Measure and Integration

Prof. Inder K. Rana

Department of Mathematics

Indian Institute of Technology, Bombay

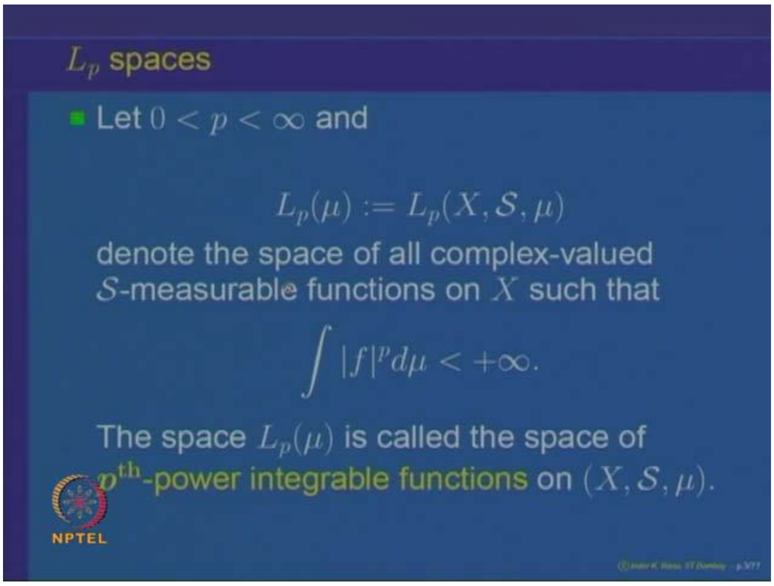
Module No.# 09

Lecture No. # 34

L P – Spaces

Welcome to lecture 34 on Measure and Integration. In this lecture, we will look at some special spaces which are constructed on measure spaces. These spaces play an important role in topics like functional analysis, harmonic analysis and so on. So, today we will be studying spaces called L p spaces.

(Refer Slide Time: 00:52)

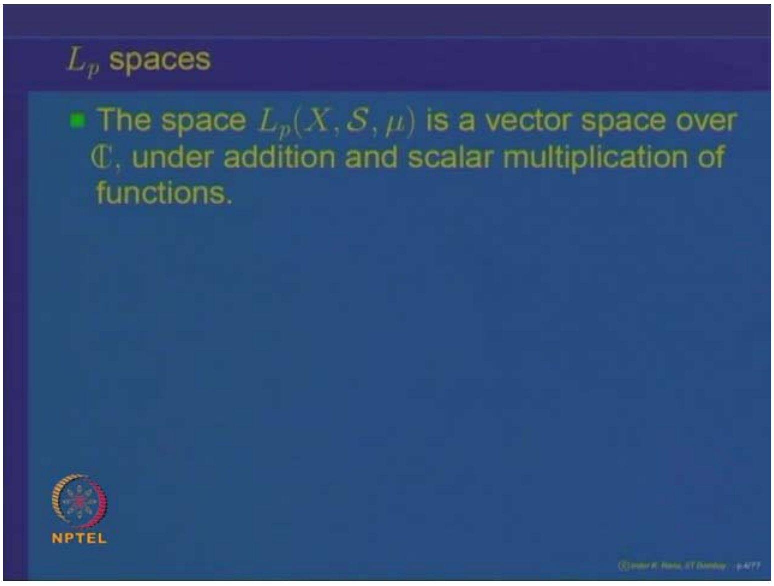


We will fix a real number between 0 and infinity p, so p is a real number between 0 and infinity and we will look at the space called L p mu which is also written as L p of X, S, mu depending on whether we want to emphasis the underlying measure space or not.

If it is clear from the context what is the underlying set X and the sigma algebra S. We will just write this space as L p mu. So, this is the space of all complex-valued S measurable functions on the space X, such that integral of the absolute value of the function f raise to power p d mu is finite. Recall in the previous lectures, we had defined the notion of function which is complex-valued on a set X and which is S measurable. We also defined the notion of its integral. So, if we take a function f which is complex-valued such that the absolute value of this function raise to power p which is a non-negative measurable function.

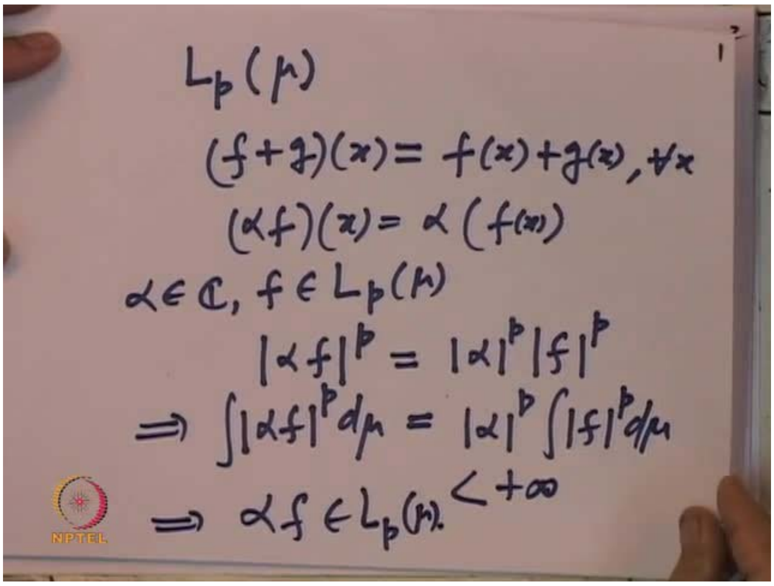
If that is integrable, integral of mod f to the power d mu is finite then, we say the function f is p th power integrable and the collection of all p th power integrable functions on the measure space X S mu is denoted by either L lower p X S mu or just L p mu. So, L p mu is the space of all p th power integrable functions on X S mu and today, we are going to study properties of this set L p X S mu.

(Refer Slide Time: 02:42)



So, the first observation we want to make is that the space L p X S mu can be treated as a vector space over the complex numbers under the addition and scalar multiplication of functions.

(Refer Slide Time: 03:01)

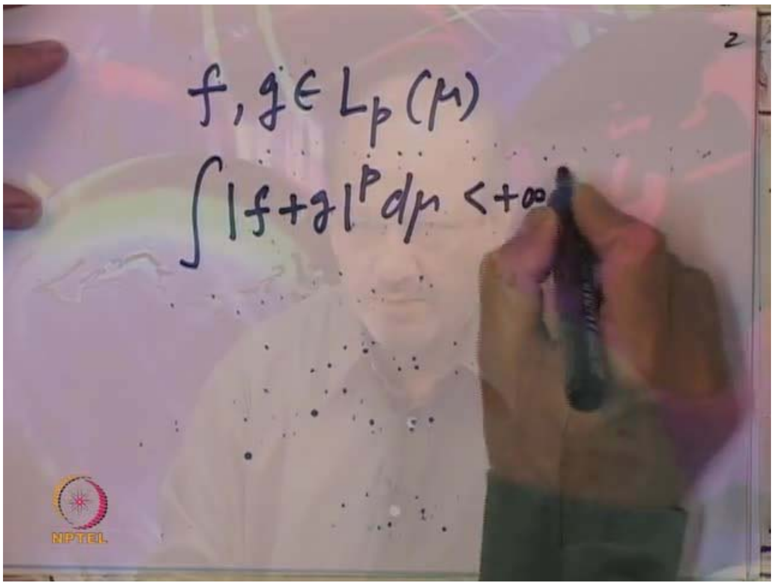


Let us observe how is that d1? We have got L p of mu, so that is the space of all p th power integrable functions. So, we want to show that if you define f plus g x to be f x plus g x for every x and alpha f x to be equal to alpha times f of x then, under this operation of addition and scalar multiplication L p mu is a vector space. So, for that we will have to show alpha times f is a function in L p of mu whenever f is a function in L p of mu.So, let us check that.

So, alpha belongs to C and f belongs to L p of mu, let us look at - we want to check alpha times f is in L p or not, so you have to look at absolute value of alpha f raise to power p and we have to show this is a integrable function, its integral is finite but, it is obvious, this is equal to mod alpha to the power p and mod f to the power p, were the property of absolute value.

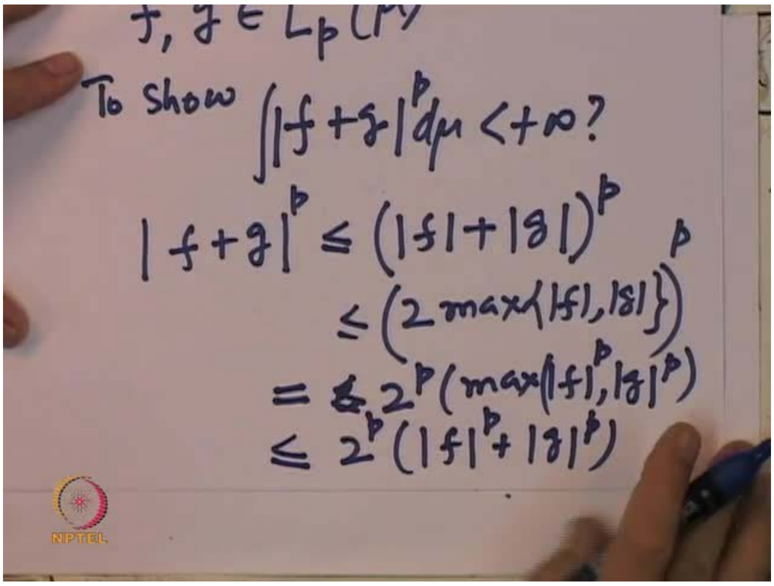
So, thus implies that integral of mod alpha f to the power p d mu is equal to integral of mod alpha to the power p into the product mod f to the power p but, integral of a scalar times a function is nothing but, the scalar times the integral of the function so by that property this is (Refer Slide Time: 04:44). Because f belongs to L p of mu, so this is finite. So, this implies that alpha f belongs to L p of mu. So, scalar multiple of functions in L p are again functions in L p.

(Refer Slide Time: 05:11)



Let us look at the second property namely the addition. So, let us take two functions f and g belonging to L p of mu. We want to show that mod of f plus g raise to power p is integral is finite but, let us observe so we want to show that this d mu is finite.

