Measure and Integration
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Module No. # 10
Lecture No. # 36

Fundamental Theorem of Calculus for Lebesgue Integral - I

Welcome to lecture number 36 on measure and integration. In today's lecture, we will be looking at the fundamental theorem of calculus for Lebesgue integrals. If you recall, we had started our lectures with a motivation saying that - for Riemann integrable functions, the importance of fundamental theorem of calculus is there and that is why it is easier to integrate functions whose derivatives are known. However, after Riemann extended this notion of integral beyond continuous functions, the fundamental theorem of calculus no longer remains true, if the integrant in the Riemann integral is not continuous.

Though the class of functions which were integrable was extended, the fundamental theorem of calculus no longer was true. So, efforts were made to extend the notion of integration and that is how we got the notion of Lebesgue integral. Today, we will show how fundamental theorem of calculus holds for Lebesgue integrable functions. Today's topic is going to be fundamental theorem of calculus for Lebesgue integral.

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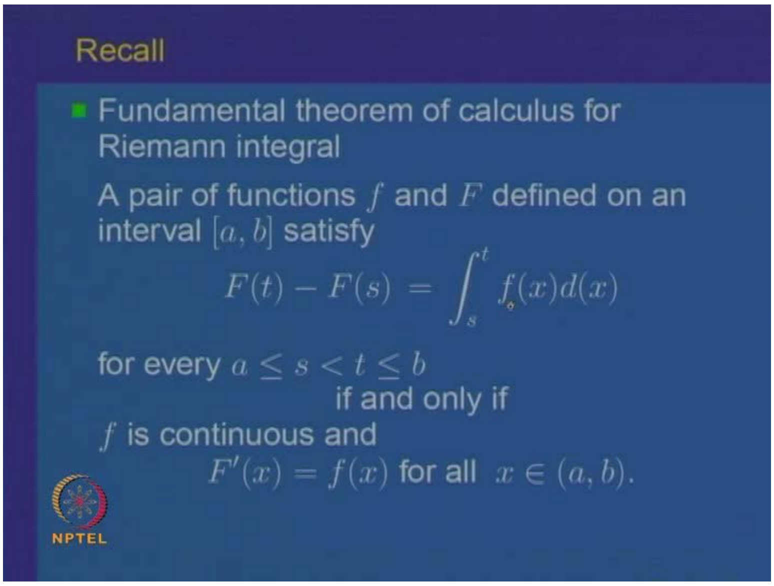

Let us just recall fundamental theorem of calculus, for Riemann integrable functions, it says - if you are given a pair of functions - small f and capital F defined on a interval a b, then they satisfy the relation that F of t minus F of s is equal to the integral s to t of f x d x for every pair of points s less than t in a b, if and only if, the function f is continuous and the derivative of capital F is small f.

This equation - F of t minus f of s is equal to integral from s to t of little f of x d x holds, if and only if this function, little f is continuous. The function, capital F is differentiable and derivative is equal to small f. So, in a sense, this relates differentiation with integration. It says that for every continuous function, indefinite integral is differentiable and if you integrate the derivative you will get back the function.

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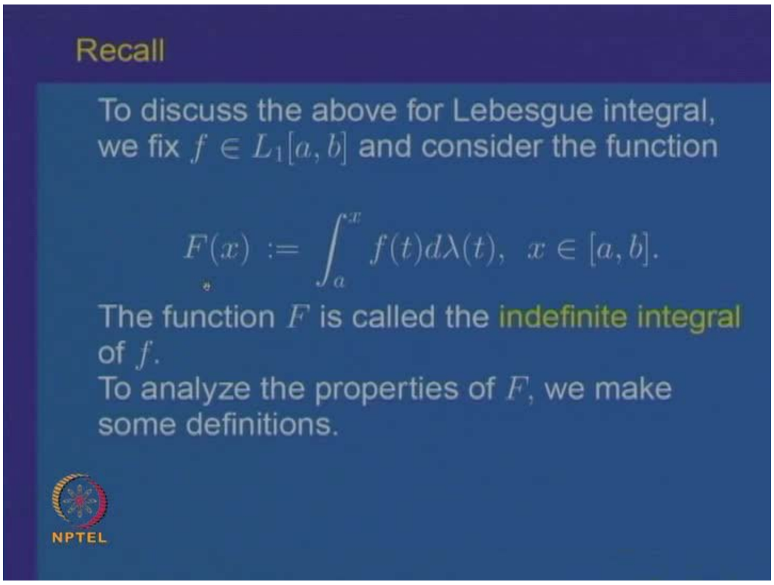

We would like to analyze this theorem for Lebesgue integrable functions. We are going to look at a function f, which is integrable on the interval [a b]. Let us fix a function f which is integrable on a to b. Look at the indefinite integral, namely capital F of x is equal to integral on the interval a to x of f t d lambda t, the integration being with the respect to the Lebesgue measure, lambda.

What we want to do is this function here? Capital F is called the indefinite integral of the function small f. We would like to analyze the properties of this function capital F in detail. To analyze these properties, we have to look at some classes of functions. We will make the definition first and then come to back to this function, capital F.

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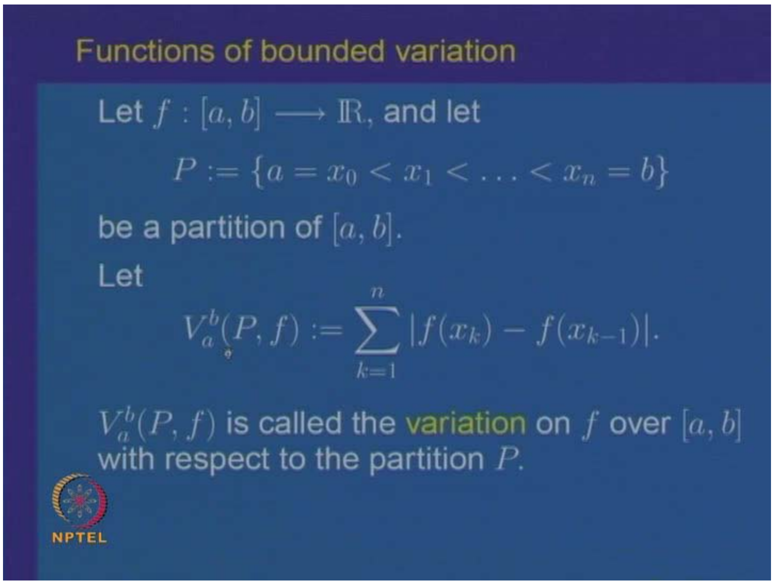

Let us take a function f which is defined on an interval [a b] taking real values. Let us take a partition P of the interval [a b] - partition being starting with a equal to x 0, the point x 1, x 2, x n equal to b. So, P is a partition with end points a and b and in between points are x 1, x 2 and x n. Let us take a partition of this interval [a b] and define what is called the variation of the function, f over this partition in the interval a to b.

This is denoted by capital V lower a upper b indicating that we are over the interval a to b with respect to a partition P of the interval [a b] of the function f. What is this quantity? You look at the variation of f on each sub interval of the partition. Look at f of x k minus f of x k minus 1 - absolute value of this. This is how the function varies on the interval x k minus 1 to x k. Look at this number, absolute value of f of x k minus f of x k minus 1 and look at the summation of all these variations 1 to n.

This is how the value of the function changes at the end points of the partition. This number is called the variation of f over the interval [a b] with respect to the partition P. It depends on the partition, the points and on the interval a to b.

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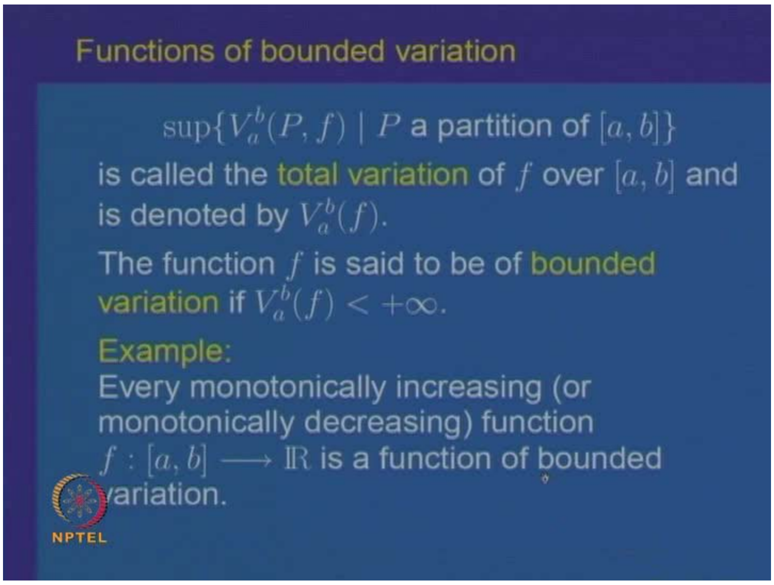

We define the supremum of all these variations of the function f over the interval [a b] with respect to various partitions of the interval [a b]. This supremum is called the total variation of f over the interval [a b] and we denote it by V a b of f. So, V a b of f is nothing but the supremum of the variations with respect to various partitions V a b of f with respect to P.

