Measure and Integration

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Module No. # 09

Lecture No. # 33

Integrating Complex-Valued Function

Welcome to lecture number 33 on measure and integration. From today onwards, we will be looking at some special topics in measure and integration. We will start with looking at how to define integral for complex-valued functions defined on measure spaces.

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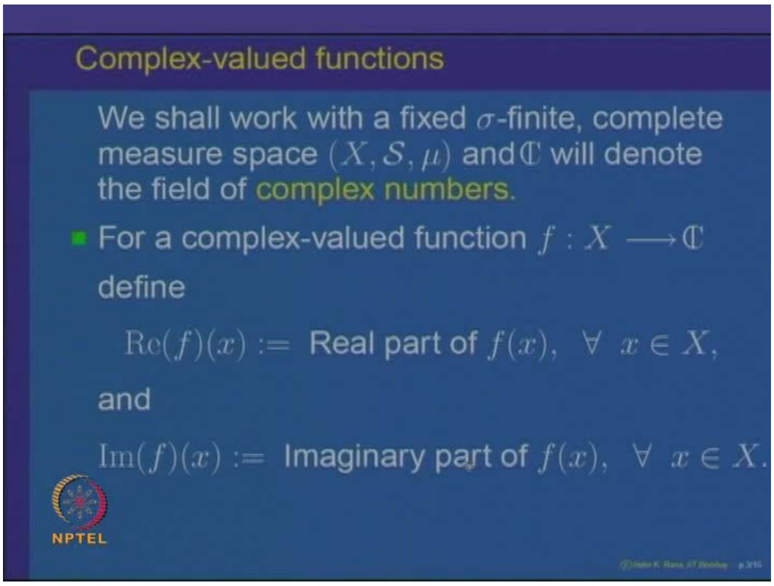

Topic for today's discussion is going to be integrating complex-valued functions. For the coming one or two lectures, we will be fixing a measure space X S mu, which is sigma finite and complete.

All the discussion will be on a fix sigma finite complete measure space X S mu. We will denote by this letter C with a line in between that is called script C to be the field of complex numbers.

For a complex-valued function f defined on X taking values in C. We say that, we define its real part and imaginary part. For any point X f of X is going to be a complex number because its values are in the complex field.

That complex number f of X has got a real part and imaginary part. We define real f at a point x to be the real part of the value f of x. Similarly, imaginary f at x to be the imaginary part of the value f of x.

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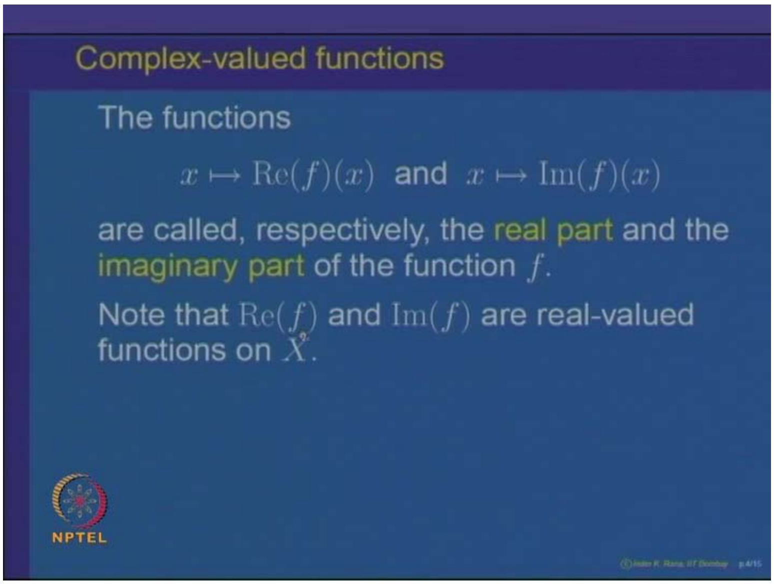

Let us observe that x going to real f x and x going to imaginary f of x are real valued functions. The first one is called the real part of the function f and Im f is called the imaginary part of the function f.

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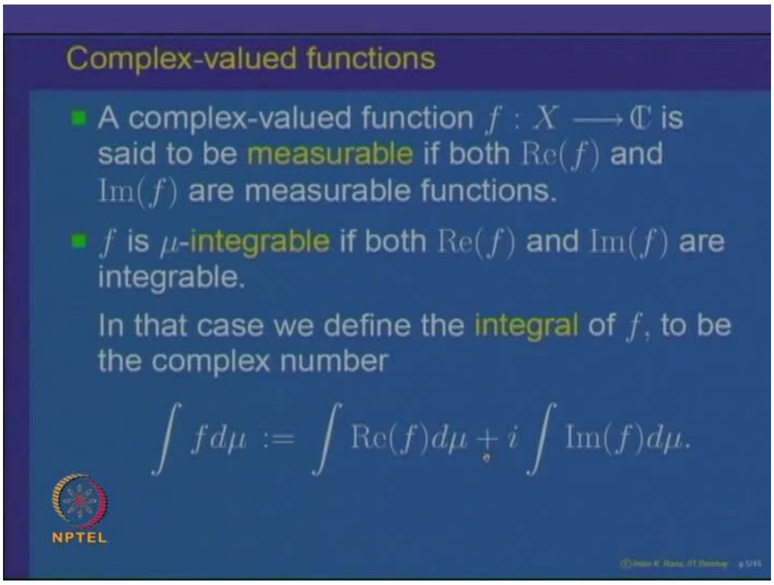

Every function f, which is complex valued has got a real part function and the imaginary part function and both are real valued functions.

Let us define, what is called the measurability of a complex valued function? A complex valued function f from X to C is set to be measurable, if both the real part of f and the imaginary part of f are measurable functions. Note that, f is equal to the real part f plus imaginary part of f. If both real part of f and imaginary part of f, which are real valued functions are measurable on x with respect to the sigma algebra S. Then, we say that the function f itself is a measurable function.

We will say, f is integrable with respect to the measure mu, if both the real part f and imaginary part f are integrable functions. Real part f is a real valued function and imaginary f is also a real valued function. If both of them are integrable as real valued functions, then we say that is the function f which is complex valued is integrable. We define the integral of f to be denoted by the symbol integral f d mu to be equal to integral of the real part of f plus i times the integral of the imaginary part of f, where this i is the square root of minus 1, which is normally used to write complex numbers.

For a complex-valued function f, we define it to be measurable if both real part and the imaginary part are measurable. Similarly, we define f to be integrable; if both the real part and the imaginary part of f are integrable.

In that case, we define the integral of f to be equal to integral of the real part plus i times the integral of the imaginary part of the function f. So, that is the definition of the integral of a complex valued function.

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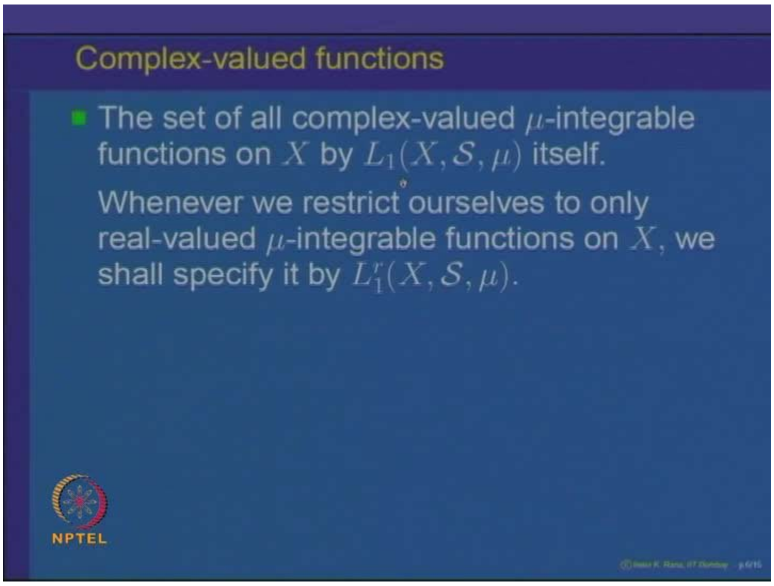

We will study properties of this integral. Let us also denote the set of all complex valued integrable functions on X by L 1 X S mu. That is the symbol; we had used to denote the real valued integrable functions. We will denote the space of complex-valued integrable functions on X S mu by the same symbol.

In case, we are referring to specifically the real valued functions, we will put a suffix r L lower 1 upper r X S mu. That will denote the space of all real valued integrable functions on X S mu.

Whenever need be and we want to specify that we are in the space of real valued functions integrable, we will use this symbol. Otherwise, the space of all complex-valued integrable functions will be denoted by the symbol L 1 X S mu.