(Refer Slide Time: 05:35)

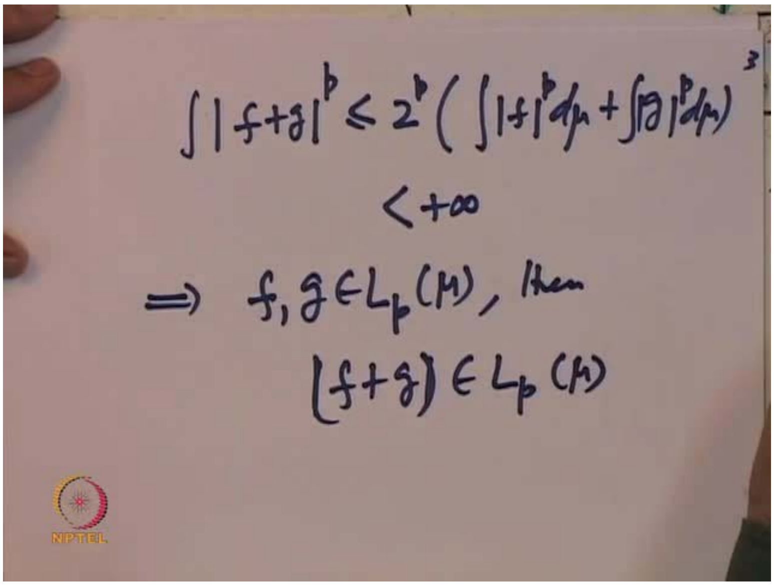


So, let f and g belong to L p of mu. To show, we want to show that f plus g belongs to L p of mu that means, integral of mod; this to the power d mu is finite. So, this is what we have to show. Let us look at the function mod of f plus g. We know that this is less than or equal to mod f plus mod g by the absolute value of by the absolute triangle inequality of the absolute value.

So, this to the power p is less than or equal to this to the power p. Now, let us observe that the right hand side, mod f plus mod g is less than or equal to 2 times the maximum value of mod f and mod g because mod f will be less than the maximum of mod f mod g and mod g also is less than maximum of mod f and mod g. So, mod f plus mod g is less than twice the maximum of mod f and mod g. So, this raise to power p but, that is same as we can take this 2 to the power p out, so this is less than or equal to 2 to the power p and the maximum of two numbers is always less than or equal to the sum of those.

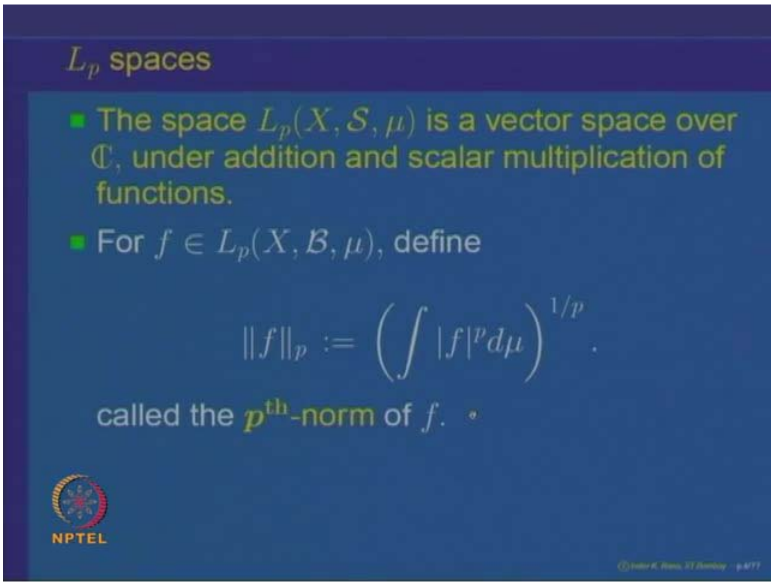
So, let us first observe that this is actually equal to 2 to the power p maximum of mod f to the power p and mod g to the power p. Now, this is less than or equal to 2 to the power p mod of f to the power p plus mod of g to the power p because maximum of two numbers is always less than or equal to the sum of the two numbers. So, what we get is that f plus g mod to the power p is less than 2 to the power p mod f p plus mod f g, so that gives the inequality.

(Refer Slide Time: 07:59)



So integrating both sides, we will get that integral of mod f plus g raise to power p will be less than 2 to the power p times integral of mod f to the power p d mu plus integral of mod g to the power pd mu. Both of them in finite so, this is a finite quantity. That implies, whenever f and g belong to L p of mu then mod of f plus g also or f plus g the function f plus g also belongs to L p of mu. So, that proves the fact that L p is a vector space over the field of complex numbers.

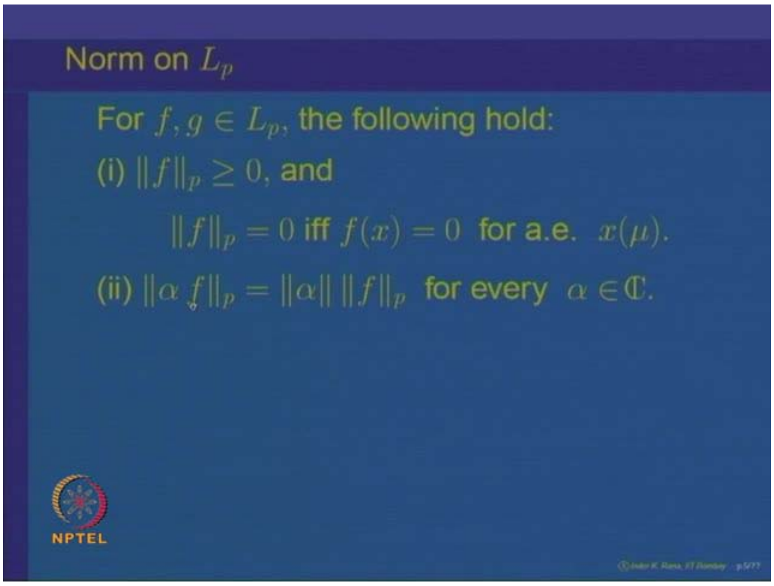
(Refer Slide Time: 08:45)



Next, let us define for a function f belonging to L p of X mu. For L p, what is called the p th norm of the function because f belongs to L p, so the integral of mod f to the power d mu is a finite number - it is a finite non-negative number - so we can take it is p th root.

So, 1 over p of this number is called the p th norm of the function f. So norm f p, so the lower index p indicates that we are taking the p th power of the function to integrate and then taking the p th root of the integral. So this is called the p th norm of f.

(Refer Slide Time: 09:28)

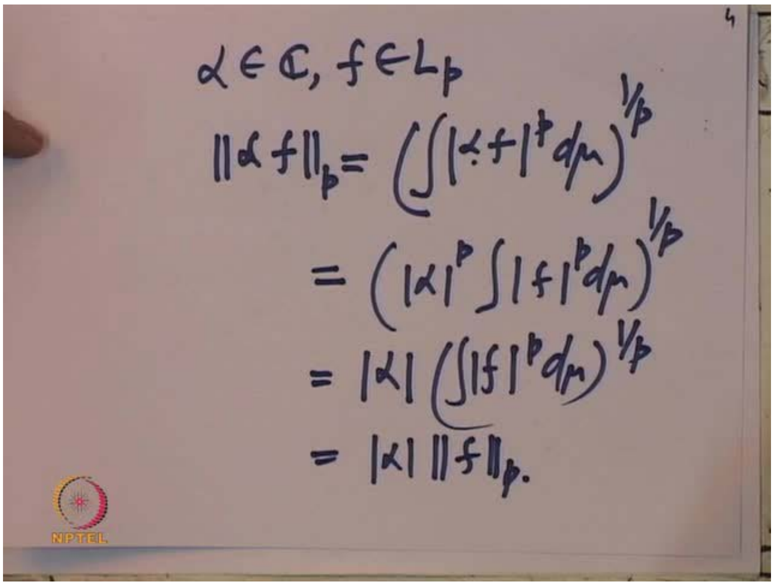


We want to show that this p th norm has the following properties namely, norm of p is always bigger than or equal to 0 that is obvious, because we are integrating a non-negative function. So, integral of mod f to the power p is always non-negative. If the function is 0 almost everywhere then of course, the integral is 0; so the norm is equal to 0. Conversely, if the norm of the function is equal to 0 - is the p th norm is equal to 0 - that means, integral of mod f to the power p is 0 and being a non-negative function that implies f of x must be 0 almost everywhere.

This property one is something similar to what we have d1 when p is equal to 1 for the space of integrable functions. So, the p th norm of the functions is always bigger than or equal to 0 and it is equal to 0 if and only if, f of x is equal to 0.

The second property that the norm of the function alpha times f is same as the absolute value of alpha times the norm of f. So, this double bar also indicates the absolute value of the scalar alpha. That is again obvious, because we take the p th norm of this function.

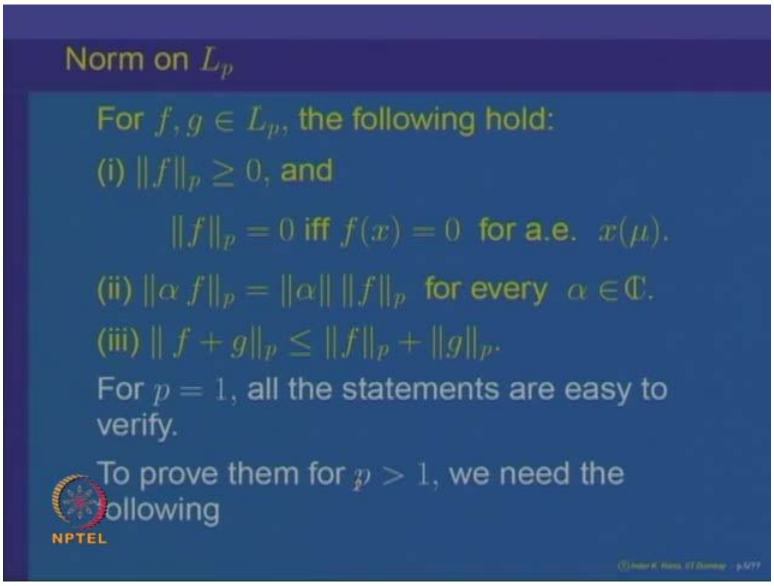
(Refer Slide Time: 11:01)



So, let us just verify this fact namely for alpha belonging to C and f belonging to L p, if we look at the norm of alpha times f, so that is equal to look at the function of alpha f take the power p integrate out with respect to mu and look at the 1 p th root of that. So but, that is equal to mod alpha f to the power p is same as mod alpha to the power p integral mod f to the power p d mu raise to power 1 by p.

Now when we open it out, so mod alpha to the power p raise to power 1 over p is mod alpha into integral of mod f to the power p d mu raise to power 1 over p which is nothing but, the norm, so this integral is nothing but, the p th norm. So that proves the property that alpha times f p th norm is equal to mod alpha times the p th norm.