Now, obviously this number is a nonnegative number. It could be equal to plus infinity because this is a supremum of nonnegative numbers. So, we say that the function has bounded variation, if this number V a b f which is a supremum of the variations with respect to various partitions is a finite quantity.

We say a function f defined on an interval [a b] is of bounded variation, if the supremum of the variations of the function f with respect to all the partitions, the supremum of this is a finite quantity. We look at some examples of functions of bounded variation. Every monotonically increasing or decreasing function is a function of a bounded variation.

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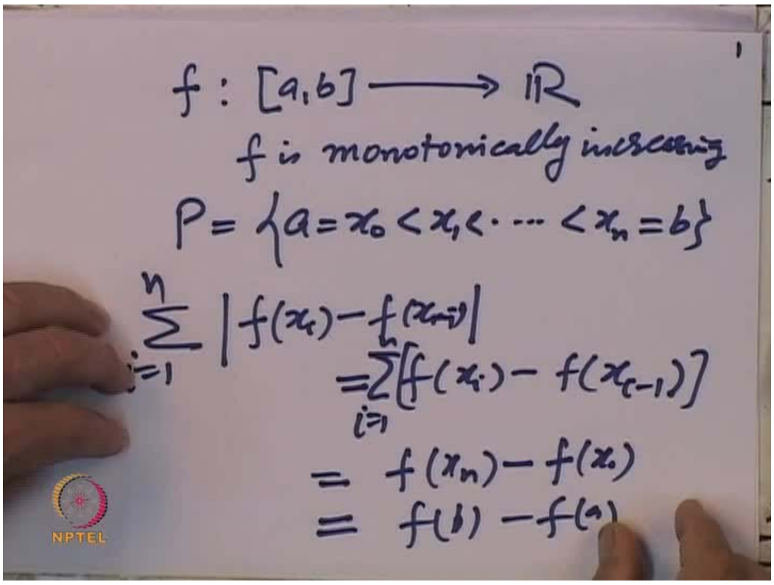

Let us just check that. Let us take a function with f defined on interval [a b] taking values in R and let us say f is monotonically increasing. Let us take a partition P of a b so that a is equal to x 0, x 1, x n equal to b. So, this is a partition of the interval [a b]. We want to look at what is absolute value of f of x i minus f of x i minus 1.

Because f is monotone, this is same as f of x i minus f of x i minus 1. Because it is monotone, this value is bigger than this. So, absolute value is same. So, if we take summation on both sides 1 to n, summation i equal to 1 to n - that will give us all consecutive terms will cancel - so that will be f of x n minus f of x 0 which is equal to f b minus f of a.

That means - this variation of the function with respect to any partition is equal to f b minus f of a. This implies that f is of bounded variation, because the supremum also will be equal to this. So, every monotonically increasing function is of bounded variation.

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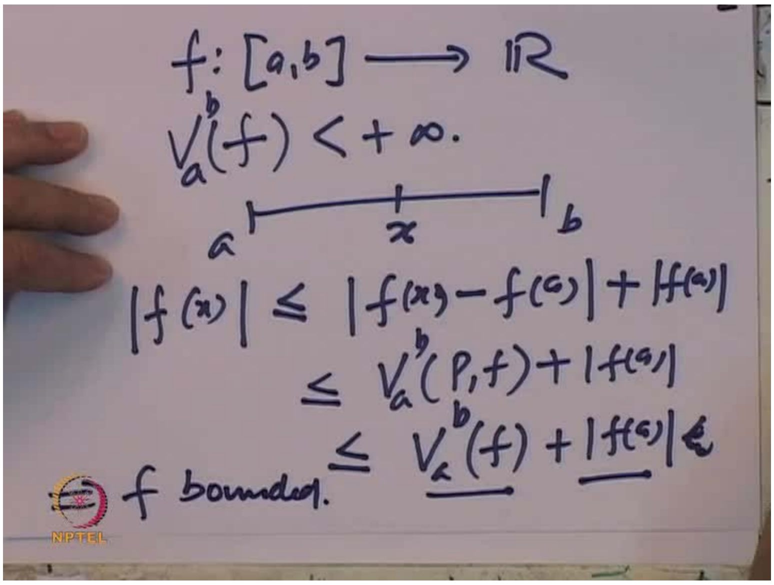

Let us check if f and g are two functions on a, b defined on the interval [a b], then the function f itself is a bounded function. Let us check also that every function of bounded variation is a bounded function. So, f on [a b] to R, we are given that the variation of f from a to b is finite. So, let us take - this is interval a, this is b, so, let us take any point x. Look at f of x; absolute value of this is less than or equal to f of x minus f of a plus f of a - by triangle inequality.

If I look at the partition of the interval [a b] by three points a, x and a, then, what will be the variation with respect to that partition? That will be f of b minus f of x - absolute value plus the absolute value of f of x minus f of a. This first term will be less than or equal to variation of the function with respect to that particular partition plus mod f of a.

What is the partition P? P is the partition of three points a, x and a. This will be one of the terms in that variation. It will be less than or equal to this and that is less than or equal to the variation over a b of f plus mod f of a. So, if f is a function of bounded variation, then, mod f of x is less than or equal to the total variation plus this quantity - which is a finite quantity. Let us call it as some number M. This implies f is bounded. Every function of bounded variation is also a bounded function.

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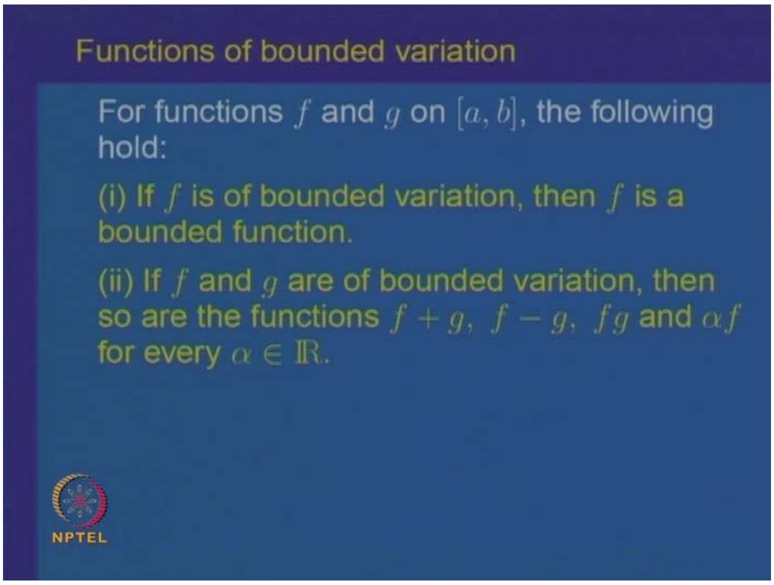

Next, let us look at the class of functions of bounded variation. We want to check if f and g are of bounded variation, so are the functions f plus g, f minus g f into g and alpha f for every alpha belonging to R. That means the class of functions of bounded variation is a nicely behaved class and these properties are all easy to check. Let us just check one of the properties - say f g.

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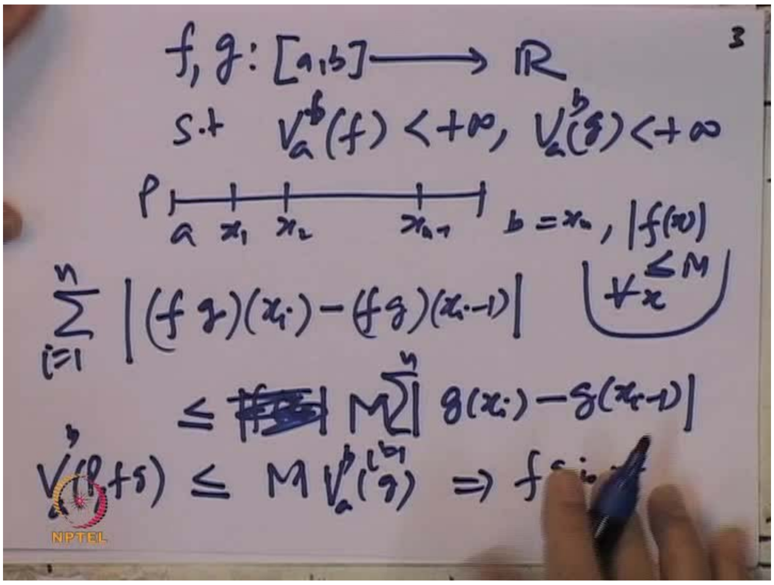

Let me check only one of them - say f and g on [a b] to R such that the variation of f is finite and g also as bounded variation. Let us take any partition - P of a to b, say - x 1, x

2, x n minus 1 and x n. Let us analyze f g of x i minus f g of x i minus 1. So, we want to look at this quantity and we are given f and g are functions of bounded variation. Since f is a bounded variation that also is a bounded function. Let us assume that mod f x is less than or equal to M for every x. Let us assume this. Because f is of bounded variation, it is bounded. Just because f g into x i is f of x i and g of x i and f of x i is less than or equal to M, I can write - this is less than or equal to M times g of x i minus g of x i minus 1, so that will be less than or equal to M times.

