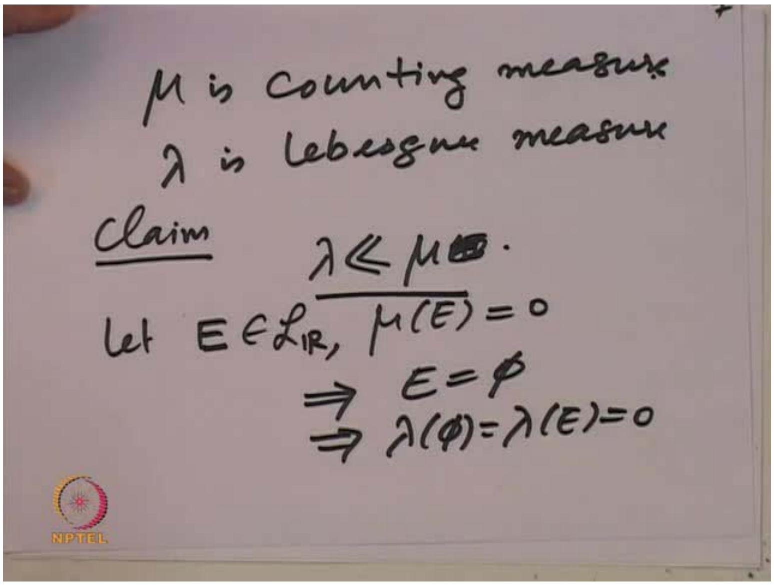

So, we want to prove the following fact. Let us observe, mu is counting measure and lambda is Lebesgue measure. Claim is that lambda is absolutely continuous with respect to E sorry, with respect to mu.

To show that, let us take a set E which is Lebesgue measurable and mu of E equal to 0 but, that means this implies what are the sets for which mu is the counting measure. Say if a counting measure of a set is equal to 0 that means E must be equal to empty set. For counting measure, empty set is the only set of measure 0.

That obviously implies lambda of phi which is same as lambda of E equal to 0. That implies obviously, that lambda is absolutely continuous with respect to the counting measure. This is happening because for counting measure only null set is the empty set.

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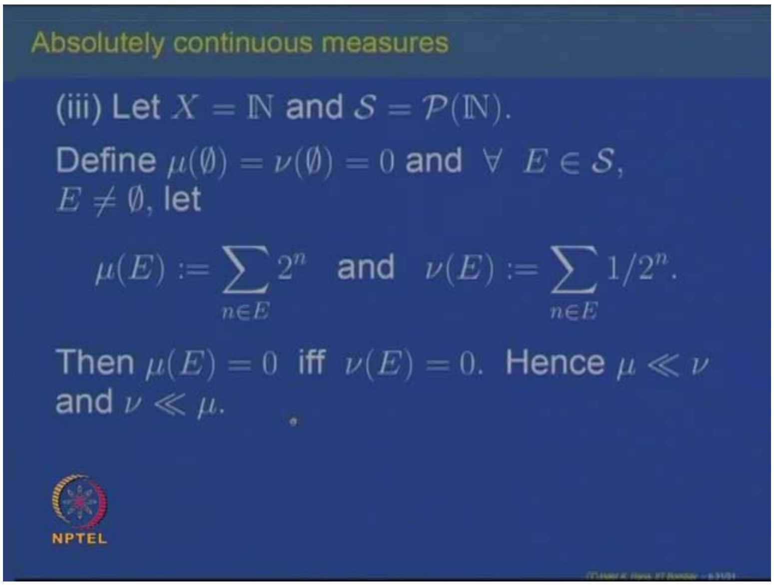

Let us take, X to be equal to the set of all natural numbers and S to be the set of all power set of natural numbers namely all subsets of natural numbers.

Let us define two measures mu and nu. Say that, mu of empty set is same as nu of empty set is 0 and for every other set which is not empty set. Let us define mu of E to be equal to sigma of 2 to the power n whenever the natural number n belongs to E and nu of E to be summation of 1 over 2 to the power n, for all natural numbers n belonging to E.

It is easy to check that these - two sets- two set functions mu and nu are measures on the measure space X and S. Then if mu of E is equal to 0 let us assume, that mu of E is equal to 0 this is possible only when this E is a empty set and similarly, nu of E is also equal to 0 then E is equal to empty set. So, mu and nu are two measures which have only empty set as the null sets. So that clearly proves, nu of E is equal to 0 if and only nu of E is equal to 0.

Hence that says mu is absolutely continuous with respect to mu and nu is absolutely continuous with respect to nu. These are some are some of the examples of absolutely continuous measures but, as you would have noticed these are obvious examples of absolutely continuous measures.

The only non-trivial way we got absolutely continuous measures was the first example, where nu was defined as integral of a non negative function with respect to a measure nu.

No wonder that we are not able get any examples, because there is a very strong powerful theorem which says that the only way one can obtain absolutely continuous measures are via integration. That means that example of when you integrate, you get absolutely continuous measures are the only ways of obtaining absolutely continuous measures.

This leads to theorem called Radon Nikodym theorem, which is very powerful and very useful and we will prove it in our next lecture.