(Refer Slide Time: 12:09)

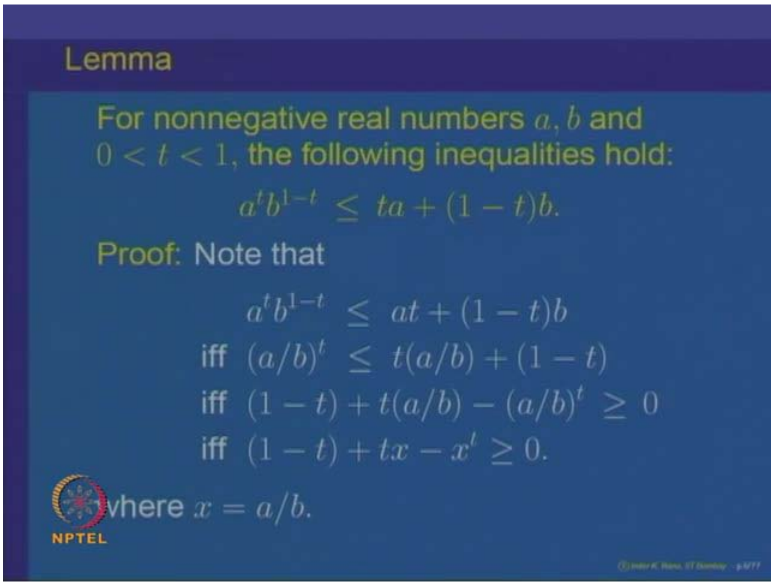


The third property, we want to prove is that the function f plus g, which we know f and g belong to L p then, the function f plus g belongs to L p. So, we want to claim that this satisfies the triangle inequality namely, norm of f plus g is less than or equal to norm of f plus norm of g.

For p equal to 1, this property was obvious, we had that followed basically because mod of f plus g is less than or equal to mod f plus mod g. So integrating both sides, we got integral of mod f plus g is less than or equal to integral of mod f plus integral mod g. So that means, the norm of f plus g is less than or equal to the norm of f plus norm of g. So for p equal to 1 this is obvious but, for p not equal to 1, we need to do some more calculations to prove this result.

So first of all, we will first prove it for the cases when p is strictly bigger than 1. So, we will be looking at the real number p which is strictly bigger than 1 and of course, less than infinity.

(Refer Slide Time: 13:30)



So, for such p, we need a Lemma which says that for every non-negative real numbers a and b, if we fix t between 0 and 1 then, the following inequality holds namely, a raise to power t b raise to power 1 minus t is less than or equal to t times a plus 1 minus t times b. If you look carefully for t equal to 1 by 2 this is just saying that the geometric mean is less than or equal to arithmetic mean. So, this is generalization of the standard inequality that the geometric mean is always less than or equal to arithmetic mean.

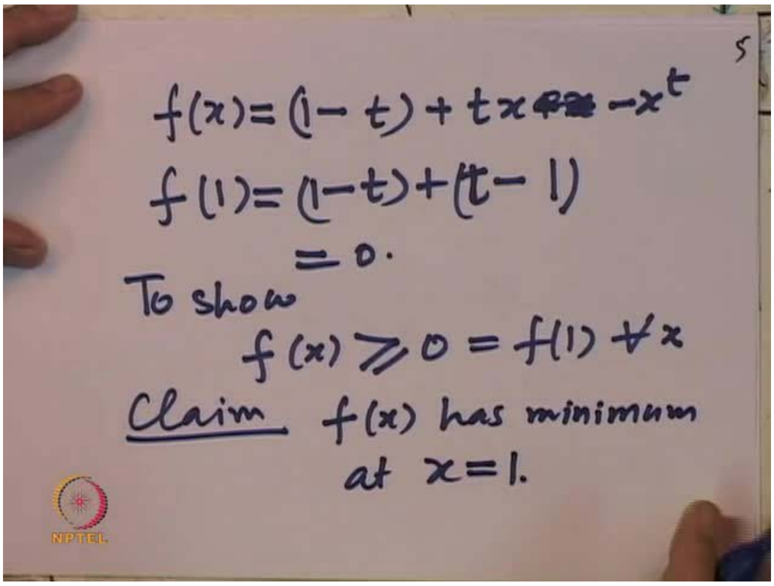
So, what we are saying is for any real number t - positive real number t - between 0 and 1. For non-negative real numbers a and b, a raise to power t b raise to power 1 minus t is less than or equal to t times a plus 1 minus t times b. Of course to prove this, let us observe that if a either a is equal to 0 or be equal to 0 then the left hand side is equal to 0 and the right hand side of also is equal to 0, so in that case it is a equality. So, if either a is 0 or b is 0 both sides are equal to 0 and there is nothing to prove.

Let us assume that both a and b are not equal to 0. So in that case, let us observe that proving this inequality that a to the power t b to the power 1 minus t is less than or equal to a into t plus 1 minus t to the times b, is same as we can rewrite this inequality as this b raise for 1 minus t is same as b time b divided by b raise to power t, so that b raise power t in the denominator. We accommodate with a raise to power t, so write this as a by b raise to power t and that b which was the power 1 we shift it to the other side, so that goes to a divided by b times t plus 1 minus t times b divided by b which is equal to 1. So the required inequality is same as proving that a divided b raise to power t is less than or equal to t times a divided by b plus 1 minus t. Now, let us just rewrite that.

So, this is same as saying bring all the terms on 1 side so that is same as saying 1 minus t plus t times a by b minus a by b times t is always bigger than or equal to 0. So, we have to show that this is always bigger than or equal to 0. Let us put this quantity a by b as x, so we have to show that for every x bigger than 0, we want to show that 1 minus t plus t x minus x to the power t is always bigger than or equal to 0.

Now realize the left hand side is a function of x and we want to show that function of x is always a non-negative function. So, 1 way of showing that would be that we look at this function f of x and realize that the value of this function at the point x is equal to 1 is equal to 0.

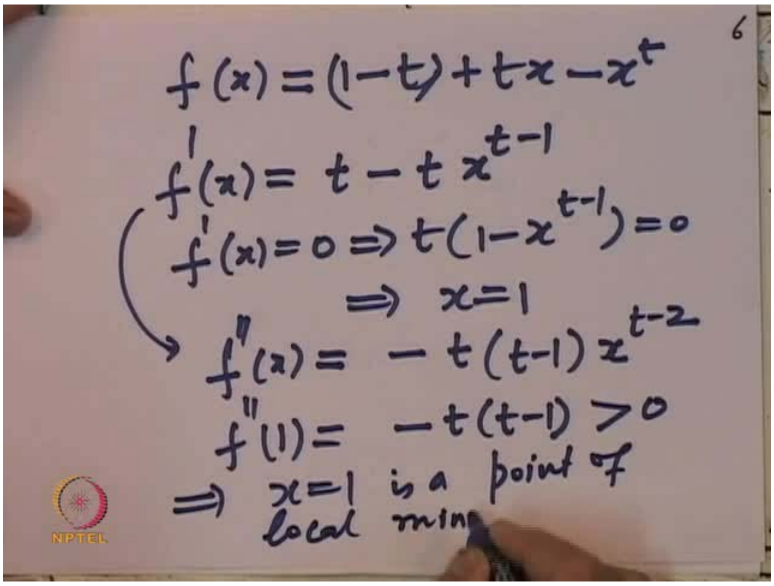
(Refer Slide Time: 17:19)



So, showing that this inequality holds is showing that f of x is always bigger than or equal to f of 1. Let us write the function f of x is equal to 1 minus t times plus t times x plus x to the sorry minus x to the power t. Then, let us calculate f of 1 which is equal to 1 minus t plus t x; t x is 1, so that is 1 minus x is equal to 1 minus t x to the power t. So, x is equal to 1, so that is 1 to the power t that is equal to so that is t times x, so that is t and this is equal to 0, so this is equal to 0. So, we want to show that f of x is bigger than or equal to 0 which is f of 1 for every x.

So, that indicates that we should try to show this function f of x has got a minimum value at the point x is equal to 1. So claim, this required inequality will be through if you can show that f of x has minimum at x is equal to 1. So, let us analyze that is d1 by using the tools of calculus.

(Refer Slide Time: 18:56)

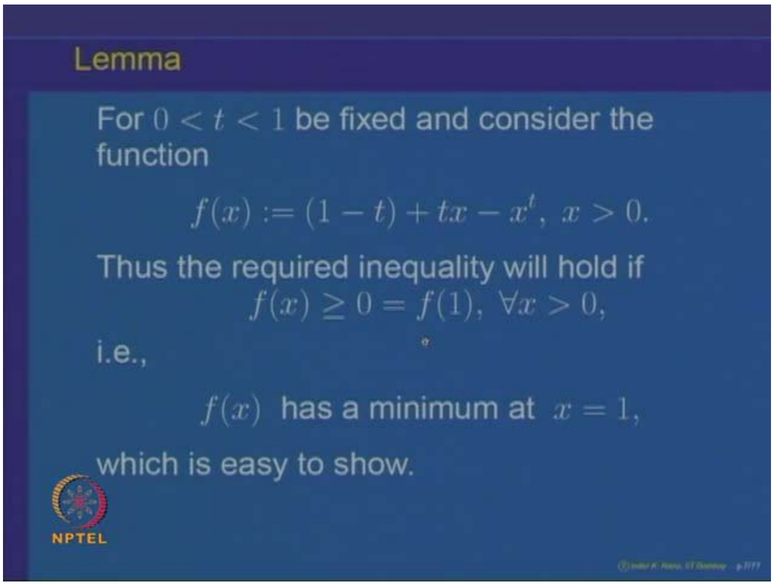


Let us use tools of calculus to analyze the maximum minimum of the function f of x which is equal to 1 minus t times plus t of x minus x to the power t. So, we realize that this function is differentiable everywhere and we calculate the derivative of this function, so that is equal to 1 minus t is a constant and derivative of t x with respect to x that is equal to t minus t times x to the power t minus 1.

So, to calculate the critical points f dash x equal to 0 implies, so this is t 1 minus x to the power t minus 1 equal to 0 and that implies x is equal to 1. So the function has a critical point at x is equal to 1 and to analyze whether it is maximum or a minimum, let us look at and apply the second derivative test. So from here, we will have f double dash of x will be equal to t is a constant, so minus t into t minus 1 into x to the power t minus 2.

