Measure and Integration

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Module No. # 08

Lecture No. # 31

Properties of Lebesgue Measure on R^2

Welcome to lecture 31 on measure and integration. In the last two lectures, we have been looking at the properties of the Lebesgue measure on the space R 2 with respect to Borel measurable subsets of R 2 and Lebesgue measurable subsets of R 2. We will continue studying these properties of Lebesgue measure on the space R 2 a bit more today.

If you recall, we had looked in the previous lecture on how the Lebesgue measurable sets and the Borel measurable sets behave with respect to the group operation and the topologically nice subsets of the plane. We had showed ((.)) aspect of the Lebesgue measure; namely, up to a constant multiple, Lebesgue measure is the only translation invariant measure on the set of all Lebesgue measurable and of course, in particular, the Borel measurable subsets of the plane.

We will be making use of that property today, but we will begin with looking at some more transformations on the plane with respect to which Lebesgue measure can change and how does it change. Let us begin by looking at today's topic; it is going to be Lebesgue measure and its further properties.

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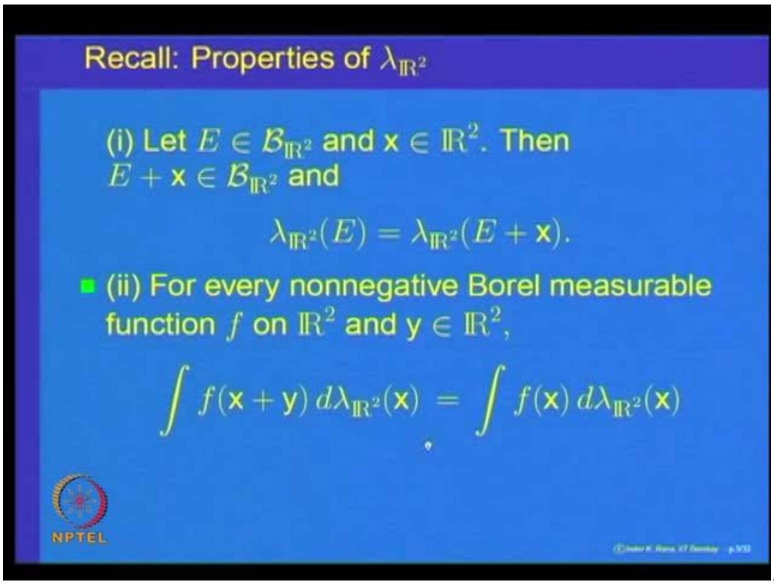

Let us just recall what we had shown in the previous lecture – if E is a Borel measurable subset of R 2, then its translate E plus x is also a Borel measurable subset in R 2 and the Lebesgue measure of the translated set is same as the Lebesgue measure of the original set; it is very much similar to the properties on the real line. We also showed that for a nonnegative Borel measurable function on R 2, if you look at the integral of the translated function, that means f of x plus y d lambda x, is same as integral of f of x with respect to x. We also showed that this is also equal to actually the integral of f of minus x. So, under reflections and translations, the integrals do not change.

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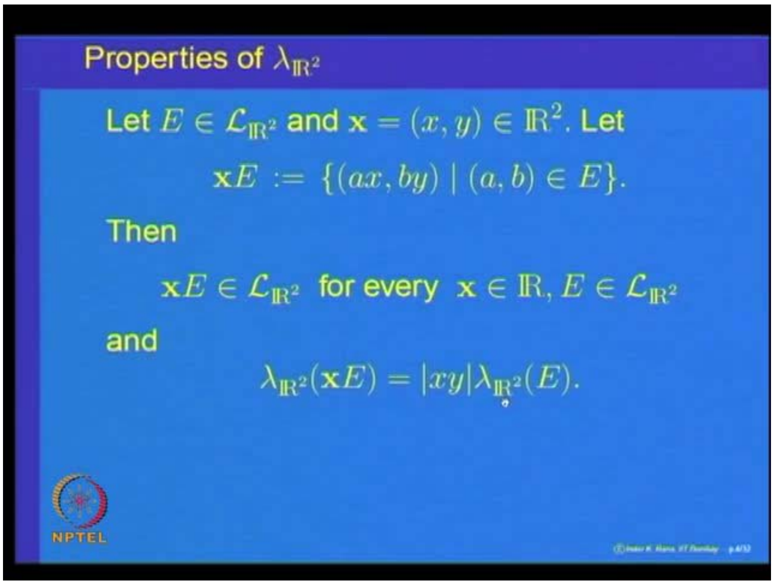

Today, we will start looking at another transformation on the plane; namely, for any vector x with components x and y in R 2, let us see how does this change if you multiply every element of a set with this vector. Let us define the vector x dot E to be equal to a x comma b y for every element a, b in E; that means the x coordinate is multiplied by x and the y coordinate is multiplied by y for every element a, b in E.

So, the first coordinate a changes by x; it goes to a x; the second coordinate which is b goes to b times y. The claim we want to prove is that for every Lebesgue measurable set E, this multiplied set x dot E is also a Lebesgue measurable set and the Lebesgue measure of this transformed set is equal to absolute value of x y (where x is the x component of the vector x and y is the second component of the vector x) So absolute value of x y times Lebesgue measure of the set E. Let us see how we prove this property.

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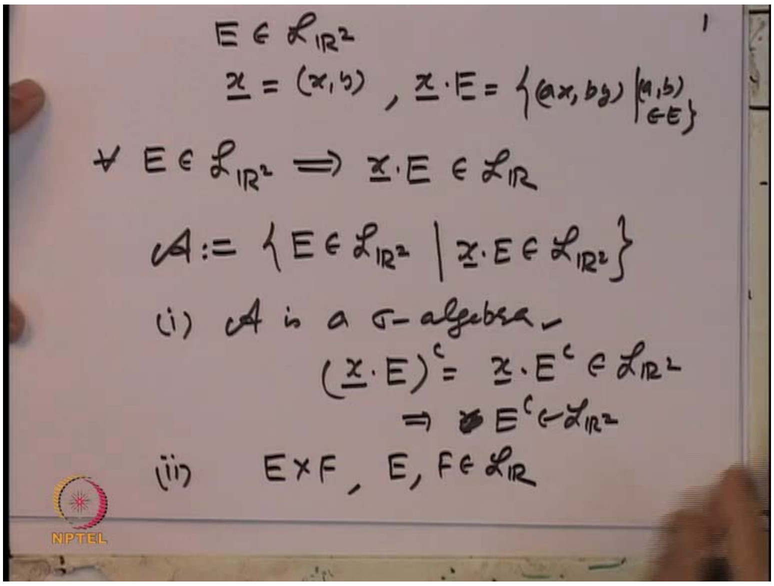

We are given a subset E which is Lebesgue measurable and we are given a vector x with components x comma y. We look at the vector x dot E which is all elements of the type a x comma b y where a, b is an element in E. We want to check that for every subset E which is Lebesgue measurable in the plane, it implies that the vector dot E is also Lebesgue measurable; that is first part of the claim.