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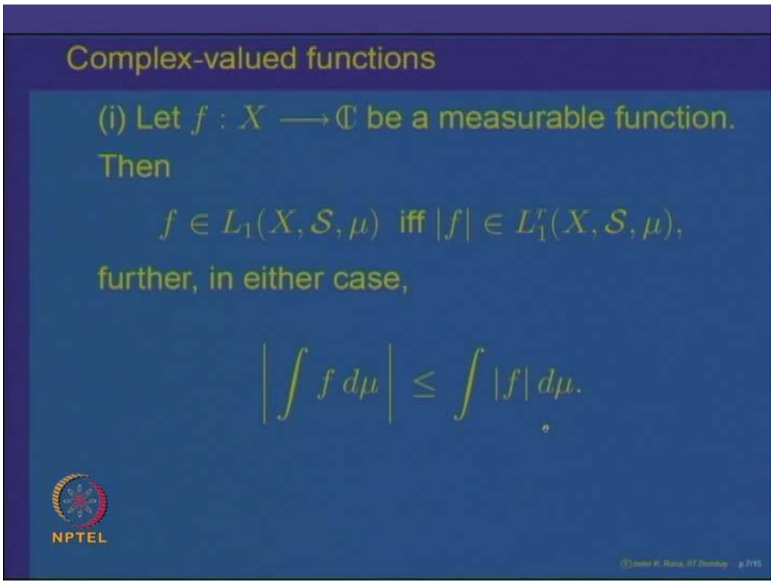

Next, we will like to study properties of integral, we will like to study the space L 1 of X S mu the space of integrable functions.

Let us start with it is a very basic property. Let us take a function f, which is a measurable function. f is a complex-valued measurable function on S, then the first property as for the case of real valued functions is that f is integrable if and only mod f, which is a real valued function is integrable with respect to is real valued integrable function. Further in that case, we want to claim that the integral of mod f modulus of the integral is less than or equal to integral of the absolute value.

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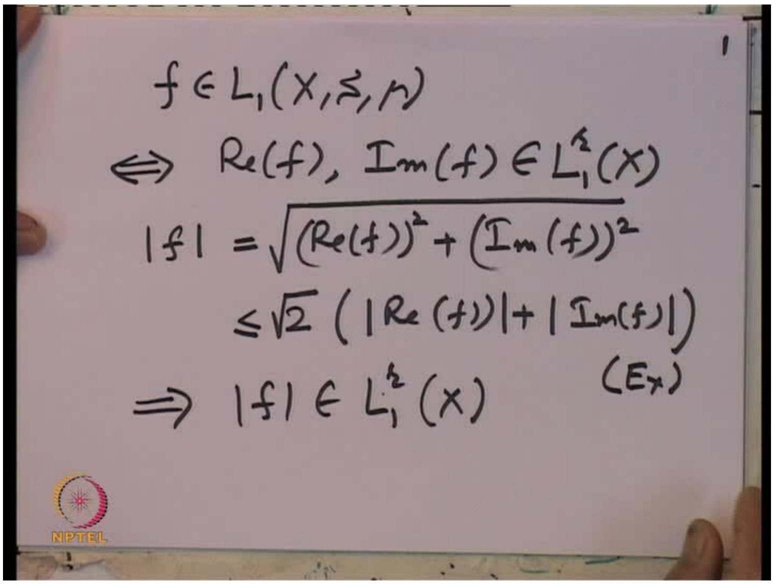

Let us look at how one proves these properties. We have got a function f, which is L 1 of X S mu. That means, what is the definition of this? That says that the real part of f and the imaginary part of f are both integrable functions L 1 of X.

Now, let us consider look at the function absolute value of f. The absolute value of f is the real part of f plus the imaginary part of f square root that is the definition of because it is a complex-valued function.

So, absolute value of f can be written this way and from here, it is easy to see that this is always less than or equal to square root 2 times the absolute value of the real part of f plus absolute value of imaginary part of f.

This is a very easy inequality about complex numbers. It is basically saying that if you take a complex number, then it is always less than or equal to square absolute value of the complex number is always less than or equal to square root of 2 times the real part plus the imaginary part. One way of looking at this would be, if you look at this definition, this is real part of f square plus imaginary part of f square.

This will be less than or equal to 2 times the maximum value of real part or the imaginary part and that is less than or equal to the real absolute value of the real part plus the absolute value of the imaginary part.

This term in the under square root you can easily see it is less than or equal to 2 times the absolute value of the real part of f plus imaginary part of f. So, square root is less than or equal to this quantity.

This is a very easy in equality to prove for complex numbers. Let me just write it as an exercise that you verify it yourself. Now, because f is integrable, real part of f is in L 1 imaginary part of f is in L 1. These are both integrable functions, mod f is a nonnegative real valued function less than or equal to 2 times an integrable function. This will imply that mod f belongs to is a real valued integrable function.

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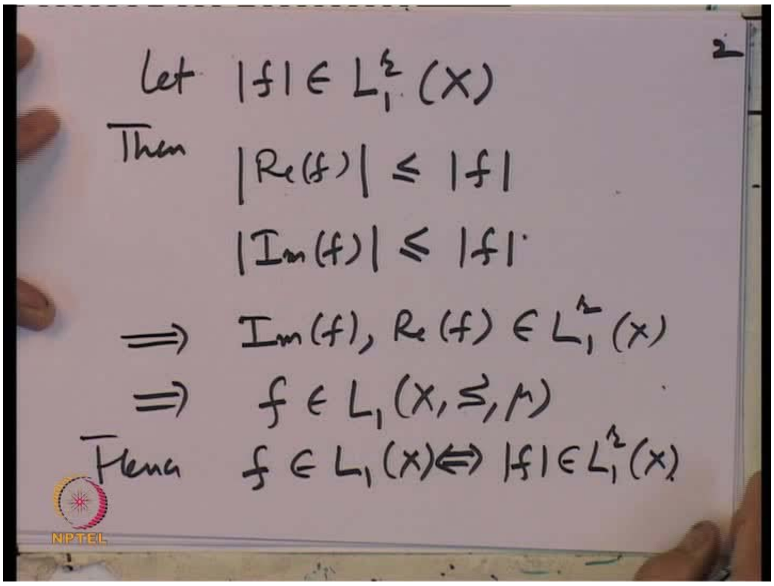

This is in equality implies that this is integrable with respect to mod f. If f is integrable that implies mod f is a real valued integrable function. Let us look, at the converse part. Let us suppose, conversely mod f be integrable so is L 1 r of X.

Then, the real part of f absolute value is less than or equal to absolute value of f. That is very simple straightforward inequality that for any complex number the real part is less than or equal to absolute value of f. Similarly, the imaginary part of f is also less than or equal to absolute value of f.

These are real valued functions and they are less than or equal to a function mod f, which is integrable. That implies that the imaginary part of f and real part of f both are L 1 real valued integrable functions and hence this implies that f is integrable.

We have proved and so this proves. Hence, f belonging to L 1 of X if and only if mod f belongs to L 1 R of X that proves the first part of the statement; f is integrable if and only if mod f is integrable.

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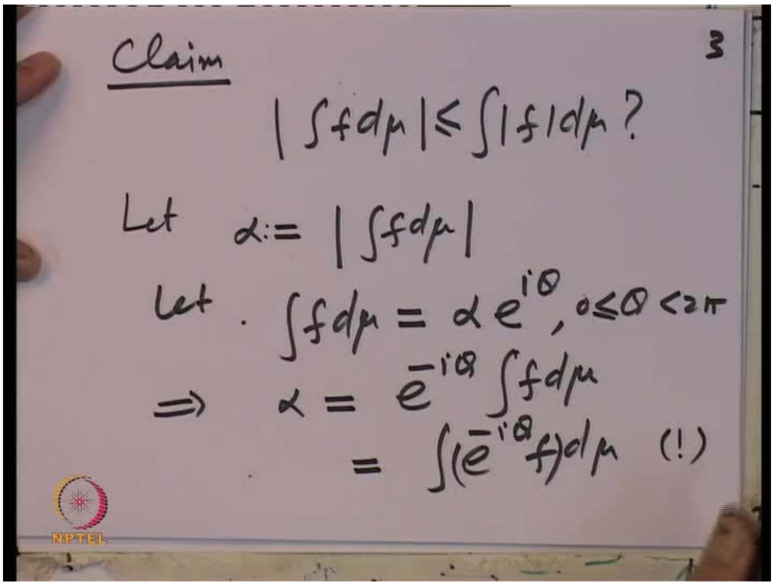

Let us look at the second part, we want to show that mod of integral f d mu is less than or equal to integral of mod f d mu. This is what we want to show. Let us write, let us denote by alpha, the number on the left hand side that is absolute value of f d mu. Note that, f d mu is a complex number and its absolute value is denoted by alpha. We can write, let integral f d mu, which is a complex number. Its absolute value alpha, you can write it as alpha times e raised to power i theta for some theta between 0 and 2 phi.