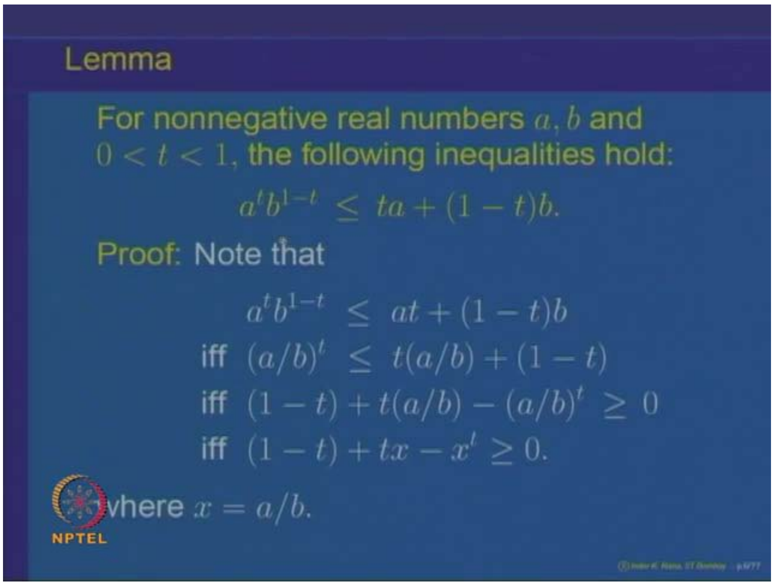
So at f dash at 1 that is equal to minus t into t minus 1 and t being a number between 0 and 1, minus t is negative t minus 1 is negative, so this is bigger than 0. So, second derivative at the critical point 1 is bigger than 0, so implies that x is equal to 1 is a point of local minimum.

(Refer Slide Time: 20:33)



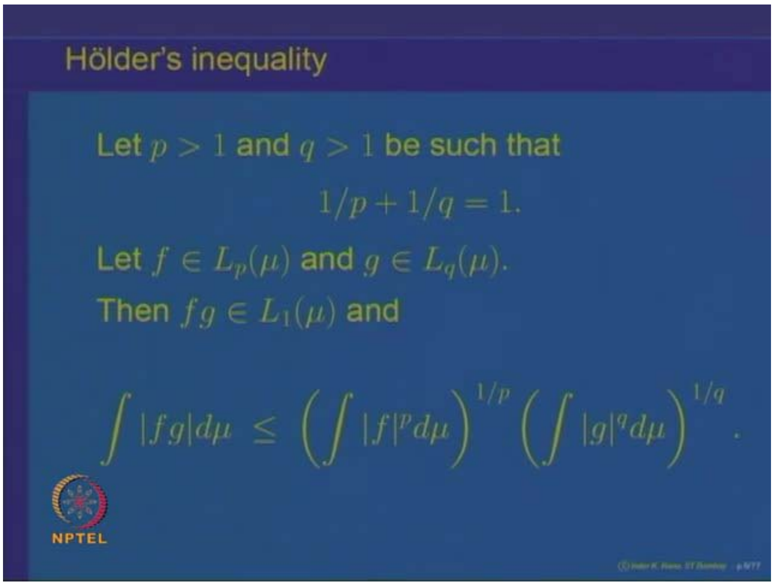
So, that proves the required property that the function has a local minimum and hence, that proves the property that the function f of x has a local minimum at the point x is equal to 1 and hence the required inequality namely, f of x is bigger than or equal to 0 is bigger than 1 holds.

(Refer Slide Time: 20:59)



So, this proves the Lemma namely, for any two non-negative real numbers a and b and for a real number t fixed between 0 and 1, a raise to power t times b raise to power 1 minus t is less than or equal to t times a plus 1 minus t times b.

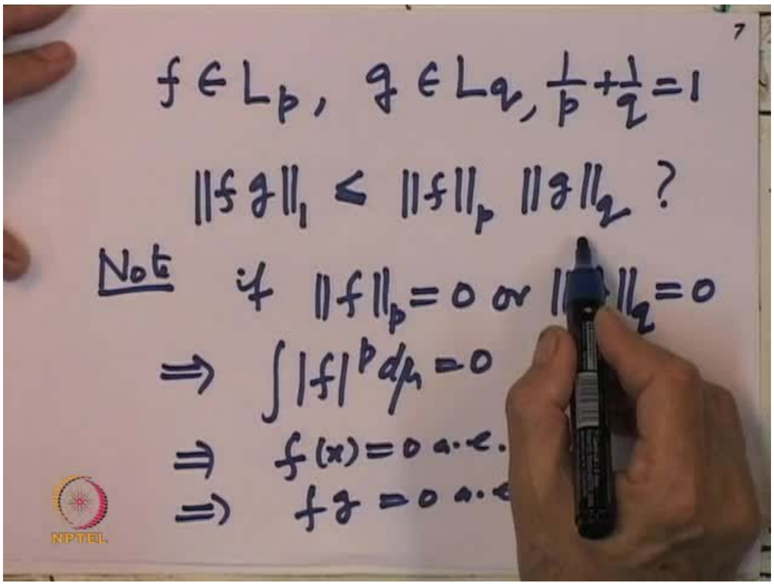
(Refer Slide Time: 21:22)



We will be using this Lemma to prove another equality for our spaces L p spaces which is called Holders inequality. So, let us state what is called Holders inequality? Holders inequality says that for real numbers p and q; p bigger than 1 and q bigger than 1 such that 1 over p plus 1 over q is equal to 1.For such numbers p and q, if I take a function f which is in L p and look at a function g which is in L q then, f times g is a function which is in L 1 and integral of f g.

The absolute value of f and g product is lesser than or equal to the integral of mod f to the power p raise to the power 1 over p and mod of g raise to power q raise to power 1 over q. So that essentially says that the function f g is integrable, so it has the L 1 norm. So, 1 can state the Holders inequality as saying that the L 1 norm of f g is less than or equal to the product of p th norm of f and the q th norm of g, so that is called Holders inequality.*

(Refer Slide Time: 22:49)

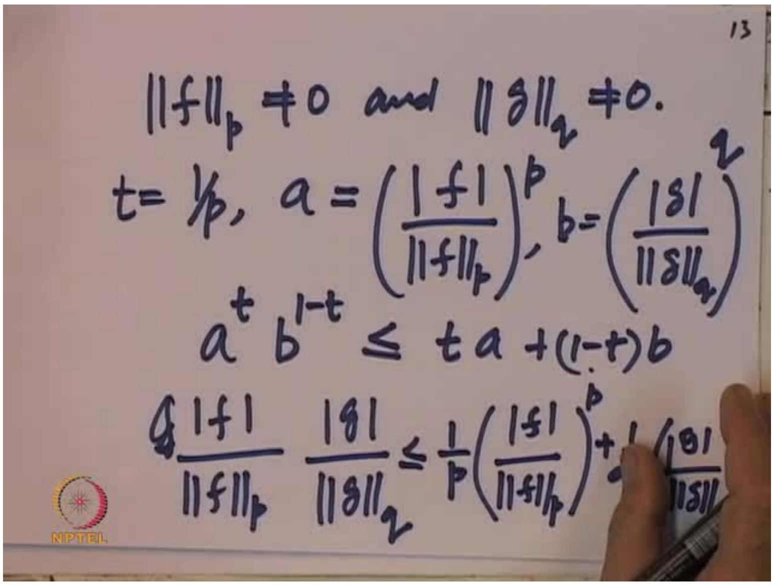


So, let us prove this Holders inequality. So, we have got two functions f and g; so f belonging to L p and g belonging to L q where, 1 over p plus 1 over q is equal to 1. We want to show that the norm f g is less than or equal to the p th norm of f and the q th norm of g, so this is the inequality we want to prove.

So let us observe first of all note let us if let us observe Let us call this, if norm of f equal to 0 or norm of g is equal to 0, if either of these two quantities are equal to 0, so what will that mean? Norm of f equal to 0 implies integral of mod f to the power p d mu is equal to 0 and that will imply that the function f x equal to 0 almost everywhere and that will imply that the function f g equal to 0, almost everywhere.

So, if norm of f is equal to 0 then, the function f g is equal to 0 almost everywhere. So, this L 1 integral of this equal to L 1 norm of the function f into g is also equal to 0. So both sides will be equal to 0. Similarly, if norm of g is equal to 0 then again both sides of the inequality will be 0 and this will be a equality. So the required inequality holds as an equality if either of norm f or norm g is equal to 0.

(Refer Slide Time: 24:46)



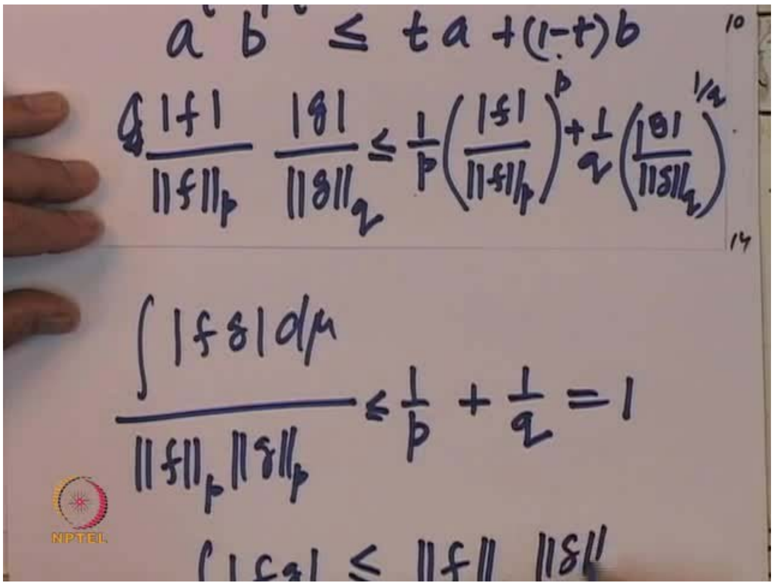
Suppose, that norm of f p is not equal to 0 and norm of g is also not equal to 0. So, we are now going to apply the Lemma. Let us consider the special case, when t is equal to 1 over p and the number a is equal to mod f divided by norm f to the power p the whole thing to the power p and b is absolute value of g divided by norm of g raise to power q.

So, we are going to apply the Lemma namely, a raise to power t b raise to power 1 minus t is less than or equal to t times a plus 1 minus t times b with t equal to 1 over p a equal to this number mod f divided by norm of f whole to the power p which is defined because norm f is not 0 and similarly, b equal to norm of absolute value of g divided by norm of g whole thing raise to power q.

So, when we do that so t raise to power 1 over t, so this is a. So, mod f whole to the power p, so that gives us mod f divided by norm of f and b raise to power 1 minus t and note 1 minus t is 1 minus 1over p which is equal to 1 over q, so that gives you norm of absolute value of g divided by the norm of g.

So that is a left hand side of a inequality is less than or equal to t which is 1 over p times a, so a is mod f divided by norm of f p whole raise to power p and similarly, 1 minus t which is 1 over q and b which is nothing but, mod g divided by norm of g to the power 1 by q.

