Measure and Integration

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Lecture No. # 32

Lebesgue Integral on R 2

Welcome to lecture number 32 on measure and integration. In the previous few lectures, we have been looking at the Lebesgue measure on the space R 2 and its various properties.

In the previous lecture we had started analyzing, how does the Lebesgue measure of a set change when we apply a linear transformation to it?

So, we had started analyzing it. Let us recall what we have done and then, we will continue analyzing this problem, and some more properties of Lebesgue measure under other transformations.

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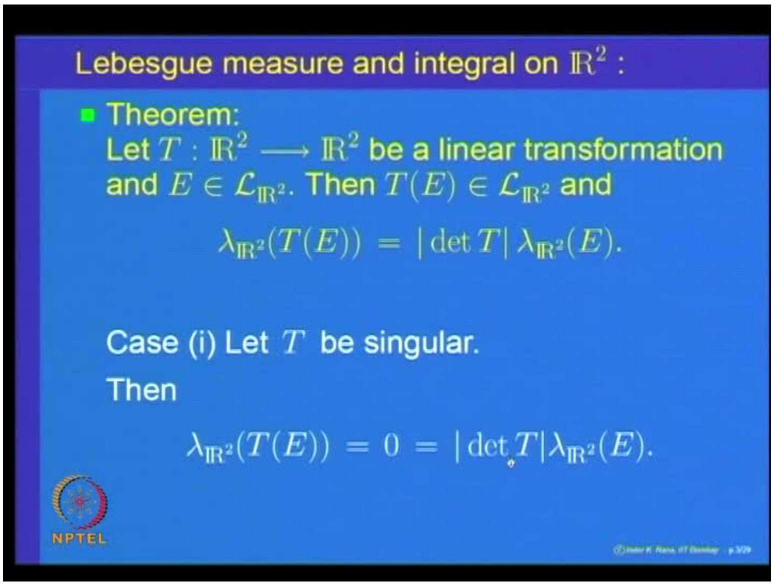

So, let us just recall, what we had done last time was that, we started looking at the theorem, namely, if T is a linear transformation from R 2 to R 2 such that, and E is a Lebesgue measurable subset, then we wanted to show, that the transformed set T of E, the image of E under T is again, a Lebesgue measurable set, and the Lebesgue measure of the transformed set is obtained by multiplying the Lebesgue measure of the original set with the constant called determinant of E. So, Lebesgue measure of T of E is equal to determinant of T times Lebesgue measure of the original set E.

So, this property, this term gets started analyzing, we had analyzed the proof of this theorem. In the 1st case, when T is a singular transformation and there we argued, that if T is a singular transformation, then T of E, the image set is going to be a Lebesgue null set and for a singular transformation determinant of E, T is also equal to 0. That is how, singular transformations are characterized.

So, in that case, both the terms, the Lebesgue measure of the translated set is equal to 0, is same as determinant of T times Lebesgue measure of the original set.

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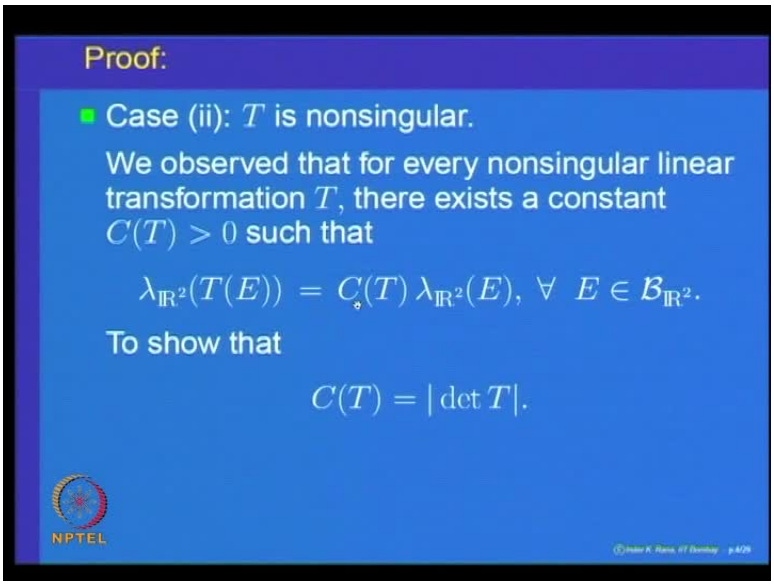

In the case when T is a nonsingular transformation, I mean, T is nonsingular. So, nonsingular means, T is invertible and here are a few facts about linear algebra, saying that T is nonsingular is same as saying as determinant is not equal to 0, and it is equivalent to saying, that as a map T is a one-to-one onto map and the inverse of course, is also a linear transformation.

So, in analyzing the proof of that we had already shown that if E is a Borel set, so we first restricted ourselves to Borel subsets of R 2. We showed that if E is a Borel set, then for every nonsingular linear transformation T of E T, the transformed set T of E also is a Borel set. That basically, follows from the fact, that every linear transformation is a continuous map and if it is nonsingular, then the inverse also is a continuous map.

So, essentially saying, that for every Borel set E, T of E is Borel, one analyzes is that when E is open set, T of E is an open set and hence, it is a Borel set. And then, one shows, that the collection of all sets for which this property is true, namely, the image is a Borel set, is a sigma algebra including open sets and hence, one concludes, that for every set E, the transformed set, the image set T of E is also a Borel set.

And we also analyze, that this, if you consider this as a measure for all Borel sets, then it is translation invariant because T is linear and that means, by the uniqueness property we got, that for every linear transformation T, which is nonsingular, the Lebesgue measure of the transformed set, namely, T of E must be a constant multiple of the original measure lambda R 2 of E and that constant will depend on the transformation T. So, this is the stage we had reached and then we wanted to analyze further, and the claim we want to prove is that this C of T is nothing but determinant of T.

So, this is the stage we had reached. So, let us continue the proof. Let us observe, that this map, T to C of T, see for every T, transformation T, we are associating a number C of T, which is nonnegative.

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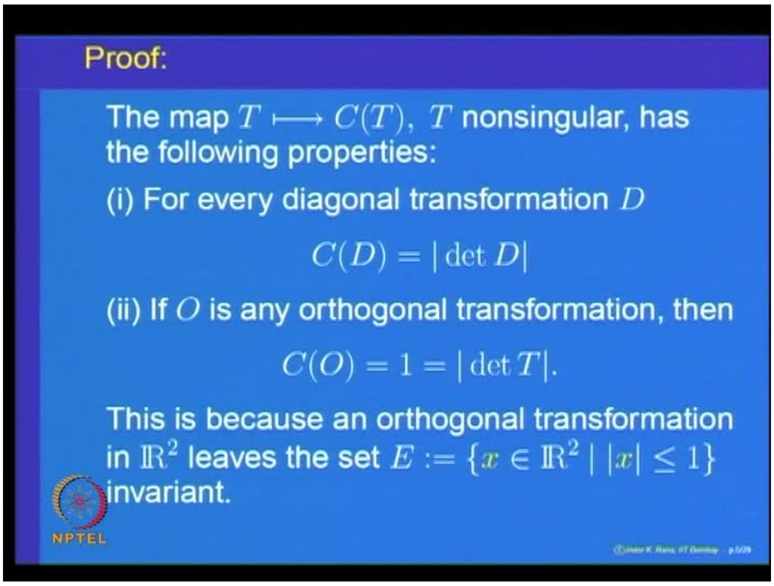

So, we get a map T going to C of T, so there is a map for every nonsingular transformation T, we are associating a number C of T. This association, this map has the following properties, namely, if T is a diagonal transformation, then we observed that was the beginning of our analysis of this theorem, that C of T is nothing but the determinant of T.

So, for diagonal transformation, C of T is equal to determinant of T. And the 2nd property is that if, if, if the linear transformation is an orthogonal transformation, then this C of O is equal to 1 is equal to determinant of T and this is because of the fact, that a orthogonal transformation on R 2 leaves the set, namely, the unit circle, you can think of it as all x in R 2, such that norm of x is less than or equal to 1 invariant.

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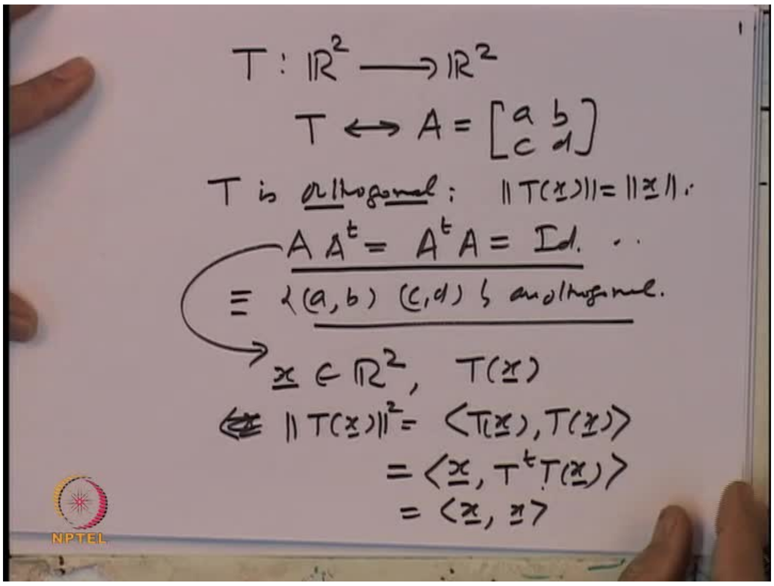

So, let us just look at this property, a slightly more, some of you may not be knowing about linear transformations. So, T is a linear transformation from R 2 to R 2 and let us, every linear transformation is given by a matrix A, so, which is a 2 by 2 matrix, so it is a, b, c and d.

So, saying that T is orthogonal is same as saying, that if you take A, and look at it as A transpose, that is same as, A transpose A, and that is equal to identity.

So, saying that T is orthogonal is characterized by this property about the matrices, matrix of the linear transformation, namely A times A transpose is same as A transpose times A, and that is equal to identity and which is also equivalent to the property, that if you look at the row vectors and the column vectors, so the row vector is a b and c d. So, these 2 vectors are orthogonal, namely their dot product is equal to 0 and each is a unit vector, so that is called orthogonal, but we are not going to use this property.

