Measure and Integration

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Module No. # 09

Lecture No. # 35

L_2 (X, S, μ)

Welcome to lecture 35 on Measure and Integration. In the previous lectures, we had been looking at the p f power integrable functions on a measure space X, S, mu and we have studied some general properties of this function spaces. Today, we will look slightly more in detail the special subspace namely, when p is equal to 2.

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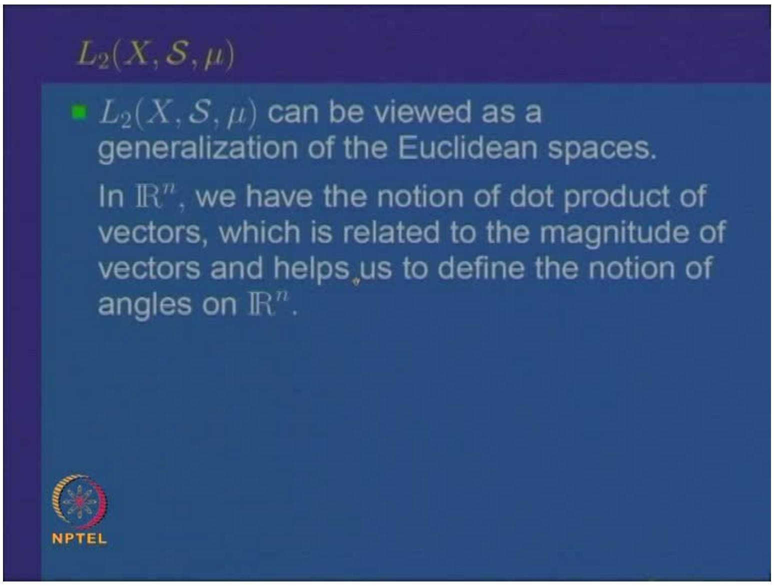

So, the topic for today's discussion would be space L 2 X, S, mu the space of complex valued square integrable functions. This is a special space in the sense that this can be viewed as a generalization of the Euclidean spaces. Recall in IR n, the Euclidean space we have the notion of dot product of vectors which is related to the magnitude of the vectors and also help us to define the notion of angles on R n.

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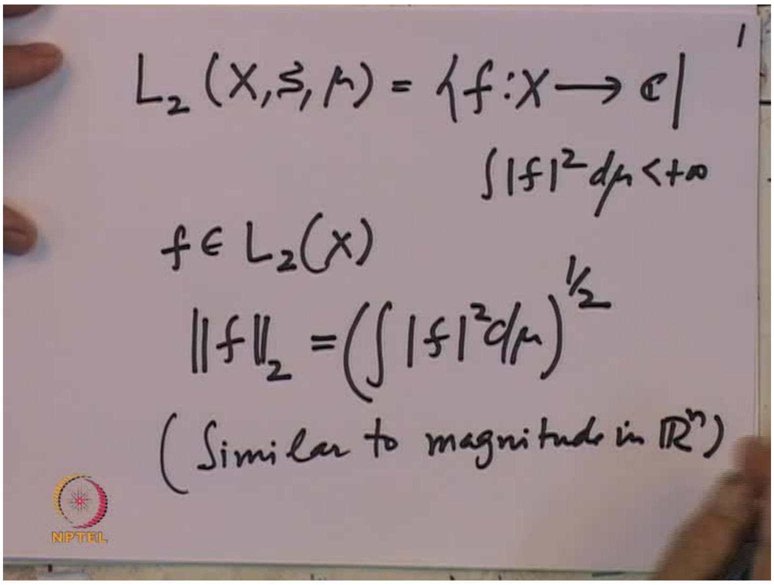

Let us see, what we can do as far as L 2 is concerned. So, we look at the space L 2 of X, S, mu to be the space of all functions, which are defined on X which are complex valued, such that mod of f square d mu is finite. For such spaces, for a function f in L 2 of X, we have defined the norm of this which is equal to integral of mod f square d mu raised to power 1 by 2. We said that this is very much like the magnitude in R n, so this is similar to magnitude in R n.

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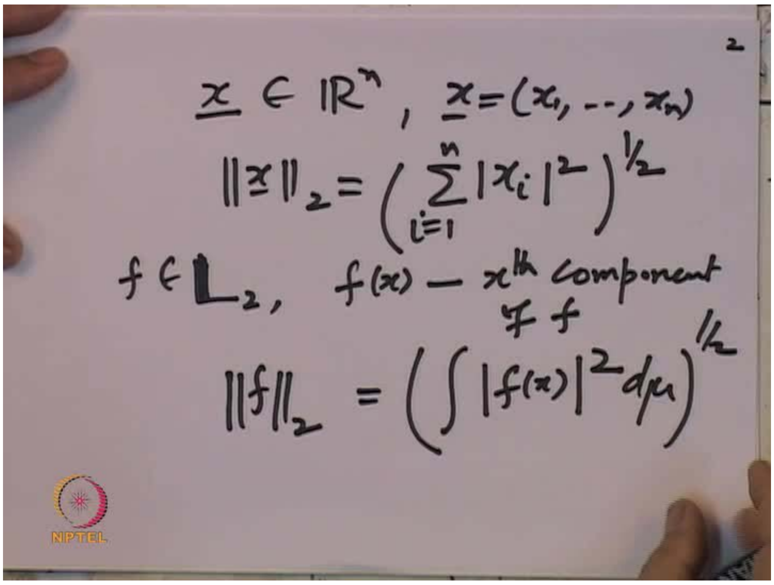

Let us just briefly recall what the magnitude in IR n was. So, for a vector say x in IR n the Euclidean norm is defined as - look at the components - if x is having components x 1 to x n then look at mod of x i square sigma 1 to n and the square root of that and for a function f in L 2; let us regard f as a vector with f x as the component, so this is the x th component of f.

How would we define the norm? Look at the x th component f x square, so this is very much similar to what we have done for Euclidean look at the component square and now sum it up; but here, the summation is over x and any x belonging to x any indexing set. So, it is nothing but integral d mu and then the power 1 by 2. In that sense this is a perfect generalization of the Euclidean norm to arbitrary spaces.

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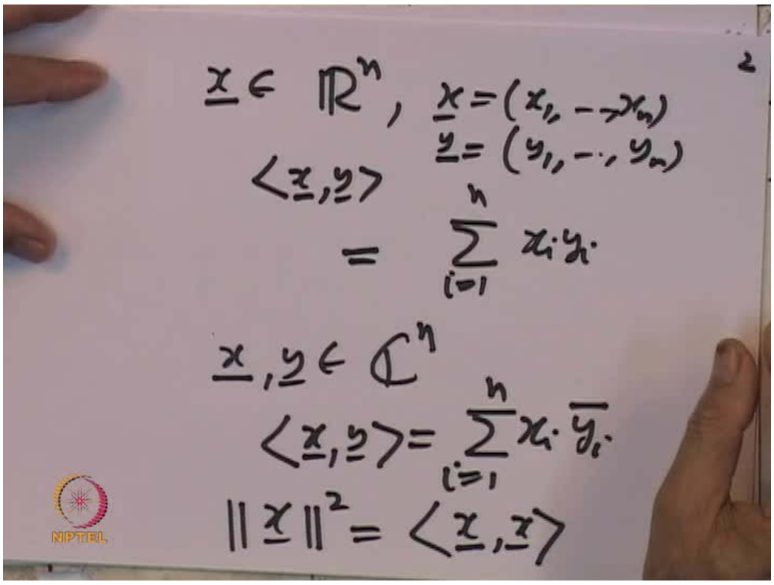

Now on L 2, for a vector x belonging to IR n with components x n, we have the notion of what is called the dot product; so dot product if x and y are two vectors with y components as y n, then x comma y the dot product is defined as x i y i and i equal to 1 to n.

If we are in complex plane, if x and y belong to C n then, the dot product x y is defined as sigma x i y i bar, i equal to 1 to n. This notion of dot product is related to the magnitude in the following way that in either R n or C n either one, the norm of x square is equal to the dot product of x with itself and we know that is the dot product in R n or C n gives the notion of the angle and orthogonality which helps us to do geometry in R n.

The basic idea of today's lecture would be on the space L 2 of x, we already have the notion of norm; the notion of distance. We will define the notion of inner product or the notion of dot product on L 2 of x and show how it is related to the notion of distance. So that helps us to define the notion of orthogonality for perpendicularity of two elements in L 2. So, we can do geometry in L 2 of X S mu.

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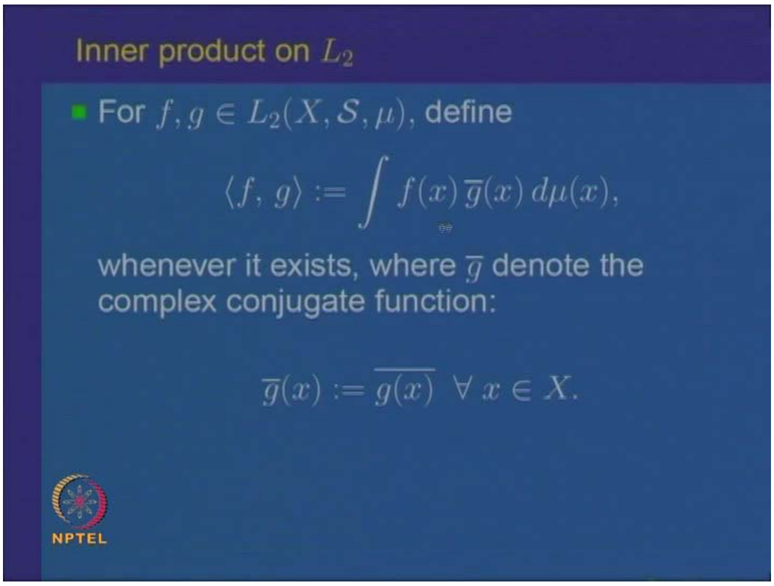

Let us define, what is the notion of the dot product in L 2? For functions f and g in L 2, keep in mind our spaces are complex valued, so define the dot product of f with g are also called the inner product of f with g as integral of f x g bar x d mu x, where this g bar is the complex conjugate of the function g, so g bar of x is g x bar. The inner product or the dot product between f and g; for two functions f and g is defined as the integral of f x g x bar d mu x, which is perfectly similar to complex C to the power n a i b i bar.

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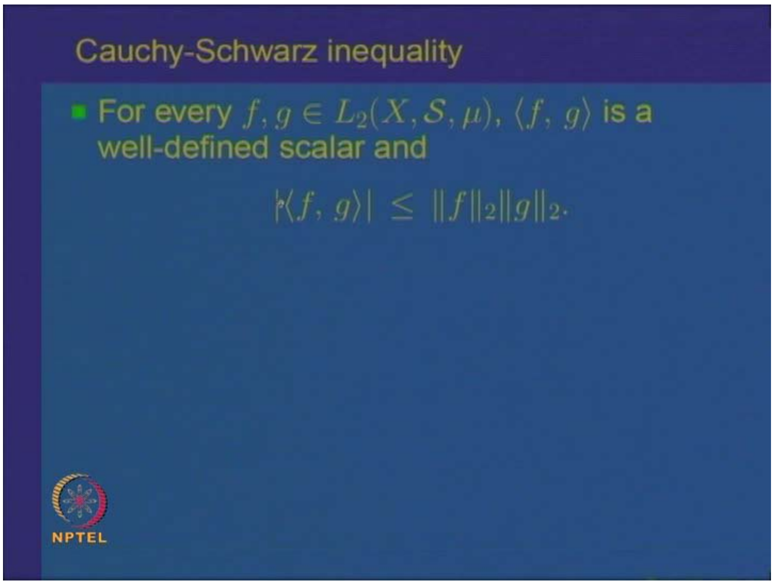

The first thing we want to say that this is well defined and that follows from the holders inequality that we had proved - recall holders inequality - it is said that if f is a function in L p and g is a function in L q then, f into g is function which is integrable and the integral of f g is less than or equal to the L 2 norm. So, let me just say it once again, what we are meaning here - that should be integral.