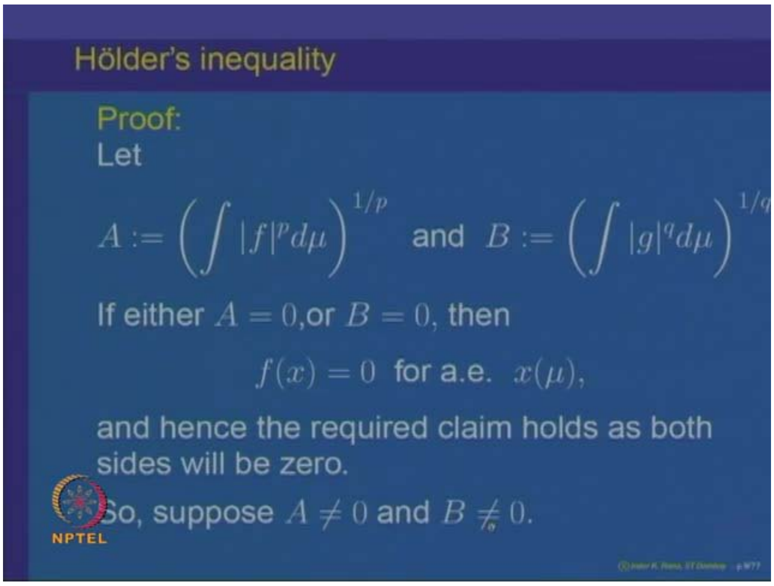
(Refer Slide Time: 27:33)



So, this is the application of that inequality and let us now simplify this a bit further and now observe. So, mod f mod g divided norm f norm g is less than or equal to 1 over p times this quantity. So, let us integrate both sides with respect to mu, so that will give you integral of mod f g d mu divided by norm of f norm of g, because these are just constants is less than or equal to 1 over p which is a scalar and mod f to the power p d mu mod f to the power integrate both sides, so that gives you the norm of f to the power p and so this is also norm f to the power p, so that gives you 1 over p plus 1 over q and that is equal to 1 that gives you.

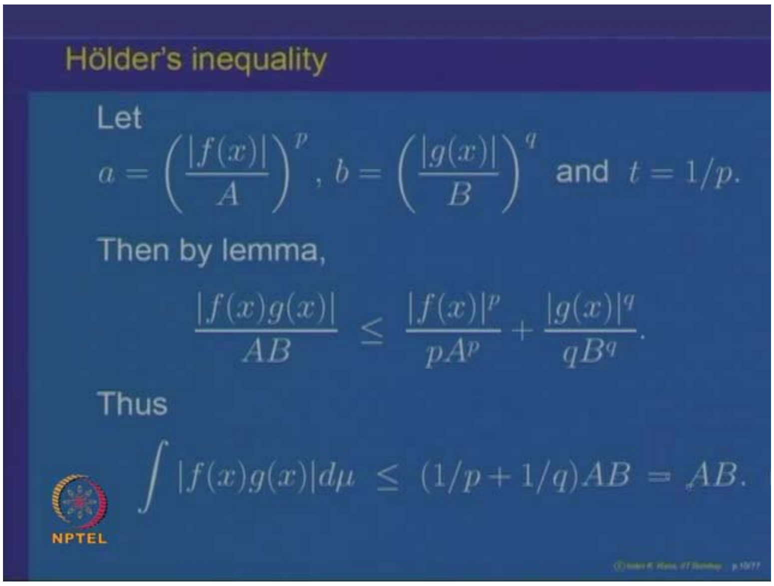
So integrating both sides, we get that norm f g is less than or equal to norm of f p norm of g q, so that is called Holders inequality. So, let us go back to revise this again what is called Holders inequality? Holders inequality says that if p is bigger than 1 and q is bigger than 1 such that 1over p plus 1 over q is equal to 1. Then, for function f belonging to L 1 L p and g belonging to L q, the product f into g is in L 1 and its integral is less than or equal to the p th norm of f into p th norm of g.

(Refer Slide Time: 29:04)



So once again, let us go to the proof. We write a as the norm of f and b as the norm of g, so if either a is 0 or b is 0, either the function f will be 0 or the function g will be 0 and both sides of the inequality will be equal to 0, so the required claim will hold.

(Refer Slide Time: 29:30)



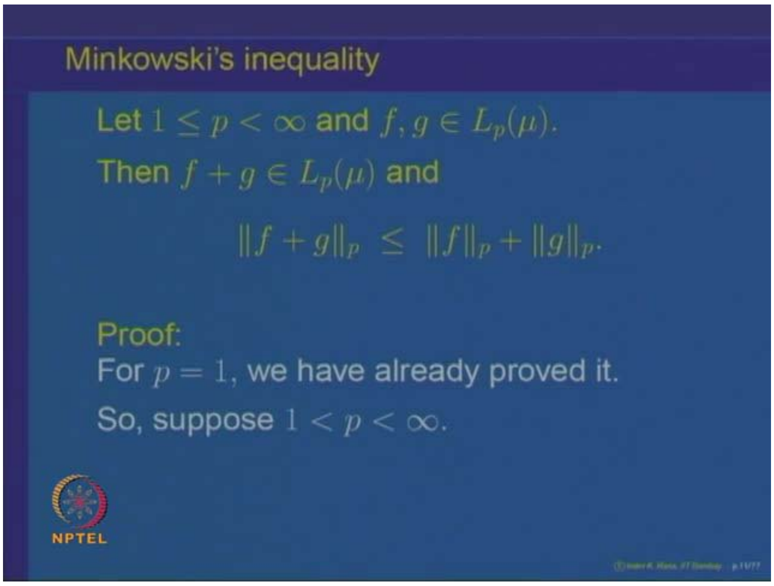
Suppose, a is not 0 and b is not 0, so then we can divide by A and B. Let us write a to be norm f, a to be absolute value of f divided by capital A. What is capital A? Recall capital A is the norm of f, so whole thing raise to power p and similarly, B is g x absolute value divided by the norm of g raise to power q and t equal to 1 over p.

So apply the Lemma, the Lemma will give us that A raise to power t is 1 over p, so mod f over A and the into g times ah into B times- B raise to power 1 minus t will give you g divided by B is less than or equal to t times so t is 1 over p applied n times A, so that is A plus 1 over q times B so that is (Refer Slide Time: 30:22).

Now integrate both sides with respect to mu, so integral of f g with respect to mu is less than or equal to integral of f p which is nothing but, A to the power p, so that cancels out. So, while integrate this cancels out; this cancels out, so this is 1 over p plus 1 over q and this A B you can take it on this side.

So, that gives you that integral of f x g x is less than or equal to A times B which is a norm of f times norm of g, so this is called Holders inequality.

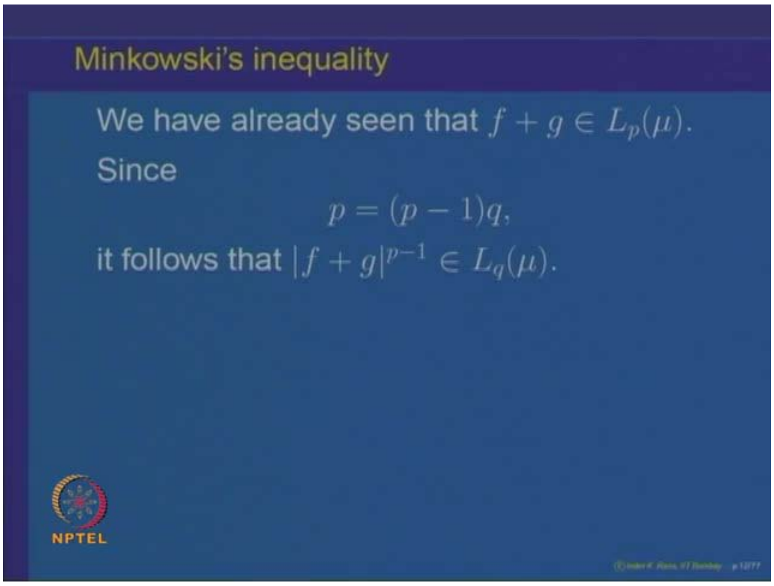
(Refer Slide Time: 31:00)



Using this inequality, we will prove another inequality which is called Minkowskis inequality which is essentially the triangle inequality for the L p norm. So, it says that f and g are in L p then of course, we have already shown that f plus g is in L p and the claim is that the norm of f plus g is less than or equal to norm of f plus norm of g.

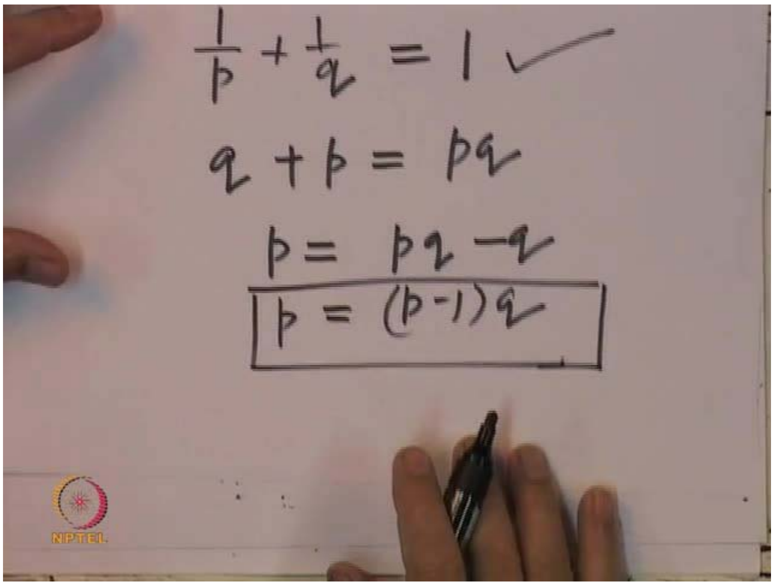
This we will prove using Holders inequality, so let us start the proof. So, for p equal to 1 we have already analyze the proof and seen it is easy.

(Refer Slide Time: 31:48)



So, let us assume p is strictly bigger than 1. So, when p is bigger than 1, let us look at the function we know f plus g belongs to L p. So, look at the function f plus g raise to power p minus 1, so note that we have the special relation between p and q namely 1 over p plus 1 over q is equal to 1 and that is same as saying the number p is also written as p equal to p minus 1 times q.