So, let us look at this property, which says, A A transpose is equal to A transpose A. So, this property gives us the following fact, namely, let us look at the dot product. So, let us take any vector x belonging to R 2 and look at the dot product of.., So, let us look at the image, the image is T of x, so x goes to T of x.

So, let us look at the dot product. So, norm of T x square, that is given by the dot product of T x with itself. So, that is the definition of the magnitude, the dot product in R 2, but this dot product can also be written as T times, this T can be written as T transpose T of x.

So, T transpose is same as A transpose basically, so you can think it as matrices and that being an identity, so this is same as x comma x. So, orthogonal transformations are also characterized by the property, that the norm of the image of any vector is equal to norm of the original vector. That is another way of characterizing orthogonal transformations; we can take that as the definition of the transformation if you like.

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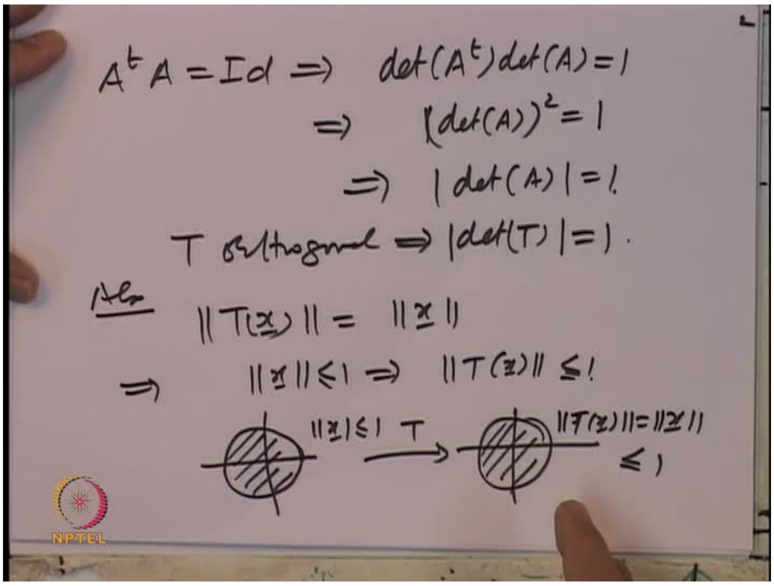

Now, from both this properties, that A transpose A is equal to identity, so that implies the following fact, that A transpose A equal to identity. This implies, that the determinant of A transpose determinant of A is equal to 1 and that implies, determinant of A transpose is same as determinant of A. So, that says, determinant of A times determinant of A, so this square is equal to 1. So, that implies that the absolute value of determinant of A is equal to 1. So, T orthogonal implies that determinant of T absolute value is equal to 1.

So, that is one fact. Also, the fact, that norm of T x is equal to norm of x implies, that if norm of x is less than or equal to 1, so that implies norm of T x is also less than or equal to 1. So, that means, if in the plane we look at the unit circle, so this is the set, where norm of x is less than or equal to 1. And if we take the transformed set, the transformed set is same, so under T this gives back to the same thing. So, norm of T x is equal to norm x less than or equal to 1.

So, that is saying that T leaves, if T is orthogonal, then it leaves the unit circle, the region inside the unit circle invariant and that essentially means, that the Lebesgue measure of this set is same as the Lebesgue measure of that set.

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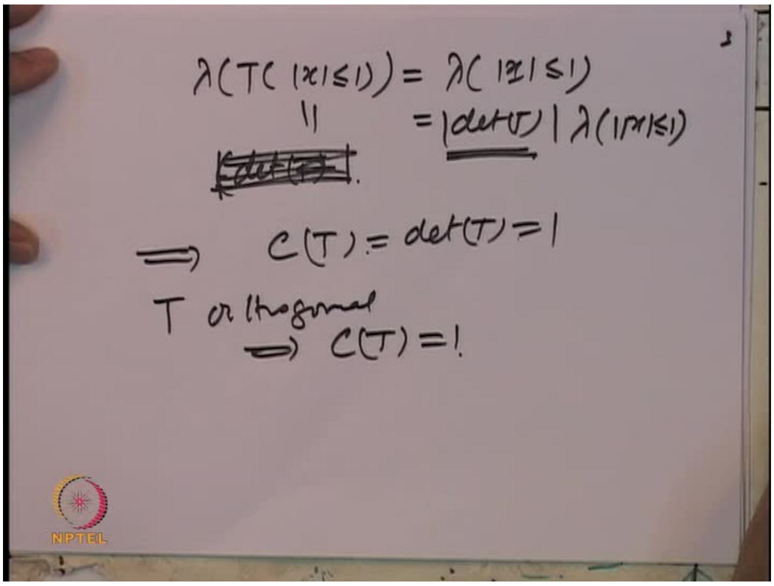

So, that implies, that the Lebesgue measure of the transformed set, so that circle mod x less than or equal to 1 is same as the Lebesgue measure of mod x less than or equal to 1. And this being equal to determinant of T, this being determinant of T, times, this being same as, so this is same as, sorry, this is same as, so that implies, but this can be written as determinant of T because that is equal to 1 times Lebesgue measure of norm x less than or equal to 1.

So, this implies, so this is the constant, so C of T, which is determinant of T is equal to. So, C of T is 1, which is same as determinant of T. So, that means, for orthogonal transformations, so T orthogonal implies C of T is equal to 1.

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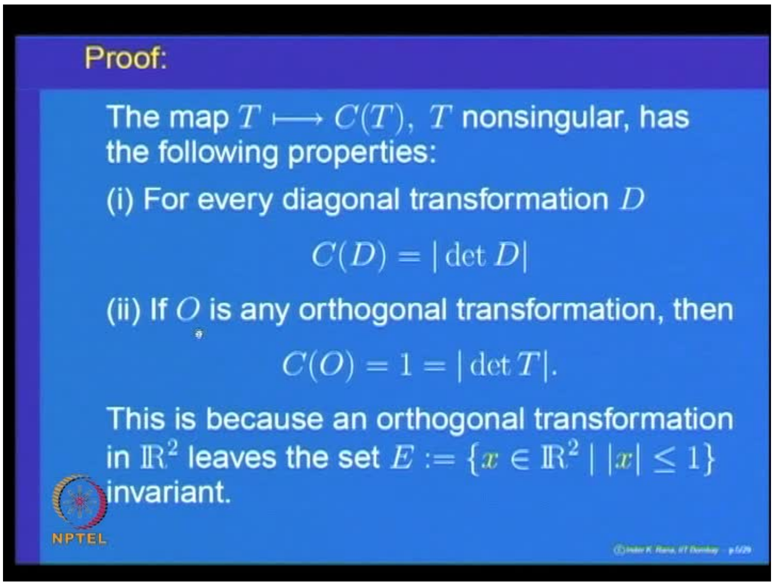

So, that is the 2nd fact that we wanted to prove, namely, if, if O is an orthogonal transformation, then the C of O, the constant, this type O here, this should have been determinant of O, is, because the unit circle, the region inside unit circle is left invariant by the orthogonal transformation.

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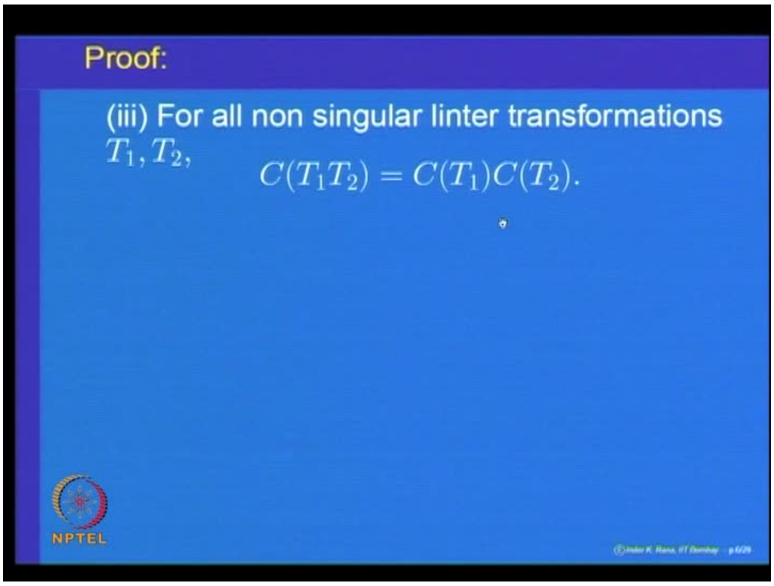

So, now, the next property we want to analyze is that for all nonsingular transformations T 1 and T 2, C of (T 1 T 2) is equal to C of T 1 times C of T 2, that means, this map is multiplicative. The map T going to C of T is a multiplicative map, namely, if I look 2 transformations, T 1 and T 2, then C of (T 1 T 2) is same as C of T 1 times C of T 2. So, let us look at a proof of that.

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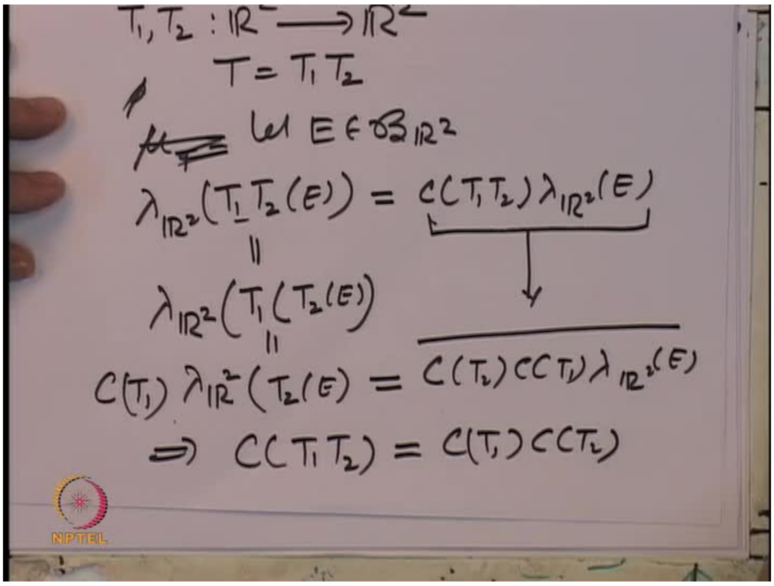

So, T 1 and T 2 are transformations from R 2 to R 2. So, we are going to look at, so let us write T is equal to T 1 T 2.