When we take the summation on both sides, summation I equal to 1 to n. This will be less than or equal to summation I equal to 1 to n; so that the variation of a to b with respect to P of f g is less than or equal to M times the variation of the function g over this. So, this implies that f g is of bounded variation.

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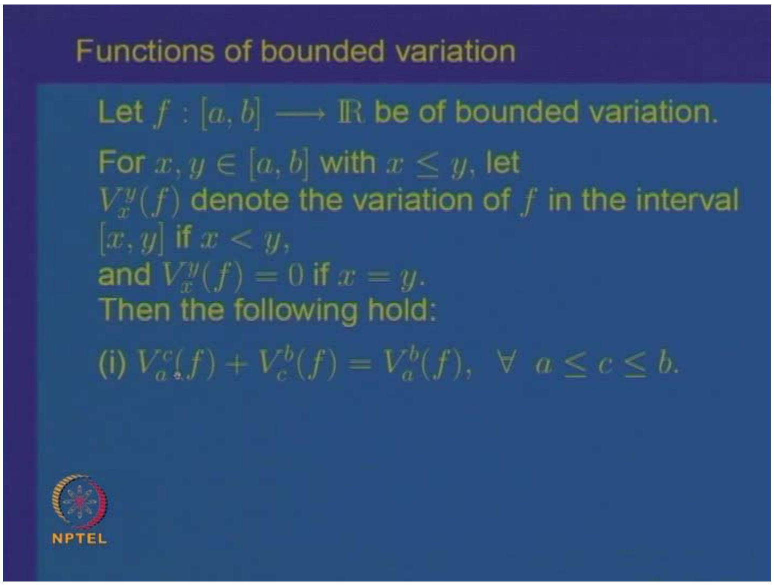

These are all easy properties to check that if f and g are of bounded variation. Then, f plus g is of bounded variation and f minus g - the product alpha times f all are functions of bounded variation. So, here are some facts about the functions of bounded variation which are of importance. We will prove one by one.

Let us take a function f of bounded variation, then, for any two points - x and y in [a b] with x less than or equal to y, let us denote the variation of f over the interval x to y; if x is less than y and if is equal, then, we denote it by equal to 0. So, V x y, V lower x upper y denotes the variation of the function f in the closed interval [x y], if x is strictly less than y. If x is equal to y, obviously we write it at equal to 0. It has the following properties. The first property is that the variation of the function over an interval is additive; that means, if we take the variation of the function over the interval a to b, then, it is same as the variation over a to c plus the variation over c to b.

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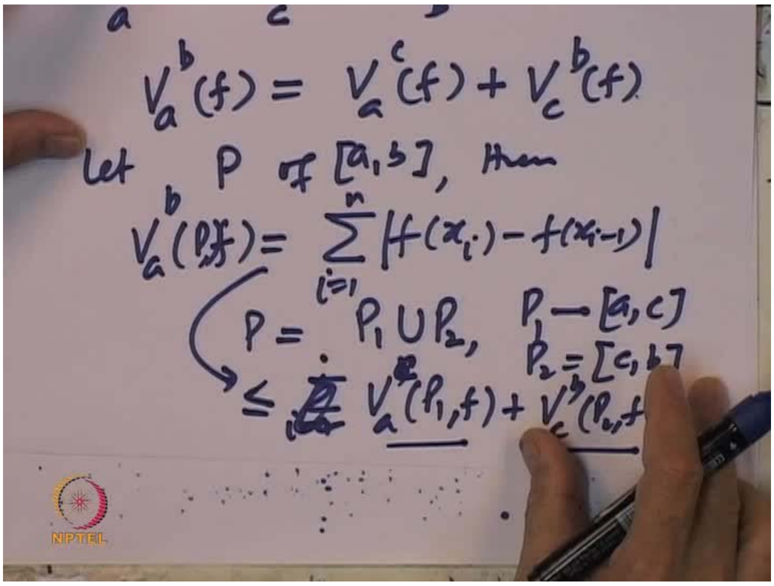

So, let us prove this property. Here is the interval a to b and here is the point c; we want to check that the variation over a to b of f is equal to variation over a to c of f plus variation c to b of f. To check that, let us take a partition P of a b, then, let us look at the variation of a to b over P of f - that will be equal to summation. Let us say, the partition points are x i minus f of x i minus 1- i equal to 1 to n, absolute value of this.

Now, this partition may or may not have the point c inside it. In any case, let us insert the point c inside this - even if it is not there - that mean let us say it lies in x i minus 1 and x i. So, introduce the point in the new partition. In that case, P can be written as P 1 union P 2, where P 1 is a partition of a to c and P 2 is a partition of c to b. We look at the partition and divide it into two parts to partition P 1 in the points lying in a to c. We had the point c here and P 2 is here.

If we introduce that point c in between, that will tell us that this quantity is less than or equal to sigma i equal to 1 to that point. Let us call this point as c as something. It is same as the variation over a to b of P 1, a to c of P 1 of f plus the variation c to b of P 2 f f less than or equal to - because when you introduce the point c here so f of x i minus f of c plus f of x i minus 1 plus, this sum - the ith absolute value sum will split into two parts with less than or equal to one term will be accommodated here, the other will be accommodated here.

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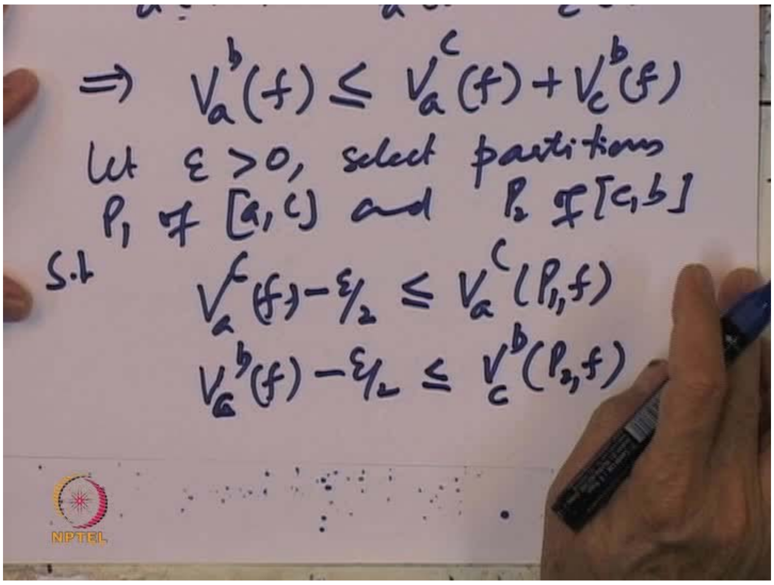

As a consequence of this, we get that. Because of this we will have that V a b of partition with respect to the partition P is less than or equal to – this is P 1 and P 2 are particular partitions - c of f plus V c b of f. So, we get the variation with respect to any partition P over the interval [a b] is less than or equal to the total variation over a to c plus the total variation over c to b.

This happens for every partition. That implies that the supremum which is nothing but the variation of f over [a b] is also less than or equal to variation over c of f plus variation c to b of f, because this happens for every partition. So, we can take the supremum and that is also here. So, we get one way inequality namely the variation of f over a to b is less than or equal to the variation a to c plus variation c to b.

To prove the other way around inequality, let us fix epsilon greater than 0 and select partitions - say P 1 of a to c and P 2 of c to b such that V a c of f is the supremum minus a small number epsilon by 2 cannot be the supremum. So, there must exist a partition P 1 of a to c such that this quantity is less than the variation a to c of P 1 of f.

Similarly, for the second one, while we are using the fact that V a c or V c b of f is a supremum, we will get a partition P 2 of c to b such that this is less than or equal to the variation of the function with respect to the partition P 2.

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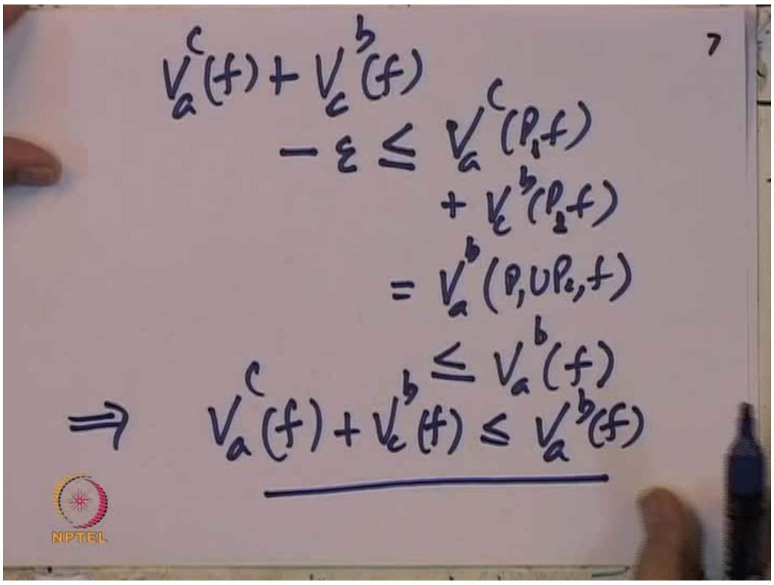

Adding these two equations will give us - V a c of f plus V c b of f minus epsilon is less than or equal to V a c P 1 f plus V c b P 2 of f. Now, we realize that the partition P 1 and P 2 put together is nothing but variation of a to b of P 1 union P 2 of f.