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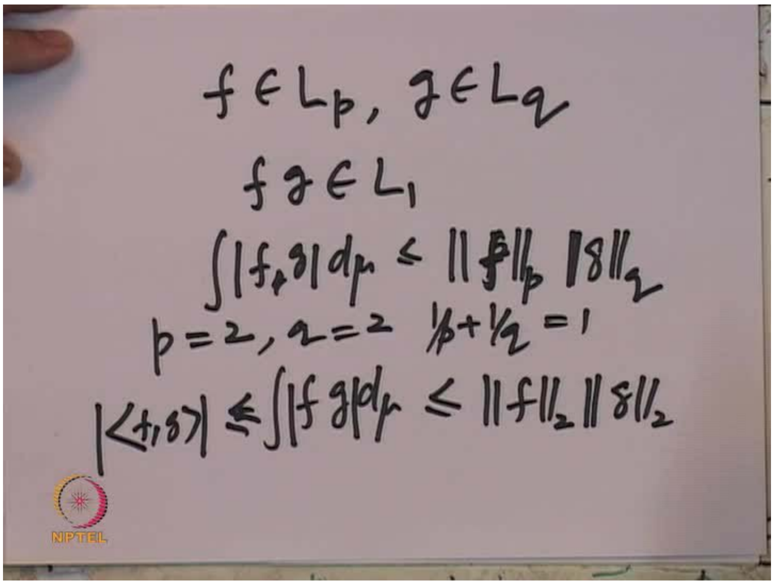

For holder inequality, we had that if f belongs to L p and g belongs to L q then, fg belongs to L 1 and integral of mod of f into g d mu is less than or equal to the p th norm of f and the q th of norm of g. For p equal to 2, so that will give us that integral of f g d mu and p is equal to 2, q is equal to 2, so we have got that 1 over p plus 1 over q is equal to 1. So, the holders inequality will give us that this is less than or equal to L 2 norm of f into L 2 norm of g. This is bigger than or equal to f g inner product, is less than or equal to this, so that gives us the Cauchy-Schwarz inequality.

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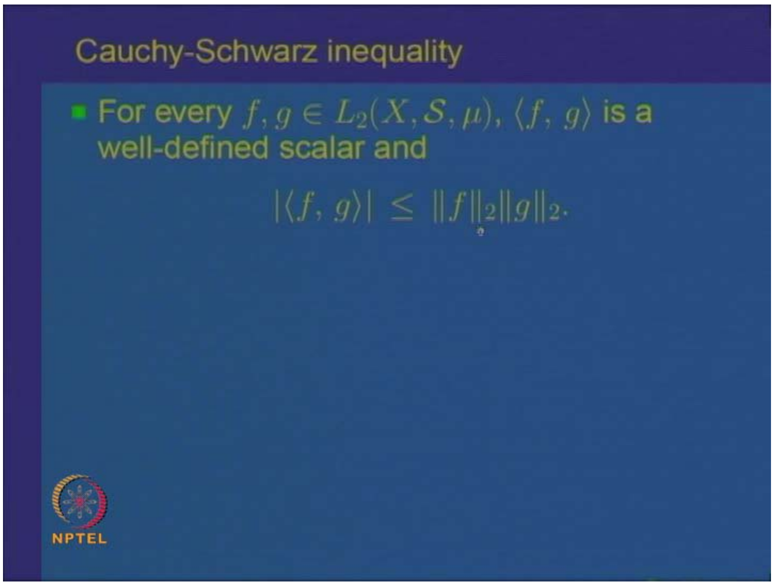

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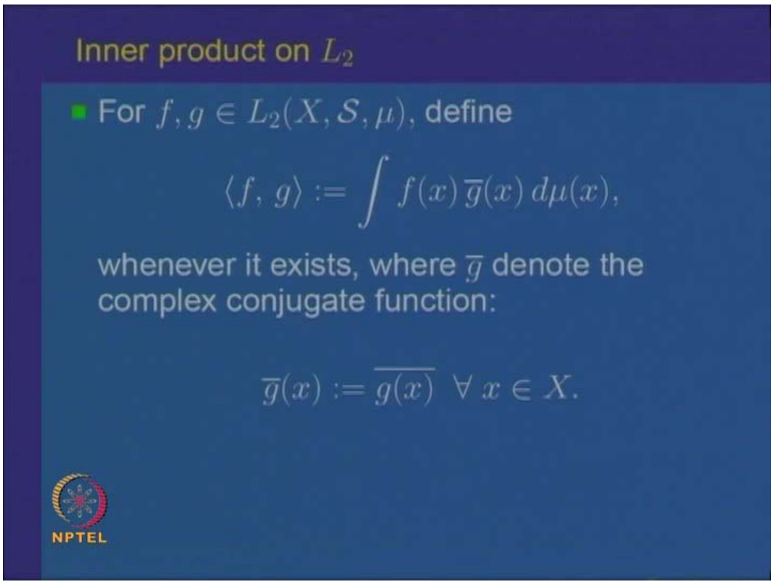

So, this is also called the Cauchy-Schwarz inequality namely, the absolute value of the inner product between f and g is less than or equal to the norm of f into norm of g. So that says that this inner product is well-defined; it is a well-defined in quantity.

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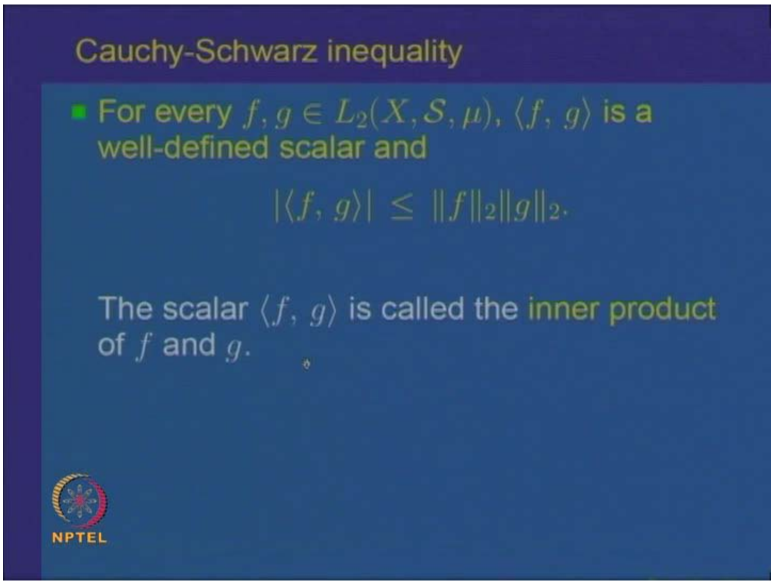

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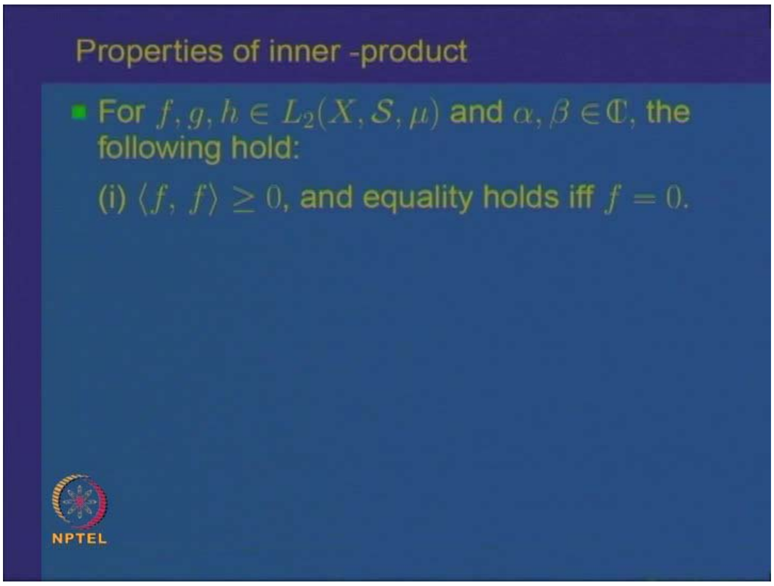

For every f and g in L 2, we have the notion of the inner product, so f comma g which is the inner product is well defined quantity. This behaves perfectly similar to that of the inner product for ordinary vectors in R n or C n that means, this is a function which is defined on L 2 cross L 2. It has the following properties namely the inner product of f with f is always bigger than or equal to 0 and the equality holds, if and only if, f is equal to 0.

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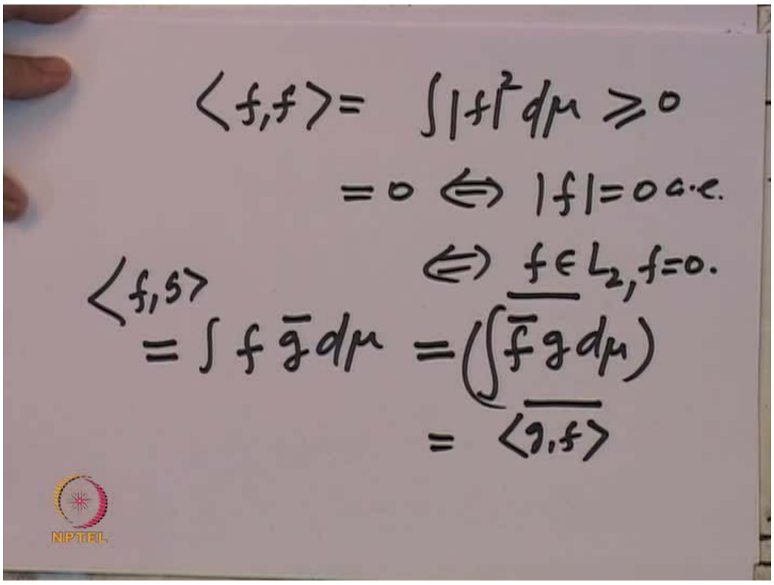

Let us look at this property that how is this true. So, inner product of f with itself is nothing but integral of mod f square d mu. This is always bigger than or equal to 0 and this will be equal to 0, if and only if, our function mod f is equal to 0 almost everywhere. So, if and only if, f is belonging to L 2 as I treated as element of L 2 f is equal to 0.

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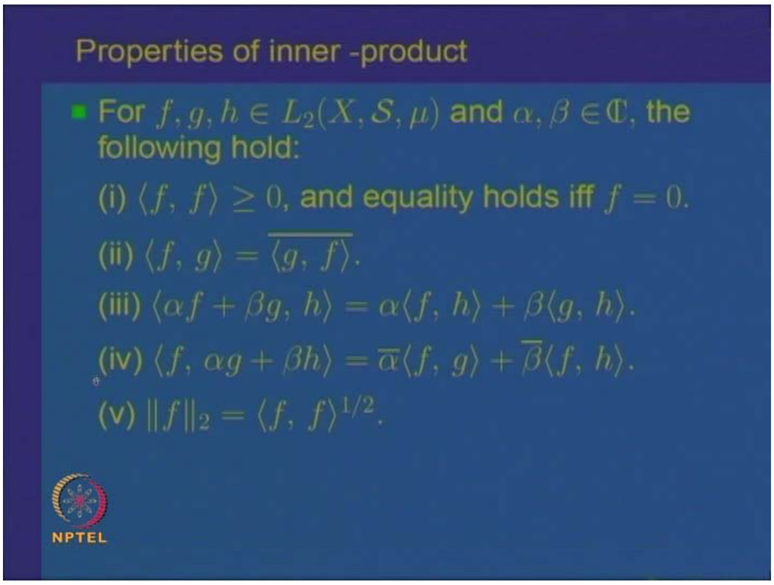

The first property is obvious; namely, the dot product of f with itself is always bigger than or equal to 0. The second property says, the dot product or the inner product of f with g is same as the inner product of g with f, so we are interchanging f comma g and the complex conjugant of it.

The inner product of f with g is same as the complex conjugate of the inner product of g with f and that is quite simple to verify from the definition. So, if we have got the inner product of f with g; so inner product of f with g is equal to f g bar d mu and that is equal to f bar g integral d mu bar and this is equal to g comma f bar. So, that verifies the property namely, the inner product of f with g is equal to the inner product of g with f bar.

Similarly, it is easy to verify using that the integral is linear. It is easy to verify that the inner product is linear in the first variable that means, alpha f plus beta g inner product with h is equal to alpha times the inner product of f with h plus beta times the inner product of g with h.