Then, we have to compute C of T 2, so we are going to look at the measure mu of T which is equal to… So, let us take any set, so let E be any set, which is a Borel set.

So, let us look at lambda of lambda R 2 of T 1 T 2 apply to E. So, let us look at this, this, by definition of the constant C of T is C of (T 1 T 2) because (T 1 T 2) is a linear transformation applied. So, this composite (T 1 T 2) is applied to E.

So, by definition of C of (T 1 T 2), that should be equal to C (T 1 T 2) of Lebesgue measure of the set E. On the other hand, we can also think of this as lambda, Lebesgue measure of R 2 of T 1 applied to T 2 of E.

So, this composition (T 1 T 2) is same as saying the linear transformation T 1 is applied to the set T 2 of E, but if you do that, then we know, that this is equal to lambda R 2 of, it is lambda R 2 of T 1 of a set. So, it is C T 1, so this is equal to C T 1 times lambda R 2 of T 2 of E. And now, that once again lambda R 2 of C of T 2 of E gives you C T 2 and the original is C T 1 into lambda R 2 of E. So, we get, that C of (T 1 T 2) times the Lebesgue measure of any set E is same as, same as C T 2 into C T 1 of Lebesgue measure of E.

So, this happens for every set E, this implies, so this implies, that C of T 1 composite T 2 is same as C of T 1 times C of T 2. So, this map is a multiplicative map. So, this is the property we wanted to prove.

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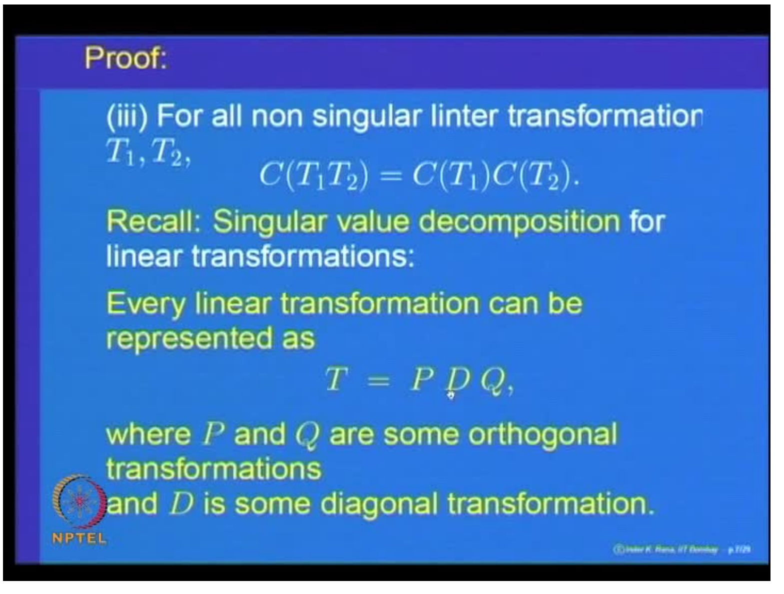

And now, we need another fact from linear algebra, namely, what is called, singular value decomposition for linear transformations.

So, in case we have not come across this theorem called, singular value decomposition for linear transformations, please look into the text book, that we have suggested, namely, an introduction to measure and integration and look at the appendix of that book, you will find a proof of this singular value decomposition for linear transformations.

So, let us take, what is the singular value decomposition? It says that every linear transformation T can be represented as a product of 3 transformations, where the 1st one P and the last one Q are both orthogonal transformations, P and Q are orthogonal transformations and this D is a diagonal transformation.

So, every linear transformation T can be represented as P times D times Q, where P and Q are some orthogonal transformations and D is some diagonal transformation.

So, this is a theorem called, the singular value decomposition in linear algebra. So, please have a look at a proof of this, in case you have not come across this theorem in the text book mentioned.

So, once we know, that for every linear transformation, T can be represented as P times D times Q, so and the property 3 just now stated to says, that the constant C of any composite is the product. So, we apply that property through this, so we get, C of T will be equal to C of P times D times Q, which is nothing but the product. So, the C of a transformation T will be C of P into C of D into C of Q.

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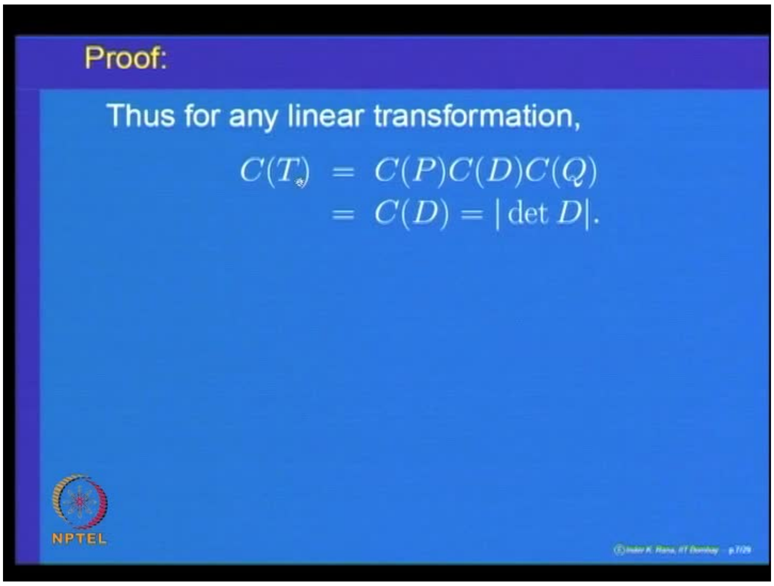

So, we get, that the constant for any linear transformation T is equal to the constant for some orthogonal transformation P times the constant for a diagonal transformation D and the constant for another orthogonal transformation Q.

But just now we observed that for orthogonal transformations, the C of orthogonal transformation, the constant associated is 1. So, C of P is 1 and C of Q is 1, so that gives you C of T is equal to C of D because both first and the last multiplicative things are 1. So, it is C of D and for a diagonal transformation we have already shown, this is equal to determinant of D.

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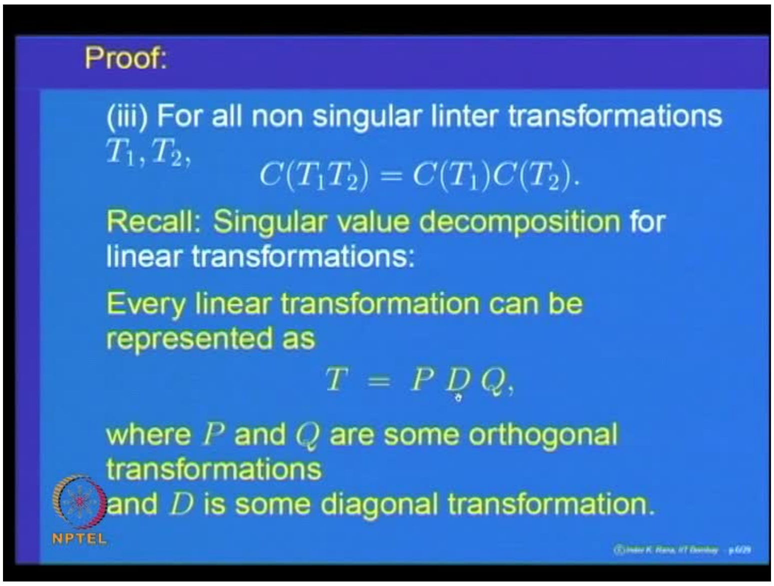

So, for the linear transformation T, the constant C of T is equal to determinant of D, where D is the diagonal transformation, which appears in the singular value decomposition T is equal to P D Q. But on the other hand, we can also look at the determinant of T from this, so determinant of T.

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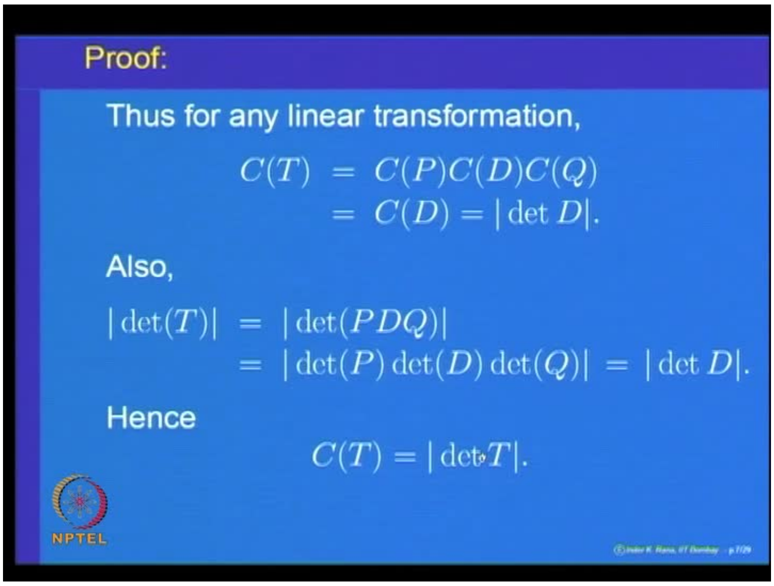

Recall, that determinant is also a multiplicative map, so determinant of T will be equal to determinant of P into determinant of D into determinant of Q. But determinant of P and determinant of Q, both are equal to 1, so that says, determinant of T is equal to determinant of D and just now we said, determinant of D is equal to C of T. So, combining these 2 we get, determinant of T is equal to determinant of D, sorry, we get determinant of, so this should be C of T is determinant of D. So, that says, that C of T should be equal to determinant of D.

So, so, here it should be, this is redundant. So, C of T is determinant of D and determinant of D is equal to determinant of T. So, this 2 combine together, gives you C of T is equal to determinant of T.

So, that completes the proof of the fact, that for a linear transformation; so, we have completed the proof, that for a linear transformation, if you take the set E and change it, transform it by a linear transformation, that is same as determinant of T times the Lebesgue measure of E. So, the Lebesgue measure of the transformed set is determinant of T times the Lebesgue measure of the set E.

So, that proves the theorem for all sets, which are Borel measurable sets. So, till now, we have proved the theorem only for Borel measurable sets. We would like to extend this theorem for Lebesgue measurable sets. So, for that let us observe the following, how are the Lebesgue measurable sets in R 2 obtained from Lebesgue, from Borel measurable sets? What is the relation between Lebesgue measurable sets and the Borel measurable sets?