We have partition P 1 on a to c. We have a partition P 2 on c to b. These two variations put together give us the variation with respect to the partition over the whole interval which is the union of the two a b and that is less than or equal to V a b of f because it is a supremum. We get for every epsilon - the V a c, the variation over a to c plus the variation over c to b minus epsilon is less than this. Epsilon is arbitrary; that implies V a c of f plus V c variation over c to b is less than or equal to variation a to b of f.

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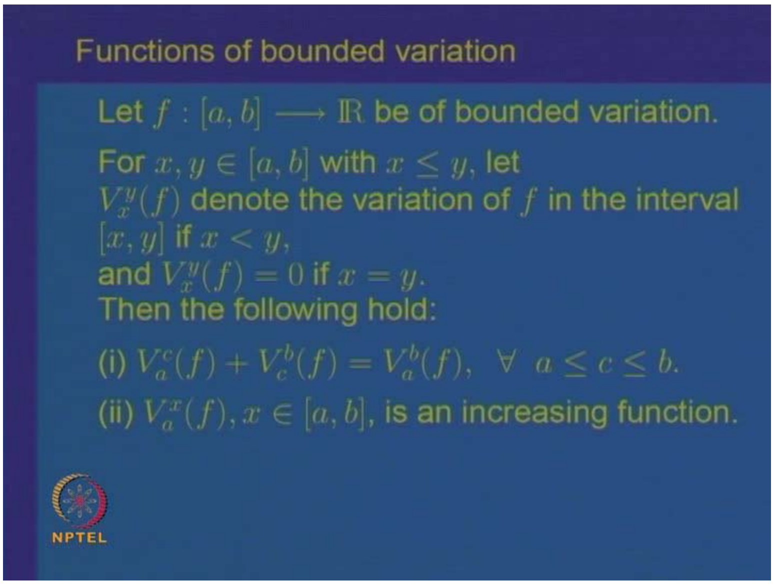

This proves the first property namely - the variation is additive over the interval which we are doing. So, variation over a to c plus variation c to b is equal to the variation over a to b of f. The next property we want to check is the following: if we take the variation of the function on the interval a to x where x is varying in the interval a to b, then it is an increasing function and that property is quite easy to verify. Let us just verify that.

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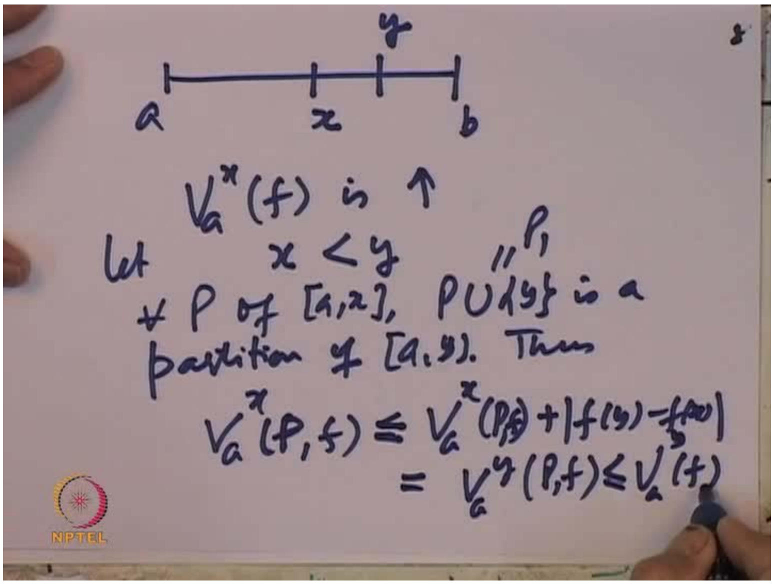

We have got the interval a to b and here is the point x. We want to check that V a x of f is increasing. Let us take a point x- x be less than y, so, here is a point y. Then for every partition P of a to x, if you look at P union the point y that gives us is a partition. Call this as P 1. It is a partition of a to y. Thus the variation over a to x of f with respect to this partition P - any partition P of a to x. We had joined the point and get a partition so that will be less than or equal to the variation a to x of p plus mod f y minus f of x, because this is variation of P with respect to f; same thing plus added.

This is nothing but the variation of a to y of P 1 of f and that is less than or equal to variation a to y of f. So, every partition P of a to x, we look at the variation a to x of that it is less than or equal to the variation a to y of x.

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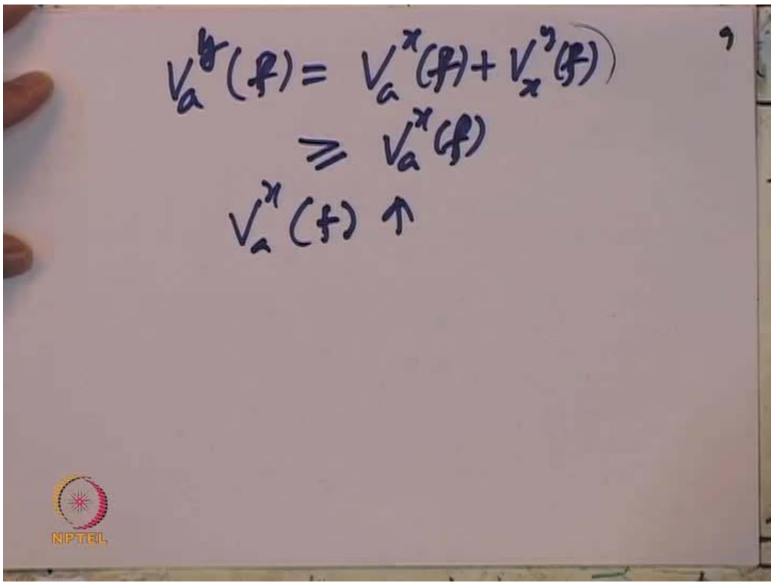

So, we can take the supremum; so that will imply that variation a to x of f is less than or equal to variation a to y of f. So, the variation is an increasing function. We could have also looked at in another way by the previous property - the variation over a to y of P is equal to variation a to x of p plus variation x to y of P. By additive property we could have done that.

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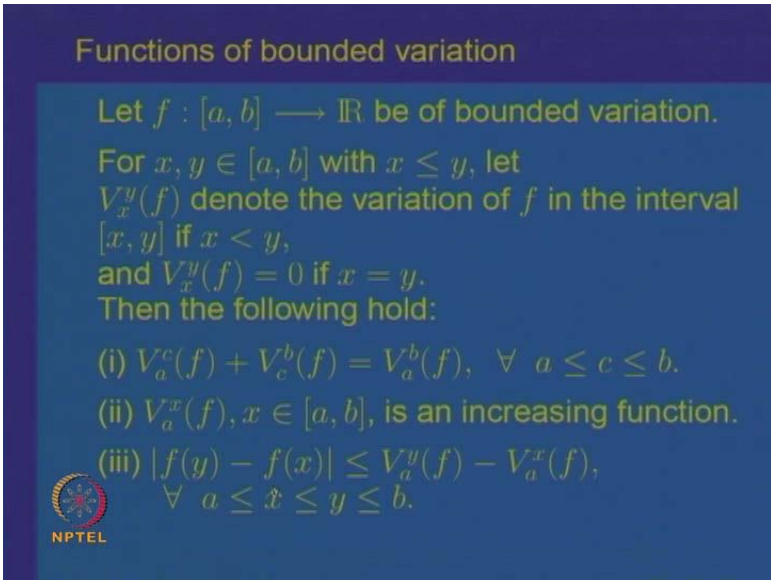

Noting that this quantity is always bigger than or equal to 0, so it is bigger than or equal to x of P, so, that is another way of proving that V a x of f is increasing. So, V a of f variation being additive, so, that will give us - this is an increasing function. So, that proves property two. Let us look at the third property which says that for any point x and y, mod of f x minus f y is less than or equal to the variation between a to y minus the variation from a to x.