Similarly, in the second variable is this conjugate linear because of the property of 2. So, f inner product with alpha g plus beta h is same as alpha bar of f g inner product f g plus beta bar of inner product of f h. In the second variable, it is not linear, it is a conjugate linear. Finally, this property is obvious namely, the L 2 norm of f is square root of the inner product of f with itself.

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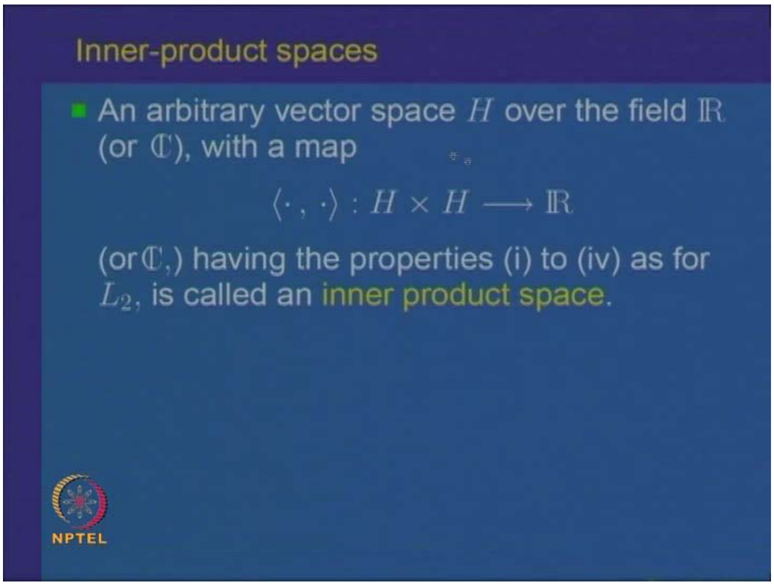

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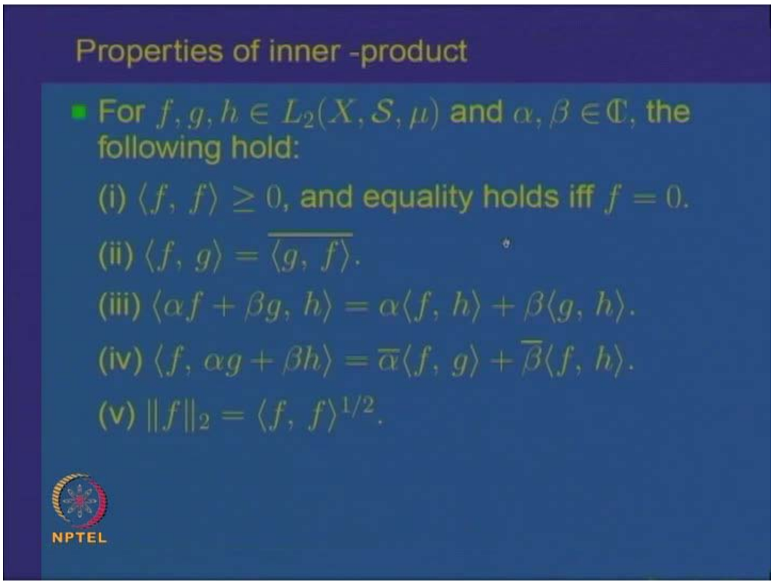

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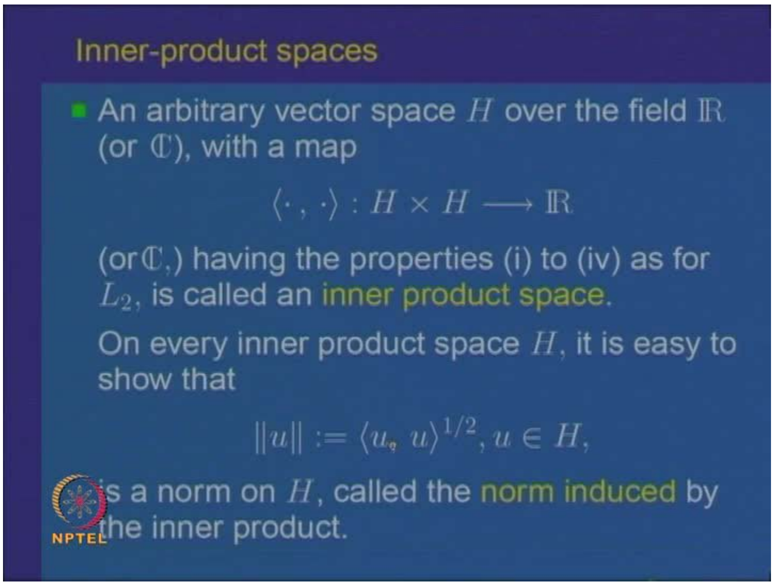

All the properties that we have for the dot product in R n or C n are defined for the inner product in L 2. This is not very special for L 2 in fact, one can look at any vector space H over the field of real or complex numbers. If one has a function which is defined on H cross H taking values in the underlying field of real or complex having properties similar to that I, II, III, IV, V of L 2, so one can define what is called an inner product space.

In general an inner product space is defined to be a vector space H on which there is a notion of inner product defined. What is an inner product? It is a function defined on H cross H to R with those properties. Once we have a notion of inner product that gives rise to notion of magnitude, by the property that the magnitude of a vector u in H is nothing but defined as the dot product of u with square root itself.

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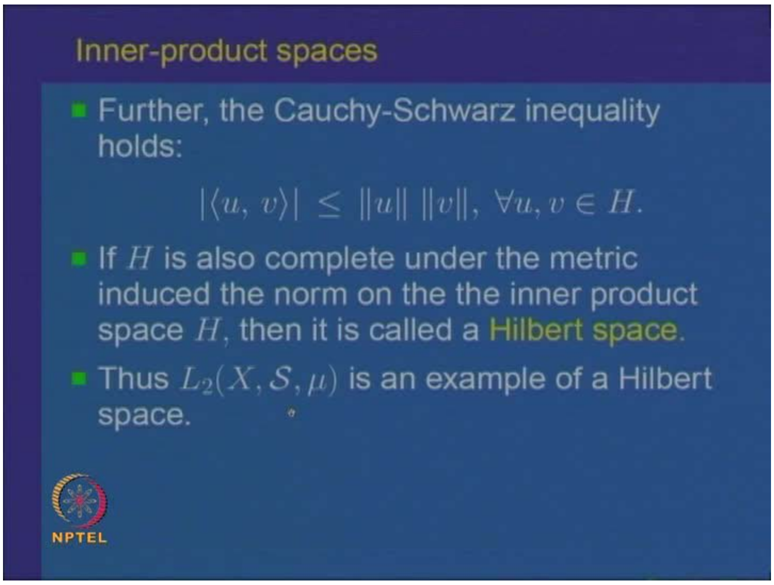

One verifies that Cauchy-Schwarz inequality holds for this kind of inner product and that means, this is a well-defined norm, so Cauchy-Schwarz inequality will say that is a norm defined on it. Once, you have the notion of norm that gives rise to a metric on the underline vector space and one can ask whether it is complete under that metric or not.

On a vector space inner part is defined, so it becomes an inner product space and inner product space gives rise to a norm and if the underlying metric induced by the norm is complete one says, H is a Hilbert space, that is a general definition of a Hilbert space.

So, our L 2 is an example of a Hilbert space, because L 2 X, S, mu is a vector space on which a notion of norm - the L 2 norm - is defined and that L 2 norm is related to the inner product, just now we have seen. We have already seen as Riesz-Fischer theorem, which said that L 2 X, S, mu is a complete metric space in the L 2 metric, so this is an example of a Hilbert space.

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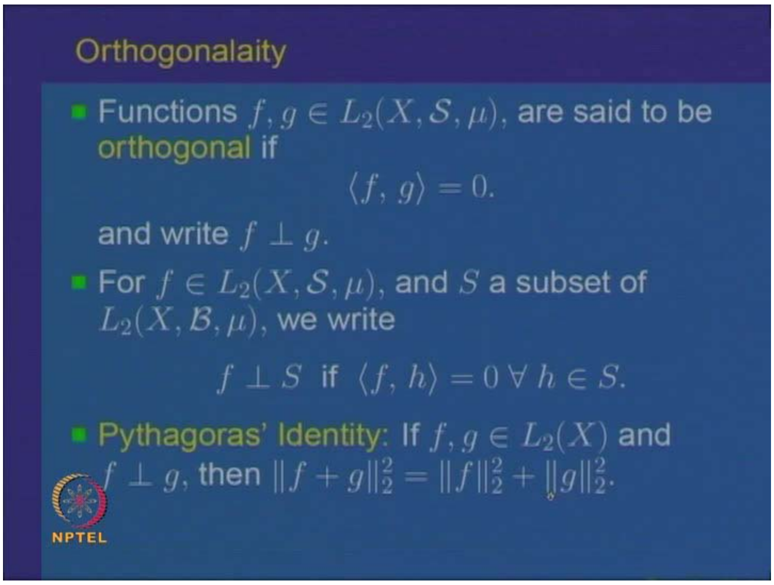

Once we have the notion of the inner product, one can define the notion of two elements in L 2 to be orthogonal or perpendicular to each other. So, we say two elements f and g in L 2 are orthogonal to each other, if the inner product between them is equal to 0 and that is what we have for vectors in R n also that the dot product is equal to 0.

We write this as f perpendicular to g; so f perpendicular to g is defined as saying that the inner product of f with g is equal to 0. We can also define the inner product if element f orthogonal to a subset S, so writing it as f orthogonal to a subset S means that f is perpendicular to every element of S.

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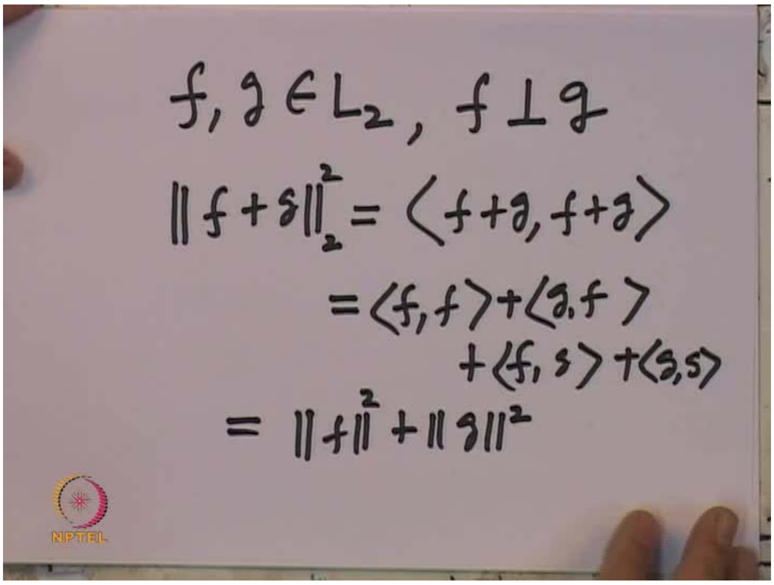

So, f is perpendicular to S will mean that f comma h, the inner product is equal to 0 for every element h in S. Similarly, we can define orthogonality of two sets also. With this one can prove what is called the Pythagoras identity namely, if f and g are two functions in L 2 and f is orthogonal to g then, the norm square of f plus g is equal to norm f square plus norm g square.

Let us just quickly verify the Pythagoras identity namely, if f and g are two elements in L 2 and f is orthogonal to g then, we want to compute the L 2 norm of square of this. By definition, this is related to the inner product, so this is inner product of f plus g with itself. Now, using the property of linearity what we will get is, this is f comma f plus g comma f plus f comma g plus the inner product of g with itself.

So, that gives you norm of f square plus norm of - the last term will give you - norm of g square, but f is orthogonal to g that means, f comma g is equal to 0 and g comma f is also equal to 0. So these two terms, the inner products of g with f and f with g both are equal to 0. So, we get what is called the Pythagoras identity namely, the norm of f plus g square is equal to norm f square plus norm g square, whenever f is orthogonal to g.