The proof of this is on similar lines as for the translation; namely, we will collect together all the sets for which this property is true and show that rectangles come inside and show that this class for which this is true forms a sigma algebra; so, everything will be inside; that is the sigma algebra technique essentially we are going to use. Let us look at that.

Let us form the collection A to be equal to all subsets E belonging to L of R 2 such that such that x dot E belongs to L of R 2; let us look at that. Let us observe; the first observation is that this collection A is a sigma algebra. This collection A is a sigma algebra; that is easy to check; I will not write; I will just orally discuss this. Let us take if a set E belongs to it, then the observation is that we want to show that complement also belongs to it; that is easy because x dot E complement is nothing but so x dot E complement is same as x dot E complement.

If E belongs to this collection A, then x dot E belongs to this collection (Refer Slide Time: 06:41). So, its complement belongs to the Lebesgue measurable sets. The complement is equal to x dot E complement. This belongs to L of R 2; it implies that E complement belongs to L of R 2. A similar computation will show that A is closed under countable union; that will prove it is a sigma algebra. Let us check the second property; namely, if I take a rectangle, if I take a set E cross F where E and F both belong to L of R, then what happens to x dot E? Let us compute that.

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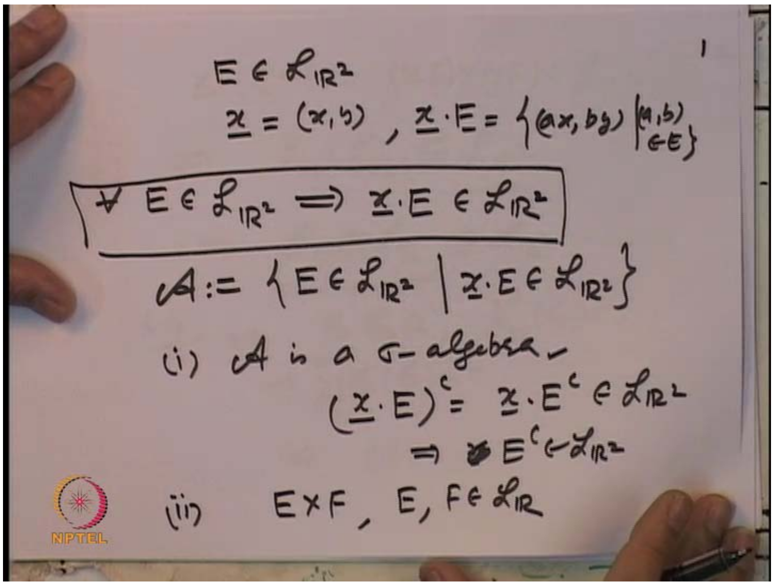

In that case, the vector x dot E cross F will look like this – the x component multiplied with E cross the y component multiplied by F; it is x E cross y F. If x is a Lebesgue measurable set, we know that x dot E is a Lebesgue measurable set and y times F also is a Lebesgue measurable set in the real line; this means this is a set in L of R 2 (Refer Slide Time: 08:13).

That means if I take a rectangle – sets of the type E cross F where E is in L of R and F is in L of R, then the rectangle satisfies the property of being in the class A; that means L_R cross L_R is inside L of R 2. Just now we observed that this is a sigma algebra; that will imply that L of R product L of R is also in L of R 2. All the elements in the product sigma algebra L_R cross L_R are inside L of R 2; L of R 2 is just the completion of this space (Refer Slide Time: 09:06). It is easy to show that if I take a null set, then x dot E also is a null set.

Also, if E is a subset of R 2 and lambda$_{R\ 2}$ of star of E is 0, then it is easy to check, I leave it an exercise, that x dot E is again a null set; then x dot E lambda star R 2 is again a ((.)). Outer measure of that is again measure 0; that implies that the sets for which the Lebesgue outer measure is 0 are also in that class. This along with the earlier fact (this fact and this fact together) imply that A is equal to L of R 2. This is basically the sigma algebra technique which is used to prove the fact that for every set E this property is true. So, x dot E belongs to L of R 2 whenever E belongs to L of R 2 (Refer Slide Time: 10:31)

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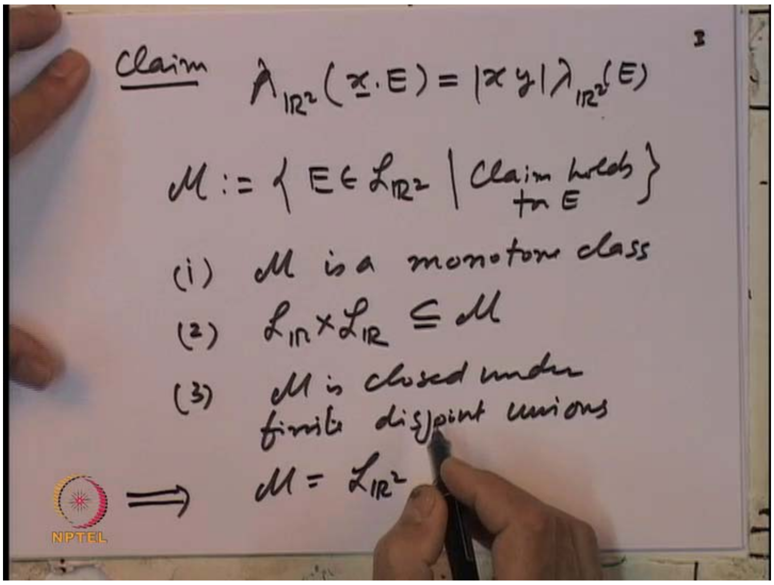

Next, we want to check that the Lebesgue measure also is preserved. That again is a proof which is similar to the earlier proof. We want to check the claim. The second part of the claim is that the Lebesgue measure of the set x dot E is equal to x y product Lebesgue measure of the set E; this is what we want to check. Let us observe the following. Basically, we are going to apply the monotone class technique here as in the case of the translation.

Let us define M to be the class of all sets E belonging to L of R 2 such that this required claim, let us put that as a star – this property, holds for the set E. The technique is to show that (i): this class M is a monotone class; that is one step – to show it is a monotone class; (2): show that the sets L_R cross L_R, the rectangles, are inside this class M; third: this M is closed under finite disjoint unions. Once these three steps are proved, this will imply that M is equal to L of R 2 essentially.

The idea is the following. These rectangles are inside M; second step says rectangles are inside M; first one says it is the monotone class; so, the monotone class generated by these rectangles will be inside M. Now, third property says that this is also closed under finite disjoint unions. Once it is a monotone class and it is closed under finite disjoint unions, that will imply that the algebra generated by L_R cross L_R also is inside M. The monotone class generated by L_R by L_R will be in ((.)). Let me just write these steps why this will imply this; basically, the reason is the following.