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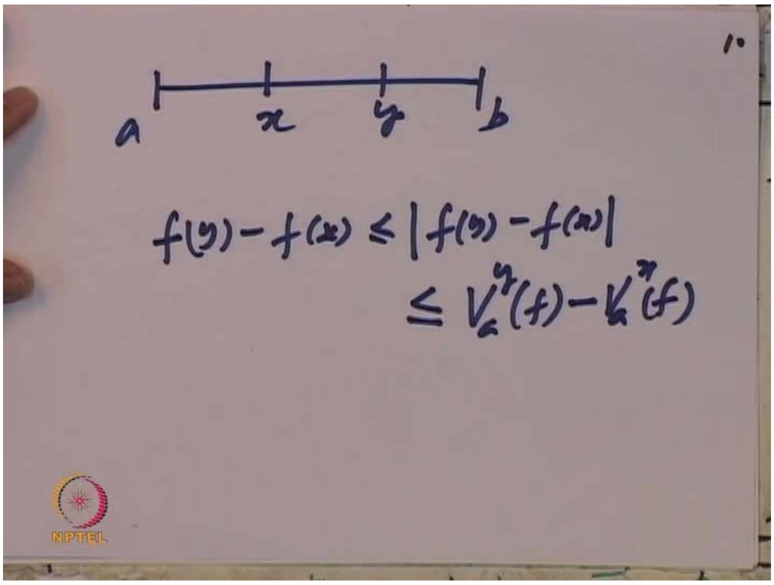

That property also is quite easy to verify. We have got the interval a to b and we have got the points x and y. Note that f y minus f of x is less than or equal to mod of f y minus f of x and that is less than or equal to v a y of f minus v a x of f. Just now we verified that. So, mod of f y minus f of x is less than or equal to this quantity. Next, let us look at the property that the function V a x f minus f of x is an increasing function.

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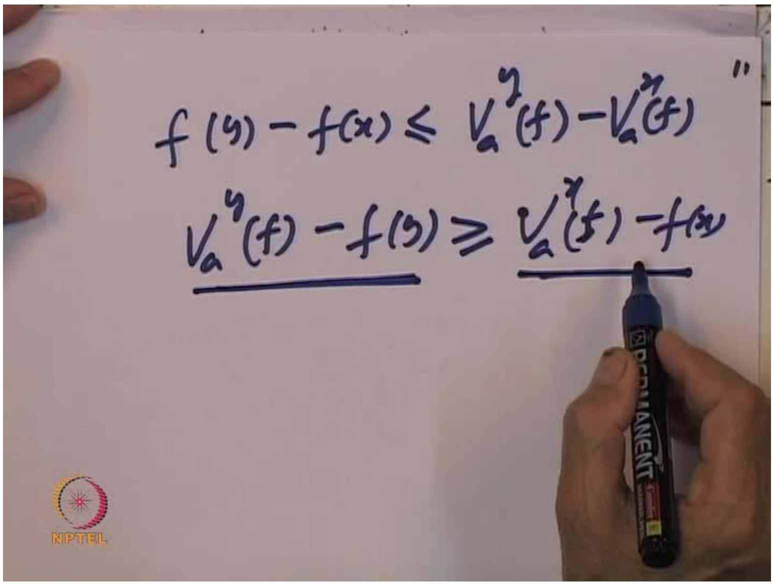

That also follows from the inequality. We just now proved the fact that f y minus f of x is less than or equal to V a y of f minus V a x of f. That says that - V a y of f minus f of y is bigger than or equal to V a x of f minus f of x. So, that says this function is bigger than or equal to this function as a function of x. So, this is an increasing function.

So, these are the properties of the variation of the function over the interval and as a consequence of all these properties, we get an important theorem namely that the characterization of functions of bounded variation. We saw in the beginning that any monotone function is a function of bounded variation. We also saw that difference and sum of two functions of bounded variation is a function of bounded variation.

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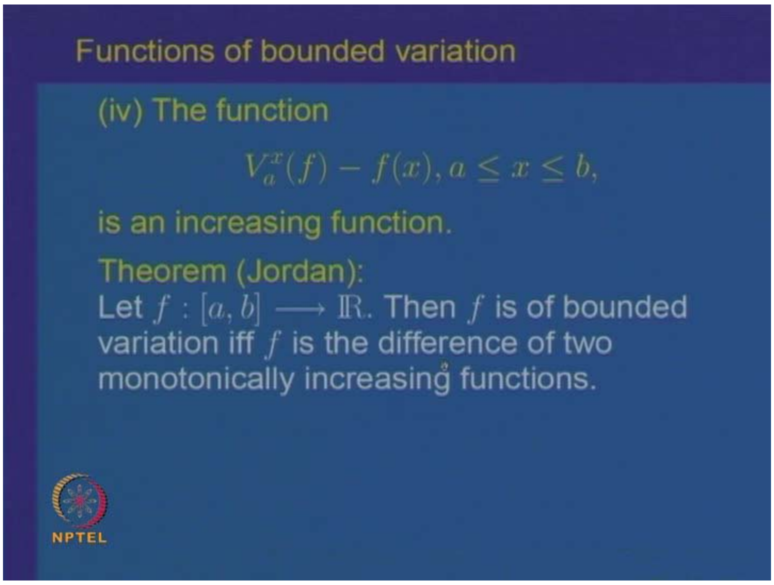

So, sum and differences of two monotone functions will again be function of bounded variation and there is a characterization by Jordan which says the following - if f is a function, then, f is of bounded variation, if and only if it is a difference of two monotonically increasing functions. So, one way of this property we have already checked. If f is a difference of two monotone functions, then each monotone function being of bounded variation, the difference will be again of bounded variation.

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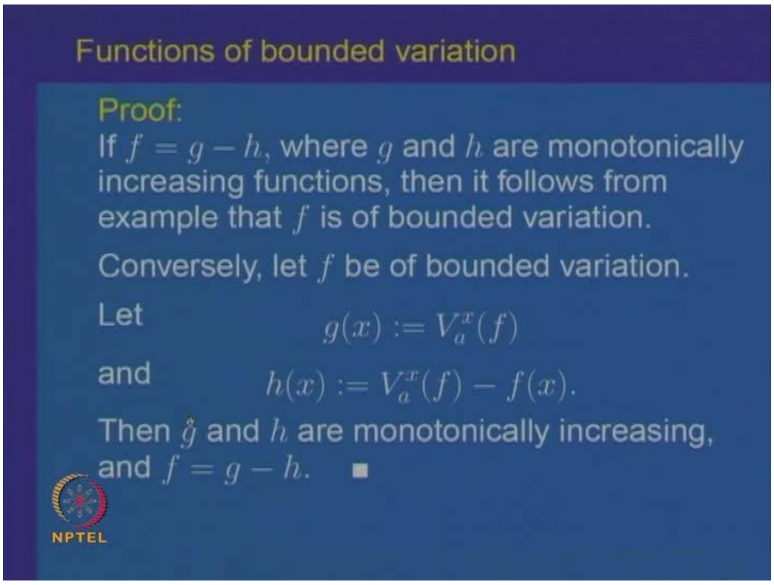

The converse is also easy to prove. We want to prove the converse namely - if f is of bounded variation, then it is a difference of two monotone functions. That is easy; because we can define g of x to be equal to the variation of the function on the interval a to x. Just now we observed that the function V a x - the variation over a to x minus f of x is a increasing function. If we take the difference of these functions g and h that is precisely f, every function f can be written as a difference of two monotonically increasing functions. That gives an important theorem of characterization of functions of bounded variation namely a function is of bounded variation if and only if it is a difference of two monotone functions.

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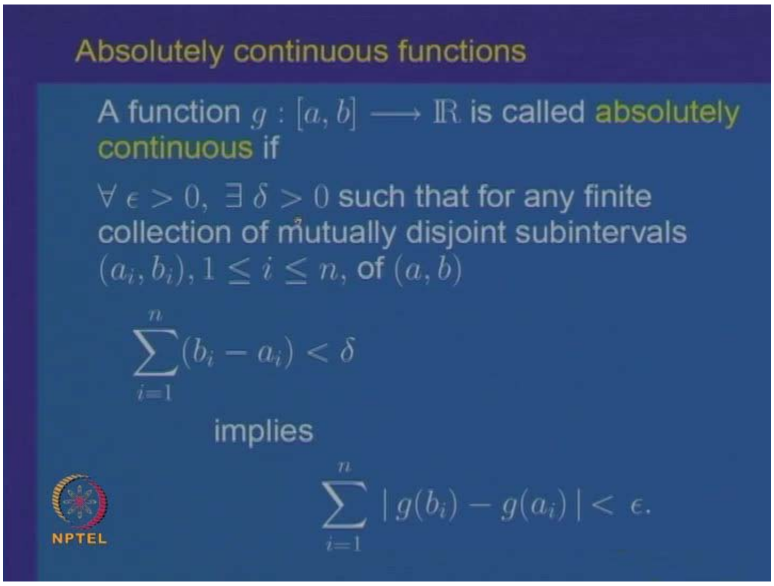

Here is an important theorem due to Lebesgue that every monotone function is a different function which is differentiable almost everywhere. Let us just recall that in your courses and analysis, you must have seen this fact that every monotone function is a function which is continuous everywhere except at countable number of points- meaning that a monotone function can have discontinuities which are at the most countably many and you know that a countable set is a set of measure 0 - is a null set.

So, one can say that elementary property about monotone functions - every monotone function is continuous almost everywhere. This is a fire reaching generalization of this fact that - not only every monotone function is a function which is continuous almost everywhere; in fact, one proves that every monotone function is differentiable almost everywhere.

The proof of this theorem is quite technical. We will not be proving this theorem. We will assume this theorem for our discussion today. Those who are interested in looking at the proof of this can look up the proof in the text book. It is given all details there. The theorem says that every monotone function is a function which is differentiable almost everywhere. So, to analyze the properties of that indefinite integral, let us define another class of functions which are called absolutely continuous functions.