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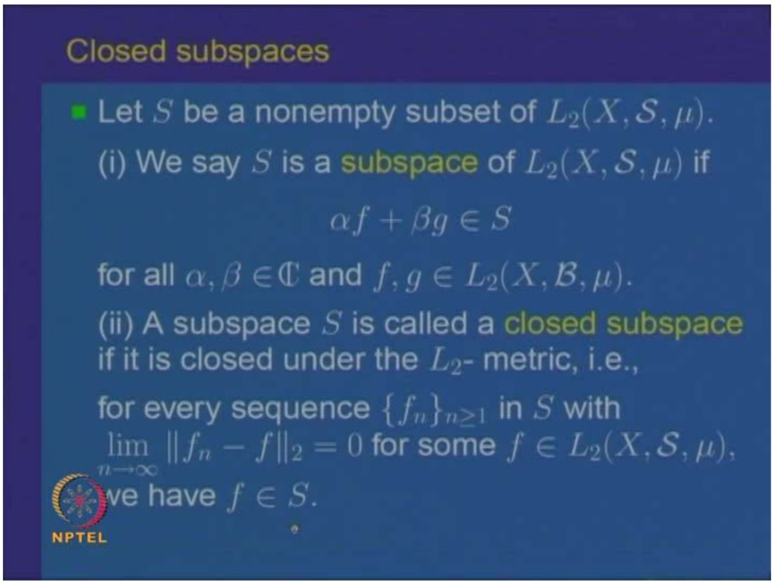

Let us carry over this idea of orthogonality a bit further; let us take S any nonempty subset of L 2. We call S a subspace of L 2, those who have done a bit of linear algebra will recognize this definition - L 2 is a vector space, so we are looking at a vector sub space of L 2. So, a set S is nonempty subset is called a subspace of L 2, if for any f and g in S, so this should be in S not in L 2 and alpha beta in complex number alpha f plus beta g belongs to S, then we say S is a subspace, so this is not L 2 it is S.

So that means, for any two elements alpha beta in S the linear combination alpha f plus beta g should be in S, in that case it is called a subspace of S. A subspace of S is called a closed subspace, if it is closed under the metric on L 2 that is L 2 metric. So, it should be a closed set that means what? That means, whenever we are get a sequence f n in S and f n converges to a function f in L 2 norm, then the limit must also be inside S. So that is what is a definition of a closed over subspace. It is a subspace and it is a closed set under the L 2 metric, so that is the notion of a closed subspace.

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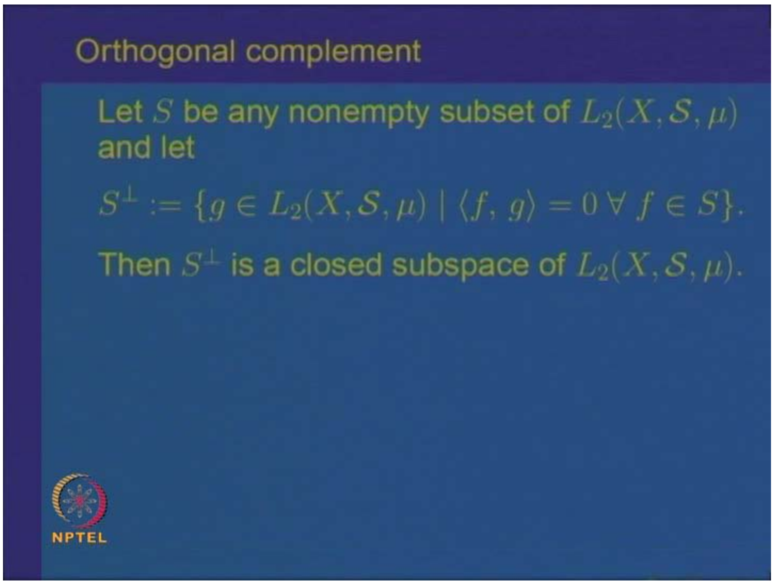

For given a set S in L 2 denoted by S upper suffix perpendicular - this is also called the orthogonal complement of S to be all elements in the space L 2 which are perpendicular - to all elements of S. Given a set S, we are looking at all elements in L 2 which are orthogonal to every element of S. So, that is called S perpendicular and this is called the orthogonal complement of S and the claim is that this is a closed subspace of L 2.

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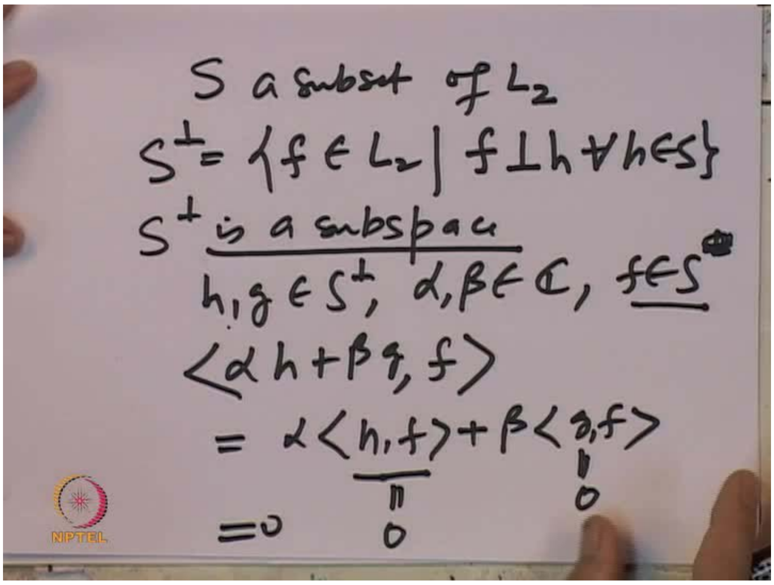

Let us verify this fact that S a subset of L 2 and S perpendicular is the set of all elements f in L 2, such that f perpendicular to h for every h in S then, claim is that first of all S perpendicular is a subspace. Let us take some h and g belonging to S particular and alpha and beta belonging to C. Then alpha h plus beta g comma - let us take an element - f, for f in S this will be equal to alpha times h f plus beta times g f using the property of linearity in the first variable for the inner product.

Now, because f belongs to S and h, g are in S perpendicular, so this quantity inner product is 0 and the second inner product is 0, so the sum this inner product is equal to 0 (Refer Slide Time: 21:42). So that says, if h and g belong to S perpendicular and alpha and beta are on the C, then alpha h plus beta g is always orthogonal to every element of S. Hence, it belongs to S perpendicular, so S perpendicular is a subspace.

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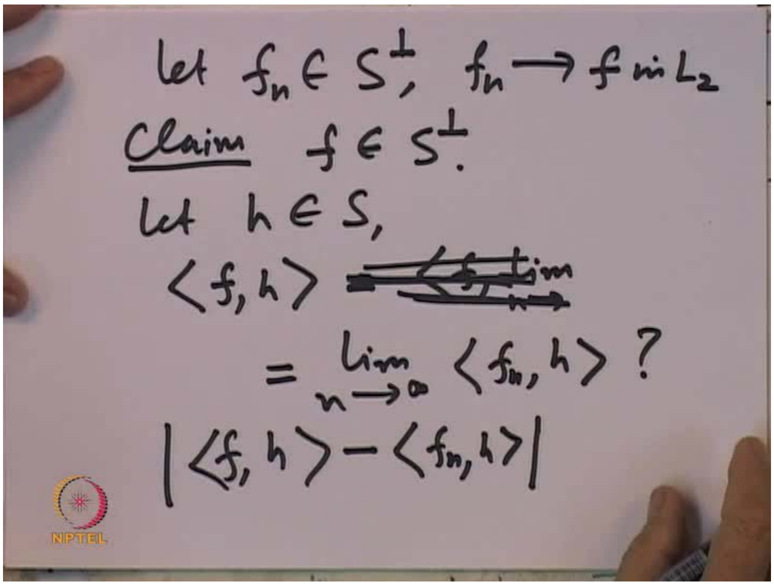

Next, let us prove that this is a closed subspace; we want to check that there is a closed subspace. So, let f n belong to S perpendicular and f n converge to f in L 2, we want to check it. So, claim that f belongs to S perpendicular, for that let us take any element h belonging to S and we want to compute f comma h and the claim is this f comma h this is equal to f comma limit n going to - so what is h? What is sorry not this is not true.

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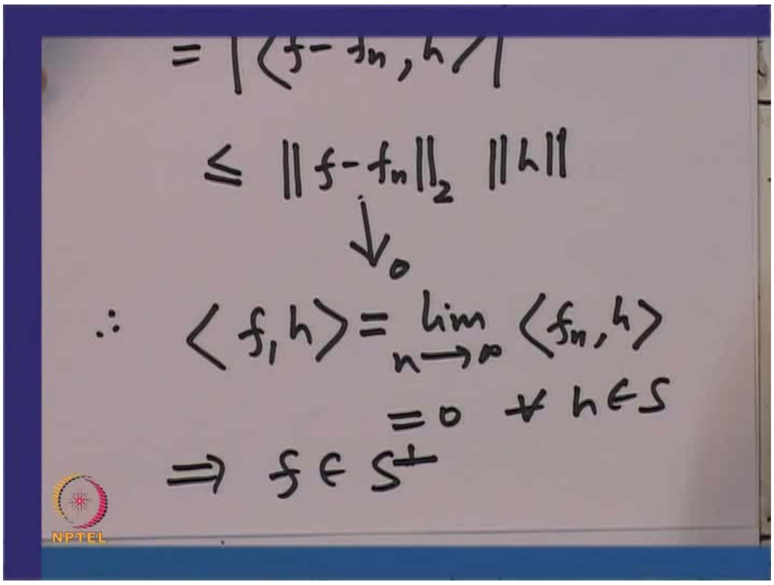

Let h belongs to S, now the claim is that since, f n converges to f in L 2, so this is equal to limit n going to infinity of f n comma h. So, this is a very simple thing to verify because, if we look at the difference of the two f and h, so why is this true? This is true because, we look this inner product of f with h and inner product of f n with h and look at the absolute value of this then, we can write this as this; so this quantity is equal to absolute value of f minus f n inner product with h. By Cauchy-Schwarz inequality, this is less than or equal L 2 norm of f minus f n and L 2 norm of h and this goes to 0.

Therefore, we get f with h inner product is equal to limit n going to infinity inner product of f n with h. Since, each f n belongs to S perpendicular h is in S, so that implies that each term is equal to 0, so this is equal to 0 for every h belonging to S. So that implies that f belongs to S perpendicular. This proves that S perpendicular for any set S, if we look at its orthogonal complement then that is a closed subspace of h, so this is also called the orthogonal complement of a set.

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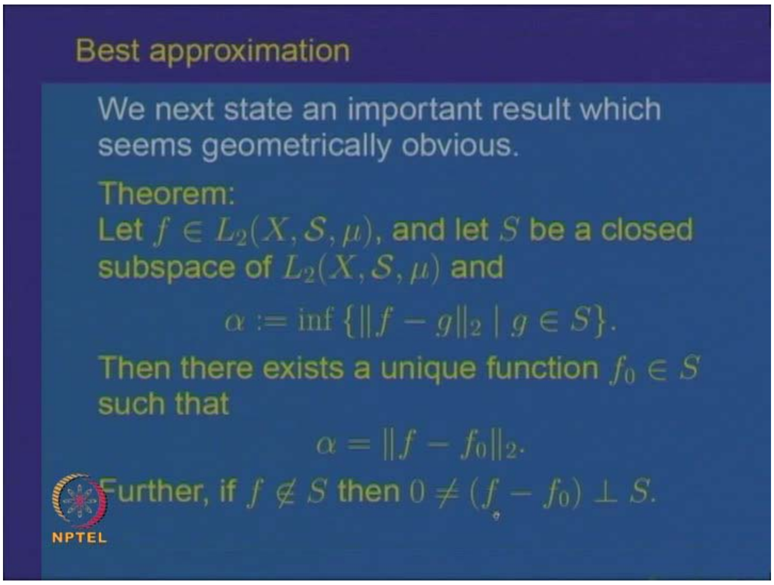

We next state an important result which seems geometrically obvious, which can be proved for any Hilbert space, so we will just look at it for our space L 2, will not prove this result, we will just assume this result the proof can be referred to the book.