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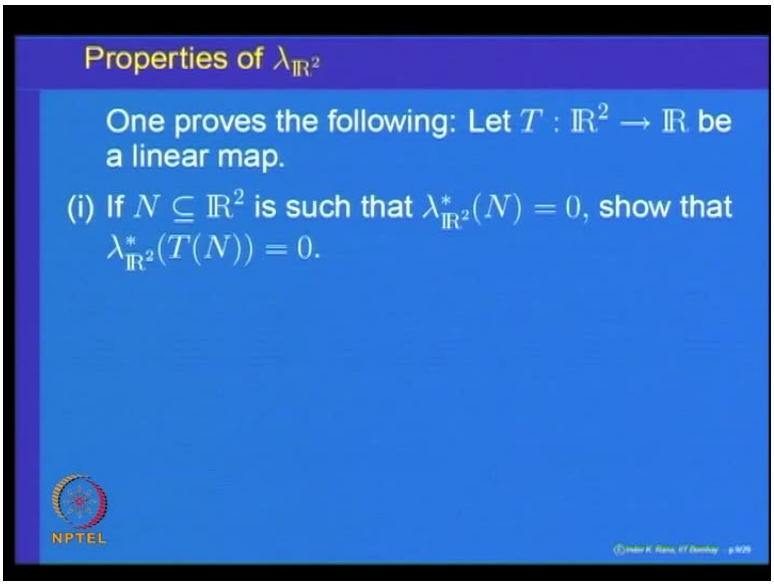

So, the first property is that, let us take any set N, which is in R 2 and such that its Lebesgue outer measure is 0.

Look at sets of Lebesgue outer measure 0 in the plane. So, we first claim, that under any linear transformation T, the image is also a Lebesgue measurable set of measure 0. That means linear transformations take sets of measure 0 to sets of measure 0 in plane. So, to prove that let us observe the following thing.

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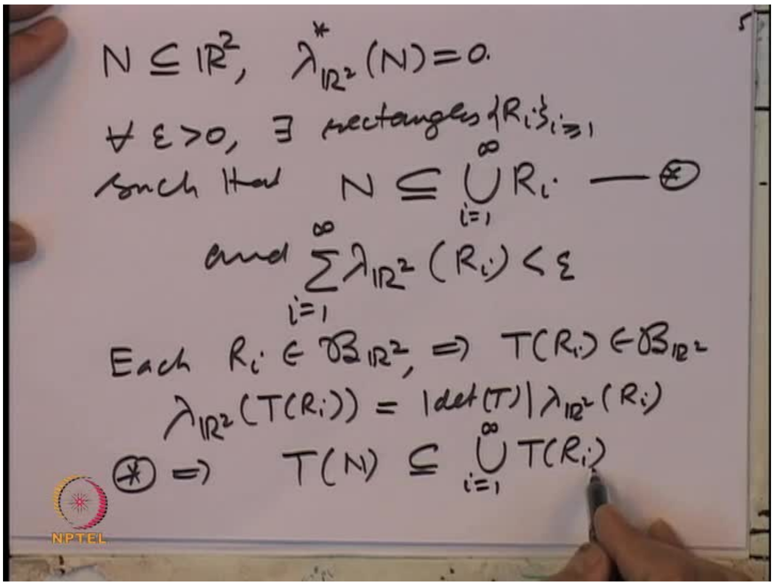

So, let us take N, a subset of R 2 and Lebesgue measure of N equal to 0. But saying Lebesgue measure of N is equal to 0 is same as saying, for every epsilon bigger than 0 there exists rectangles, there exists rectangles, say R i, i bigger than or equal to 1, such that the set N is contained in the union of this rectangles R i, and the Lebesgue measure of the rectangle R i added together is less than epsilon. So, saying a set as a null set is same as saying it can be covered by rectangles, such that the total measure of the rectangles put together is less than epsilon.

But note, now, so each R i is a rectangle, so it is a Borel set, is a Borel subset of R 2. So, that implies, that T of R i also is a Borel subset of R 2 for a nonsingular linear transformation T, if T is nonsingular.

And the Lebesgue measure of T of R i, by what we have proved just now, is equal to determinant of T times Lebesgue measure of, Lebesgue measure of R i, that is just now we have proved. For Borel set this property holds, so now the fact, that N is covered by union of R i's is, implies, so this fact star implies, that T of N is covered by union of T of R i i equal to 1 to infinity.

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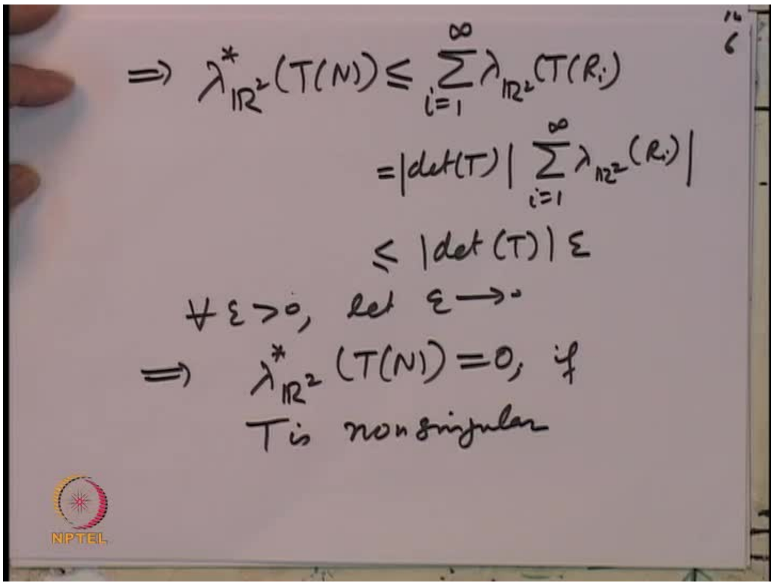

So, this is contained by T of R i, so that implies, by the countable sub additive property of the Lebesgue measure, outer Lebesgue measure R 2 of T N is less than or equal to summation i equal to 1 to infinity Lebesgue outer measure of T of R i. And Lebesgue measure of T of R i is determinant, so this is equal to determinant of T absolute value times summation of i equal to 1 to infinity Lebesgue measure of R i and that is less than epsilon.

So, it is less than or equal to absolute value of determinant of T times epsilon and since this property holds, so this holds for every epsilon bigger than 0. So, let epsilon goes to 0, so that will imply, Lebesgue measure of outer measure of T of N is equal to 0 when T is, if T is nonsingular and for a singular transformation we know, T of R 2 itself is 0, so T of N will be 0.

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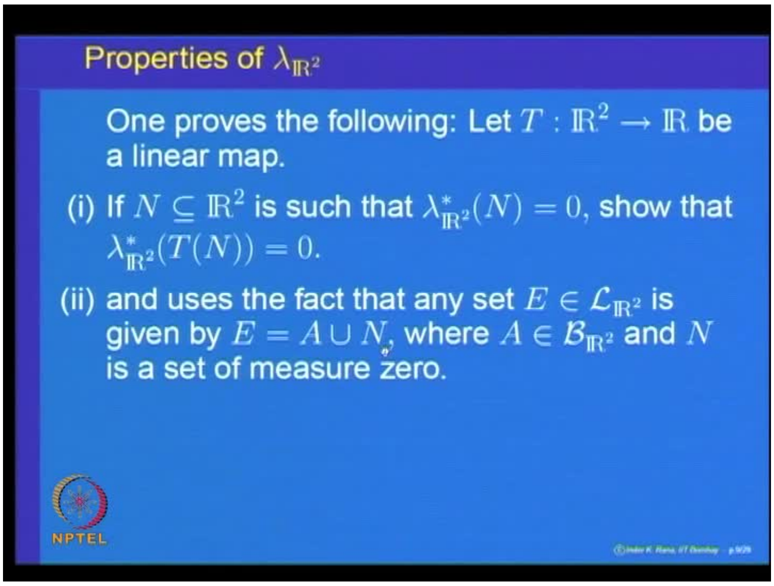

So, this proves the fact, that for every set N, which is of Lebesgue outer measure 0, lambda, the image T of N under any linear transformation is again a set of measure 0.

And the second fact, between Lebesgue measure and, Lebesgue measure and, Lebesgue measurable sets and the Borel measurable sets is the following - every Lebesgue measurable set E can be represented as a union of 2 sets, one Borel set A and a null set N and the Lebesgue measure of E is same as the Lebesgue measure of the set A, N is a set of measure 0. So that is the 2nd fact one has.

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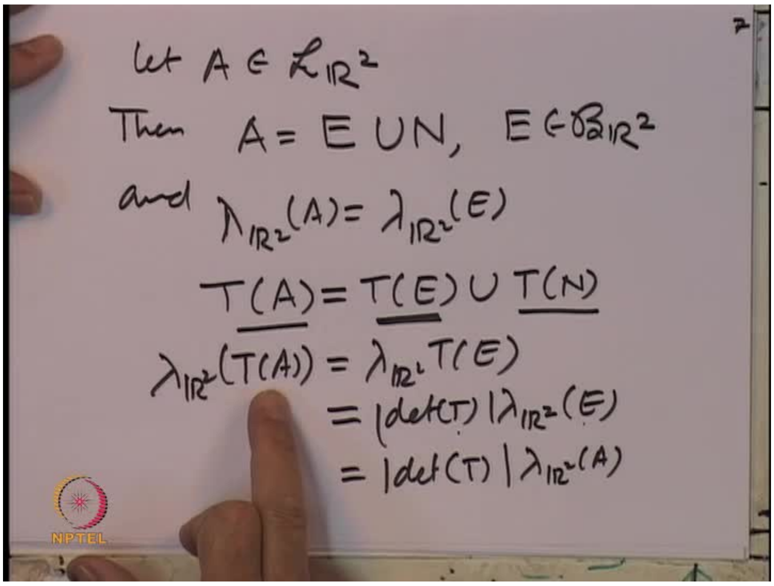

So, let us take, so let A be a Lebesgue measurable, let us take Lebesgue measurable set in R 2, then the set A can be written as E union N, where E is a Borel set in R 2 and the Lebesgue measure, so lambda, lambda R 2 of A is same as lambda R 2 of E.