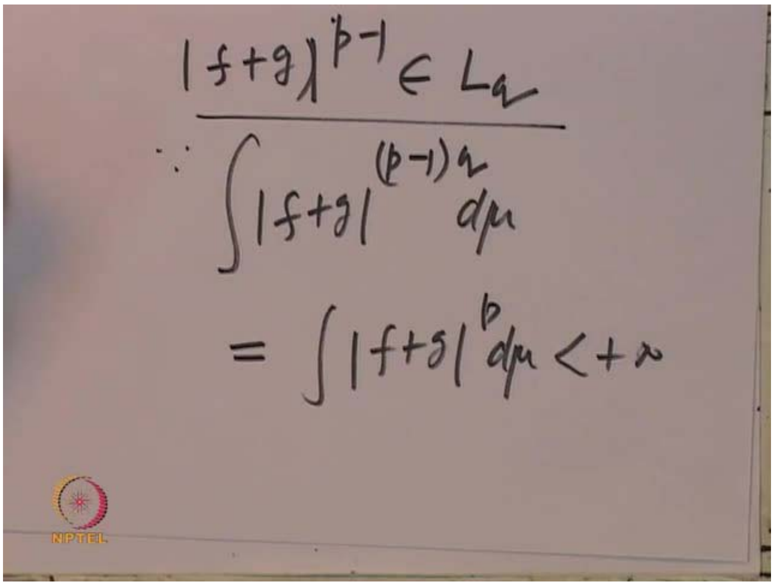
(Refer Slide Time: 32:31)



So that is following from This is from the inequality, so this is coming from the equality, so let us just recall that we have seen that 1 over p plus 1 over q is equal to 1. So that says cross multiply that says, q plus p is equal to p q and that says that p is equal to p q minus q, so that is same as saying from here q is common, so p minus 1 times q.

So that is one observation that if p and q have the relation 1 over p plus 1 over q is equal to 1 then p can be written as this (Refer Slide Time: 33:09).

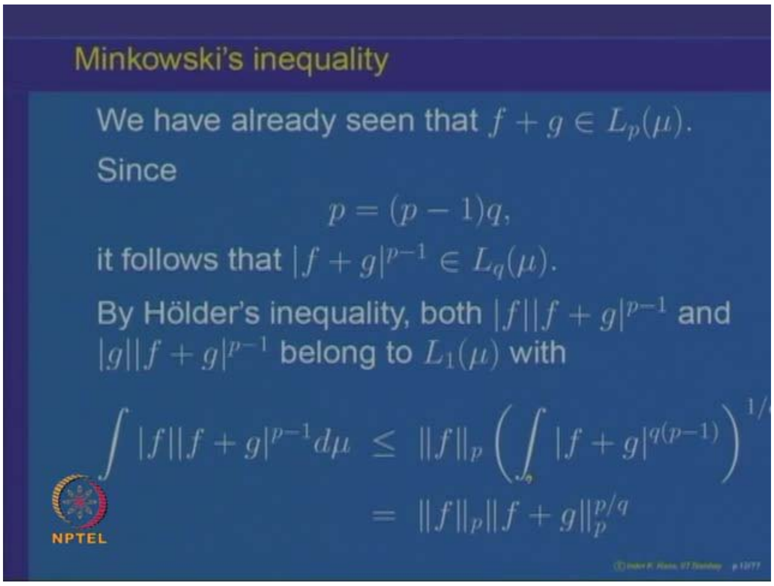
(Refer Slide Time: 33:21)



So once that is true, let us look at the function. So, consider the function which is mod f plus g raise to power p minus 1. We want to claim that this belongs to L q. So for that because the reason is mod f plus g raise to power p minus 1, so we now raise it to the power q integral d mu. So what is that? That is equal to integral of f plus g, p minus 1 into q that we have already seen p minus 1 into q is p, so that is equal to p d mu and that is finite.

So that proves that if f and g are in L p then mod f plus g raise to power p minus 1 is L q. So this observation will be use soon.

(Refer Slide Time: 34:21)

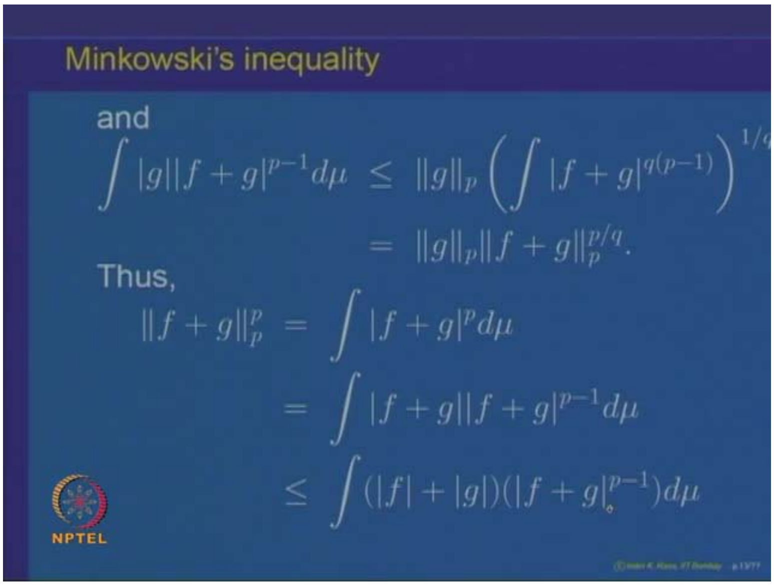


So let us write consider the Holders inequality with the functions mod f into f plus g raise to power p minus 1 and mod g into mod of f plus g raise to power p minus 1. Note, f is in L p this function is in L p and f plus g raise to power p minus 1 is in L q so by Holders inequality this function is integrable; its integral is less than or equal to norm of f plus norm of this function. Similarly, g is in L p and f plus g raise to power p minus 1 is in L q, so once again this product will be in L 1and Holders inequality will apply.

So, we start the proof by observing that mod f times mod of f plus g raise to power p minus 1; this function begin in L q, so this is in L p; this is in L q, so the product is L 1 so that will be less than or equal to the p th norm of f that is the p th norm of f plus the q th norm of this function mod f plus g raise to power p minus 1. So what is the q th norm so it is f plus g the function is to the power p minus 1 for the q th norm raise it to the power q the whole thing raise to power 1 minus q that is same as this p minus 1 so this is equal toah this 1 we have already seen it is equal to p so, this thing is norm of f plus g raise to p raise to power 1 over q.

So, this integral is nothing but, norm of f plus g raise to power p by q. So we get the inequality using Holders inequality namely, the product of the function mod f and mod f plus g raise to power p minus 1 is less than the p th norm of f times the p th norm of f plus g raise to power p by q.

(Refer Slide Time: 36:35)



A similar application of Holders inequality to the second function will give us that the mod of g times mod of f plus g raise to power p minus 1 is less than the norm of g times the p th norm of the second function which is nothing but, mod f plus g raise to power p by q.

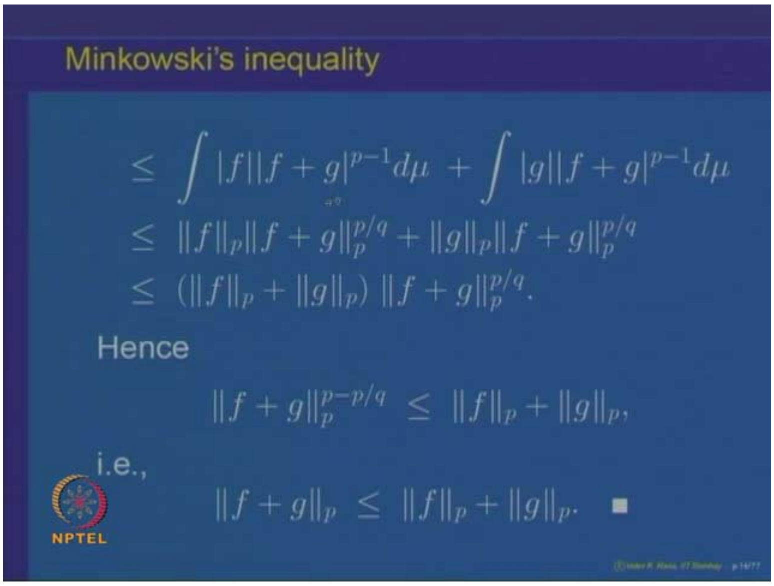
Now let us use these two to calculate the norm of f plus g. So to calculate the norm of f plus g let us raise it to power p, so the p th power of the norm of f plus g is nothing but, integral of mod f plus g raise to power p. Now the trick is that this power p, we write it as p into p minus 1.

So this integral is nothing but, integral of f plus g raise to power 1 into the same thing raise to power p minus 1, so this number mod f plus g absolute value is written as mod f plus g and times mod f plus g raise to power p minus 1.

Now the first absolute value f plus g we use triangle inequality this is less than or equal to mod f plus mod g, so this is by triangle inequality from here.

Now this integral can be split into two integrals so the p th power of the p th norm of f plus g will be less than or equal to integral of mod f times mod f plus g raise to power p minus 1 plus integral of mod g times that.

(Refer Slide Time: 38:17)

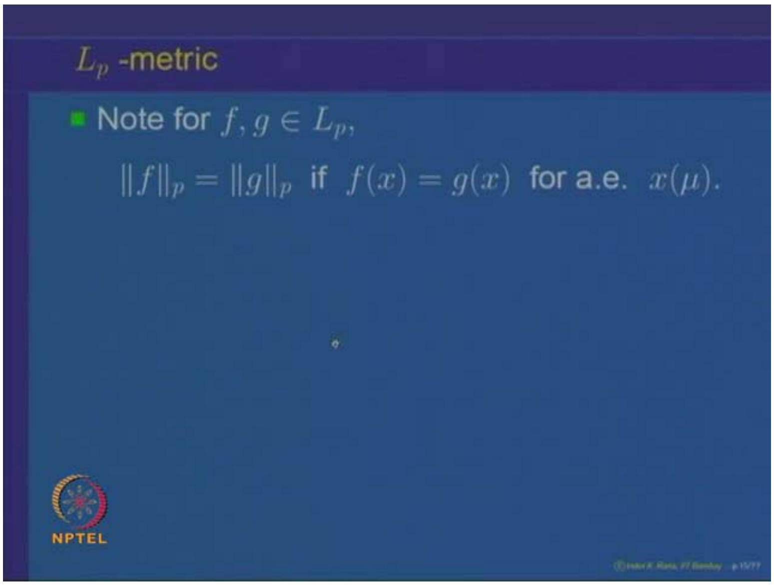


Then, we will use the earlier obtained bounds. We have got this is less than or equal to this integral plus this integral. Here we were using that Holders inequality the bound we have obtained, so this integral mod f times mod f plus g raise to power p minus 1 is less than or equal to the p th norm of f and the q th norm of this function and the q th norm of this function is mod f plus g norm raise to power p by q. Similarly, the second term is less than norm of g times the norm of f plus g raise to power p by q.

Now, this norm of f plus g raise to power p by q is common, so let us take it out. So this is less than or equal to norm f plus norm g times norm of f plus g raise to power p minus p by q. So on the left hand, if you recall we had the norm raise to power p and now on the right hand side, we have got the norm the one term is norm raise to power p by q. So take it on the other side so we get norm of f plus g raise to power p minus p by q is less than or equal to norm of f plus norm of g.