Every complex number can be written as its absolute value at times e raised to power i theta or some theta between. So, that implies that alpha is equal to e raised to power minus i theta times integral f d mu.

We will just show in the next property that integral of f a scalar multiple is same as I can write this as e raised to power minus i theta times f d mu. We can write this as this. We will just in a minute, we will prove this property that for any scalar complex scalar times the integral of f is same as integral of scalar time’s f whenever f is integrable.

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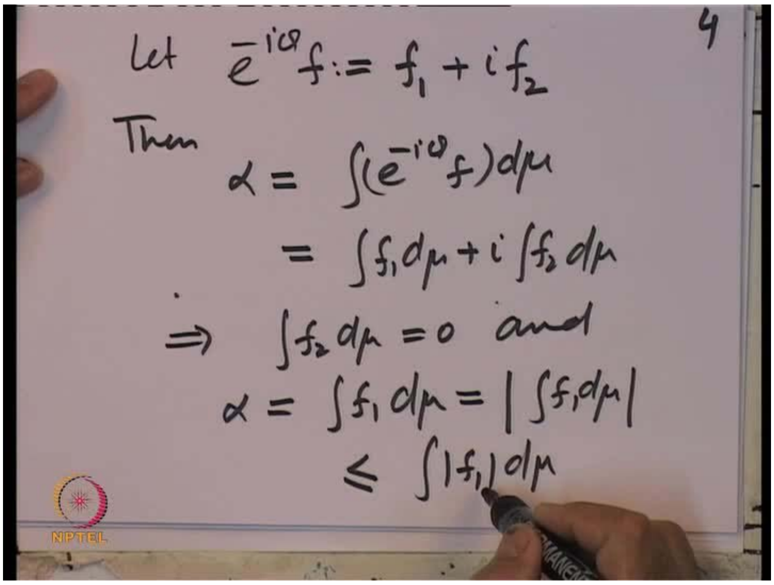

By using that property, we will just put it that we are going to prove this property soon. This is equal to this. Let us write, e raised to power minus i theta f, it is a complex-valued function. So, it will have the real part f 1 plus i time the imaginary part f 2. Let this be equal to this.

Then, alpha which is equal to e raised to power minus i theta f d mu. Its integral, I can write it as integral of f 1 d mu plus i at times integral of f 2 d mu because this function e raised to power i theta f has got real part f 1 imaginary part f 2. So, its integral must be equal to integral of the real part plus i times integral of the imaginary part.

Note, alpha is real because what was alpha? alpha was nothing but the absolute value of the integral of f d mu. It is a nonnegative real number; because it is a nonnegative real number and hence, we are writing it as an integral f 1 d mu plus i times integral f 2 d mu. The imaginary part must be 0. That implies that integral f 2 d mu is equal to 0 and alpha must be equal to integral of f 1 d mu and alpha is nonnegative. That implies, I can write this also equal to absolute value of integral f d mu because alpha is a nonnegative real number.

That is equal to f 1 d mu and that for real valued functions, we know this is less than or equal to integral of mod f 1 d mu.

Here, we are using the property that for a real valued integrable function, integral of f 1 d mu absolute value is less than or equal to absolute value of the integral. So, that is this.

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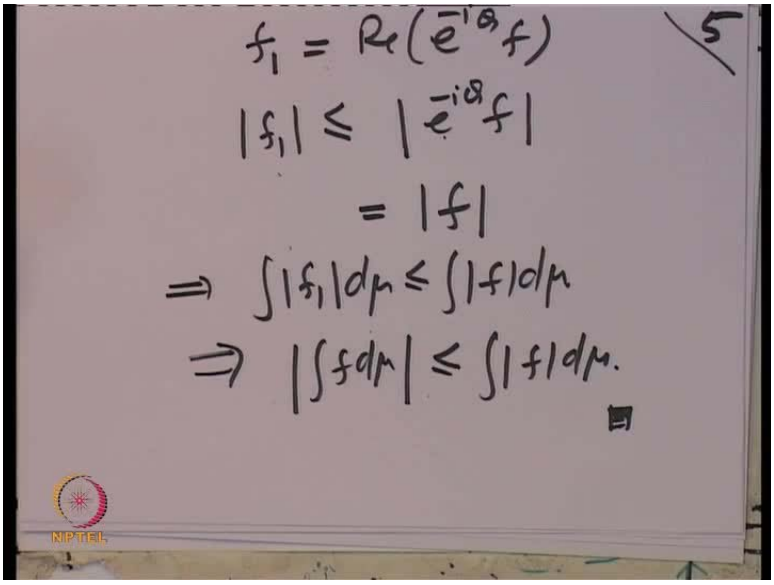

As a next step, let us observe f one was real part of e raised to power i theta times f that was real part of this function. That means, absolute value of f 1 is less than or equal to absolute value of e raised to power minus i theta of f. Real part of any complex number is less than the absolute value of that complex number but e raised to power i theta is a complex number of absolute value 1. So, this is same as mod of f.

Absolute value of f 1 is less than or equal to absolute value of f that implies that integral of mod f 1 d mu is less than or equal to integral of mod f d mu. Now, let us combine these two facts that alpha was equal to less than or equal to integral of f 1 d mu and integral f 1 d mu is less than or equal to integral of mod f d mu. That implies, these two facts together imply that integral of f d mu, which was equal to alpha is less than or equal to integral of mod f d mu.

That proves the fact that f is integrable if and only if mod f is integrable and integral of absolute value of the integral f d mu is less than or equal to integral of the absolute value of the function.

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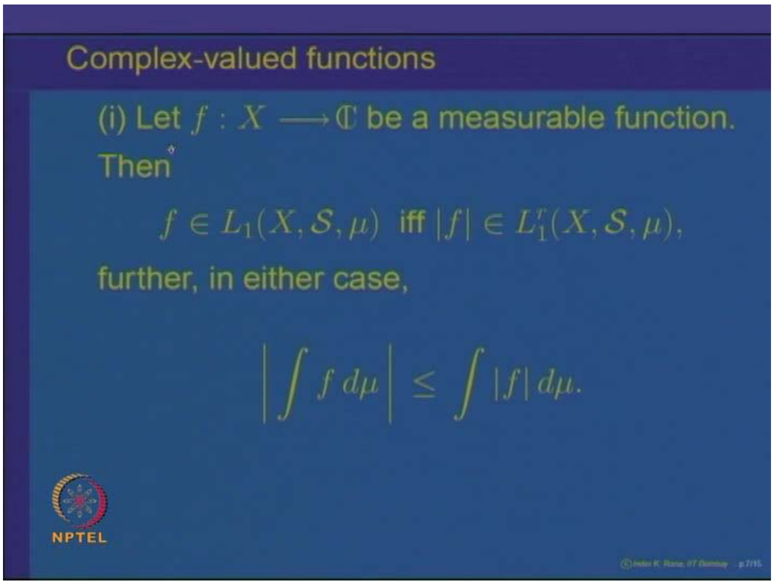

This proves the first property about integrals that for a integrable function, a complex valued function f is integrable if and if only absolute value of this complex-valued function, which is a real valued function is integrable. Absolute value of the integral f d mu is less than or equal to integral of f d mu.

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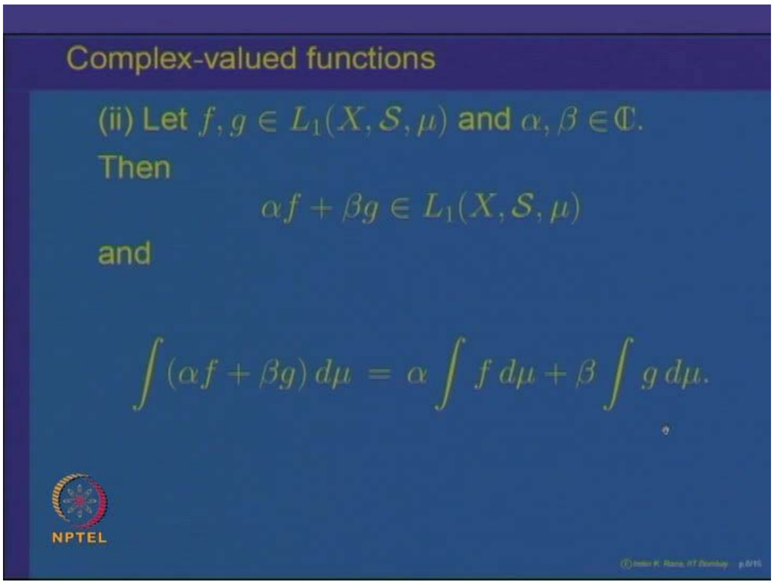

That proves the first property. Next, we want to prove the linearity property. Part of which, we have used already in the previous proof that if f and g are complex-valued integrable functions and alpha and beta are complex numbers then alpha f plus beta g, the linear combination is also integrable and integral of alpha f plus beta g d mu is equal to alpha times integral f plus beta times integral g.