Now, if we look at T of A, so if we look at the set T of A, then that will be equal to T of E union, T of, T of N, if T is, say nonsingular. And now, this is a Borel set and this is again a null set, so Lebesgue outer measure of T of A is equal to Lebesgue outer measure of T of E, which is equal to determinant of, because this is a Borel set, so determinant of T times Lebesgue outer measure, Lebesgue measure of the set E and which is same as the Lebesgue measure of the set A. So, this is same as Lebesgue measure of T times Lebesgue measure of A.

So, we have used the fact, that if A is a Lebesgue measurable set, then A can be written as E union N, where E is a Borel set and N is a null set. That means the Lebesgue measure of R 2 of the set A is same as the Borel, Lebesgue measure of the Borel component of it, that is, R 2 of E.

So, now, if I apply transformation T to it and say T is nonsingular, then T of A will be equal to T of E union T of N, and just now we observed, that T of N is a null set and T of E is a Borel set. So, Lebesgue measure of T of A will be nothing but the Lebesgue measure of T of E, which by the earlier case is determinant of T times Lebesgue measure of E and Lebesgue measure of E is same as the Lebesgue measure of A.

So, that proves, for a, for a, that proves for a Lebesgue measurable set, the Lebesgue measure of the transformed set T of A is same as determinant of T times the Lebesgue measure of the set A itself.

So, that proves the theorem completely namely; so, that proves the theorem completely, namely, that if E is any Lebesgue measureable set, then T of E is also Lebesgue measurable and the Lebesgue measure of T of E is equal to determinant of T times the Lebesgue measure of E. So, this is how the, this is how the Lebesgue measure of a set E in the plane changes with respect to linear transformations.

Now, we will give some applications of this now because there are many nice linear transformations in the plane.

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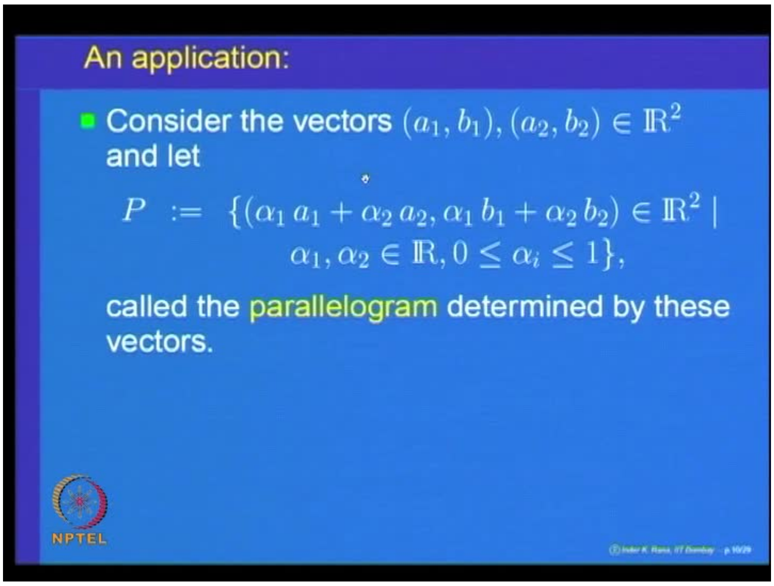

So, let us look at the first application of this, namely, so let us take 2 vectors a and b, (a 1, b 1), (a 2, b 2) in R 2 and look at the set, which are all sets, set P of all vectors in the plane of the type, where the 1st component is alpha 1 a 1 plus alpha 2 a 2, and the 2nd component is alpha 1 a 1 plus alpha 2 b 2, where alpha 1 and alpha 2 are numbers between 0 and 1. This is called the parallelogram determined by the vectors (a 1, b 1) and (a 2, b 2); in the picture it is the following.

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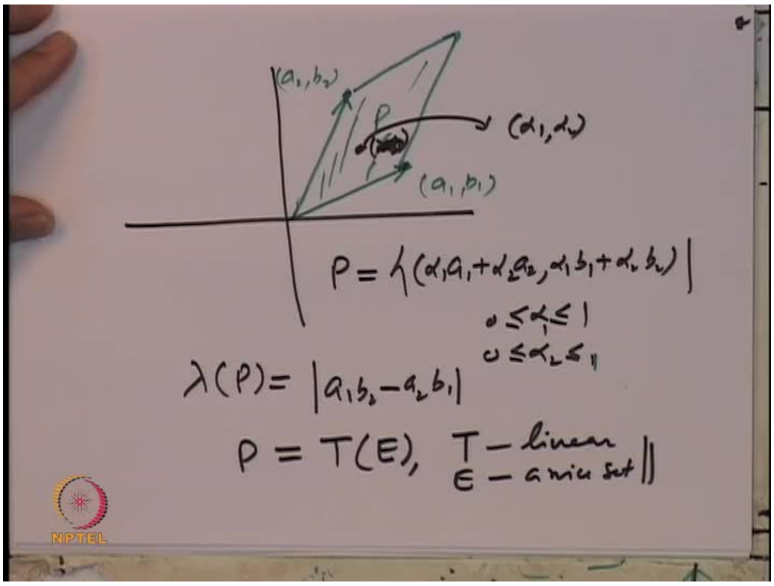

So, if we have the plane, let us take 2 vectors, one vector is this vector and the other vector is this. So, this is the vector, which is (a 1, b 1) and this is the vector, which is (a 2, b 2). Then, they determine a parallelogram, the geometric object, so let us see what is that parallelogram, so that is nothing but this parallelogram.

So, that is the parallelogram P determined by these 2 vectors (a 1, b 1) and (a 2, b 2), and any vector in between, so this vector, so this parallelogram P is characterized by, that this any vector inside here, say, call it as (x, y), so P is then x, looks like alpha 1 a 1 plus alpha 2 a 2 and 2nd one looks like alpha 1 b 1 plus alpha 2 b 2, where this alpha and alpha 2 have the property, their numbers between 0 and 1. So, alpha 1, alpha 2 between 0 and 1, so this is the point, this is a point with components alpha 1 and alpha 2.

So, this is what the parallelogram, so one can check geometric fact, that if I take 2 vectors a 1 and b 1 and look at the geometric picture of this parallelogram, then if I take any alpha 1, alpha 2 between 0 and 1 and look at this, then this is nothing but the parallelogram given by these 2 vectors. The claim, we want to is that, we want to show, that the Lebesgue measure of this parallelogram is same as the absolute value of, a 1 b 1, (a 1, b 2) minus (a 2, b 1).

Say, these are the components, (a 1, b 1) and (a 2, b 2) are the components of the vectors, which we started with, so the claim is, so the claim, that Lebesgue measure of P is equal to the absolute value of (a 1, b 2) minus (a 2, b 1).

So, to prove this, we are going to show, that this P is equal to, T of, T of a set E, where T is a linear transformation, T linear and E is a nice set. And it is not difficult to guess what is T and what is E, so, let us just look at that.

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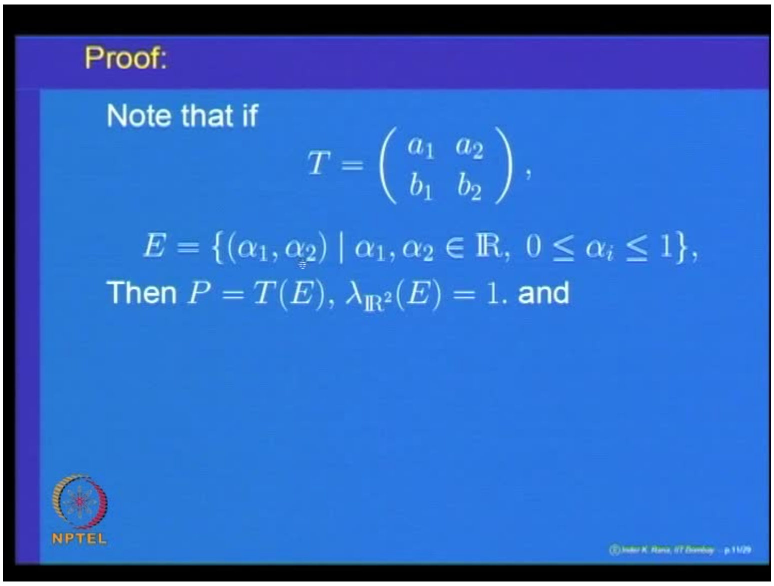

So, let us observe that if T is the matrix with the components (a 1, a 2, b 1, b 2), so rows are the vectors, which are given vectors, are (a 1, b 1) and (a 2, b 2). So, the 1st column is the vector (a 1, b 1), 2nd column is the vector components of the vector (a 2, b 2).

If I look at this transformation T and look at the set E with components alpha 1, alpha 2 in, where (alpha 1, alpha 2) are real line and they are between 0 and 1. So, what is this set E? This set E is nothing but, this set E is nothing but a rectangle, actually a square in the plane with sides (0, 1) to (0, 1) and if I look at T of E, so any set (alpha 1, alpha 2), so what will be this?

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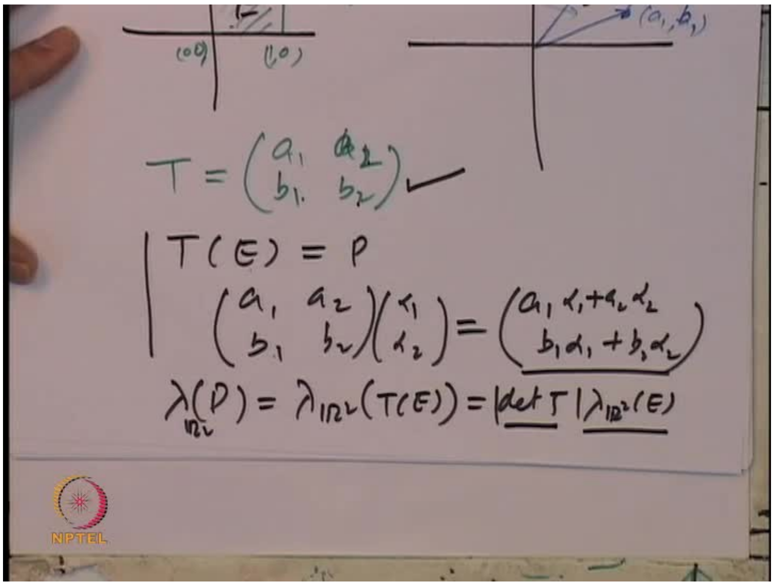

So, what we are saying is the following, that if I look at the set E, so here is, here is the set E, which looks like, so this is the set E (0, 0), this is (1, 0), (1, 1) and (0, 1).