Now, using the fact that 1 over p plus 1 over q is equal to 1 realize that this number is nothing but p common. So 1 minus 1 over q that is equal to 1 over p, so that cancels so this number is equal to 1. So we get, what is called Holders inequality which is also same as the triangle inequality for the p th norm namely, norm of f plus g is less than or equal to norm of f plus norm of g, so that is called Minkowskis's inequality.

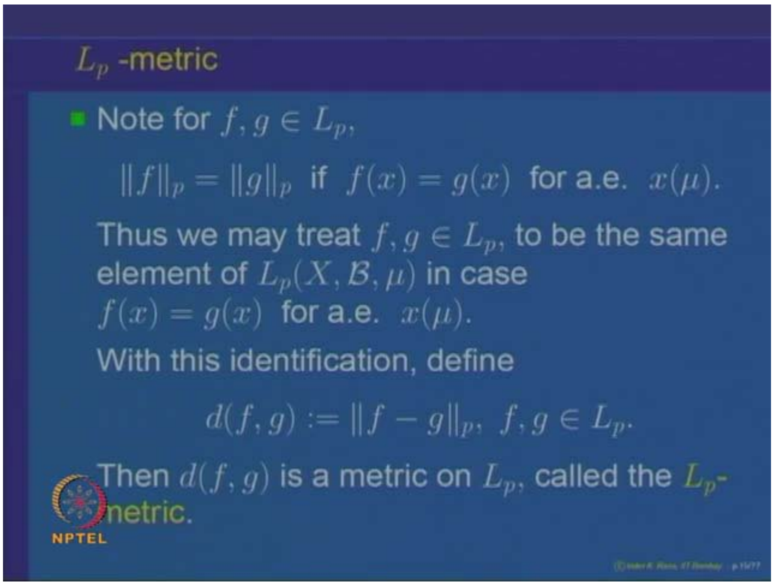
(Refer Slide Time: 40:08)



So, what we have shown is the following that for the space of p th power integrable functions it is a vector space one and secondly for every function f in this space we can define its L p norm which is the integral of absolute value of the function mod f to the power p the whole thing raise to power 1 over p. We have just now shown it has the three properties namely, the norm L p norm of a function is bigger than or equal to 0 and it is equal to 0 if and only if, the function is 0 almost everywhere.

Second property that the L p norm of alpha times f is equal to absolute value of the alpha times the norm of p and third the triangle inequality namely, norm of f plus g is less than or equal to norm of f plus norm of g.

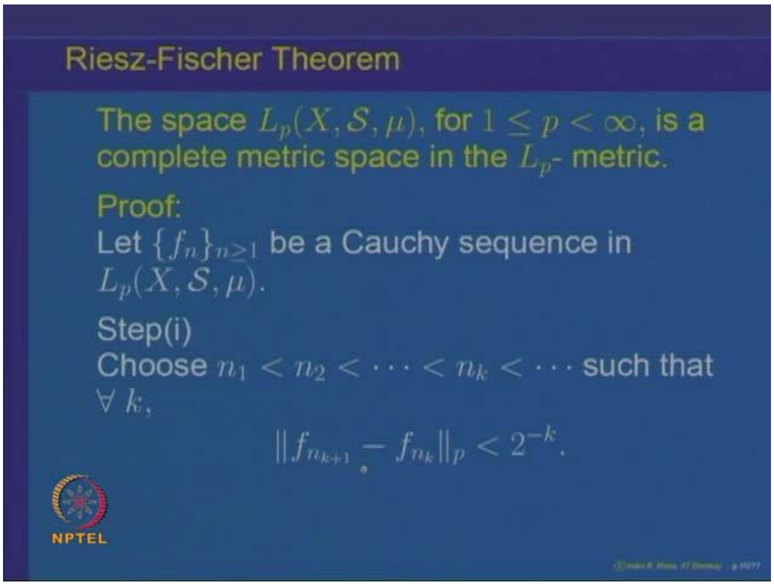
(Refer Slide Time: 41:21)



So as in the case of L 1, let us identify functions which are equal almost everywhere. So for f and g in L p, if we identify functions which are equal almost everywhere then, you observe that the norms of these two functions are same. So, norm of a function f is equal to the norm of a function g if f and g are equal almost everywhere. So, if we identify functions which are equal almost everywhere will get their norms to be same.

So, we will do that, so in L p we will not distinguish between functions which are equal to 0 almost everywhere. So with that understanding, let us define the distance between two functions in L p to be d f g equal to norm of f minus g with respect to p. So, once we do that this becomes a metric on L p, so d f g is a metric on L p because d f g equal to 0 if and only if f is equal to g almost everywhere and we have identify function which are equal almost everywhere.

(Refer Slide Time: 42:35)

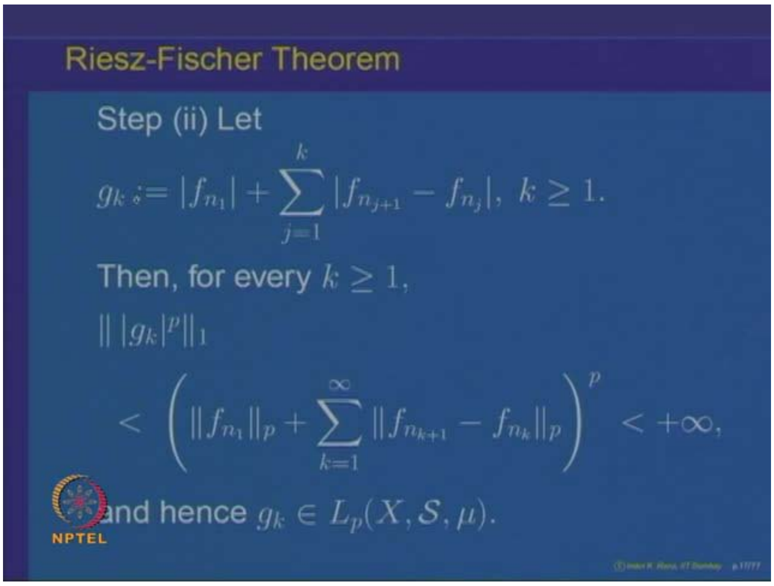


So, this becomes a metric and it is called the L p metric on the L p space. Like L 1 claims what is called Riesz-Fischer theorem namely, the space L p is a complete metric space under this metric L which is called L p metric. The proof of this theorem is verbatim same as the proof for saying that L 1 of a b is complete. So, we will just sketch the proof and ask you to verify the steps which will also help you to revise the earlier proof and if you still have difficulty you can refer to the text book.

So, the steps of the proof are as follows. Let us take a Cauchy sequence in L p, we want to show that this Cauchy sequence is convergent in L p. So, for that the first step is because this sequence is a Cauchy sequence, we can select integers positive integers n 1 less than n 2 and less than n k such that.

The consecutive difference between f n k plus 1 and f n k is less than 2 to the power minus k and because Cauchy has says, the elements of the sequence are going to come closer and closer as you go to infinity. So, using that we can select inductively integers n 1 less than n 2 less than n k and so on such that. The difference between f n k plus 1 and f n k is the norm is less than 2 to the power minus k.

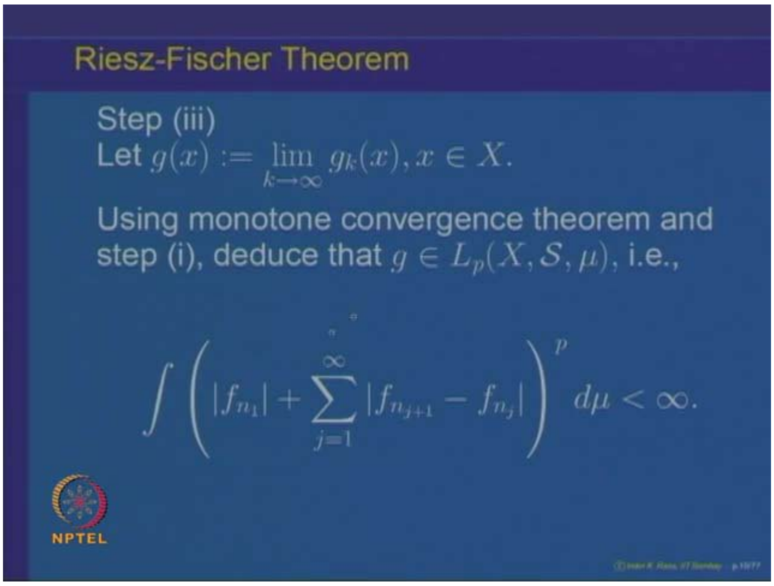
(Refer Slide Time: 44:17)



Once that is done, let us define g k to be the function mod f n 1 plus the sums 1 to k of the differences f n j plus 1 minus f n j for every k, we define the sequence. Then, this g k is a function claim is that if you take the power of this to the power p of this then this function g k to the power p L 1 norm is actually, we should be defining this modification. We should make f n 1 to the power p plus f n j plus 1 minus f n mod to the power p. So this one will require that you define, so there is a misprint here this should be mod f n to the power p plus mod of f n j plus 1 minus f n j absolute value to the power p, so that is g k.

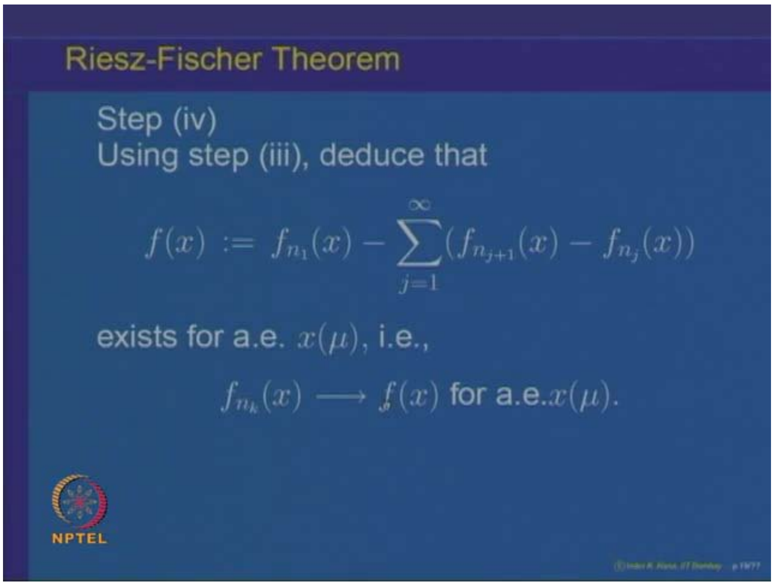
So once that is there, if you integrate both sides then we get that this is less than or equal to this power, p is here; so integral of the power p, so that gives the norm of the function f n 1 to the power p and this to the power p that is finite. Hence, this g k s will belong to L p space.