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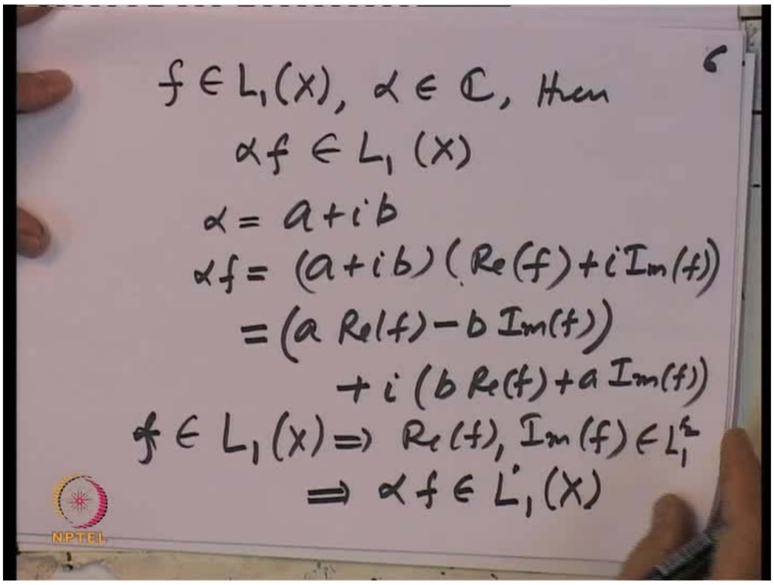

To prove this inequality, we will split the proof into two parts. As a first part, let us prove that if f belongs to L 1 of X and alpha is a complex number then alpha times f belongs to L 1 of X. Let us prove this part first because alpha times let us look at proof of this so let us write alpha as a plus ib it is a complex number.

Let us write it as a plus ib and alpha f is equal to a plus ib times real part of f plus i times imaginary part of f. Which on expansion, I can write as a times real part of f minus b times imaginary part of f plus i times from here, I will get b times real part of f and from the other one we will get a times imaginary part of f.

So, the complex-valued function alpha f is written as its real part is a real f minus b imaginary f and its imaginary part is b real f plus a imaginary f.

Since, f is integrable, which implies that real f and imaginary f are both real valued integrable functions.

They are real valued integrable functions. So, real f imaginary part f is integrable and a times real f is integrable, b times imaginary f is integrable. So, the difference is integrable. That means the real part of the function alpha f is also a real valued integrable function.

Similarly, the imaginary part of alpha f, which is b times real f plus a times imaginary f is also integrable because real f and imaginary f are real valued integrable functions.

It implies that alpha f is L 1 function and what is the integral of it so I can write now the integral of this function, integral of alpha f d mu.

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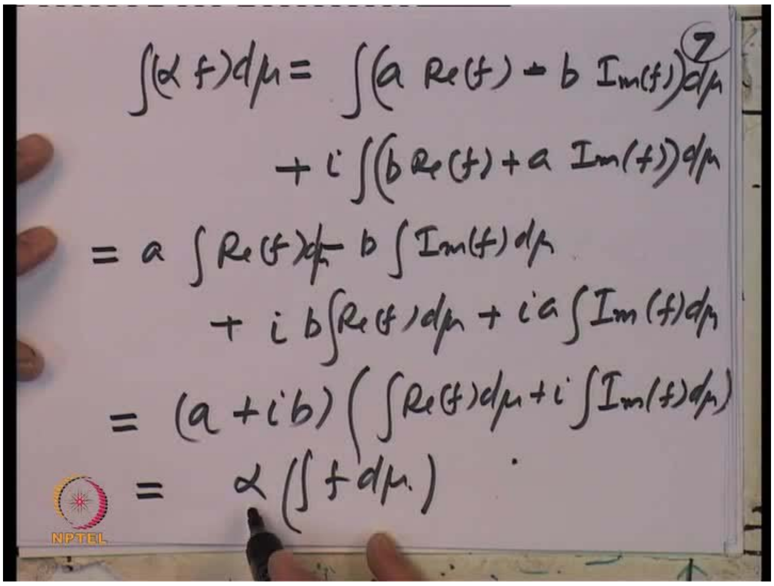

By definition, it is integral of the real part plus i times the integral of the imaginary part. It is integral of a real f minus b times imaginary f d mu plus i times integral of the imaginary part of the function, which is b times real f plus a times imaginary f d mu. So, that is integral of alpha f.

Let us use the properties that integrals is a linear operation for real valued function, the first integral I can write it as a times integral of real part of f minus b times integral of imaginary part of f d mu. That is the first one plus i times b real part of f integral plus i times a integral of imaginary part of f d mu.

Using linearity, we have split it into four parts but now it is easy to check that this is nothing but a plus ib times integral of real part of f d mu plus i times integral of imaginary part of f d mu. Just open off the bracket and that is same as this. This means that is equal to alpha times integral of f d mu.

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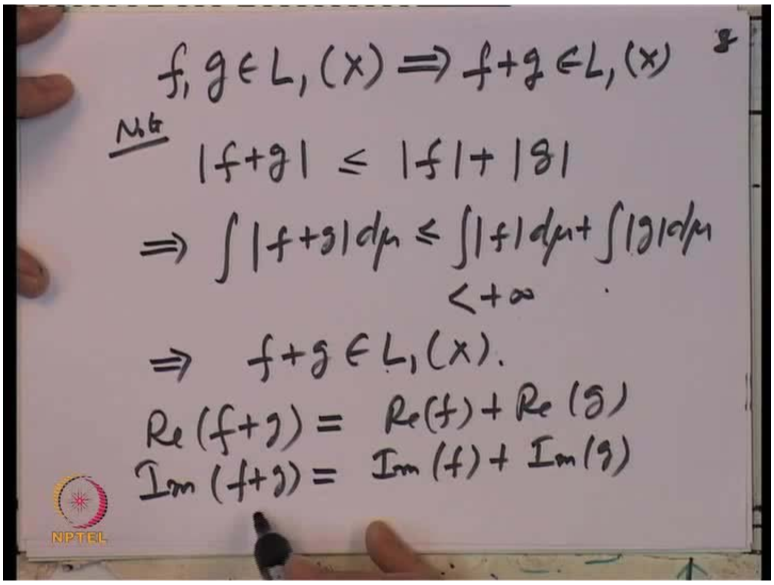

We have shown that integral of alpha f d mu is equal to alpha times integral of f d mu. That is the first part of the linearity property that if I take a function f which is L 1 and multiply it by scalar alpha. Then that is also integrable and integral of alpha f is equal to alpha time's integral of f.

Now, let us look at the second part of requirement if f and g belong to L 1 of X. Then that implies that their sum is also in L 1 of X and the integral of f plus g is equal to integral f plus integral g. So, that is easy to verify because mod of f plus g we want to show that is integrable.

Let us look at note that, as in real case is less than or equal to absolute value of f plus absolute value of g. By triangle inequality property, integral absolute value of f plus g is less than or equal to absolute value of f plus absolute value of g.

That implies these are all real valued functions nonnegative that will imply integral of f plus g d mu is less than or equal to integral mod f d mu plus integral mod g d mu and both of them are integrable. That is finite, which implies that f plus g is an integrable function.

To compute the integral of f plus g that is simple thing. Let us observe to compute the integral; we note that the real part of f plus g is nothing but real part of f plus real part of g that is easy to verify. The imaginary part of f plus g is equal to imaginary part of f plus imaginary part of g.

Because f is integrable, the first one will give you that real part f plus real part g is a real valued integrable function. So, real part of f plus g will be integrable. Similarly, imaginary part of f plus g is also integrable and further we can write down the integrals.

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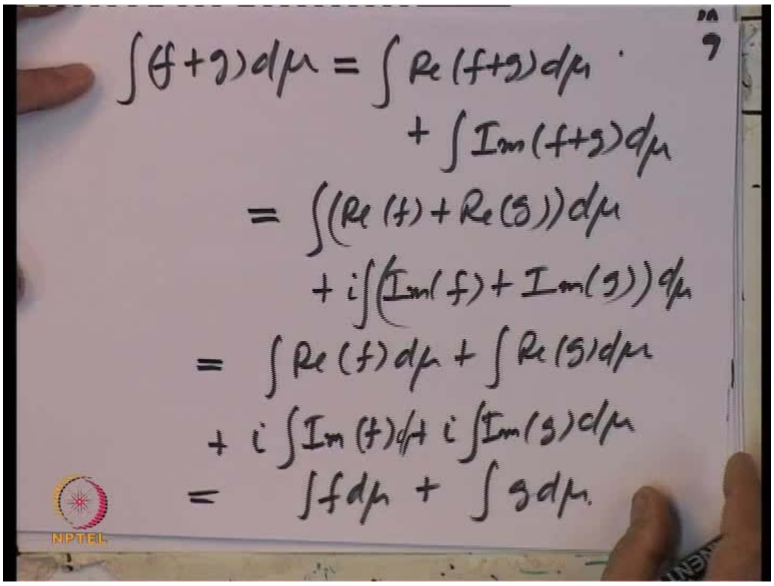

The integral of f plus g d mu by definition is equal to integral of the real part of f plus g plus integral of the imaginary part of f plus g. But real part of f plus g is real part f plus real part g , the first one is real part of f plus real part of g d mu plus integral imaginary part of f plus imaginary part of g d mu.