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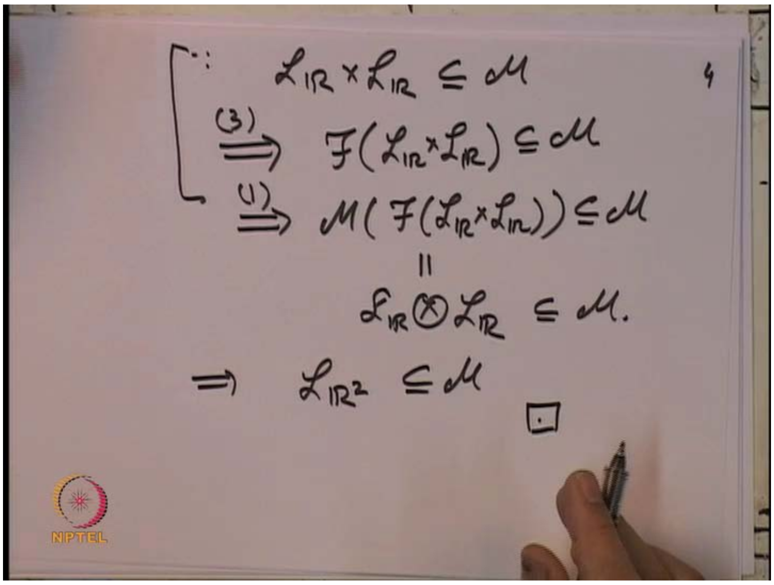

Here is the reason why this will happen. L_R cross L_R, the rectangles inside M, will imply by the step (3) finite disjoint unions that the algebra generated by these rectangles will also be inside the class M, because it is closed because the algebra generated by a semi-algebra is nothing but the finite disjoint unions. This will imply by (1) that M is a monotone class and this algebra is inside it; that will imply that the monotone class generated by this algebra – that is L_R cross L_R – will also be inside M.

Our monotone class theorem says that this is nothing but the sigma algebra generated by this class. This will imply this is the product sigma algebra – Lebesgue measurable sets cross Lebesgue measurable sets – is inside M. Once again, one shows that this is also true for null sets; it will hold for completion; that will imply that L of R 2 which is a completion of this is also inside M; that will prove the required result (Refer Slide Time: 14:47).

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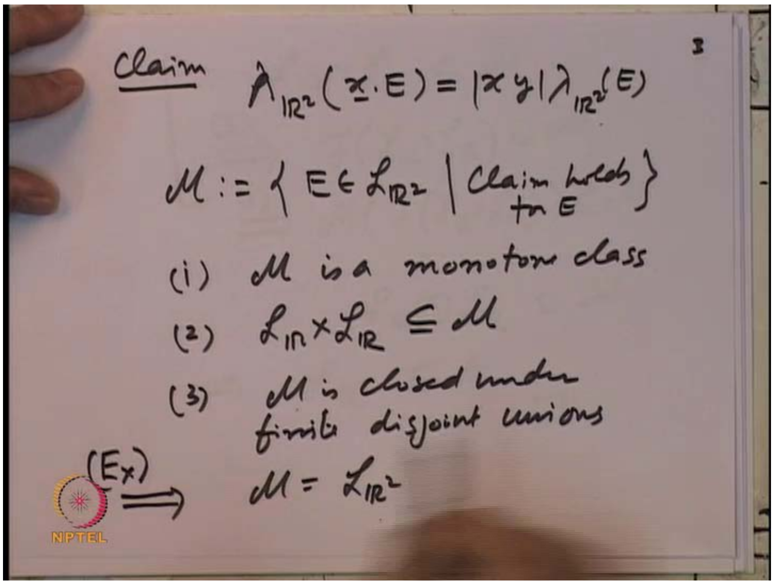

These steps that this collection M is a monotone class, includes rectangles and is closed under finite disjoint unions – this proof is similar to that of the proof when we had translation of a set E by a vector. I will suggest that you try this as an exercise yourself on the same lines as the earlier proof, because it is a repetition of the same idea again and again; it is better to get used to it by doing it yourself.

That will prove the required claim; this property for the Lebesgue measure will be proved (Refer Slide Time: 15:31). If E is a Lebesgue measurable set, then the product x times E is also a Lebesgue measurable set and its Lebesgue measure is equal to the Lebesgue measure of the set E multiplied by absolute value of the components of x that were x comma y. This property relates ((.)) something ((.)) multiplication. Now, we would like to rewrite this property in a slightly different way.

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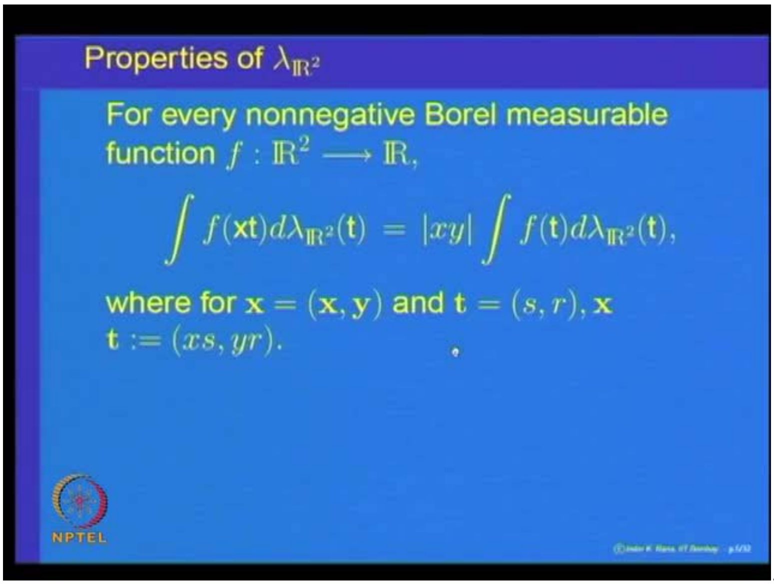

Before that, let me just state the corresponding result for integrals; if f is a nonnegative measurable function on R 2, then integral of x times t, this multiplication as defined now, is equal to the absolute value of x y times the integral of the function f. That is how if I multiply the function f each by a vector x, then the integral changes by the value mod of x minus y.

The proof of this is once again an application of sigma algebra monotone class; simple function technique, I am sorry. I would like to leave it as an exercise once again; copy the proof; try to copy the proof for the translation x plus t saying that the integral is invariant and here the multiplication comes. The steps are essentially first take f to be the indicator function of a measurable set; just now we have proved that result.

Once it is true for indicator functions, this being an equality involving integrations it will hold for finite linear combinations; that means this will be true when f is a nonnegative simple measurable function (Refer Slide Time: 17:33). For a general nonnegative measurable function, one takes limits of nonnegative simple measurable functions and essentially applies monotone convergence theorem to get that this result is also true. The standard simple function technique will give you a proof of this; I will suggest that you have a look at that proof yourself.