So, we say a function g is absolutely continuous if for every epsilon bigger than 0, there is a delta bigger than 0 - such that whenever any finite number of collections of mutually disjoint subintervals a i b i are given in a b such that the total length of these intervals is less than delta, then, the total variation of g over the end points of these intervals -namely summation of g b i minus g of a i is less than epsilon.

This definition looks very much like continuity; it is very much similar to absolute continuity. So, if we take when i is equal to 1, whenever two points x and y a and b a i b i are close, then, it will imply f of g of b i minus g of a i is close. So, it is a generalization of the notion of uniform continuity. Let us look at once again. It says that - for any given epsilon, there must exist a delta such that whenever there are disjoint intervals a i b i of total length less than delta, then, the variation g b i minus a i summation 1 to n is less than epsilon.

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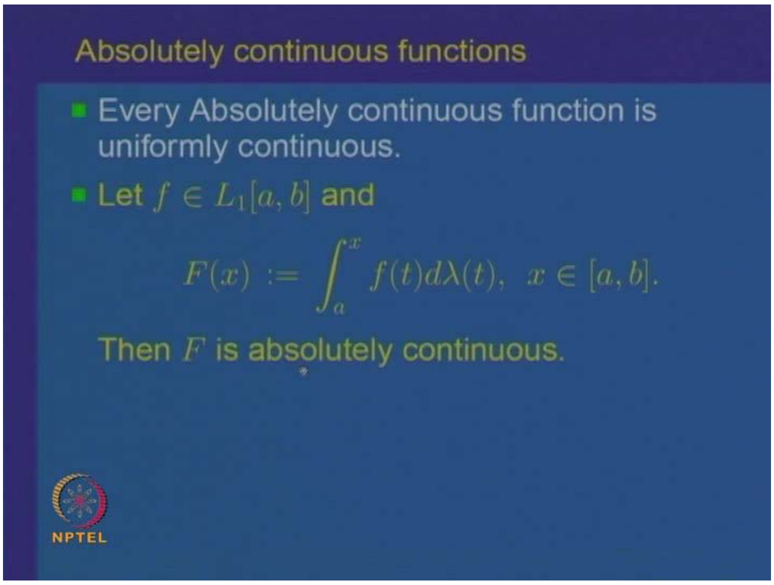

So this property or function is said to be absolute continuity of the function. Clearly every function absolutely continuous function is also uniformly continuous.

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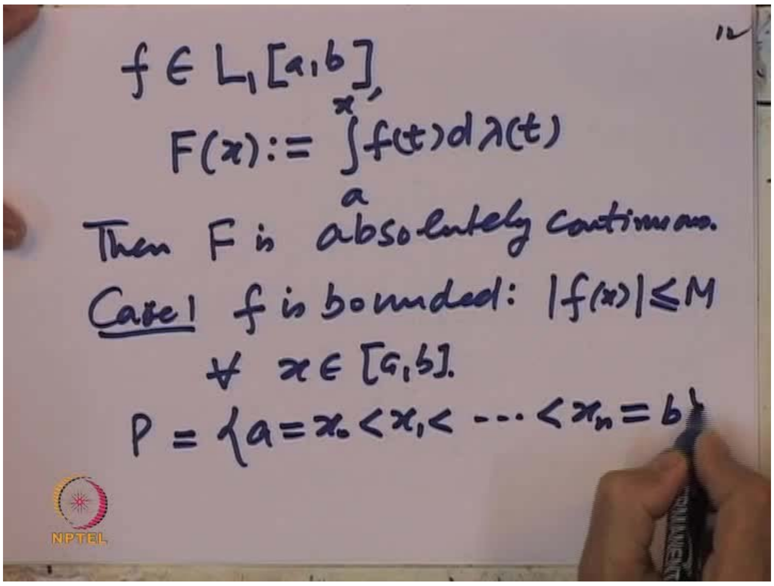

Let us look at the indefinite integral of a Lebesgue integrable function on a to b f x is equal to a to x f t d lambda t. We want to prove that this function is absolutely continuous. So, let us prove the fact that - if f is L 1 of a b and we define capital F of x a to x f t d lambda t, then, this function capital F is absolutely continuous.

So, to prove this, we will do it in two steps. Let us first assume the case, case 1- namely f is a bounded function. Let us say, mod of f of x is less than or equal to M for every x belonging to [a b]. Let us take a partition P of a b, so, a equal to x 0 less than x 1 less than x n equal to b. Then, let us observe - what is absolute value of F of x i minus F of x i minus 1. That is equal to absolute value of integral a to x i f d lambda minus integral a to x i minus 1 f d lambda

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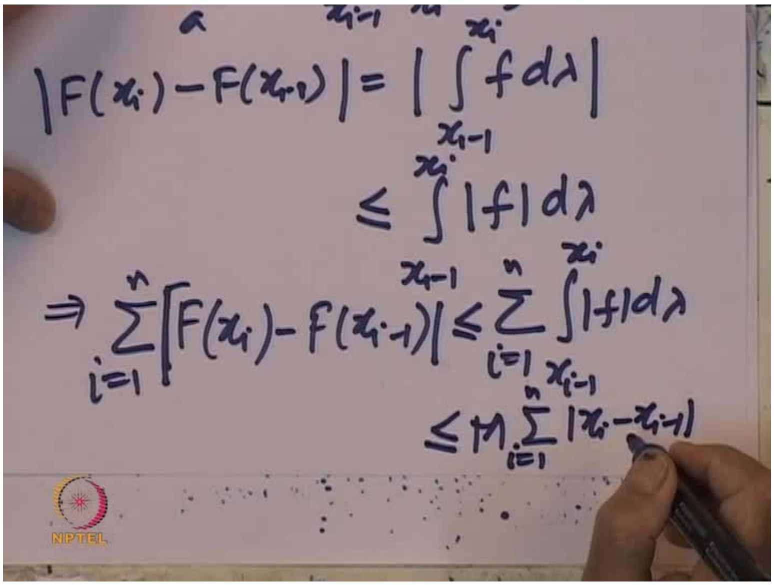

Let us note that this right hand side - this integral a to x i minus a to x i minus 1.This is x i minus 1 and this is x i and this is b. So, integral a to x i minus the integral a to x i minus 1 is nothing but the integral over this portion. So, this we can write – absolute value of F of x i minus F of x i minus 1. We can write this is equal to absolute value of integral - x i minus 1 to x i f d lambda.

Now, using the fact that absolute value of the integral is less than or equal to integral of the absolute value, we can write this as integral over x i minus 1 to x i mod of f d lambda. So, for every I, this implies that summation i equal to 1 to n of f x i minus f of x i minus 1 absolute value will be less than or equal to summation i equal to 1 to n integral over x i minus 1 to x i mod f d lambda.

This is for any points x i y i - will have this property and now because f is bounded, so, we can say this is also less than or equal to mod f is less than or equal to M; so, m times sigma i equal to 1 to n mod x i minus x i minus 1.

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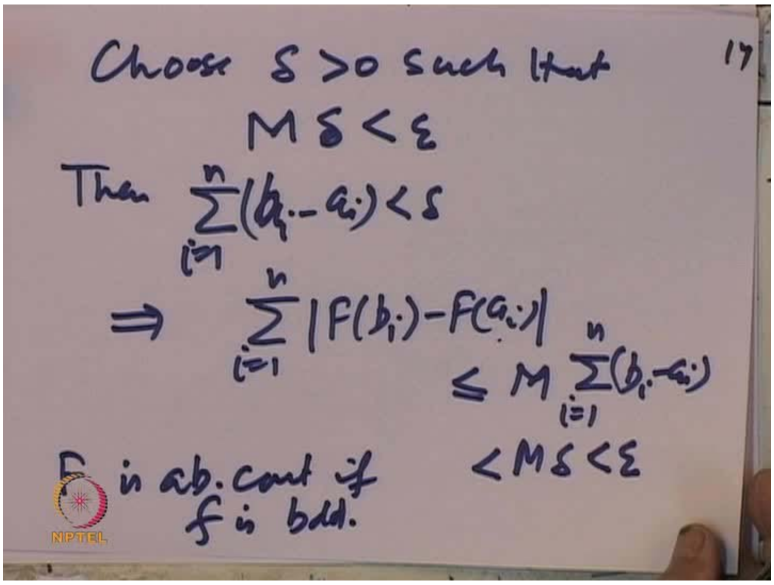

So, for any set of points, we get this property. As a consequence, let us choose delta bigger than 0 such that M times delta is less than epsilon. So, for a given epsilon, we will select delta such that this is true. Then, sigma of b i minus a i if less than delta will imply that - just now we have showed that i equal to 1 to n mod of F of b i minus F of a i will be less than or equal to M times summation i equal to 1 to n b i minus a i which is less than M times delta less than epsilon.

Whenever we have intervals a i b i say that sigma of the lengths of the intervals is less than delta, the variation at the endpoints of this intervals F b i minus F a i is less than epsilon. So, that will show that F is absolutely continuous if small f is bounded, so that is case one.