That follows from the simple result that we have just now shown. Now, these are all real valued functions, integration is linear. That splits into four integrals, integral of real part of f d mu plus integral of real part of g d mu. There is i here because real and imaginary plus i times integral of the imaginary part of f plus i times integral of imaginary part of g d mu.

Now, the first and the third term combined together will give you so that is equal to integral of f d mu plus integral of g d mu.

Integral of f plus g is equal to integral f plus integral g. That proves property 3 completely namely integral of f plus g is equal to integral f plus integral g.

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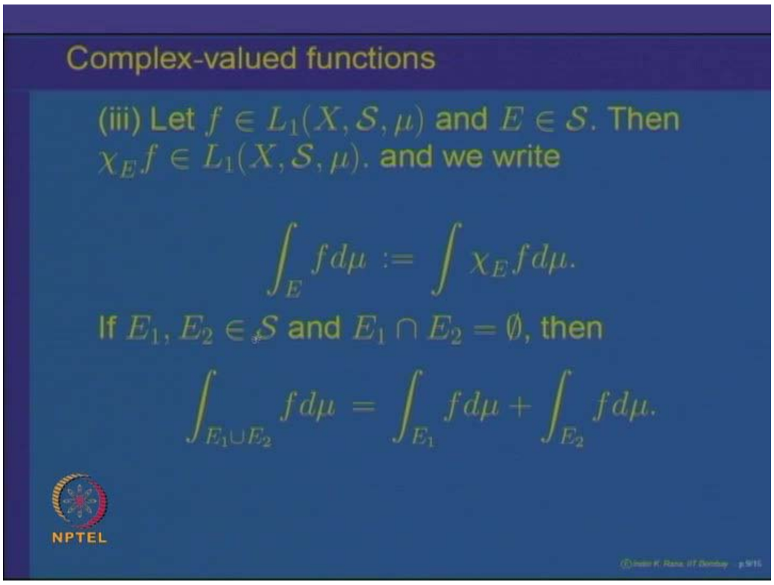

Let us look at some more properties. Let us look at the third property, if f is a integrable function and E is any measurable set then the indicator function of E times f is also a integrable function. We write this as integral of the indicator function of E time's f d mu is written as integral of f over E.

That is one property and we want to show something more that if E and f are disjoint measurable sets. Then, integral of f over E union E 2 is same as integral over E 1 plus integral over E 2.

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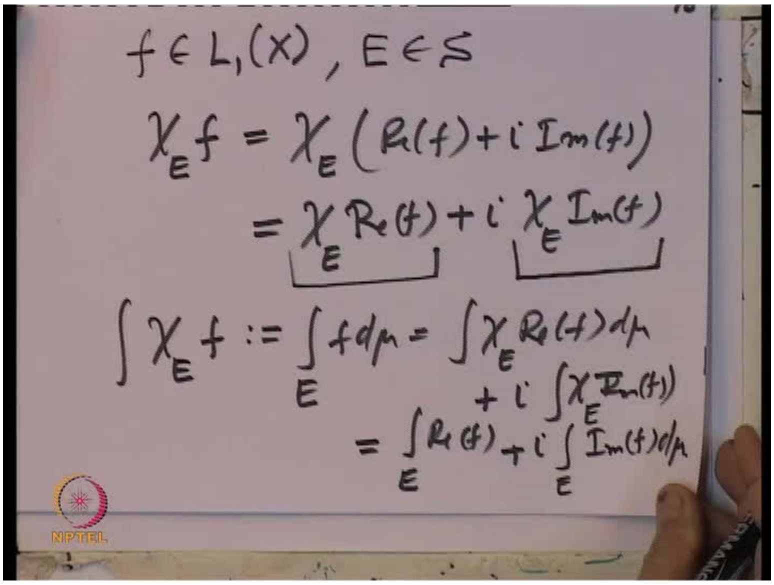

Let us prove this property. Let us take a function f, which is L 1 and E is a set, which is measurable.

Now, we want to look at indicator function of E times f. That is same as indicator function of E times real part of f plus i times imaginary part of f, which I can write it as indicator function of E times real part of f plus i times indicator function of E multiplied with the imaginary part of f.

What we are saying is that for the function indicator function of E times f the real part is indicator function of E times E f part of f and its imaginary part is indicator function of E times the imaginary part of f.

Real part of f is integrable imaginary part of f is integrable so multiplying with the indicator function of E also leaves some integrable. Both real part of the indicator function of E times f is integrable and the imaginary part of indicator function of E times f, which is indicator function of E times imaginary part of f that are real valued integrable functions.

Hence, we can write the indicator function of E f, which we are denoting by integral over E of f d mu to be equal to integral of the indicator function of E of the real part of f plus i times integral over E of the imaginary part of f.

If you recall, we had denoted it by integral over E of the real part of f plus at i or sometime also called iota integral over E imaginary part of f d mu.

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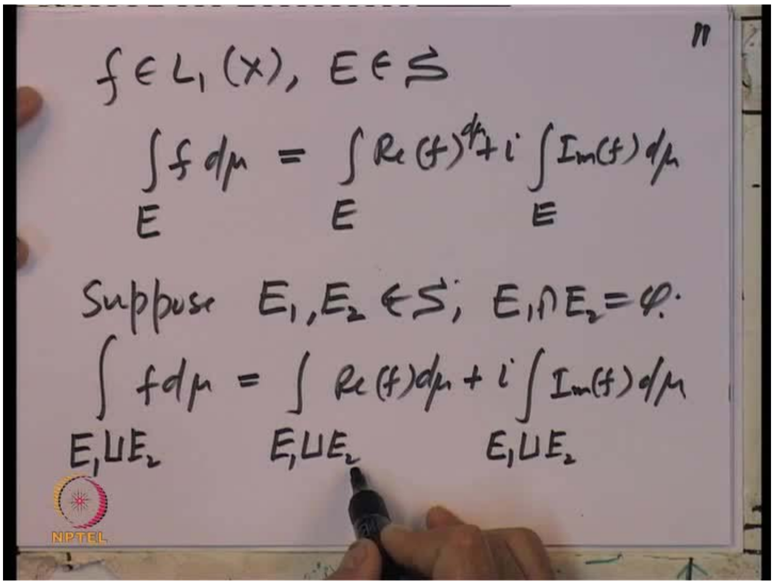

What we are saying is? The following for a integrable function for f integrable and E belonging to a measurable set integral of over E of f d mu is well defined and is nothing but integral over E of the real part of f plus iota times the integral of imaginary part of f over E d mu here also d mu.

Now, let us come to the second part. Now, suppose E 1 and E 2 are two sets, which are measurable and they are disjoint E 1 intersection E 2 is empty. In that case, by the above claim integral over E 1 union E 2 of f d mu will be equal to integral over E 1 union E 2 of real part of f d mu plus iota times integral over E 1 union E 2 of imaginary part of f d mu.

Now, let us recall that for real valued integrable functions integral over E 1 union E 2. Whenever, E 1 and E 2 are disjoint is nothing but the integral over E 1 plus the integral over E 2.

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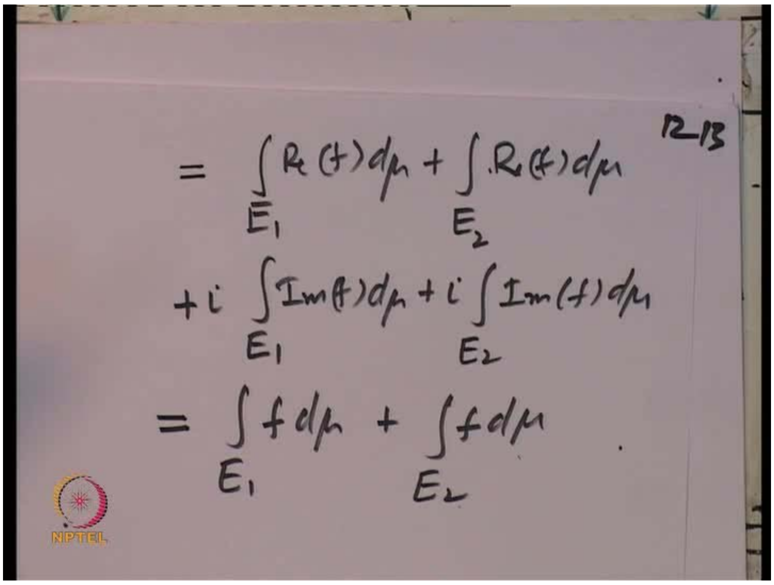

This right hand side, I can use that property and write it further as integral over E 1 of real part of f d mu plus integral over E 2 of real part of f d mu that is the first one plus i times the second integral gives me integral over E 1 of imaginary part of f d mu plus iota times integral over E 2 of the imaginary part of f d mu.