So, if I look at this set and look at the transformation given by T where T is equal to a 1, b 1 and a 2, b 2. If I look at this, then the image of this, under this is precisely, is precisely, that parallelogram P, where this is a 1 and a 2 and this is (a 1, b 1) and this is (a 2, b 2). So, this transforms to this parallelogram.

So, if T is this and E is this set, then T of E is equal to P, that is because what is…, So, let us look at (a 1, a 2, b 1, b 2) and a vector is (alpha 1, alpha 2), so what is that?

So, that is, a 1 alpha 1 plus a 2 alpha 2 and that is, gives you, b 1 alpha 1 plus b 2 alpha 2. So, that says, the vector here (alpha 1, alpha 2) goes to the vector there given by this and that is precisely the parallelogram.

So, under the, this is a linear transformation, so under the linear transformation T given by this matrix, the unit square changes to the parallelogram and once that is true, so this will imply, that the Lebesgue measure, so this will imply, the Lebesgue measure of P is same as the Lebesgue measure, of the, in the plane of T of E and that is equal to determinant of T absolute value times Lebesgue measure of E. But Lebesgue measure of E, that is, the area, that is a rectangle, so it is an area, that is equal to 1 and determinant of T is (a 1, b 2) minus (a 2, b 1).

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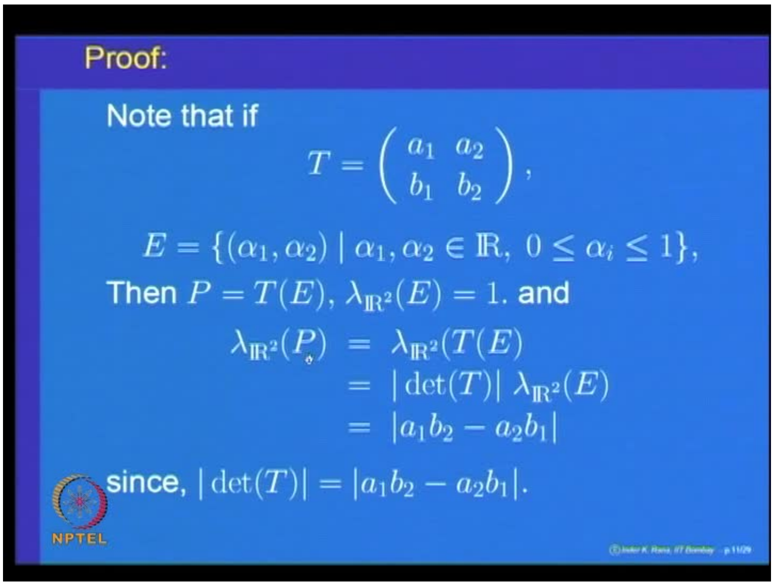

So, that gives us the result, that the Lebesgue measure of P is same as the Lebesgue measure of the transform set T of E, which is equal to determinant of T times the Lebesgue measure of E and that Lebesgue measure of E being equal to 1. So, that gives us determinant of T, which is nothing but (a 1, b 2) minus (a 2, b 1). So, this gives us, that Lebesgue measure of the parallelogram is the determinant of the, given by the vector, so that is (a 1, b 2) minus (a 2, b 1).

So, this is one of the results, that one proves normally in geometry, in linear algebra, that the determinant is nothing but a measure of the parallelogram, area of the parallelogram determined by the vectors.

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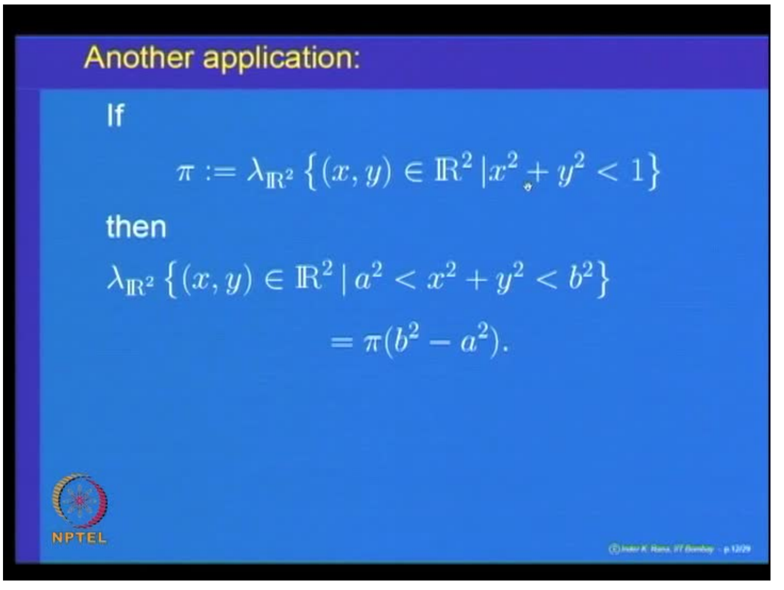

Let us look at another application of this formula, how the linear transformation changes? So, let us look at the Lebesgue measure of the unit circle, area, region enclosed by the unit circle, so Lebesgue measure of all vectors (x, y) in R 2 such that, x square plus y square is less than 1. So let us look at that.

So, let us call the Lebesgue measure of this to be equal to a number pi, we are not assuming anything about pi, we are just saying that the Lebesgue measure of this unit circle is a finite quantity.

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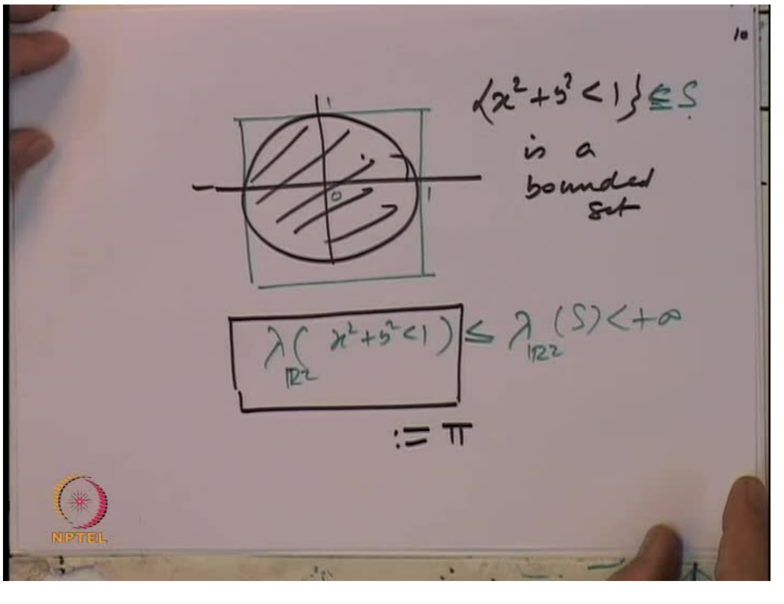

So, let us see, why it is finite? So, here is the unit, it is a bounded, so this is x square plus y square less than 1, so that is that set, so this is a bounded set.

So, for example, this is enclosed inside this rectangle, inside this square of, this is 1 and this is 1, so this is 0 and this is 1 and this is 1. So, it is enclosed inside this square of length, so that means, that the area, so this is less than or equal to, it is a subset of the square and square is a bounded thing. So, that means, the Lebesgue measure of the points, say, that x square plus y square is less than 1, will be less than or equal to the Lebesgue measure in the plane of the square, which is finite quantity.

So, that means, that the Lebesgue measure of the region enclosed by the unit circle is a finite number and this finite number, we are just calling it by the number, we are denoting it by the symbol pi. So, pi is the Lebesgue measure of the region enclosed by the unit circle.

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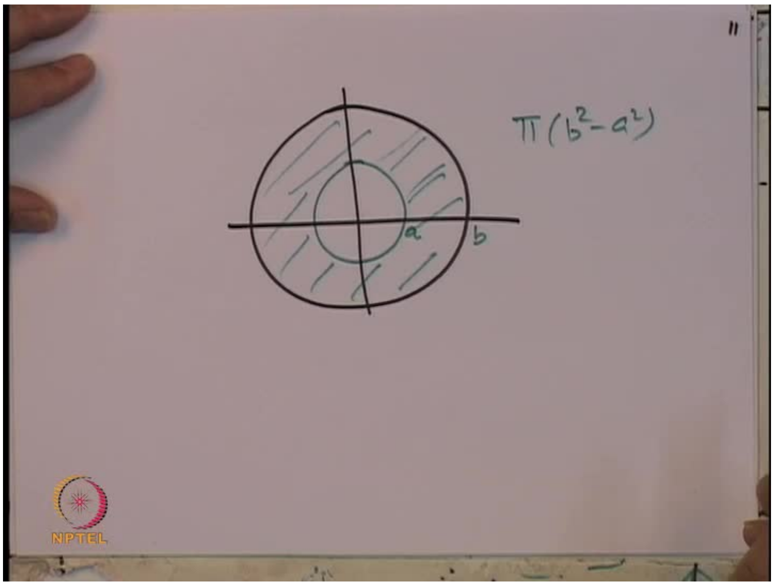

So, then the claim is that if we look at the annulus region, that is, x square plus y square, bigger than a square and less than b square, then its Lebesgue measure is pi of b square minus a square. So, what we want to prove is that if I look at, here is the bigger circle and here is the smaller one and this radius is a and this radius is b.

So, we are saying, the Lebesgue measure of this portion is nothing but pi b square minus a square; that is what we should be expecting from our ordinary geometry, that we have been learning in schools namely, the area of the circle is equal to pi r square.

So, we will first prove that the area of a circle of radius r is equal to pi r square and from there we will deduce this fact. So, let us observe, that, so the first thing is, let us take the linear transformation T, which is diagonal, which is given by (a, 0, 0, a).

So, we are looking at the diagonal transformation (a, 0, 0, a) and look at the unit circle, area enclosed by the unit circle, so that is E. So, E is the set of all (( )) vectors (x, y) in R 2, say, that x square plus y square is less than 1.

Then, if we look at any point here and transform it according to this T that will look like, so let us look at what will that look like?