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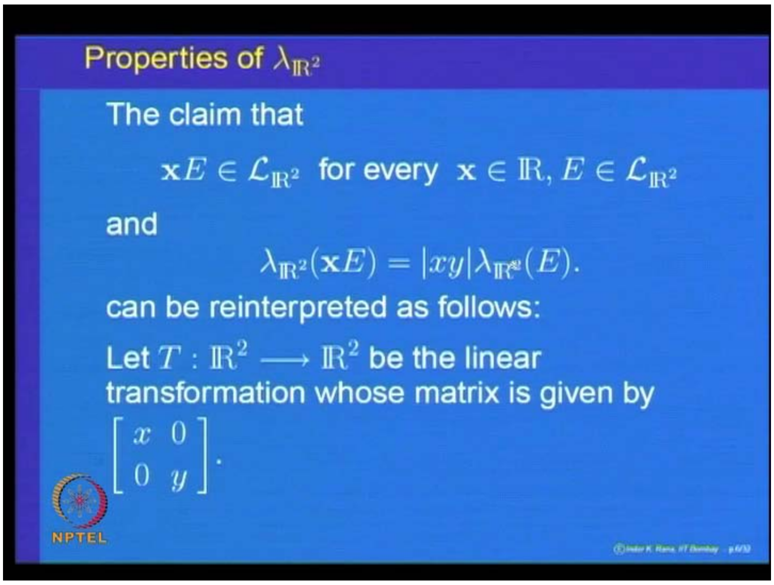

Let us go to some more properties of Lebesgue measure. Just now we proved that for a Lebesgue measurable set E and a vector x in R 2, this should be R 2, if you multiply, lambda$_{R\ 2}$ of x E is equal to the absolute value of the product of the coordinates of the vector x with which you are multiplying into the Lebesgue measure of E. One can reinterpret this result as follows; let us look at a transformation T from R 2 to R, a linear transformation whose matrix is given by the diagonal matrix x, 0, 0, y. Here, some knowledge of linear algebra is required; let me state a few things about some facts about linear algebra that we are going to use. Here are some facts about linear algebra that we are going to use.

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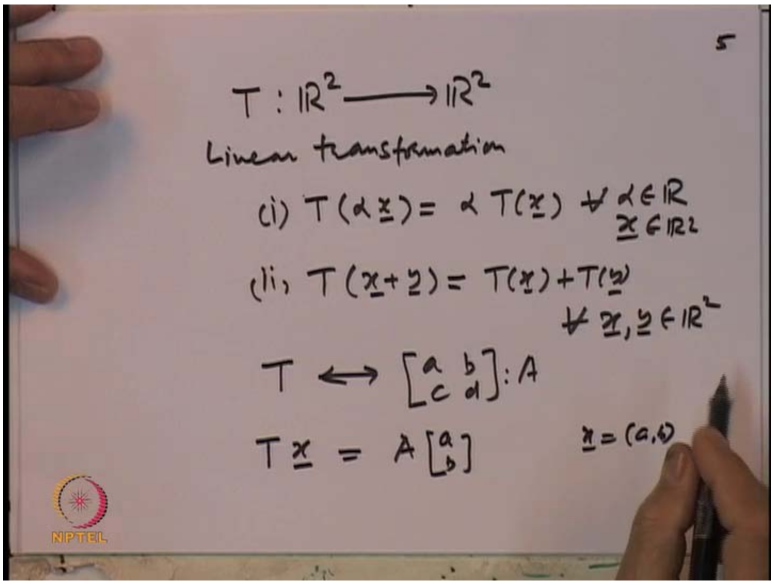

The first thing is that a linear transformation is in linear algebra; I will be looking at linear transformations on R 2 ((.)) only. Let us look at linear transformations on R 2. What is a linear transformation? A linear transformation is a function T from R 2 to R 2 which has the following property that if I take T of alpha times a vector x, it is alpha times T of x for every alpha belonging to R and the vector x belonging to R 2; that is one property. The second is that for any two vectors x and y if I look at image of T of x plus y, that is equal to T of x plus T of y for every x y belonging to R 2; such a map is called a linear transformation and any such linear transformation T is also given by a two-by-two matrix; let us call it as a, b, c and d.

What is the meaning of this? That means that T applied to a vector x is same as… If you call this as the matrix A, that is a matrix A applied to the component of x, let us call it as a, b, applied to a, b, where x is equal to a comma b. This is a very standard thing in linear algebra that linear transformations are described by matrix multiplication where A is called the matrix corresponding to the linear transformation and that is obtained via ((.)) of R 2. I will not go into detail; that is what essentially we will require.

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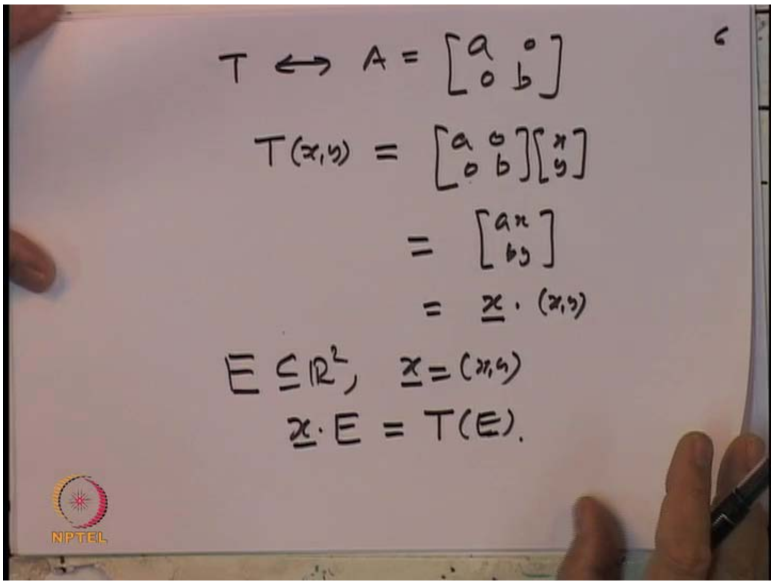

Let us look at a special linear transformation, namely, T which comes from a matrix which is the diagonal matrix a, 0, 0, b. What is the effect of the linear transformation T? T applied to a vector with components x, y is same as the matrix a, 0, 0, b applied to the column vector x, y; matrix multiplication says it is just a x and b y; a into x plus 0 into y plus 0 into x and b into y; that is same as, in our notation, the vector x multiplied with the vector with components x, y.

In our notation, we had this: for any set E subset in R 2 and a vector x with components x, y, saying that we are multiplying x with E, the property that we studied just now, is same as looking at the image of the set E under this linear transformation E; that is what that interpretation is. You can say that this multiplication by the vector x is transforming the set E by the linear transformation T which is a map from R 2 to R 2.