That is using the properties that for real valued functions integral over E 1 union E 2 whenever, they are disjoint splits into two into the sum of that. Now, I can combine real f 1 and imaginary f over E 1 together.

So, I can write that integral over E 1 of f d mu and the second combination will give me integral over f d mu over E 2.

What we have shown is that integral over E 1 union E 2, whenever they are disjoint of a function f is integral over E 1 plus integral over E 2. That proves the required property 3 completely.

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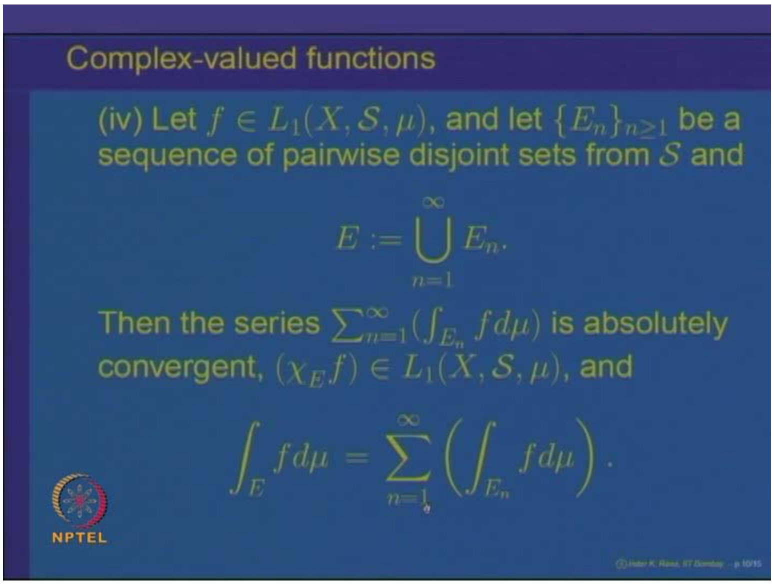

We will extend this property bit further. The next property is an extension of this property, which says let f be a integrable function and let us take a sequence E n be a sequence of pair wise disjoint measurable sets.

Let us write E, as the union of the sets E n then the claim is the series look at the complex series summation 1 to infinity integral over E n of f d mu. This series is absolutely convergent and the sum of the series and implies as a consequence that integral of chi E times f is a integrable function and the integral of f over E is nothing but the sum of this complex series sigma 1 to infinity integral of E n f d mu.

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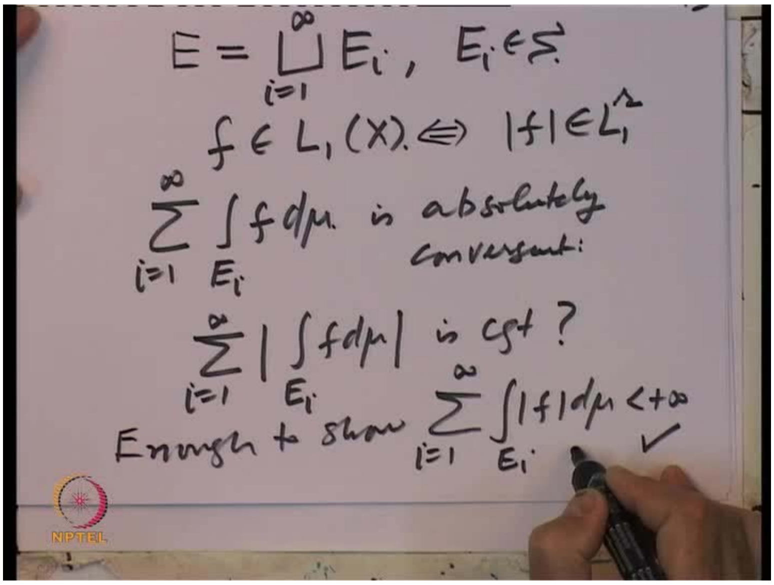

To prove this property, we will be using the corresponding property for the real valued integrable function let us write E is equal to union disjoint sets E i 1 to infinity, where E i's belong to S and f is a integrable function.

Let us look at the series, the series we are concerned with this sigma i equal to 1 to infinity integral over E i of f d mu. We want to show that is absolutely convergent.

That means, we want to show that the series i equal to 1 to infinity absolute value of integral over E i of f d mu is convergent. This is what we want to show.

Note that, just now we proved that the absolute value of the integral is less than or equal to integral of the absolute value.

To prove it is convergent, it is enough if you prove that the series i equal to 1 to infinity integral of absolute value of f d mu over E i is finite.

But for that, let us just observe that absolute value of f is a real valued integrable function and that we have already observed, while discussing the integral of nonnegative measurable functions that this is finite, if mod f is integrable.

This property is true, by the property of real valued integrable functions because f in L 1 that is same as saying that mod f is a real valued integrable function and E i being disjoint, we know this series is convergent. As a result, we will have that the series summation over i integral over i E i's of f d mu is an absolutely a convergent series.

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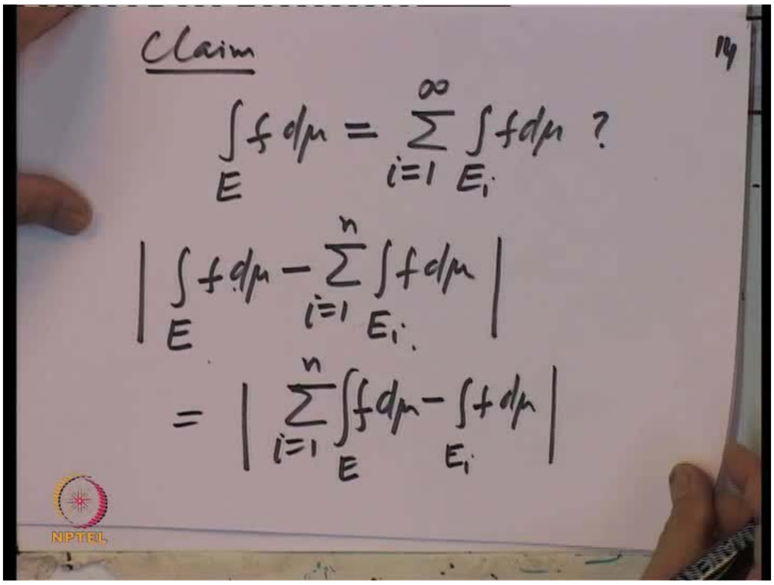

Now, we only want to prove the claim. The claim is that integral over E of f d mu is equal to summation i equal to 1 to infinity of integrals over E i of f d mu.

This is what we want to check. Let us look at the partial sums of this series. Let us look at integral over E of f d mu minus summation i equal to 1 to n integral over E i of f d mu. Let us look at this absolute value of this.

This is same as the integral of absolute value of the integral minus this sum. This we can write it as equal to absolute value of summation i equal to 1 to n of f over integral of f over E d mu minus integral over E i of f d mu. So, that is equal to this.

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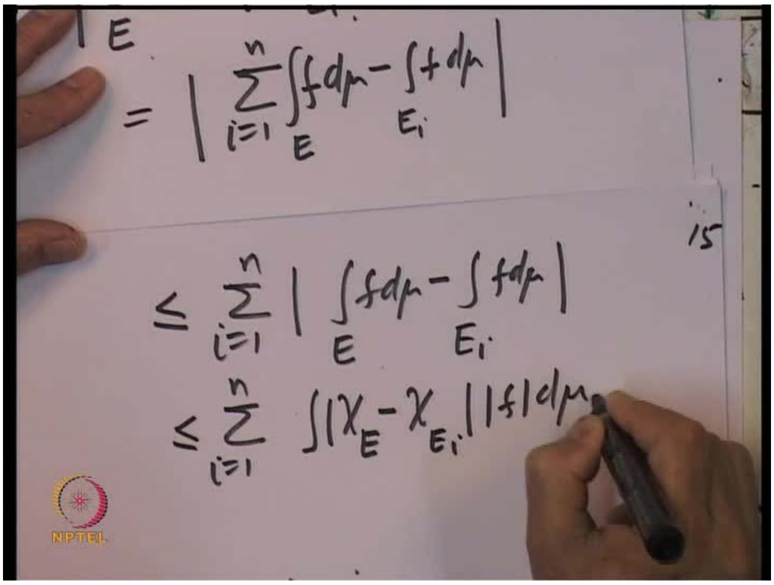

Now, using the triangle inequality for absolute value, I can write this as or we can write this as less than or equal to summation i equal to 1 to n absolute value of integral over E f d mu minus integral over E i of f d mu.