(Refer Slide Time: 45:49)



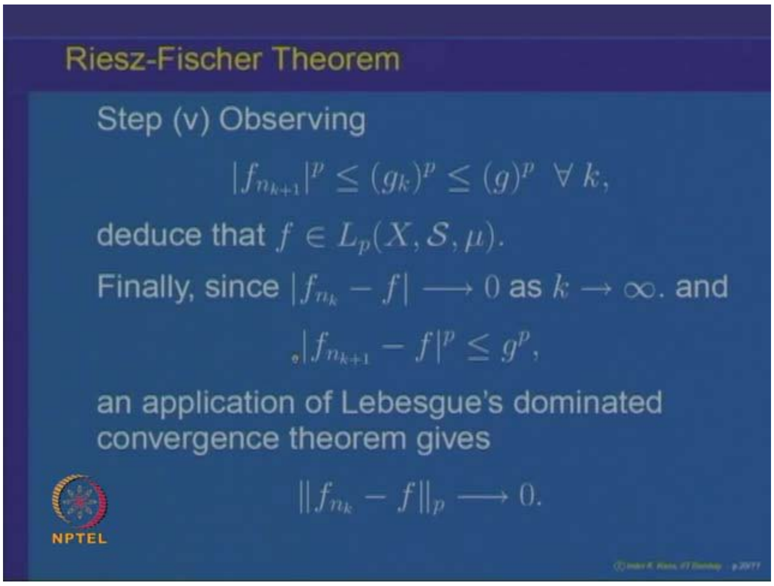
Now as a consequence of monotone convergence theorem, the series form if we define g x then, using series form of the dominated convergence theorem one shows that this function g also is in L p.

(Refer Slide Time: 46:18)



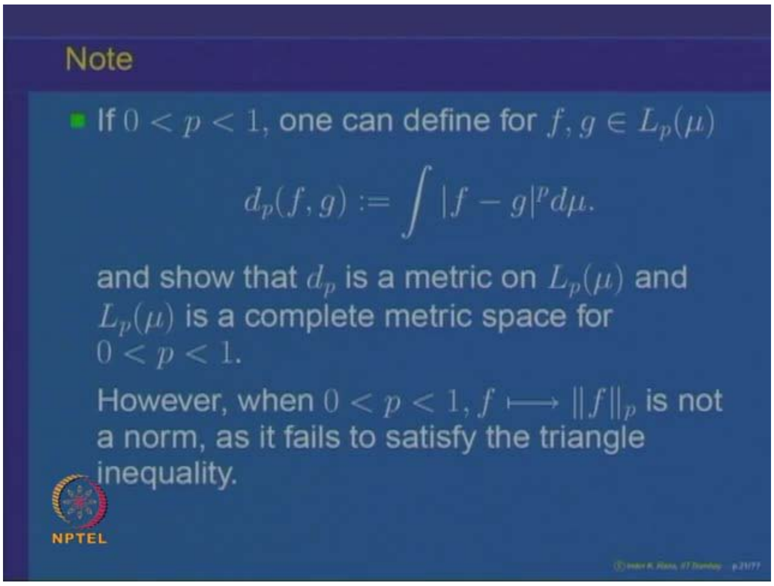
Hence, deduce that this integral of mod f n 1 so g k this is finite and hence the function f x which is the partial sum which is the sum of the series f n 1 x minus this is exist almost everywhere. So as a consequence, we will get this f n k because the partial sums are just f n k that converges to f of x.

(Refer Slide Time: 46:39)



Now an application of dominated convergence theorem again gives you that f belongs to L p and the integral of mod f n k plus 1 minus f to the power p converges to 0, so norm converges ((to 0)). So, the proof is essentially the same as that of L 1. So, I have just outlined the steps try to prove this steps yourself and convince that it is ok. So, this is what is called the Riesz-Fischer theorem, so this is we have proved it for p bigger than 1 and less than infinity.

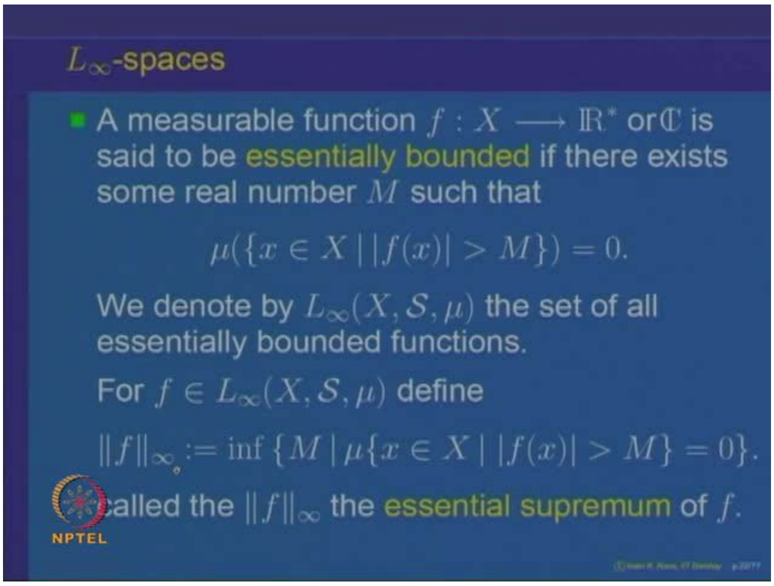
(Refer Slide Time: 47:15)



One can ask the question that what happens for this number p between 0 and 1? For p between 0 and 1, one can define these spaces called L p spaces and show they are metrics spaces. However there is a problem that if you try to define the norm as integral of mod f to the power p raise to power 1 over p that does not satisfy the triangle inequality.

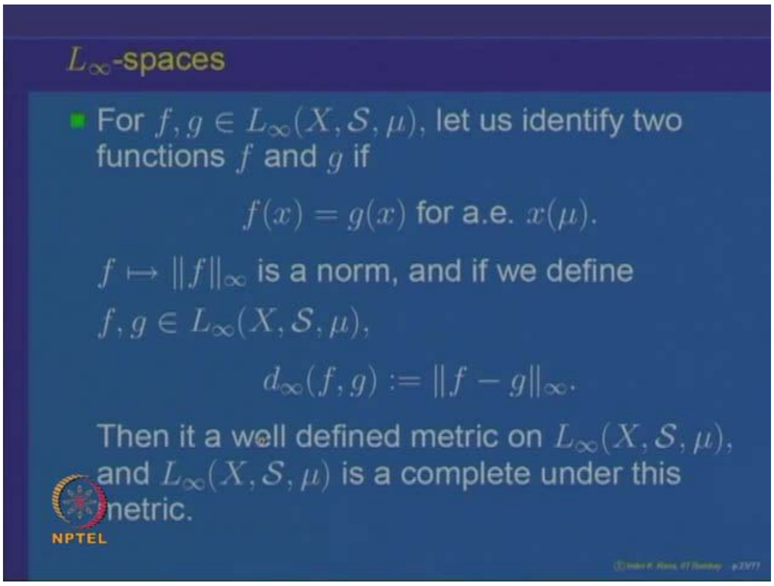
So, one can define a metric d f g to be for p between 0 and 1. One can define the metric to be the integral of the absolute value f minus g to the power p but, that does not really help. It becomes a metric, one can show it is complete also but, the problem comes that f going to norm if you define that norm; it is not a norm, so integral of mod f to the power p is not a norm. So the triangle inequality fails. So, that is why for p between 0 and 1 these spaces are not very interesting for applications point of view.

(Refer Slide Time: 48:30)

**


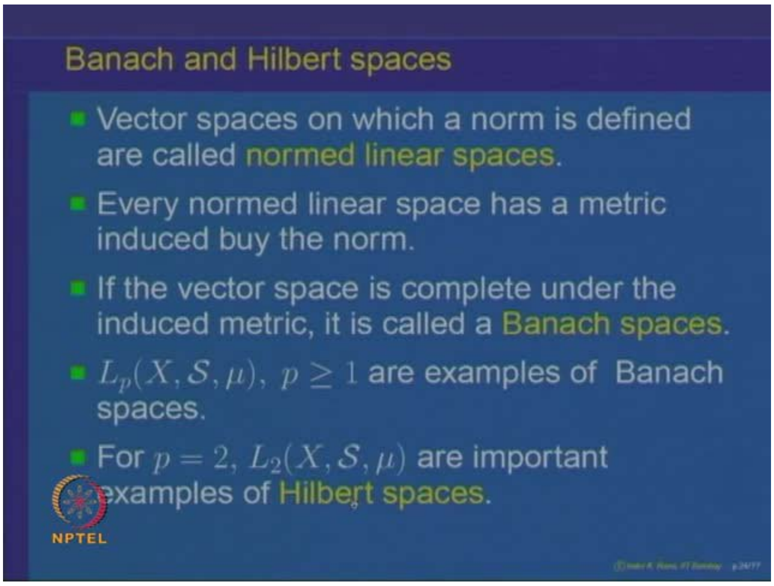
Another observation, one can also define what are called L infinity spaces namely, you look at - say function f defined on x is essentially bounded if there exists a real number m such that the measure of the set where mod f x is bigger than m is equal to 0. So you collect together functions which are essentially bounded and call them as L infinity. So, one can show that L infinity is a vector space and if L defines for a function f in L infinity what is called the infinity norm L infinity norm to the infimum of these constants m such that the measure of the set where f x is bigger than m is equal to 0.

(Refer Slide Time: 49:23)



Then, one can show that this indeed is a norm on L infinity it is called the essential supremum of f and one shows that if you identify functions f and g to be same if equal almost everywhere then, this f going to L infinity is a norm. So, that gives a metric namely, the distance between f and g to be norm of f minus g. This indeed is a metric and one can show that L infinity becomes a complete metric space like L p for p bigger than or equal to 1.

(Refer Slide Time: 49:59)



So, these are examples of metric spaces which arise out of measure theory. So, the importance of this lies in the following fact, so whenever we have 1 given a vector space and a norm is defined on that space is called a Norm linear space. Every norm give raise to a metric, so the metric being the distance between f and g to be the distance, the norm of f minus g, so that gives a norm. If under that norm- under that induced metric induced by that norm if this vector space becomes complete metric space then the space is called a Banach space.

So, L p space is p bigger than or equal to 1 including L infinity are examples of Banach spaces. For p equal to 2 is a very special space then, one can even define the notion of angle and relate distance and angle so, L 2 give the examples of a Hilbert space.

So today, we have looked at L p spaces which give very important examples of norm linear spaces in fact Banach spaces and also L 2 gives examples of a Hilbert space. All these spaces play an important role in the subject of functional analysis and in harmonic analysis. So, if you go for higher studies, you will come across these spaces again in your studies. So, let me stop here today. Thank you.