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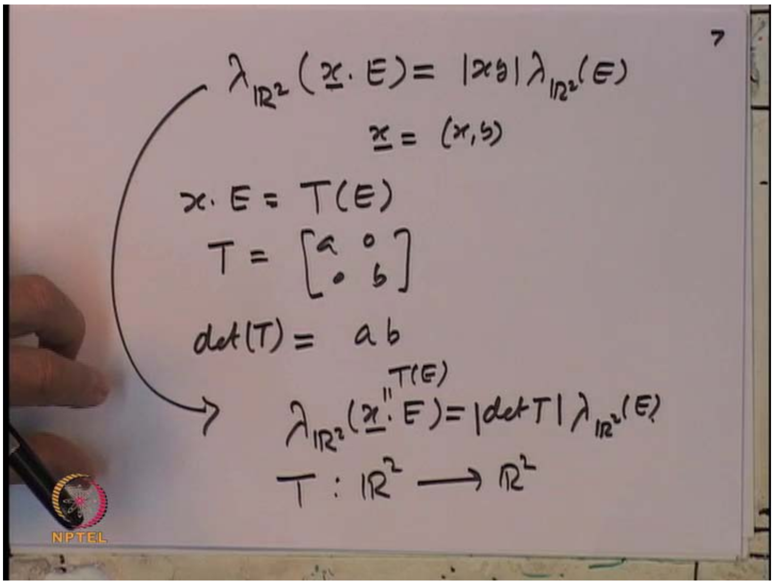

The result we proved just now says that the Lebesgue measure of R 2 of the set x dot E is equal to the absolute value of x y times the Lebesgue measure of the set E where the vector x is equal to components x and y. We said x dot E is the image of E under the linear transformation T; also, this T is given by the diagonal matrix a, 0, 0, b. For this diagonal matrix, there is a notion of what is called determinant of this transformation T. What is a determinant? Determinant for two-by-two is cross-multiplying and subtracting the values; so it is a times b for this determinant.

The absolute value of x y when you multiply by x… Here, our vector is with component x and y. This result can be interpreted as lambda$_{R\ 2}$ of x dot E is equal to this absolute value of x y is nothing but the determinant of T times lambda$_{R\ 2}$ of E. Our result that under multiplication that is how the value changes of the Lebesgue measure can be interpreted in terms of linear transformations that if I take the linear transformation T from R 2 to R 2 which gives this multiplication as interpreted earlier, then the Lebesgue measure of the translated set, this is T of E (Refer Slide Time: 25:08), Lebesgue measure of the transformed set is determinant of T times the Lebesgue measure of the original set.

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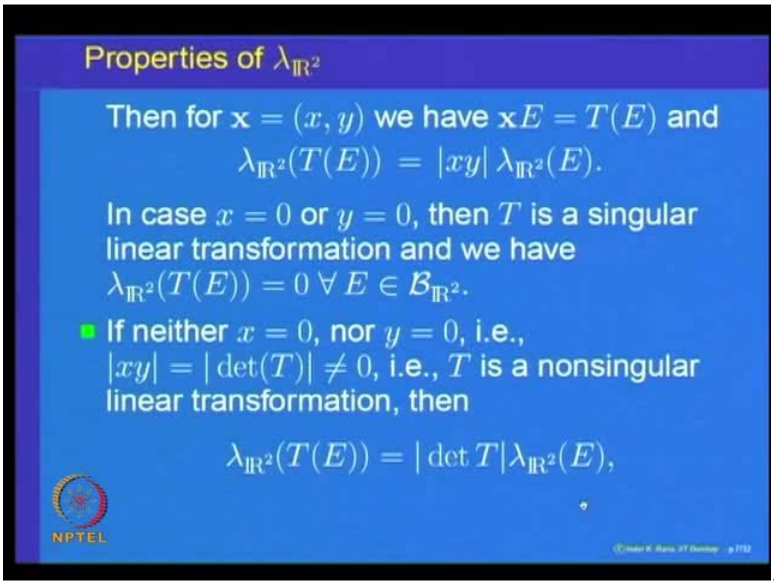

This is the property interpreted in terms of maps. Let us also observe something called when T is singular; in that case, determinant of T is 0; we are basically saying that if x or y are 0, then both sides are 0; if neither x or nor y is 0, then determinant is not 0 and T is nonsingular. Let us just look at these facts a bit more seriously.

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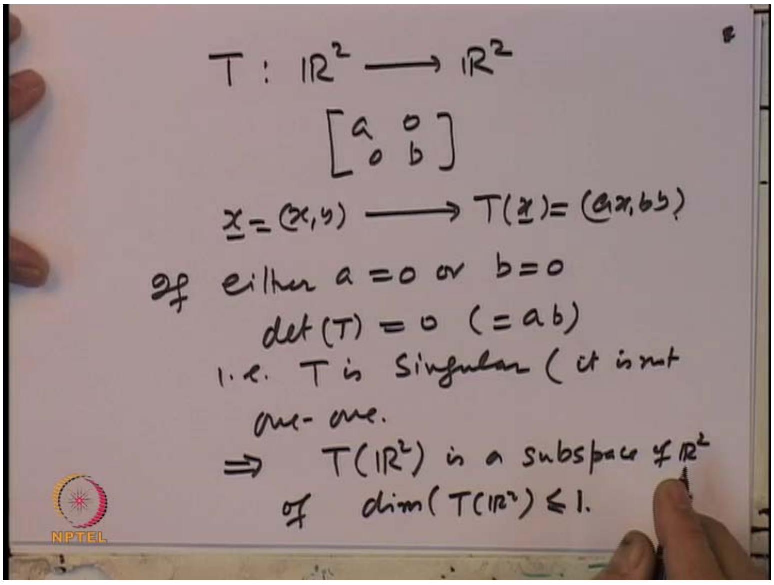

We have a map T from R 2 to R 2. This is given by the diagonal matrix a, 0, 0, b; this is the matrix. That means a vector x comma y goes to T of x which is nothing but a x comma b y. Now let us observe a thing here: if either a is 0 or b is 0, then determinant of

T is equal to 0, because determinant of T is equal to equal a times b; that means that is T is singular; whenever determinant of a linear transformation is 0, T is singular; that means in terms of functions it is not one-one; it is not one-one.

T is a linear transformation which is singular; it is not one-one; that implies that the image T R 2 of the whole space is a subspace of R 2, because under linear transformations the image is always a subspace of dimension of T R 2 has to be less than or ((.)); it cannot be 2 because then it will be one-one and because it is not one-one, it is less than or equal to 1. Here, I am discussing a bit of linear algebra because that will be required here.