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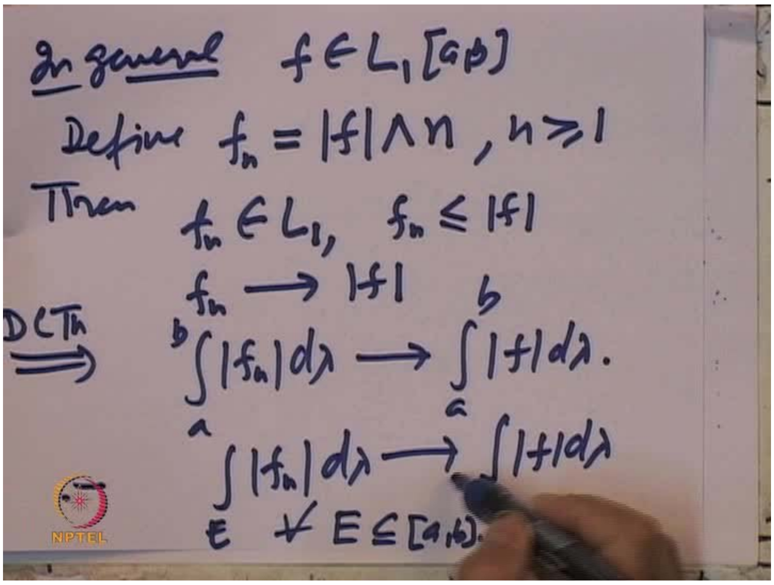

Let us look at the second general case. In general, when f is in L 1 of [a b] we can approximate it by bounded function. Let us define f n as the minimum value of mod f and the number n. So, whenever the graph of mod f goes above M, we take the value as n- cutting the graph at the line equal to n. Then, this is for every n bigger than or equal to 1.

Then, each f n belongs to L 1. Each f n is less than or equal to mod f and f n converges to mod f. These are all easy to verify because n increases, so, f ns will converge to mod f. This implies - by dominated convergence theorem that integral of mod f n d lambda a to b converges to integral a to b of mod f d lambda.

We can even write this happens not only over the whole of it; in fact this is also true that integral over E of mod f n d lambda converges to integral over E of mod f d lambda for every set E Lebesgue measurable set inside the interval a to b.

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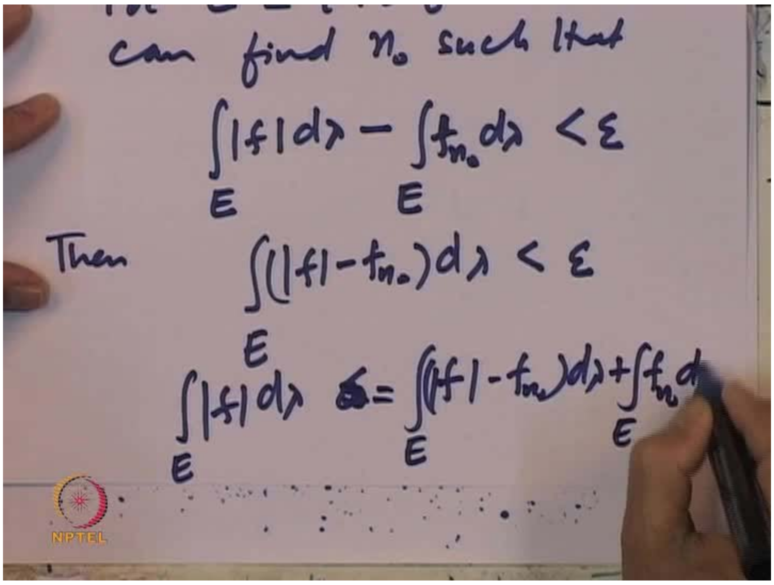

We will use this fact and note that each f n we have defined is a bounded measurable function; so, using this fact, let us fix. So, for E contained in a b fixed, we can find n naught such that integral of mod f over E d lambda minus integral over E of f n naught d lambda is less than epsilon. So, for a given epsilon we can find n naught and say that this is true.

So, then that is same as saying that integral over E mod f minus f n naught d lambda is less than epsilon. Let us analyze the integral over E of mod f d lambda; that is equal to - we can write it as integral of mod f minus f n naught d lambda over E plus integral over E of f n naught d lambda.

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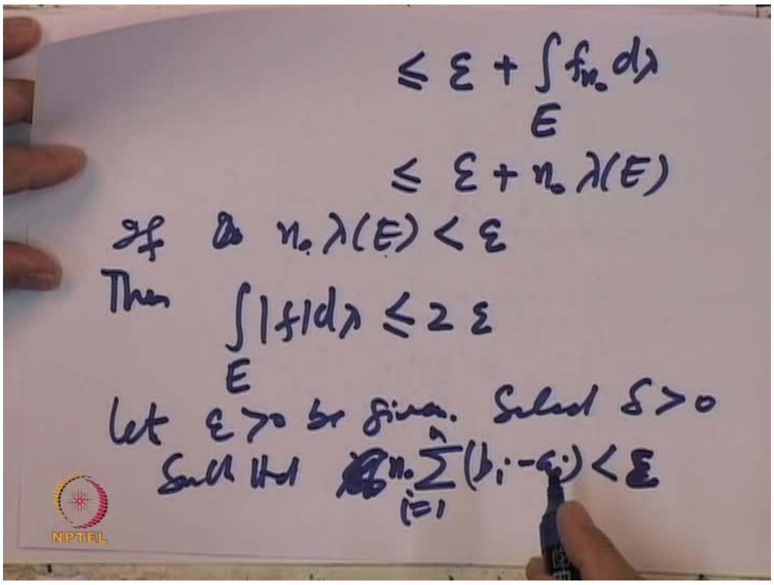

No, let us observe the first integral is less than epsilon so it is less than epsilon plus integral over E of f n naught d lambda and f n naught is a bounded function. Actually f n naught is bounded by the number n naught so that says that this is less than or equal to epsilon plus n naught times lambda of E.

So, what does it the mean? This means that if n naught lambda of E is less than epsilon then integral over E mod f d lambda will be less than 2 epsilon. This is for any set E. So, now we will apply it to our case. Let us take - let epsilon be given, select delta bigger than 0 such that summation b i minus a i i equal to 1 to n is less than delta.

Let us select delta bigger than 0 so that this lambda naught of E - let us such that this is less than n naught of this is less than epsilon. So, what we are saying is in a sense that take the set E to be the finite disjoint union of those intervals a i b i to be less than epsilon. Then what will happen? Whenever these intervals are disjoint and their total length is less than less than delta, then, we will have that integral of mod f will be less than epsilon.

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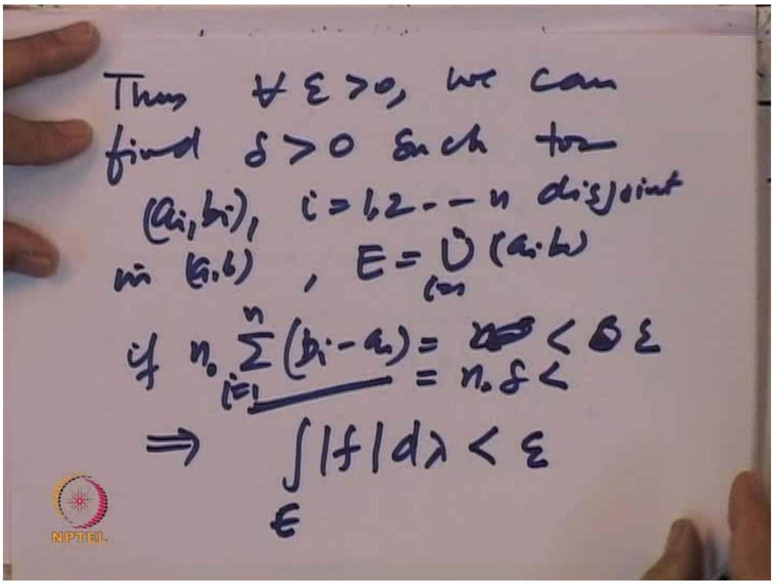

So, this will give us the following fact namely - for every epsilon bigger than 0, we can find delta bigger than 0 for intervals a i b i i equal to 1 2 up to n disjoint in (a b) if we take E to be that set union of a i b i over i equal to 1 to n, then, that will give us say for disjoint intervals - if the sigma b i minus a i n naught is less than epsilon. That will imply integral of mod f over E d lambda is less than epsilon.