The result says that if f is a L 2 function and S is a closed subspace of L 2 then, look at the number alpha which is the infimum of all the L 2 distances of f from g where g is any element in S. Then, the theorem says that this infimum is attained at some point in S, so that means there exists not only it is attain then there is a unique function f 0 belonging to S such that this infimum alpha is equal to norm of f minus f 0. Further, if f does not belong to S then, look at the difference of f minus f 0 that is always going to be perpendicular to S, so that is the claim of the theorem.

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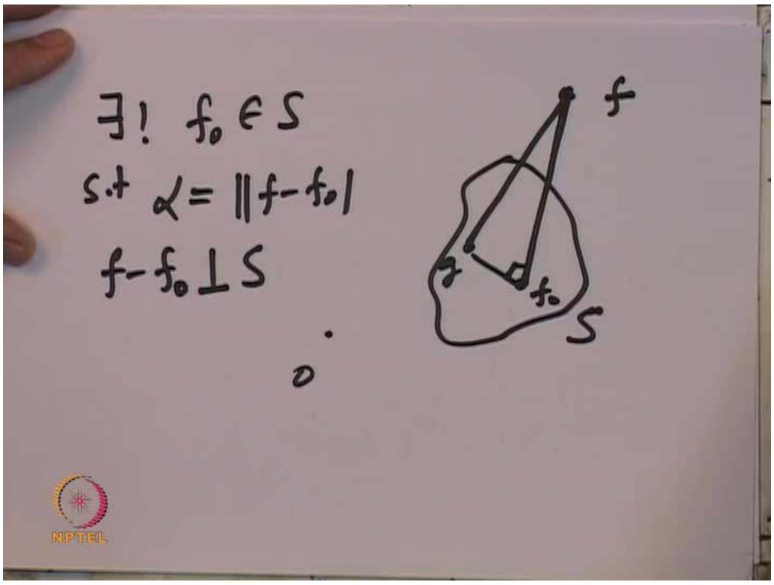

We will not prove it, we will just geometrically analyze this result a bit. So, look at the closed subspace S of L 2, this is a closed subspace of L 2 and we have got a function f, which is outside it, what we are going to do is? We are going to look at any point inside S point g and look at the L 2 distance of this, so it says that there is a value called f 0, such that L 2 distance of f from it is the minimum. If this is 0 says, if I look at f minus f 0 so that is going to be orthogonal to it, so that is the theorem.

There exist a unique point f 0 belonging to S. Such that alpha the infimum is equal to the distance of f minus f 0 and f minus f 0 is perpendicular to S. Look at this vector f minus f that is always orthogonal to this S. Geometrically in a sense given a point and given a subspace it is kind of the projection gives you the minimum. So, this is a generalization of the projection theorem for finite dimensional spaces.

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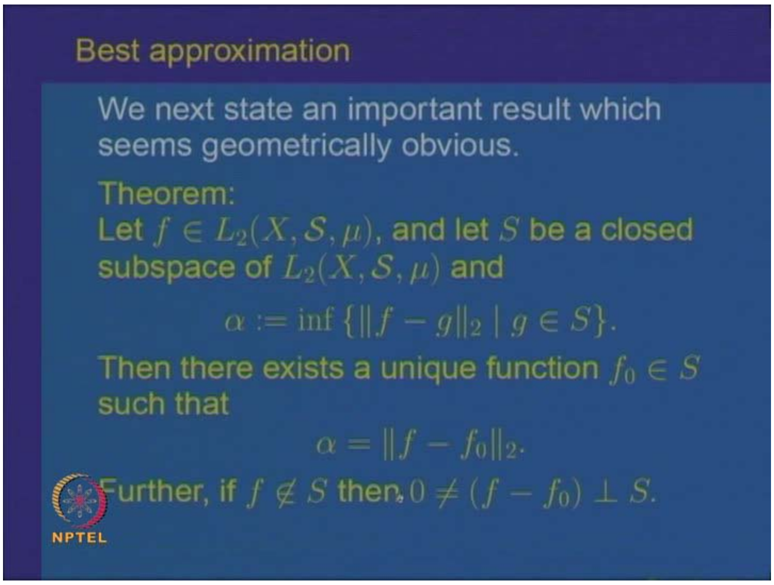

This is also called the best approximation theorem for Hilbert spaces. Let us look at once again, it says that if given a closed subspace of L 2 - a function f in L 2 - look at alpha the minimum of the distances between f and elements of g says, there is value there is a function in S, where this value is attained.

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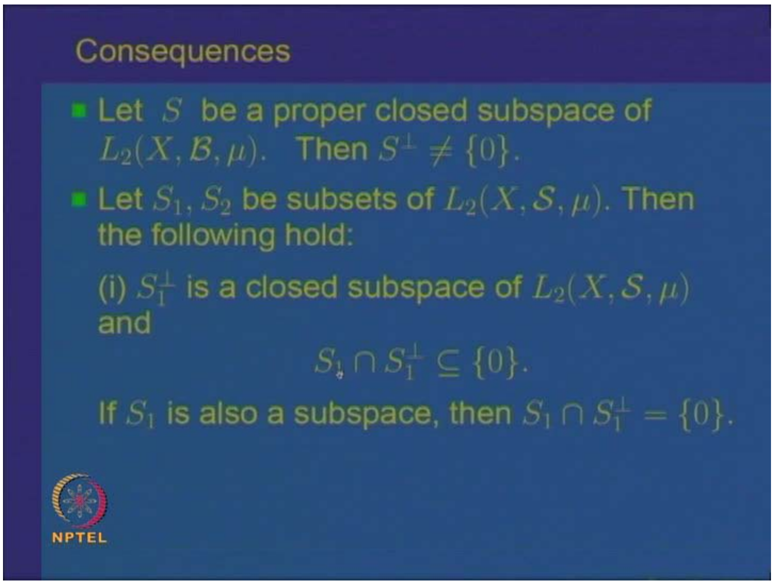

As a consequence of this, it also says that if S is proper close subspace of L 2 then, its perpendicular cannot be 0, because if it is proper then there is an element f minus f 0 which is not 0, it is orthogonal to it.

So, as an immediate consequence that if S is proper closed subspace of L 2 then its orthogonal complement cannot be 0 it has to be something else. Also means that if the orthogonal complement of something is 0 then S must be equal to L 2, another way of stating the same thing is this. As I said, we will not be proving this theorem but, we will give some applications of this today. So, let us look at some properties of orthogonal complement before we go and to prove some general facts.

Let us take S 1 and S 2 be subsets of L 2 then the following properties hold namely, S 1 perpendicular is a closed subspace that we have already shown and S 1 they intersect only at the most at 0, they are just sets.

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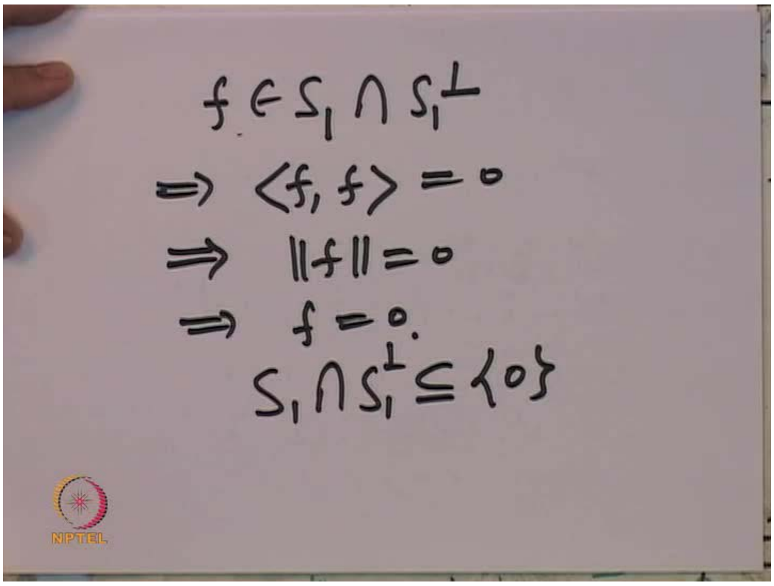

S 1 perpendicular is a subspace because S 1 may not be a subspace. So, it says that if S intersection S perpendicular is always inside 0 and that is obvious, because if f belongs to S 1 intersection S 1 perpendicular then that means, the inner product of f with itself because f belongs S 1 and it also belongs to S perpendicular that must be equal to 0. So, that implies norm of f is equal to 0 and that implies f must be equal to 0.

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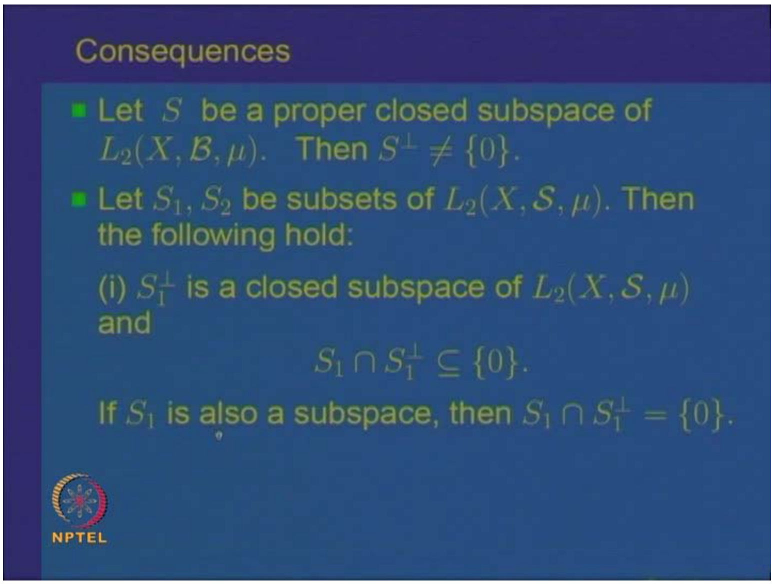

Hence, S 1 intersection S 1 perpendicular is inside 0. As an obvious consequence, if S 1 is also a subspace then S 1 intersection S 1 perpendicular, they do not have anything common other than the vector.

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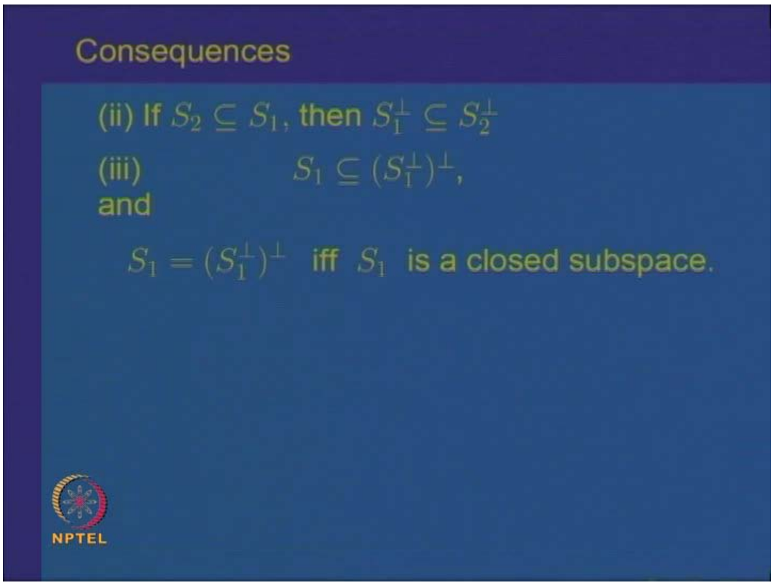

The second property says that if S 2 is a subset of S 1 then, S 1 perpendicular is a subset of S 2 perpendicular and that is obvious, because if we take any element say h in S 1 perpendicular then, the inner product of h with every element of S 1 is equal to 0 and in particular with S 2 is equal to 0 that also belongs to S 2, so this property is obvious.