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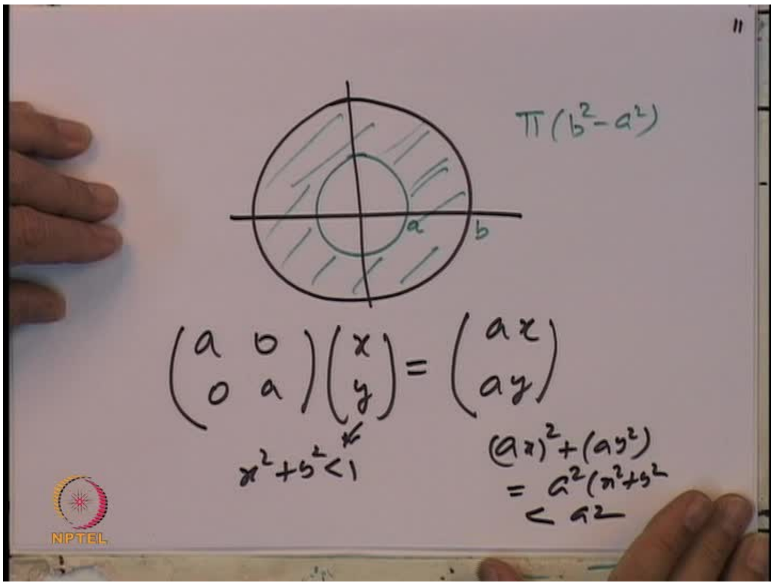

So, let us look at the transformation (a, 0, 0, a) and let us look at a vector (x, y), so that gives us the vector (ax, ay).

So, if this vector had the property, that x square plus y square is less than 1, then the transform vector (ax, ay) has the property, that ax square plus ay square is equal to a square times x square plus y square, so which is less than a square.

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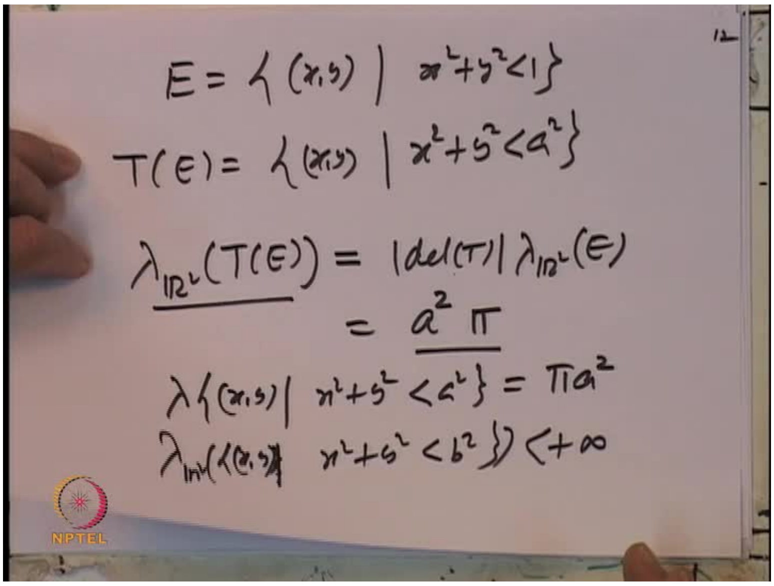

So, that shows, that the unit circle, so if E is the unit circle, that is (x, y) x square plus y square less than 1, then T of E is the circle, the region enclosed by the circle, x square plus y square less than a square.

So, that is what we know, so that means, now we apply, apply our theorem of linear transformations. So, look at the Lebesgue measure R 2 of the transform set E, so that is equal to, by the property of that theorem it is determinant of T times Lebesgue measure of the set E, but determinant of T is equal to, it is a diagonal transformation, so that is a square and Lebesgue measure of E, which is unit circle, is pi.

So, Lebesgue measure of the transformed set is equal to pi a square. So, that means, the Lebesgue measure of all the points (x, y) say, that x square plus y square is less than a square is nothing but pi a square. So, that is the magnification that we are getting and so as a consequence of this, let us deduce for the annulus region.

The area is the required Lebesgue measure, is pi of b square minus a square. So, for that we have to just observe, that if I look at the circle, so the set of points (x, y), such that x square plus y square is less than b square is a set, which is a bounded set and Lebesgue R 2 of that is finite.

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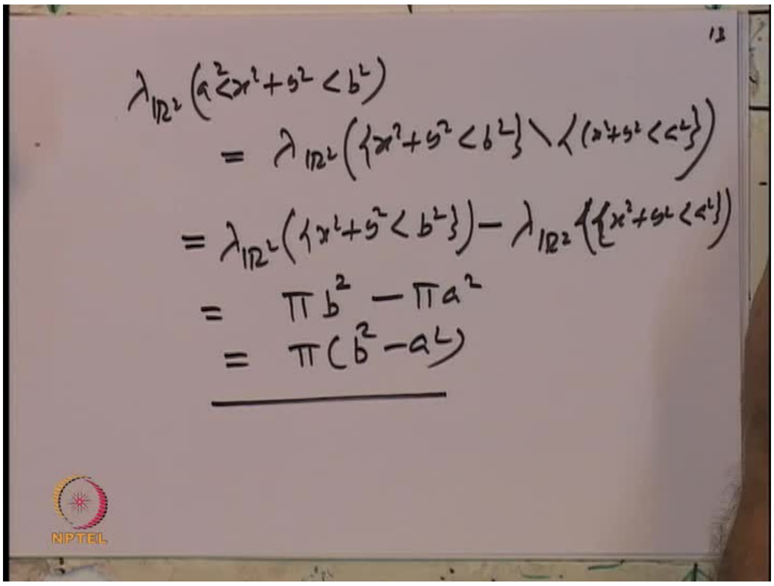

So, it is set of finite outer measure, finite Lebesgue measure, so I can write, that the Lebesgue outer measure of x square plus y square bigger than a square and less than b square, is nothing but the Lebesgue measure of the set x square plus y square less than b square (( )) minus the inner circle. So, that is x square plus y square less than a square.

And now everything being finite, I can write this as the Lebesgue measure of the region enclosed by the outer circle, so that is x square plus y square less than b square minus the Lebesgue measure of x square plus y square of Lebesgue measure of x square plus y square less than a square.

This is possible because everything is a finite quantity. So, the measure of the difference a minus b, measure of a minus b is measure of a minus measure of b whenever b is a subset of a, and everything is finite.

So, that property gives you this and this is pi of b square. Just now we saw pi of a square, so that is pi of b square minus a square, so that proves. So, what we are saying is these simple properties help us to confirm, that the Lebesgue measure on the plane, that we have defined is essentially extending the notion of area in the plane to a bigger class of subsets. And the usual formulas for the areas, that you have been using priory without any justifications, are now being justified by the Lebesgue measure.

I will just point out one more extension of this result, namely, that the area, Lebesgue measure of the annulus region is pi of b square minus a square to something, in integration, which looks like the change of variable formula in multiple integrals, namely, if you have a double integral, then and you change to Cartesian to polar coordinates, then the d x d y, normally we have that formula, that when you change d x d y, it should be r d r d theta.

So, more rigorous, we are saying, that for a particular class of functions, I want to state and give an outline of the proof, we will not be proving it fully, we will give an outline of the proof.

So, let us go to the next application of, so this is what I just now said, that the Lebesgue measure of the annulus region is Lebesgue measure of the outer circle minus Lebesgue measure of the inner circle, that is, pi b square minus a square.

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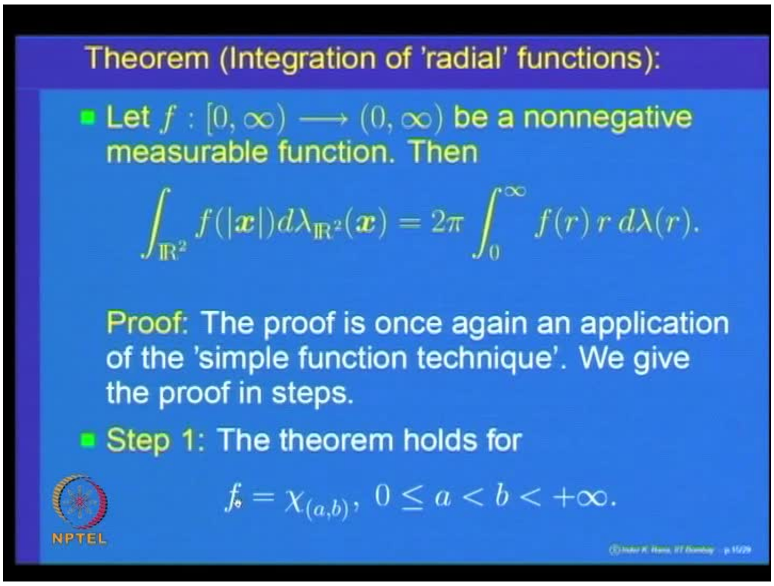

So, this is another application or extension of this result, just now we proved, is called the integration of the radial functions.

So, let us look at, the theorem says, let us look at a function defined from (0, infinity) on the positive, on the nonnegative part of the real line, taking values in, nonnegative values, so that is 0 to infinity. So, it is a nonnegative measurable function defined on the nonnegative part of the real line.

Then, the claim is, that if I look at the double integral, integral over R 2 of f absolute value of x, x is a vector, so absolute value means the norm, the magnitude of the vector x, so look at this, so this is like a composite function, the vector x goes to the magnitude, that is a nonnegative real number and f evaluated at that.

So, the double integral, integral with respect to R 2 is given by 2 pi times f r d r d theta. So, that is the claim, that this integral is equal to this integral and what is the meaning of a nonnegative radial function? f is a nonnegative, I should have said here, it is a radial function. That means, it depends only on the absolute value of the function, it does not… So, f is a nonnegative, so this is a radial function.

So, this you can think it as f composite, the magnitude is the radial function, is the radial function, it, the value of this composite function depends only upon the magnitude of the vector and not on the position of the vector.

So, to prove such a result, the proof is a typical application of a simple function technique. So, one tries to prove, that for simple measurable functions this is true and then apply monotone convergence theorem and so on. So, I will just outline the steps, for a detailed proof you may consult the text book.

So, let us look at the first step, let us look at the first step when this function f is the indicator function of an interval (a, b) when f is the indicator function of an interval (a, b), where (a, b) is an interval in the nonnegative part of real line. So, a less than b, bigger than 0, so when f is the indicator function, let us compute this both sides and what it that look like? So, when f is the indicator function of (a, b), so here is the indicator function, so this is 0 to infinity.