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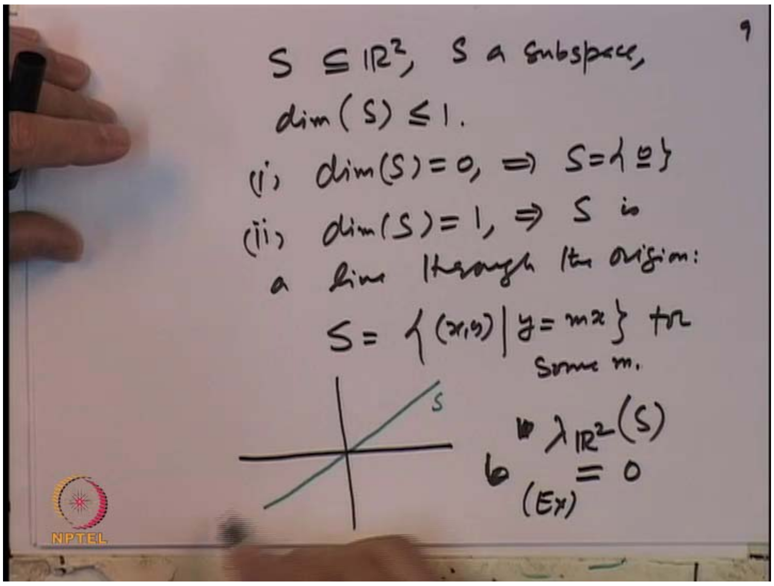

The question arises what is subspace S of R 2? S is a subspace and dimension of dimension of S is less than or equal to 1. One possibility is dimension of S is equal to 0; that implies that S is just the zero vector. Secondly, dimension of S is equal to 1; that will imply that geometrically S is a line through the origin or mathematically I can write S as all x comma y where a subspace S should look like y is equal to m x for some m.

That means in R 2, a subspace has to be nothing but a line through the origin; a subspace S has to be a line though the origin. Once it is a line through the origin, what is going to be the Lebesgue measure of this line? Let us look at the Lebesgue measure of R 2 of this line S. Obviously, the guess is this is going to be equal to 0. There are various ways of proving this. To prove that Lebesgue measure of any line is equal to 0, what one can do is try to approximate this line by small rectangles. I think this is a good exercise; I leave it as an exercise to check that the Lebesgue measure ((.)) saying that the area of the line is equal to 0. Let us use these facts.

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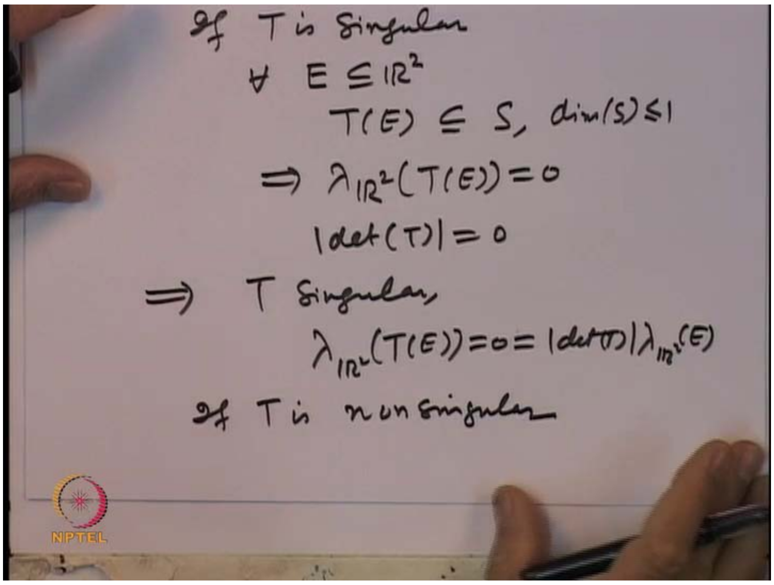

If T is singular, then for every subset E contained in R 2, T of E is going to be a subset of dimension 1; it is a subset of S; dimension of S less than or equal to 1 implies that the Lebesgue measure of R 2 of this set E is going to be equal to 0. On the other hand, we also know the determinant of T is also equal to 0. That implies for singular transformation T singular, the Lebesgue measure of T of E equal to 0 equal to determinant of T which is again 0 times Lebesgue measure of E; that property holds when T is singular. If T is nonsingular, that means…

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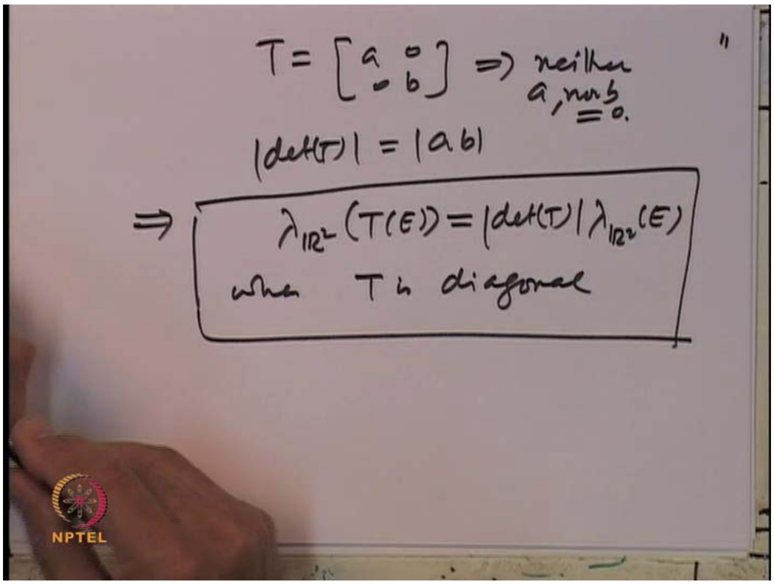

T is given by a, 0, 0, b; that implies that neither a nor b is equal to 0. Determinant of T is equal to a b; that once again implies that (our earlier result implies that) lambda of R 2 of T of E is equal to determinant of T times Lebesgue measure of the set E when T is diagonal. For diagonal transformations, we have this result: if you take a Lebesgue measurable set E and transform it according to a linear transformation, then the transformed set has got Lebesgue measure which is determinant of T absolute value times the original Lebesgue measure of R 2. (Refer Slide Time: 33:09) This is for nonsingular determinant.