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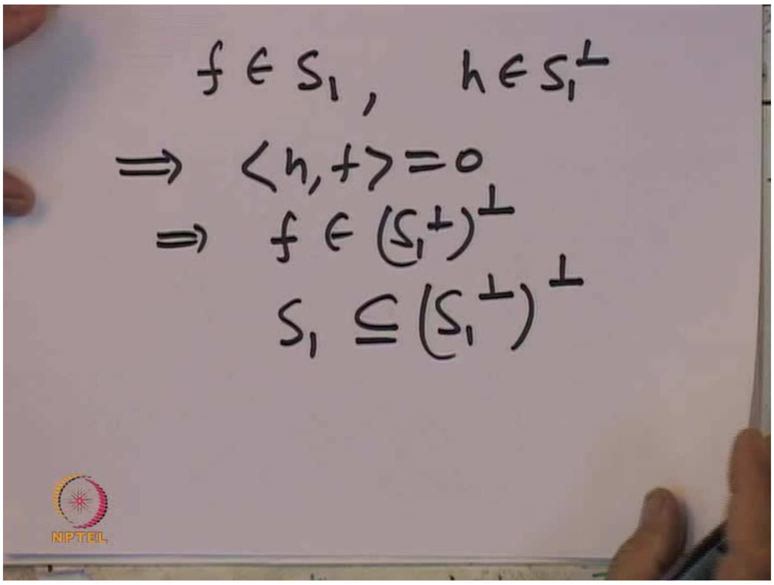

The third property says that S 1 is a subset of S 1 perpendicular perpendicular. So, orthogonal component of the orthogonal complement always includes S 1. So that property is again obvious, because if we take f belonging to S 1 and h belonging to S 1 perpendicular then that implies h comma f the dot product is equal to 0, because f belongs S 1 and h belongs to S 1 perpendicular and that is equal to 0. That means, h is perpendicular to f; this means that for every h in S perpendicular f comma h or h comma f is equal to 0 that means, f is belonging to S 1 perpendicular perpendicular. So, S 1 is always a subset of S 1 perpendicular perpendicular.

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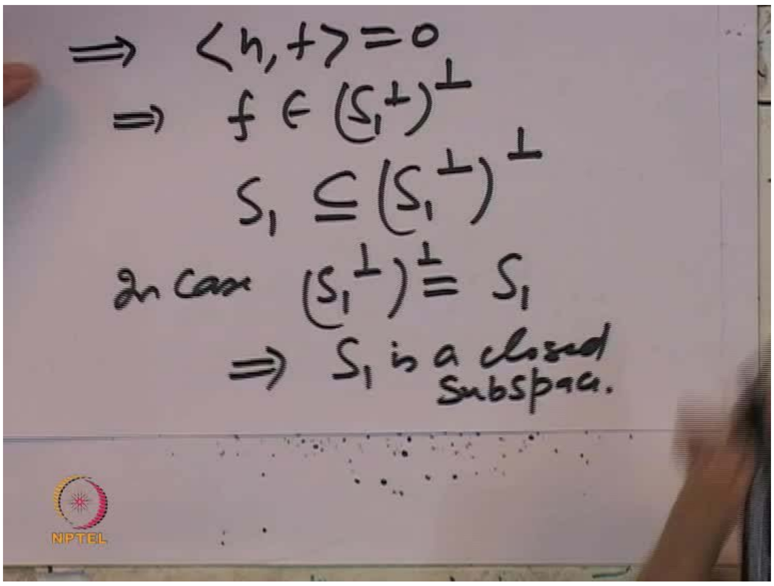

We want to show that incase S 1 perpendicular perpendicular equal to S 1, these two are equal then, the left hand side is an orthogonal compliment of a subspace; so this is the closed subspace - this implies S 1 is a closed subspace.

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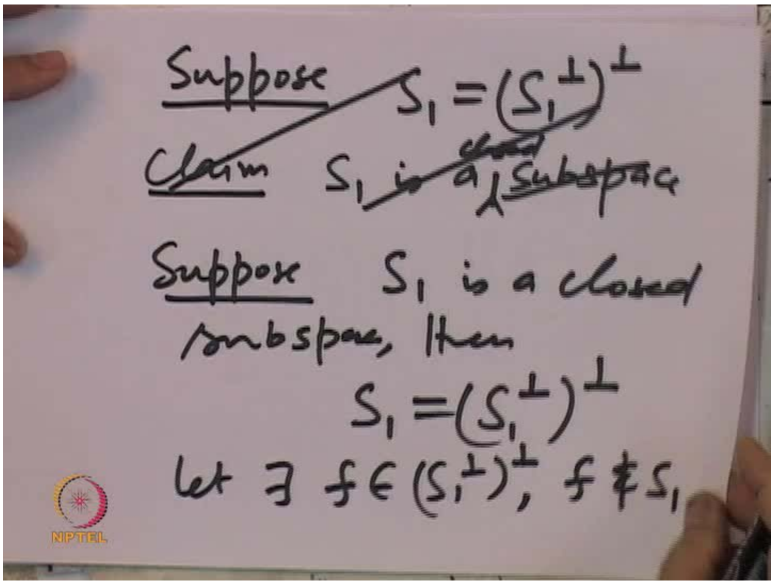

Let us prove the converse part namely, the converse is also true. Suppose, S 1 is equal to S 1 perpendicular perpendicular then the claim that S 1 is a closed subspace, so that we have already shown. We want to prove the way round, so this is not what we want to do is. Suppose S 1 is a closed subspace, so converse is if S 1 is a closes subspace then we want to show S 1 is also equal to S 1 perpendicular perpendicular.

To prove this, let us take there exist f belonging to - this is subset of this anyway - let us assume there is a S 1 perpendicular perpendicular f not in S 1. In that case, let us apply our best approximation theorem, so implies by the theorem just now we stated which we did not prove, that there exist and element f naught belonging to S 1 such that f minus f naught is perpendicular to S 1. So that means, f minus f naught belongs to S 1 perpendicular.

So there is an element, f is not in S 1; so there is an element in f naught in S 1 such that the difference is perpendicular to S 1. Now, let us observe that this element f naught belongs to S 1 which is contained in S 1 perpendicular perpendicular.

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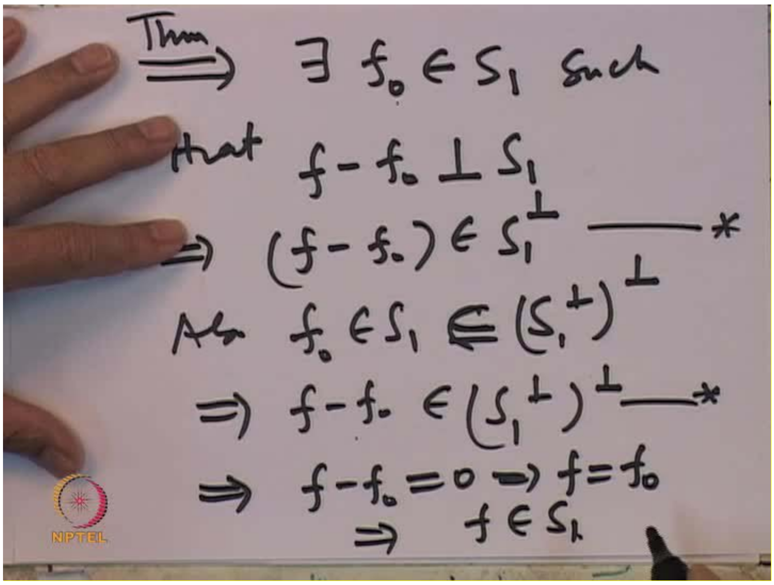

So that implies so we have got f belonging to S 1 perpendicular perpendicular and f naught also belonging that means f minus f naught belongs to S 1 perpendicular perpendicular and f naught is also in the same thing and this being a subspace the difference must also belong to S 1 perpendicular perpendicular.

Now, the element f minus f naught belongs to S 1 perpendicular perpendicular and it also belongs to S 1 perpendicular. From here and here, it belongs to a subspace and orthogonal complement of it that means f minus f naught must be equal to 0, implying f is equal to f naught (Refer Slide Time: 36:33).

So that means what? That means f belongs to f naught f and where is f naught; f naught is in S 1, so this f also belongs so we would certify the f in S 1 perpendicular perpendicular and we are getting that f is equal to f naught, where f naught is an element in S 1, so that implies that f belongs to S 1.

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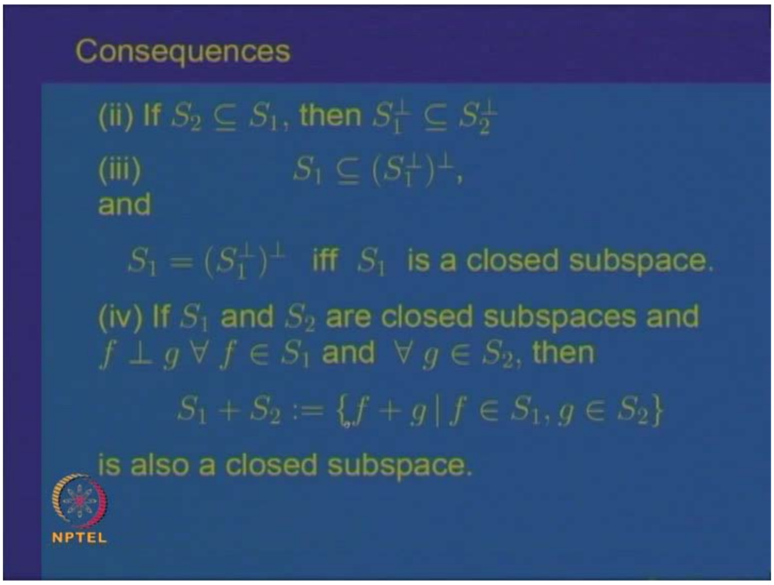

What we are shown is, whenever f belongs to S 1 perpendicular perpendicular it also belongs to S 1, so these two are equal. That proves the fact that if S 1 is equal to S 1 perpendicular perpendicular then S 1 is a closed subspace of it. Next, let us observe, the fact that if S 1 and S 2 are two closed subspaces and S 1 is perpendicular to S 2 then S 1 plus S 2 is also a closed subspace.

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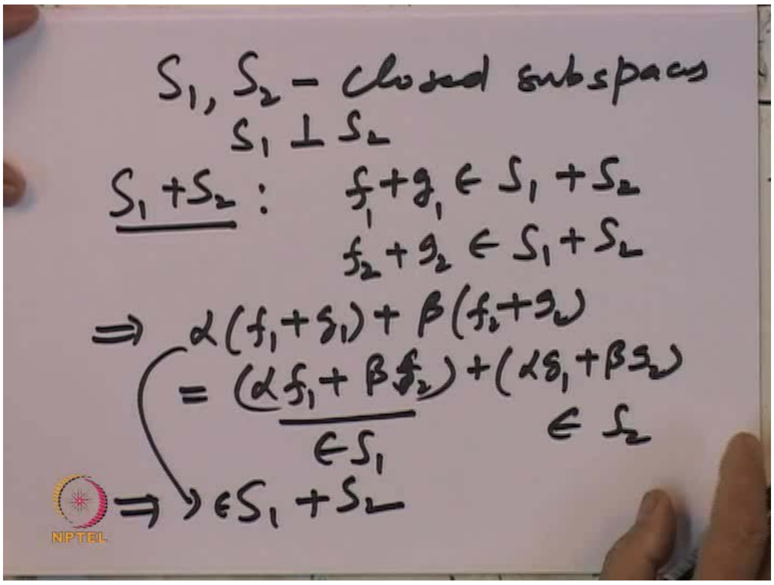

Let us observe that S 1, S 2 is closed subspaces and S 1 is perpendicular to S 2. Let us take two elements; so let us look at S 1 plus S 2, we want to show it is a subspace. Let us take an element say f plus g in S 1 plus S 2, where f belongs to f 1 g 1, where f 1 belongs to S 1 g 1 belongs to S 2.

Let us take another element, f 2 plus g 2 also belonging to S 1 plus S 2, where f 2 belongs to S 1 and g 2 belongs to S 2. Then, for every alpha and beta alpha times f 1 plus g 1 plus beta times f 2 plus g 2 is equal to alpha f 1 plus beta g 2 plus alpha g 1 plus beta g 2. Now, because S 1 is a subspace (Refer Time: 39:00) alpha f 1 plus beta f 2- I made it as beta f 2 and alpha g 1 plus beta g 2, so this element belongs to S 1 and this element belongs to S 2. So, implies that this element belongs to S 1, so this element belongs to S 1 plus S 2.