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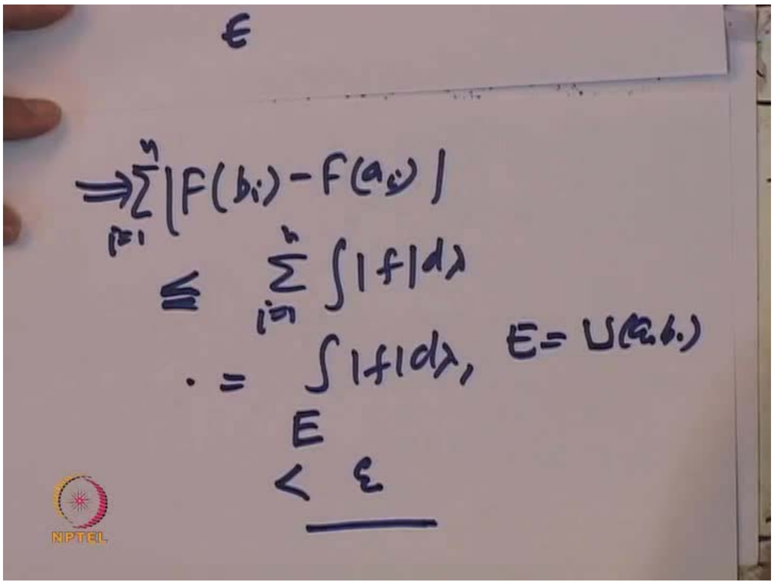

This will give us for the unbounded case - that will imply that if we look at F of b i minus F of a i summation i equal to 1 to n that we know is less than or equal to summation i equal to 1 to n integral over mod f d lambda. So, which is nothing but integral over the set E mod f d lambda where E is the disjoint union of a i b i and that is less than epsilon so that will prove that the function is absolutely continuous.So, this will prove in both cases that the function capital f is absolutely continuous.

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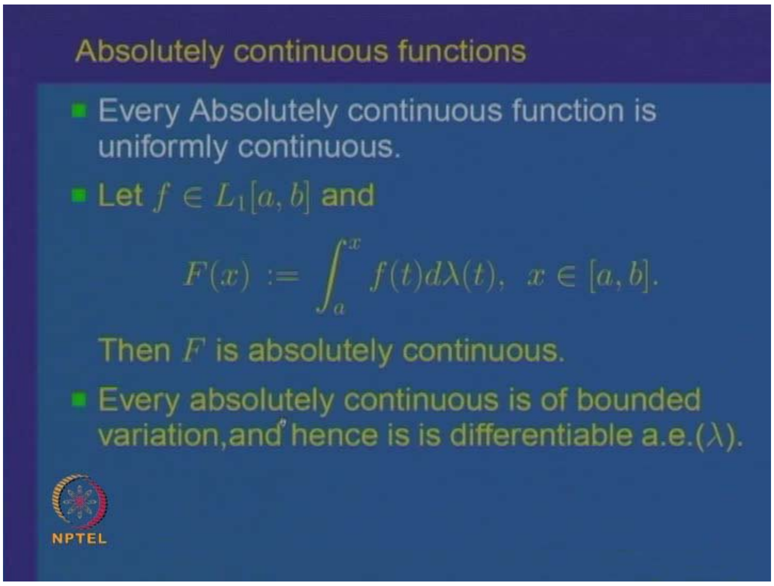

This is an important example of absolutely continuous functions that if a function f is in L 1, then, f of x the indefinite integral is absolutely continuous and every absolutely continuous function is a function of bounded variation. This is another obvious fact that every absolutely continuous function is a function of a bounded variation.

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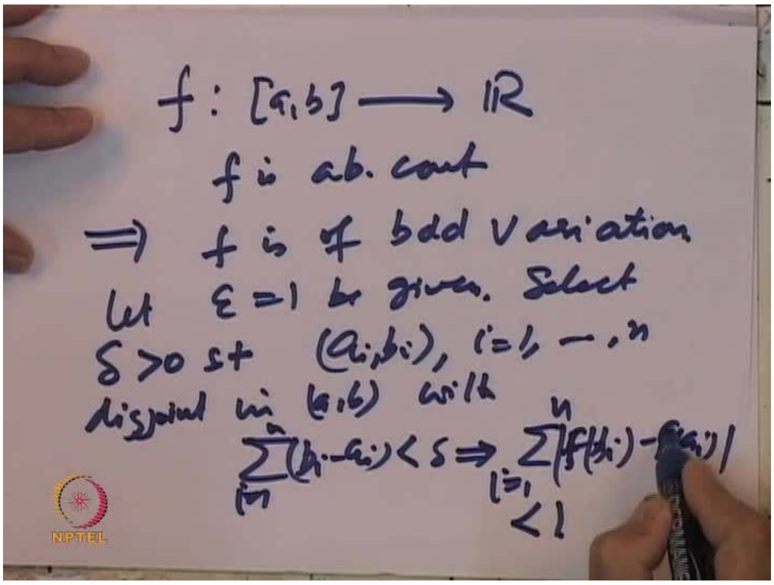

So, let us prove that. If f is [a b] to R and f is absolutely continuous that implies f is of bounded variation. So, let epsilon equal to 1 be given, select delta bigger than 0 such that whenever a i b are disjoint, i equal to 1 to n - disjoint in a b with sigma b i minus a i 1 to n less than delta that will imply that the variation F of b i minus F of a i i equal to 1 to n is less than 1.

Essentially that means that here is the interval a to b, so, whenever we have disjoint intervals in a length at the most delta - then the variation is less than 1. So, let us divide this interval; let P equal to a equal to x 1 x n equal to b - b - a partition of a b such that x i minus x i minus 1 - the difference is less than or strictly less than delta.

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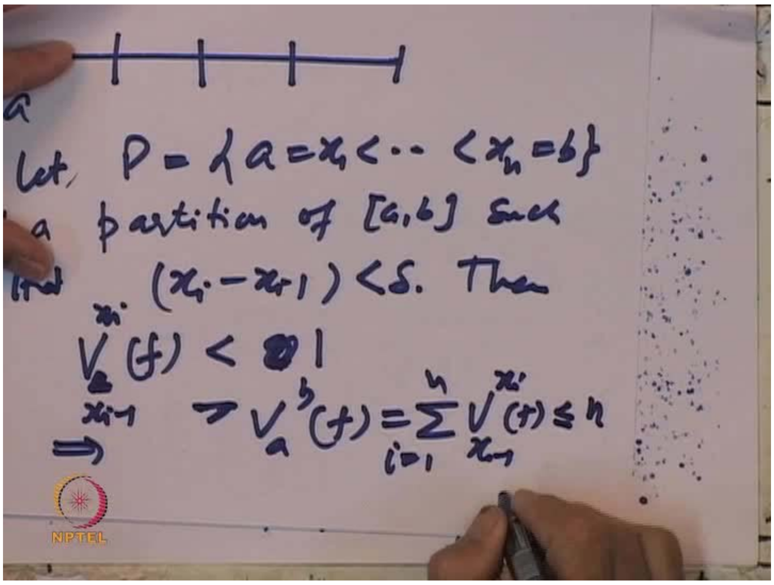

Then, by absolute continuity, the variation of the function f in x i minus 1 to x i is less than epsilon. Wherever we take any partition of this any sub interval then with the variation inside that -by absolute continuity- is less than is less than 1 - epsilon equal to 1. So, that implies that the variation over a to b of f which is summation i equal to 1 to n variation x i minus 1 to x i of f which will be less than or equal to - if these intervals are n – so, that will be less than n which is finite.

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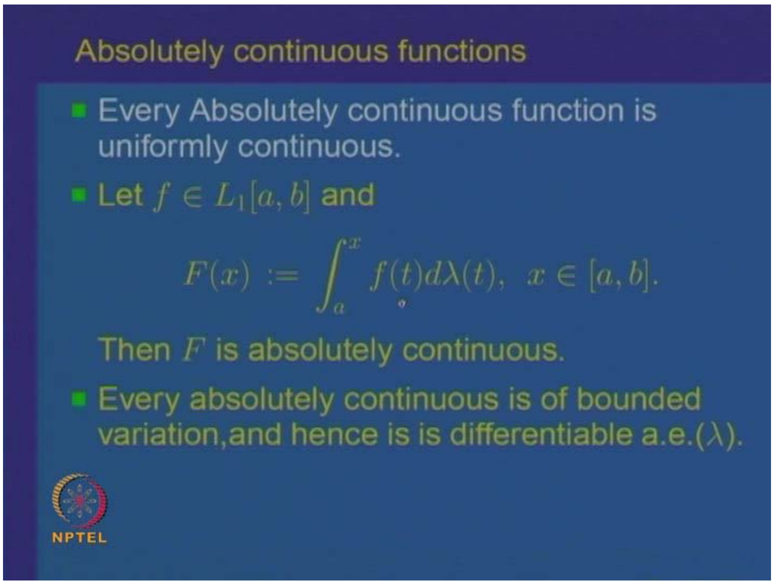

So, that will prove that every function which is absolutely continuous is also of bounded variation and every function of bounded variation is a difference of two monotone functions and every monotone function is differentiable almost everywhere. So, we will get that this function f which is an indefinite integral is also differentiable almost. This is the beginning of the fundamental theorem of calculus for Lebesgue integrals.

Today, we have looked at the property namely - if f is a integrable function and look at its indefinite integral capital F of x, then, we have shown this function is absolutely continuous function and hence it is a function of bounded variation and as a consequence of bounded variation is a difference of two monotone functions and hence differentiable almost everywhere.

So this is the first part of the fundamental theorem of calculus saying that the indefinite integral is differentiable almost everywhere. We will continue this in the next lecture and we will show in fact that not only the indefinite integral is differentiable almost everywhere, the derivative is actually equal to the function - the integrant namely the function small f almost everywhere. So, that will be the first part of the fundamental theorem of calculus. Thank you.