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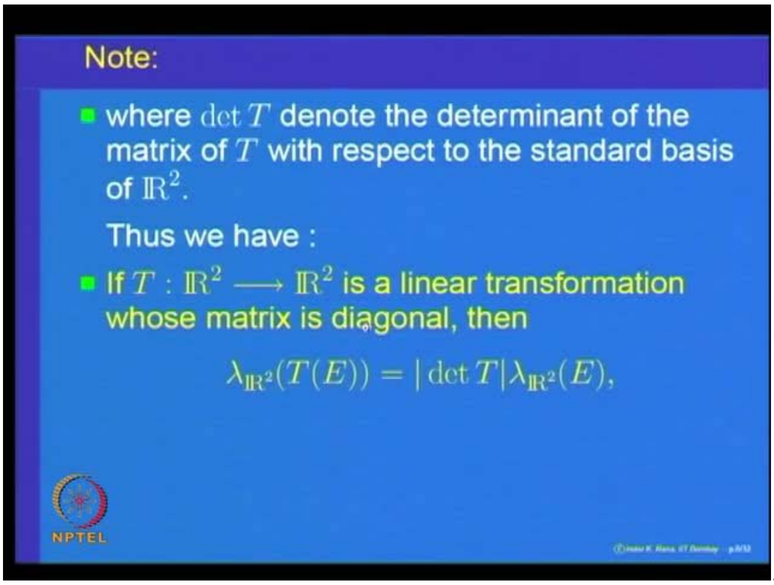

The question arises: can we say that this result is true for all linear transformations? We have got a result for diagonal transformations; namely, if T is a diagonal transformation from R 2 to R 2 and we transform a Lebesgue measurable set according to this, then the Lebesgue measure of the transformed set is absolute value of the determinant of T times the original measure. The question is: can we say this result is also true for arbitrary linear transformations of the plane? We are going to prove yes, that is true; this result holds for all linear transformations in R 2; that is what we want to prove.

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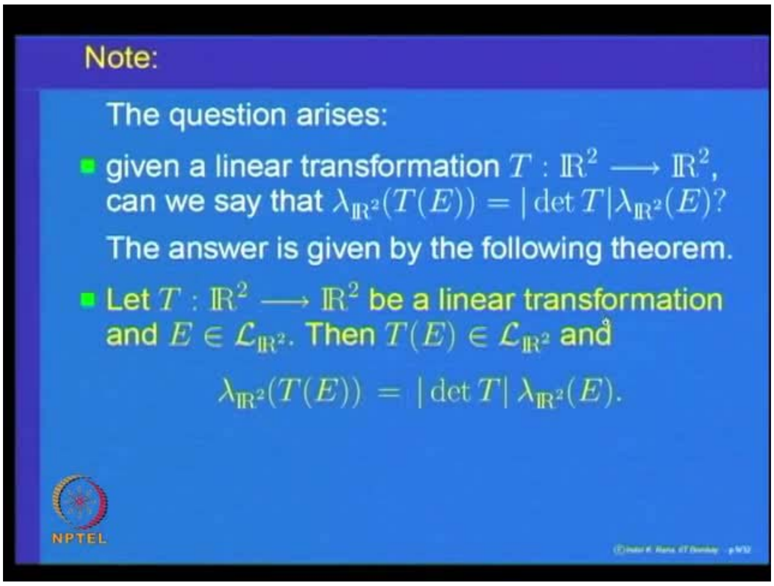

The theorem says for all linear transformations in R 2, one can say that the Lebesgue measure of the transformed set is determinant of the absolute value of the determinant of T times the Lebesgue measure of E.

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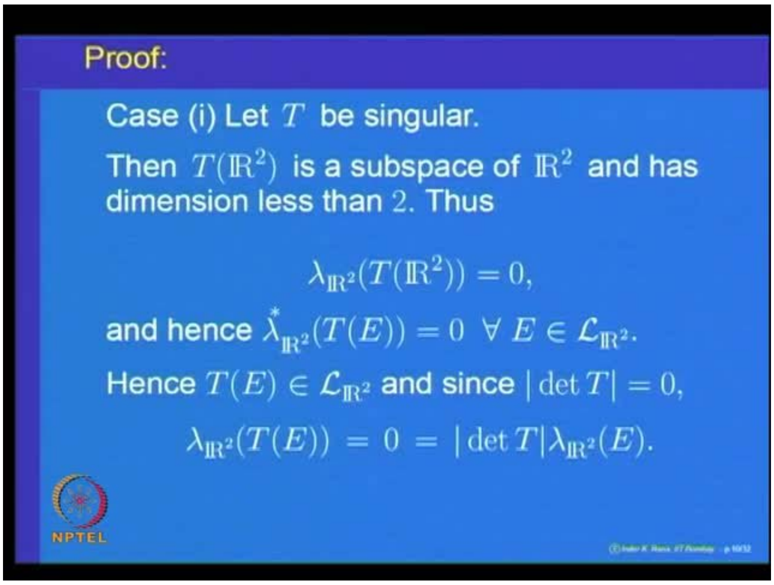

As before, we will assume T is singular. We have just now observed when T is singular this result is true because T of R 2 is a subspace of R 2 of dimension less than 2; so, it will be either a line or a single point – the origin vector. In either case, the Lebesgue measure of the transformed set is equal to 0; it is an outer measurable set of measure 0; it will belong to the Lebesgue measurable is a Lebesgue measurable set. What we are saying is that if E is a Lebesgue measurable set and T is the singular transformation, then T of E is a set of outer Lebesgue measure 0 in the plane and hence it is Lebesgue measurable; since determinant of T is also 0, the required property holds when T is a singular transformation.

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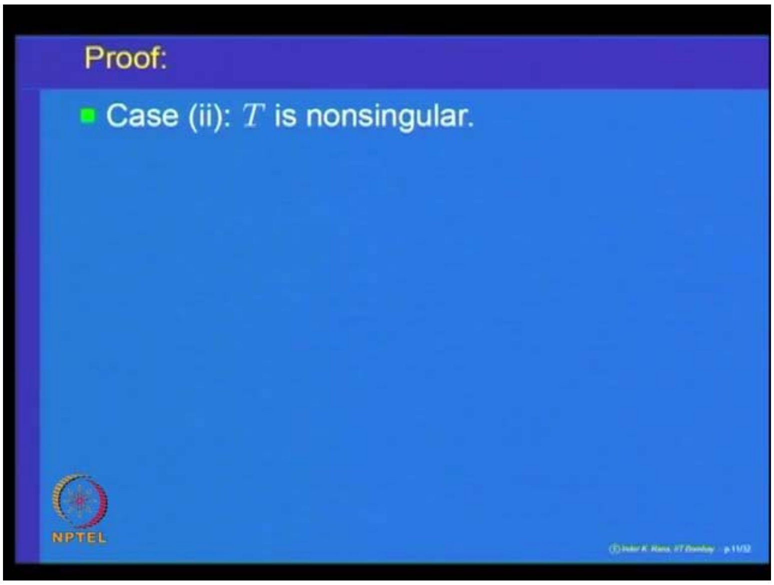

The second case is when T is a nonsingular. Let us look at the case when T is nonsingular.