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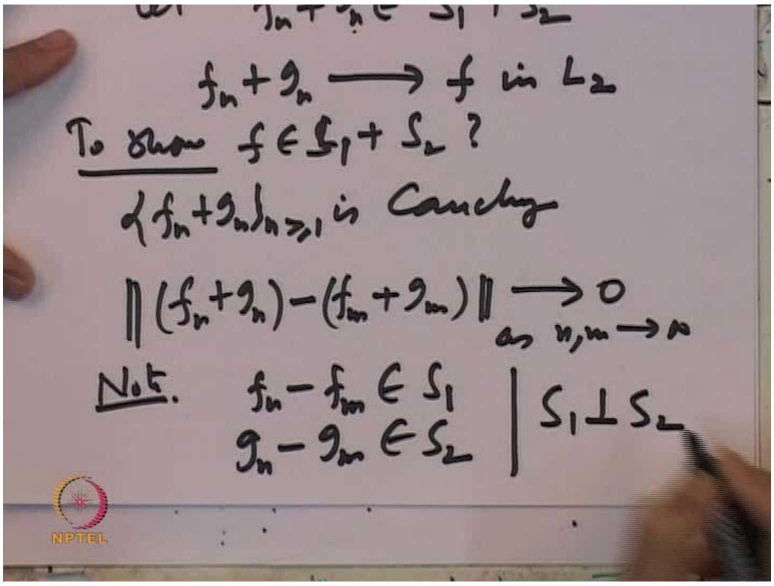

So that proves S 1 plus S 2 is a subspace. To prove it is a closed subspace let us observe, let f n plus g n belong to S 1 plus S 2, where f n belongs to S 1 and g n belongs to S 2 and f n plus g n converge to f in L 2. To show S 1 plus S 2 is closed that means we have to show that f belongs to S 1 plus S 2, so that is what we have to show.

Now, let us observe that f n plus g n being convergent is Cauchy; f n plus g n is a Cauchy sequence. Let us look at f n plus g n minus f m plus g m, so this norm goes to 0 as n and m go to infinity.

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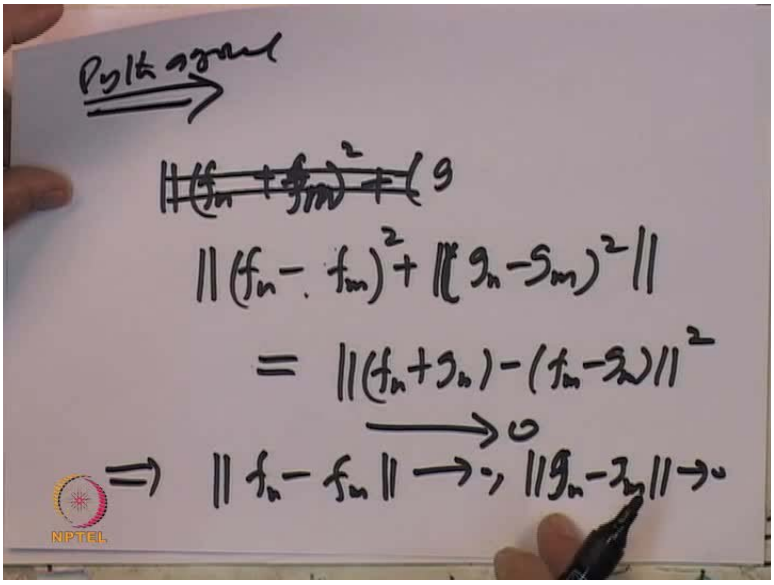

Now, note that f n minus f m belongs to S 1 and g n minus g m belong to S 2, so this implies by Pythagoras theorem that the norm of f n minus f m square plus norm of g n minus g m square, this is equal to norm of f n plus g n minus f m minus g m square.

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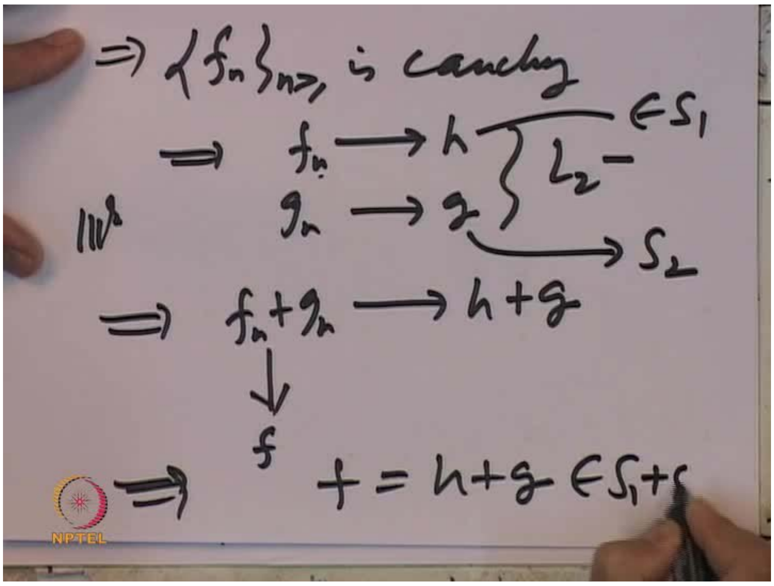

So that is by Pythagoras theorem and this goes to 0, so that implies that norm of f n minus f m goes to 0 and norm of g n minus g m goes to 0. So, meaning what? This says that f n itself is Cauchy and g n itself is Cauchy. So that implies that f n is a Cauchy implying that f n must converge to some h.

Similarly, g n must converge to some g, all in L 2. Similarly that implies that f n plus g n converges to h plus g and we know that this converges to f, so this implies that f is equal to h plus g. Now, note that f n is a sequence in S 1 and S 1 is closed, so this h belongs to S 1 and g belongs to S 2, so this belongs to S 1 plus S 2 (Refer Slide Time: 43:00).

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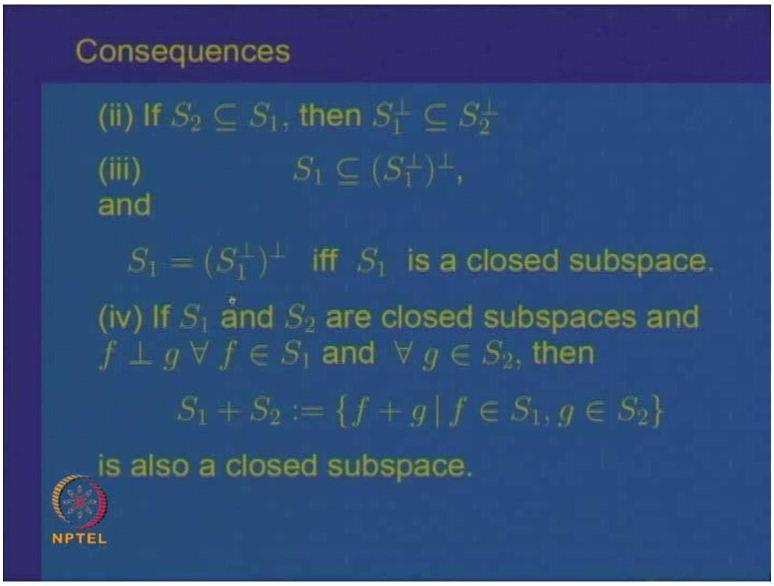

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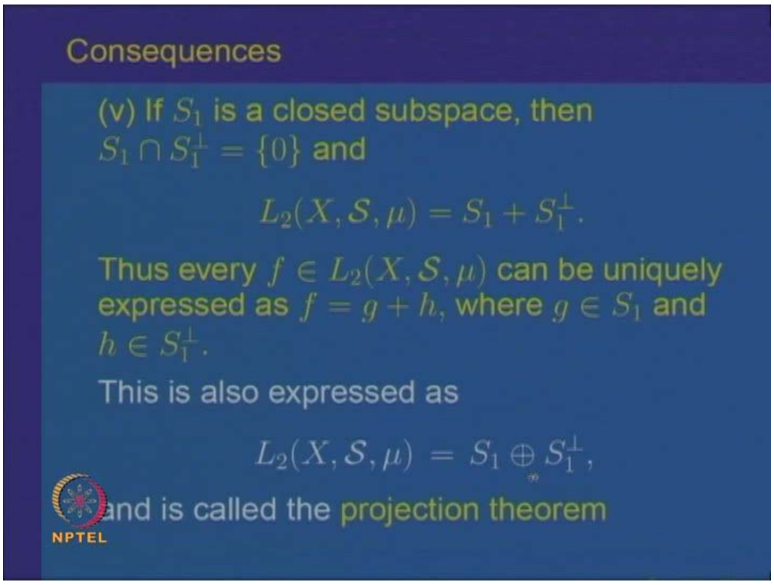

This completes the proof that if S 1 and S 2 are closed subspaces and S 1 is perpendicular to S 2, then S 1 plus S 2 also is a closed subspace. Finally, we can claim that S 1 is a closed subspace and then we know that S 1 intersection S 1 perpendicular is

0. In that case, L 2 is equal to S 1 plus S 1 perpendicular and the reason for that is because this intersection is 0, so there cannot be S 1 plus S 1 perpendicular is a closed subspace if it is not whole then there must be an element outside, which is not true. So that means for every closed subspace S 1 of L 2; L 2 can be expressed as S 1 plus S 1 perpendicular. That means, every element of L 2 can be represented as an element in S 1 plus an element in S 1 perpendicular and this decomposition will be unique because S 1 intersection S 2; S 1 is a subspace so there is nothing common between them.

This is also called sometimes the projection theorem that means, for every closed subspace S 1 of L 2; L 2 can be represented as S 1, one writes as a direct sum of S 1 perpendicular namely, these two are equal and the intersection of these two subspaces is equal to 0.

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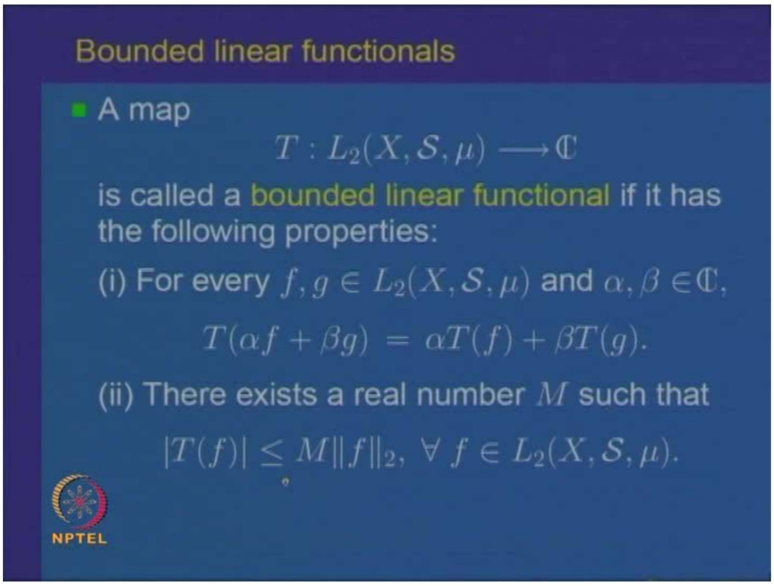

Next, let us come to analyzing maps on the space L 2 which is a vector space. As on any vector space you can analyze linear maps on the vector space taking values in the underlying field, here our vector space is L 2 actually it is a Hilbert space; so look at a map, which is a linear map T from L 2 to C. We say it is a bounded linear functional if it has to following properties first of all, it should be linear; so T is a linear map as a vector space L 2 to C. Secondly, we want that it is bounded in the sense that if there is a constant M such that norm of T f is less than or equal to M times the norm of f 2.

So, this called the boundedness of the linear map T. We say T is a bounded linear functional if T is linear on L 2 and an absolute value of T f is less than or equal to a constant M times f where M is a constant fix and this happens for every f in L 2. It is quite to this condition, boundedness actually implies that T is also continuous, because if f n converges to f then T f n absolute value is less than or equal to M times norm of f n minus f and that will go to 0.