Using the fact that absolute value of the integral is less than or equal to integral of the absolute value. I can write this is less than or equal to summation i equal to 1 to n and this is nothing but integral of chi E minus chi of summation. I think we may have to slightly modify the proof but let us see this may also work out. So, E i absolute value times absolute value of f d mu this I think will not work out.

So, let us modify the proof a bit, because we want to say that as n goes to infinity this goes to 0. This may not exactly happen in this case.

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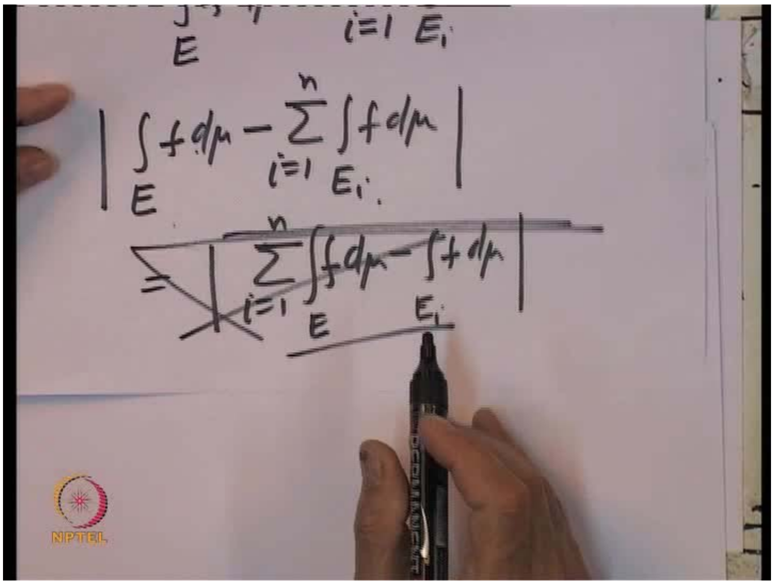

So, let us modify the proof slightly. Let me go back to this step. So, this is what we want to analyze.

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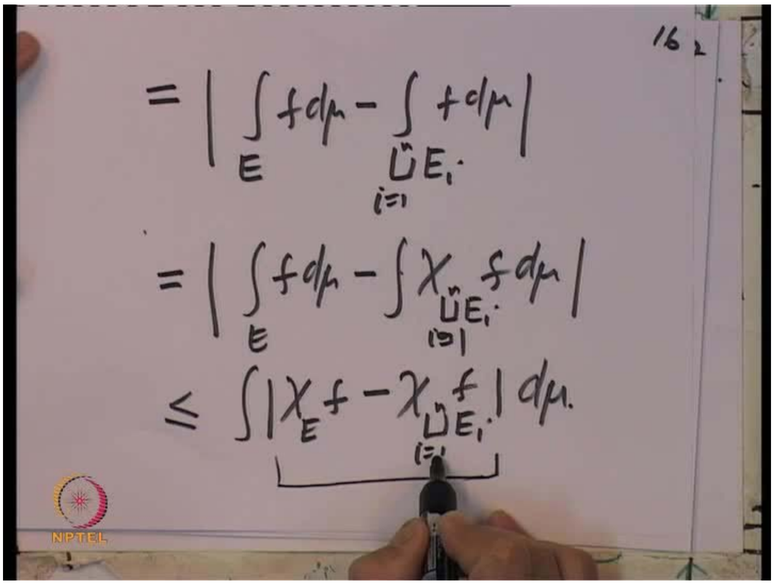

To analyze this, we proceed as follows. Let us first write, this is equal to this is a finite sum i equal to 1 to n integral over E i and that we have already observed is equal to integral over f d mu minus. This, I can write it as integral over union of E i's i equal to 1 to n of f d mu. I can write it that.

Let us write this as absolute value of integral over E f d mu minus this sum. Let me write the integral over the indicator function of union E i, i equal to 1 to n of f d mu and this, I can write as the indicator function of E.

So, this is less than or equal to integral over chi E of f minus chi union of E I, i equal to 1 to n f d mu. By using the fact that absolute value of the integral is less than or equal to so. This is minus, the union and the union I can write as the indicator function of E times f and absolute value of the integral is less than or equal to integral of the absolute value. Using that i come up to here.

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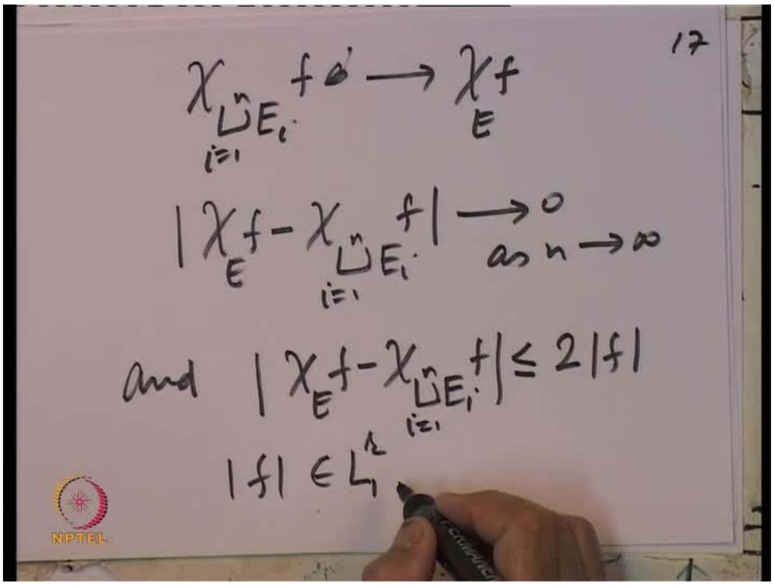

Now, let us observe that this is a sequence of functions and where does it converge. Let us observe this point that indicator function of union E i, i equal to 1 to n and f converges to the indicator function of E of f. This converges to this function point wise. So, if I look at the difference, this difference the chi E of f minus chi union of E i i equal to 1 to n f d mu f this goes to 0 as n goes to infinity.

That is one observation. The integrant goes to 0 as n goes to infinity and this is less than or equal to and this is absolute value of chi E f minus indicator function of union E i's 1 to n f d mu f is less than or equal to absolute value of the first one plus absolute value of the second function and both are less than or equal to f so 2 times mod f.

Mod f is a function goes to 0 and this function is dominated by 2 times mod f, which is a integrable function and mod f is a real valued integrable function.

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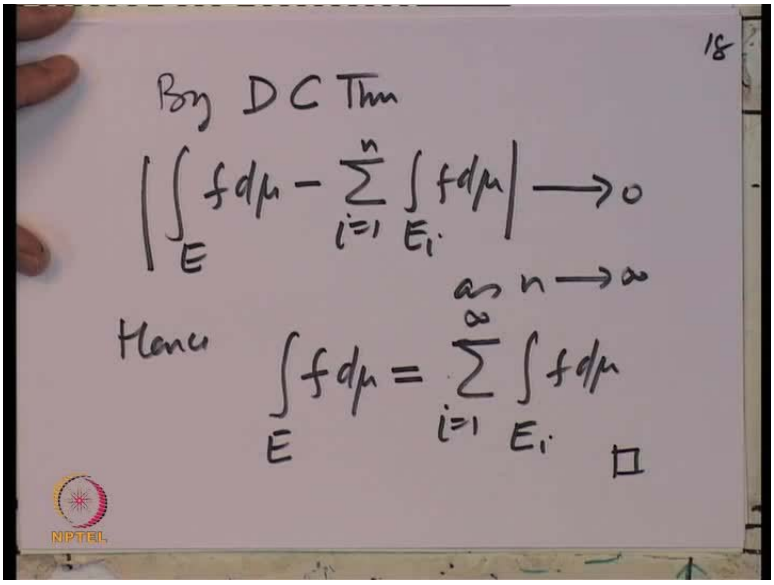

By the dominated convergence theorem, we will have that this goes to 0 as n goes to infinity. What we are saying is, an application of dominated convergence theorem will tell me that integral over E of f d mu minus summation i equal to 1 to n, which we are trying to analyze absolute value of that integral over E i of f d mu absolute value of that.