So, 0 to infinity means, this will give you, indicator function will give you only a to b. So, this will be a to b of the function f r, the function is indicator function, this value is 1, so x r d r. So, when you integrate r d r you get r square by 2 between a and b and when you put the values, you get b square minus a square by 2, so that is equal to pi of b square minus a square. So, this side is nothing but pi of b square minus a square, and what is this f?

So, the indicator function of (a, b) evaluated at the absolute value means, you are integrating in the annulus region between the limits a and b. So, it is pi b square minus pi a square, so in that, then it is just equal to pi of b square minus a square.

So, these both sides are nothing but the result, that we discussed just now, that the area of the annulus region is equal to pi b square minus a square. So, step 1 is for indicator function is that result.

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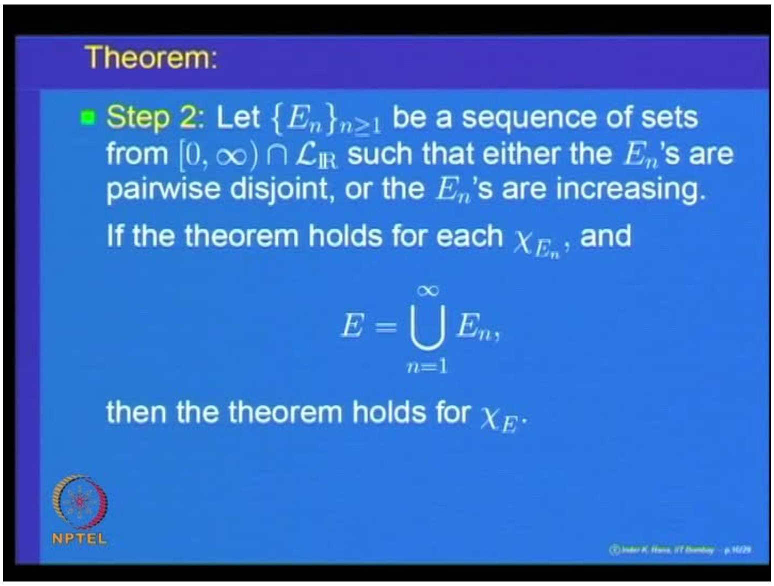

The next thing is we look at sets, which are either sequences E n's, which are sequences of sets, which are Lebesgue measurable of course, either are pair-wise disjoint or an increasing sequence. And supposing for the indicator function of each set E n this result holds, then the claim is, it also holds for the union of E n's.

So, if for each E n the result holds, then the result, sorry, for each indicator function of each E n it holds, then it also holds for the indicator function of the set E and that essentially, is an application of the monotone convergence theorem to the earlier result.

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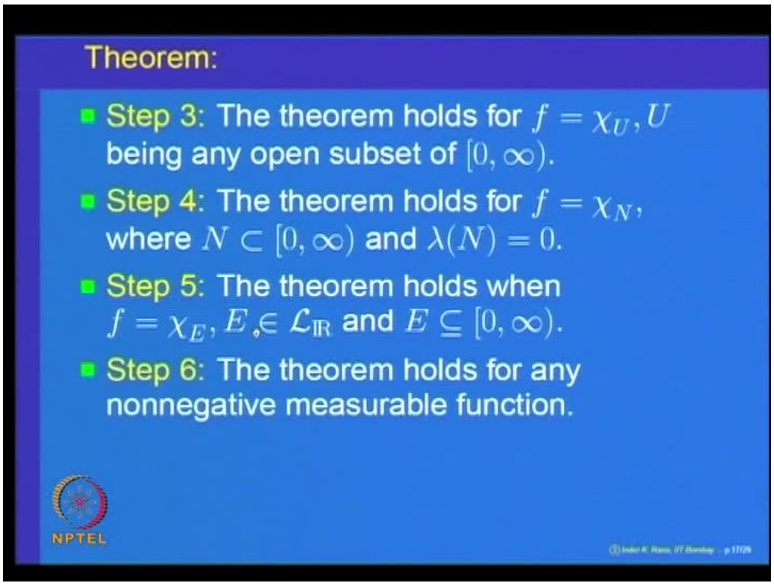

And then the step 3 is, from such things one comes to open sets by the fact, that each open set is a countable disjoint union of, countable disjoint union of intervals. So, for intervals, that property holds, so this will hold for every open set and from the open sets and null sets, so one shows the corresponding property also holds for null sets.

So, open sets and null sets, one goes to the indicator function of any Lebesgue measurable set because any Lebesgue measurable set can be written in terms of open sets and null sets. So, and then from this one applies usual monotone convergence theorem technique from the indicator function to nonnegative measurable function. So, these are the steps one follows to prove theorem of this kind.

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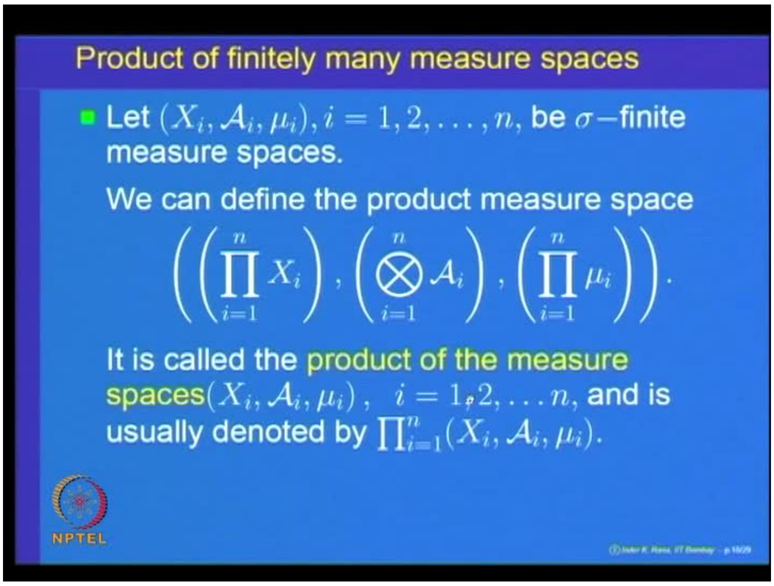

I just want to conclude today's lecture by saying what we have done it for product spaces, of 2 product spaces, can also be extended to any finite number of product spaces.

So, namely, if you are given a finite number of measure spaces (X i, A i, mu i), we define the product of 2 of them, can be extended by, sort of, doing one at a time, by iteratively, we can define the product of the space, this X spaces, X i's 1 to n, you can define the product sigma algebra A i's 1 to n and you can also define the product measure, inductively one can do that. And so, this is called the product space, I will not go into the details of it, but this is useful.

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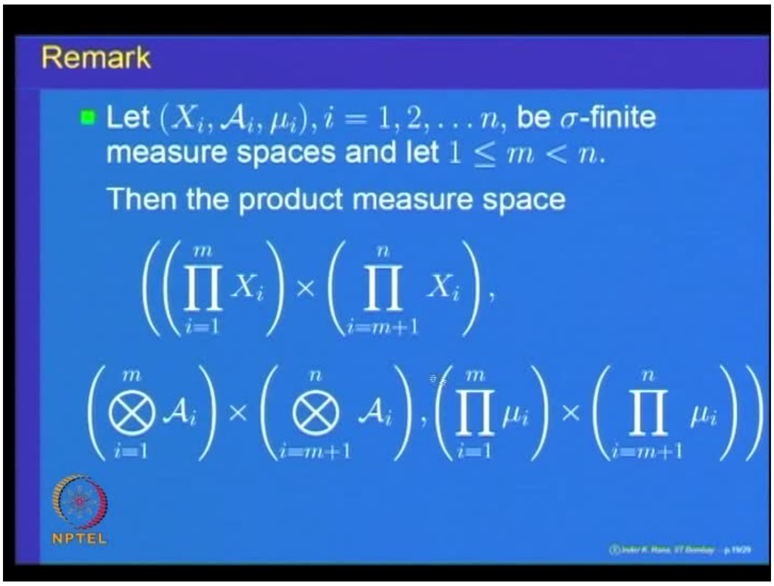

And one can also show, that if you take product of say, some finite number, m number, and take product of some n number of copies and then take the product again, that is same as the product of them put together. So, it is same as the product of X i's from, n, 1 to m plus n. So, these are same, so one can, same usual identity.

So, basically it is saying that what we have done it for 2, product of 2 measure spaces, can be benefit for a finite number of them.

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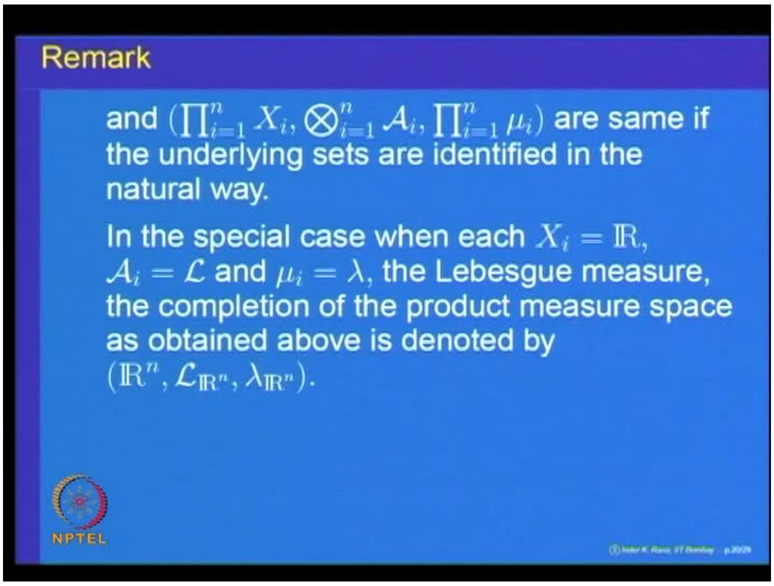

So, as a consequence one can define the notion of Lebesgue measurable subsets in R n and the notion of Lebesgue measure in R n; so, this can be done. So, again, those who are interested should refer the textbook for more details.

So, what we have done today is we have completed the study of product measure spaces and with that we have essentially completed what is called the basic concepts in measure theory, namely, we have done the extension of measure, then integration of measure, then measure and integration on product spaces.

This is the core of the subject and from now onwards I will be looking at some special topics in our subject of measure theory.

Thank you.