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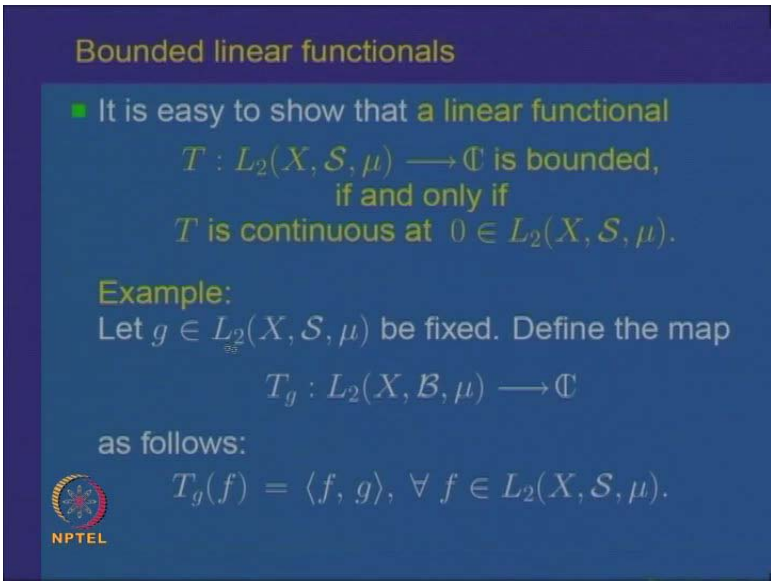

So, it is easy to verify that T if a linear map is bounded, if and only if it is continuous and because you on vector space continuity at 0 are enough. So, one can verify easily that every bounded linear map is continuous at 0 is equivalent to it. One way of defining a continuous linear map is the following, fix any g in L 2 and look at the map T lower g defined on L 2 to be T g at f is equal to f comma g for every f.

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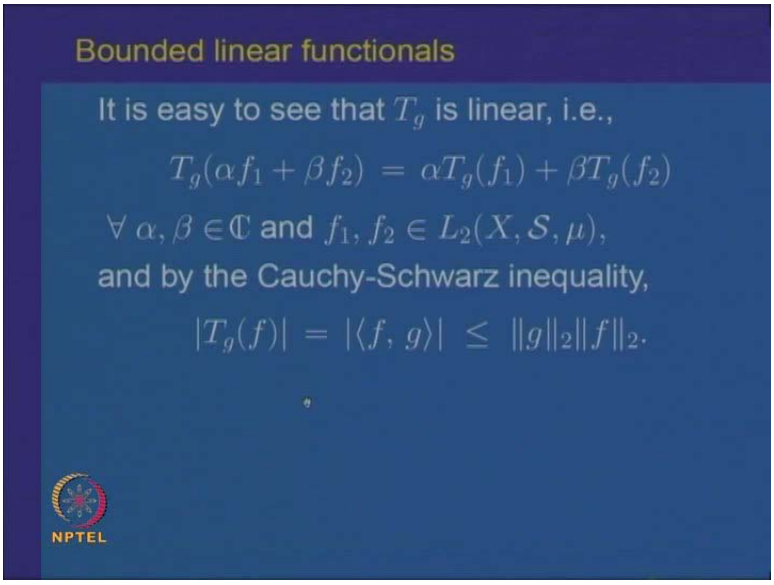

So that means, the value of T g at f is defined as f g inner product of f with g for every f. It is easy to see that this is a linear map because of the inner product is linear in the first variable, that will give it is a linear map and it is bounded because of the Cauchy-Schwarz inequality. So, this is linear and by the Cauchy-Schwarz inequality it is a bounded linear map.

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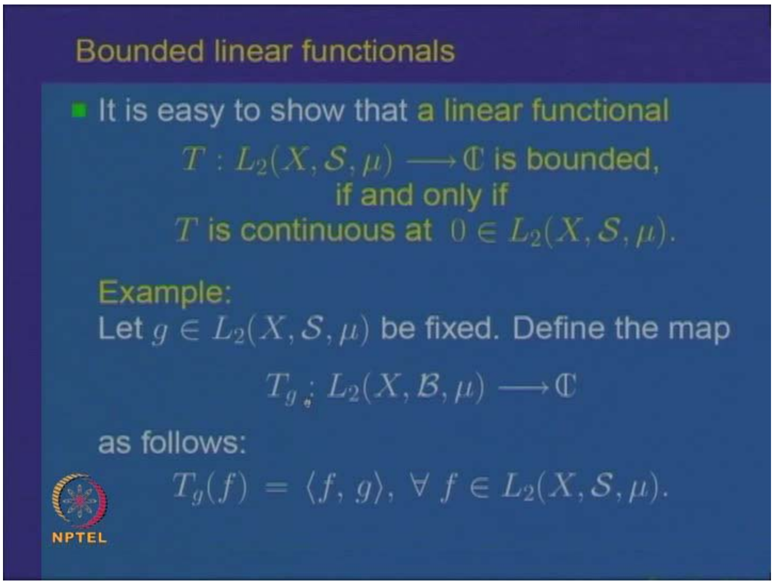

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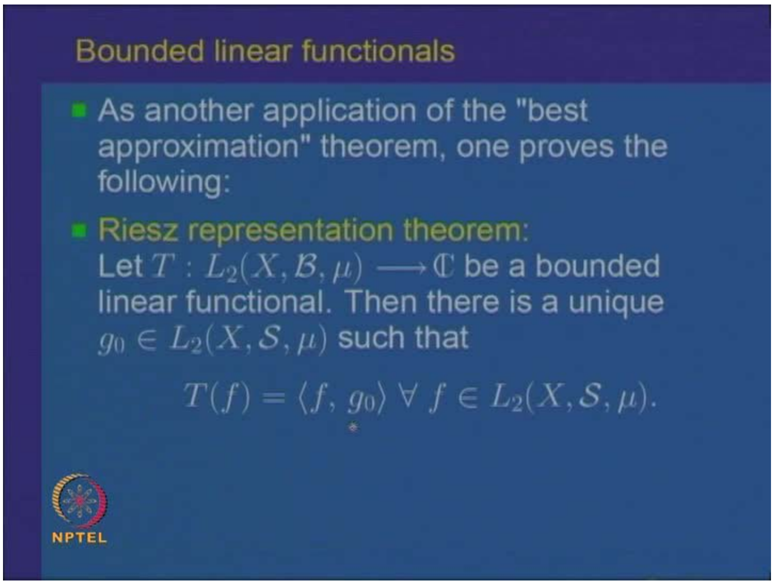

One way of constructing bounded linear functionals on L 2 is by taking the inner product of any element f with a fixed element g. This is important theorem called Riesz representation theorem, which says that this is the only way of constructing bounded linear functionals on L 2. It says that if T is any bounded linear functional then, there is a unique g 0 belonging to L 2 such that T f is equal to f g naught. That means, every linear function T on L 2 arises via inner product of f with a fix element g 0 and this g 0 is also unique.

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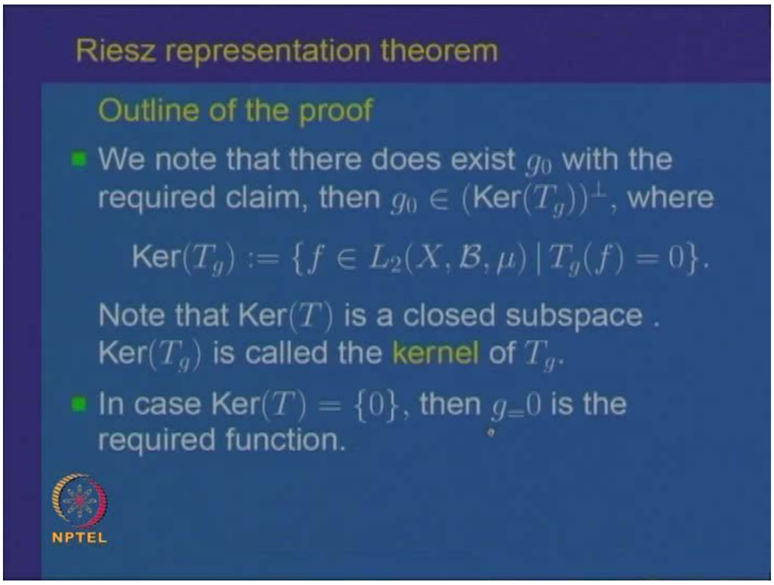

Let us just outline the proof of this. First of all, let us observe that there are two cases look at suppose that there exists any g 0 with the required claim, then what will happen? Then, g 0 must belong to what is called the Kernel of T of g that means what? A Kernel of g is all elements such that which are map to 0 and this is a closed subspace of Kernel of a bounded linear functional is a closed subspace.

So, if there is no g then this will be so; that means that our required claim will hold with T equal to 0. So, that is essentially saying the Kernel of T g is a closed subspace of g and if one possibility is Kernel of T g is equal to the whole space then it is equal to 0, because if Kernel of T g equal to the whole space then T g will be identically 0, so any g 0 equal to 0 will satisfy.

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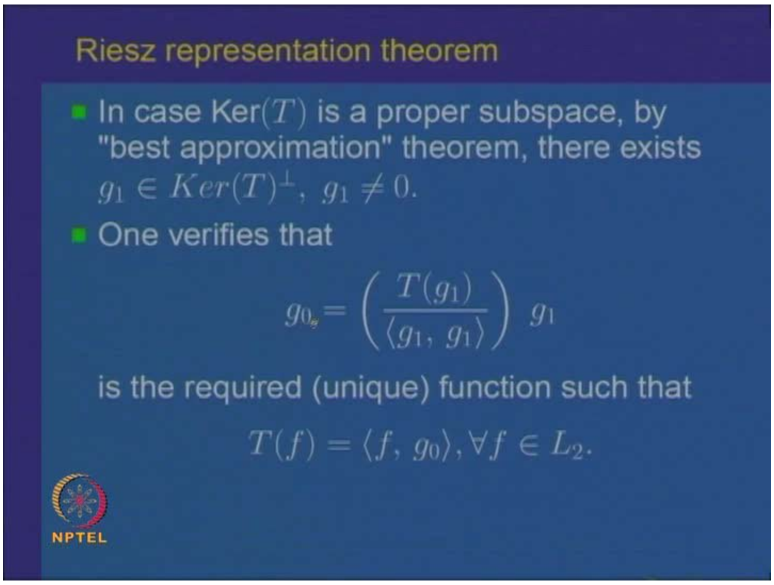

Let us assume that the Kernel of T 0 is a proper closed subspace of it. Then by the best approximation theorem g 1; there is a g 1 in kernel of T perpendicular that is the consequence of the best approximation theorem and thus, g 1 will not be equal to 0. In that case, one verifies that if we take g 0 to be equal to T g 1 divided by g 1 comma g 1 into g 1, if this selection of g 0 is the required unique function such that T of f is equal to f g 0 for every f in L 2.

Essentially, one implies the best approximation theorem to get an element g 1 in Kernel of T perpendicular. Why one is looking at Kernel of T perpendicular is because, if the required condition is to hold then that function g 0 has to belong to Kernel of T perpendicular, because if it is g comma f and that means for f in kernel that must be 0 so the required g 0 has to be from here. Let us pick up any element and then modify it and show that is required. So, this is what is called the Riesz representation theorem; so this is the Riesz representation theorem.

Today, what we have looked at is, the space L 2 is a perfect generalization of the space of the R n or the space C n that means there is a notion of an inner product defined on it, which is related to the norm and which gives the notion of perpendicularity.

We have proved; we stated one important theorem namely, if S is any closed subspace of L 2 and you take an element f in L 2. There is a best approximation then there is an element g 0 in the closed subspace, which best approximates within minimum distance from f. As a consequence of this; one consequence is the projection theorem namely every closed, if s is any closed subspace of L 2 then L 2 is a direct sum of S perpendicular.

The second consequence is characterizing all bounded linear functionals on the Hilbert space L 2 namely, the only way bounded linear functionals can be constructed on L 2 is via the inner product. That means T of f; if T is a boundary linear functional and T of f must be equal to the inner product of f with an element g 0, for some element g 0 with the inner product. So that is characterization of bounded linear functions. Thank you.