This is what we wanted to show goes to 0 and this we got is less than or equal to this integral and the integrant goes to 0 and we showed that and is dominated by an integrable function. So, Lebesgue dominated convergence theorem says, this goes to 0 as n goes to infinity. Hence, that proves integral over E of f d mu is going to be equal to summation i equal to 1 to infinity integral over E i of f d mu.

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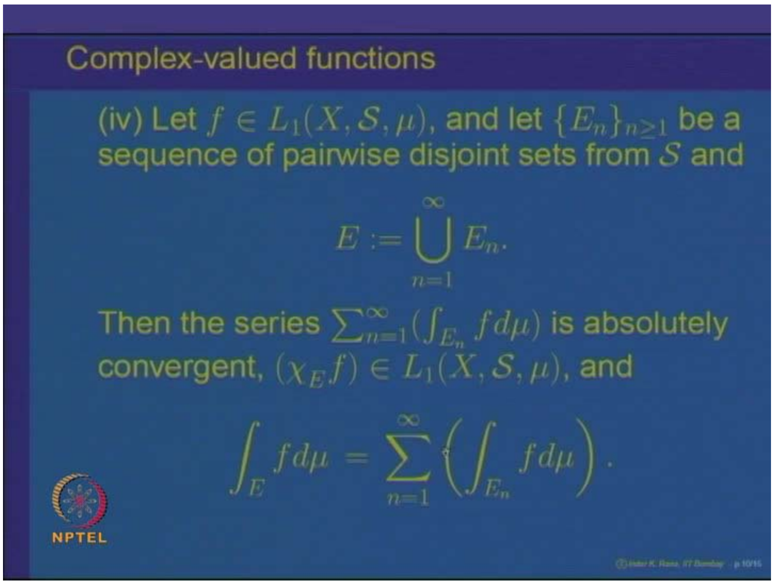

This complex series convergence and this is the sum. That proves the result for complex-valued functions that if E is a disjoint union of measurable sets and f is integrable. Then, integral of f over E is equal to summation of integrals over E n’s. These are the basic properties of complex-valued integrable functions and one can actually also prove dominated convergence theorem for complex-valued functions.

Recall that for real valued functions, we had three important theorems one was Monotone convergence theorem, another one was Fatou’s lemma and the third one was Dominated convergence theorem.

For modern convergence theorem was true for nonnegative measurable functions and Fatou’s lemma also is valid for nonnegative real valued measurable functions but for complex-valued functions no way of saying that a function is nonnegative or something. So, you cannot expect monotone convergence theorem and Fatou’s lemma to hold for complex-valued functions but it is fortunately enough the dominated convergence theorem, which is true for real valued integrable functions and it is also true for complex-valued functions and let us look at a proof of that.

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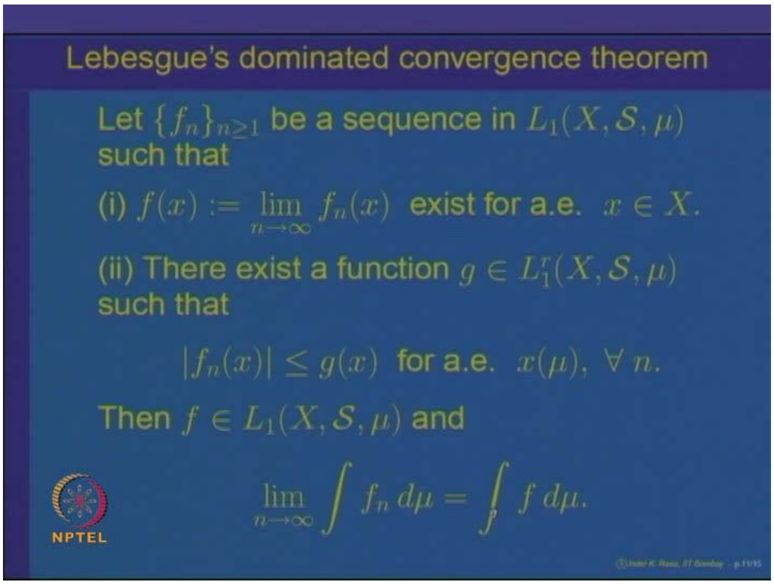

The theorem says that if f is a sequence of actually measurable functions is enough complex-valued measurable functions, such that f converges to the limit f n's converge to a limit f point wise almost everywhere.

There is a function g, which is real valued integrable function such that mod of absolute values of f n's are dominated by g. Then, the f is the limit function is integrable and its integral is equal to limit of the integrable.

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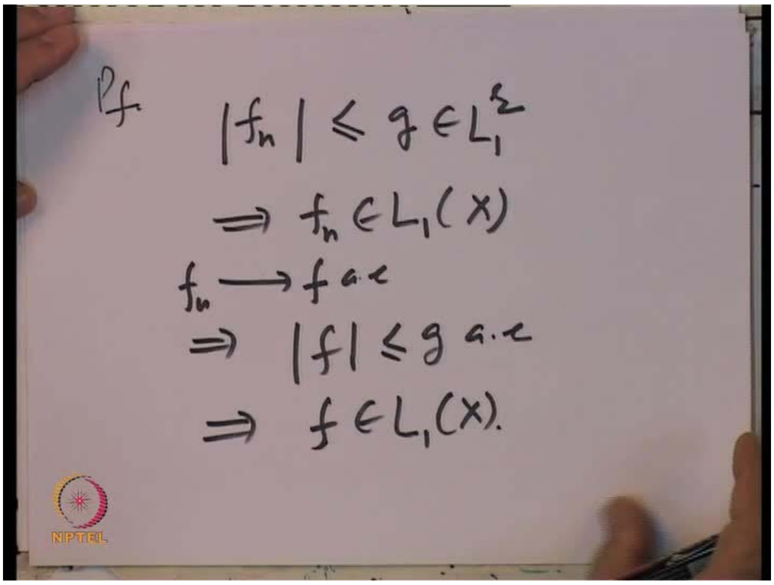

Basically, this is a straightforward application. Let us look at the proof of this. We are given that f n's is a sequence of measurable functions and mod f n's are dominated by g which is in L 1 of r that automatically implies it is a nonnegative function.

This implies that f n is L 1 of X because integral of f n will be less than or equal to integral of g for real valued functions and that implies f n is L 1. Also, because f n x converges to f almost everywhere and it will imply that mod f is less than or equal to g almost everywhere and that also will imply that f is also an integrable function.

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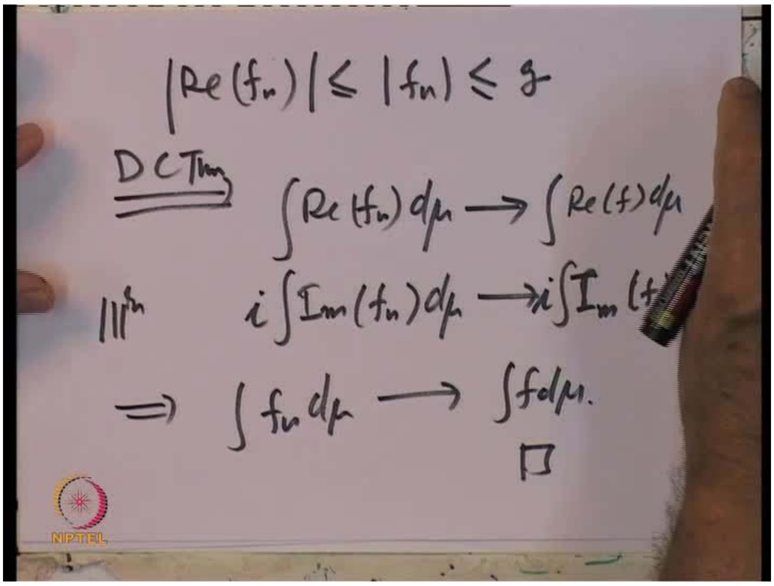

Now, we want to prove that the integrals will converge. Let us look at the integral; look at the real part of f n, real part of f n is dominated by mod f n is dominated by g. By dominated convergence theorem, for a real valued functions we get that the real part of f n integral d mu converges to integral of real part of f d mu.

Similarly the imaginary part, integral of the imaginary part of f n d mu converges to integral of imaginary part of f d mu. Now, we already shown that for complex scalar multiplication the corresponding result hold i times that will converge to i times that and adding these two implies that integral of f and d mu converges to some of these limits. So, that is equal to integral of f d mu. What we have shown is that for complex-valued functions monotone convergence theorem also holds.

So, what we have done today is we have extended the notion of measurability and integrability from real valued functions to complex-valued functions. We have shown that the integral for complex-valued functions has same properties as that of real valued functions and monotone convergence theorem also gets extended.

So, we will be using these results too in the next lecture to define what are called the p th power integrable functions and their properties. Thank you.