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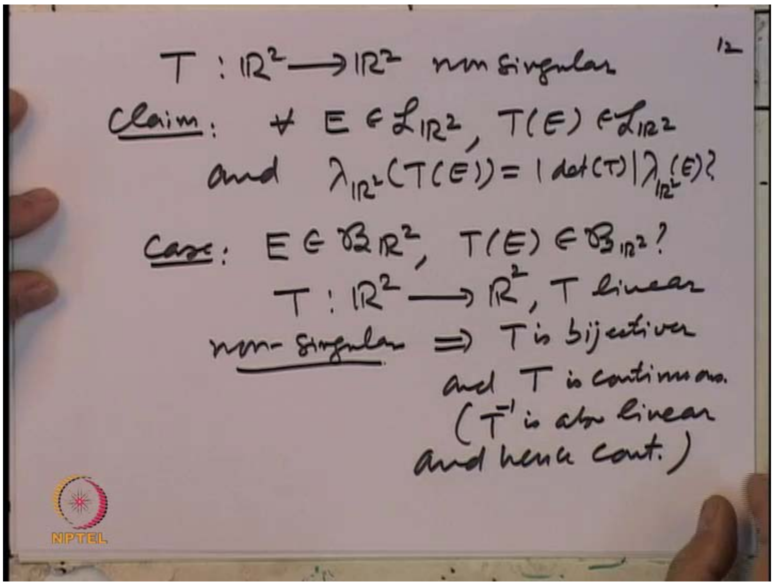

T from R 2 to R 2 is nonsingular. The claim is for every E belonging to L, Lebesgue measurable set; T of E is also Lebesgue measurable; the Lebesgue measure of the transformed set T of E is absolute value of determinant of T times Lebesgue measure of E; that is what we want to check. We will first do it; let us assume for the time being that the set E is a Borel subset in R 2. When E is a Borel subset of R 2, we want to check that T of E is also a Borel subset of R 2; that is the question we want to first analyze.

For that, we observe that if T is a linear transformation from R 2 to R 2, T linear and nonsingular, the nonsingularity implies T is bijective; every linear transformation which is invertible, of course, has to be bijective. Secondly, on the plane we have got the notion of topology convergence; one can easily check that every linear transformation is continuous.

We are using two things here; namely, a linear transformation is nonsingular if and only if actually it is bijective and every linear transformation on R 2 to R 2 is continuous. Actually, it is true for R n to R n, but we are only constructing it for R 2 to R 2; so, T is continuous. Not only is T continuous, but the inverse map bijective T inverse is also linear and hence continuous. Any nonsingular linear transformation T from R 2 to R 2 is a continuous map and the inverse also is a continuous map.

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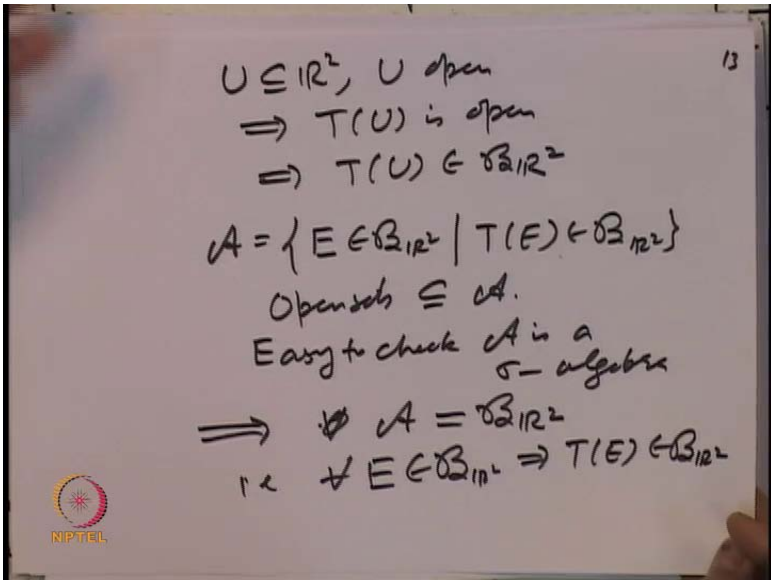

Once that is true, let us take a set U contained in R 2 and U open. That will imply T of U is open because T inverse is continuous. This implies the T of U is a Borel set in R 2. What we are saying is if I look at the collection of all Borel subsets in R 2 such that the image is a Borel subset in R 2, then open sets are inside this collection A and it is easy to check that.

It is easy to check that this collection A is a sigma algebra basically because B R 2 is a sigma algebra and T is a bijective map. A is a sigma algebra; that will imply that this collection A is actually equal to B R 2; that is, for every set E which is a Borel set in R 2 implies T of E is also a Borel subset of R 2. So, T preserves the collection of all Borel sets.

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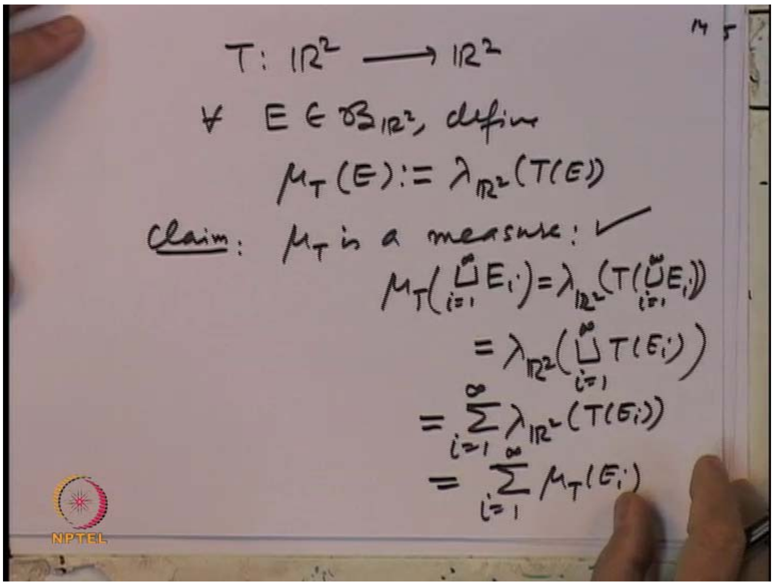

T is a map from R 2 to R 2. For every set E which is a Borel subset of R 2, let us define a new measure mu of T as follows: mu T of E is equal to Lebesgue measure of the set T times E. The claim is that mu T is a measure; that is easy to check because if I take the disjoint union of sets E i 1 to infinity and then look at mu T of that, that is going to be lambda of T of the disjoint union of the sets E_i.

Now, let us observe that T is a one-one onto map. T of the union is going to be disjoint union of T of E_i, i equal to 1 to infinity; lambda being a measure, this is equal to sigma i equal to 1 to infinity $\text{lambda}_{R\ 2}$ of T of E_i. This is the same as the summation i equal to 1 to infinity $\text{lambda}_{R\ 2}$ of T of E_i is mu T of E_i. That proves the fact that mu_T is a measure. The second property we want to check for this measure mu_T is that mu_T is translation invariant; let us check that.

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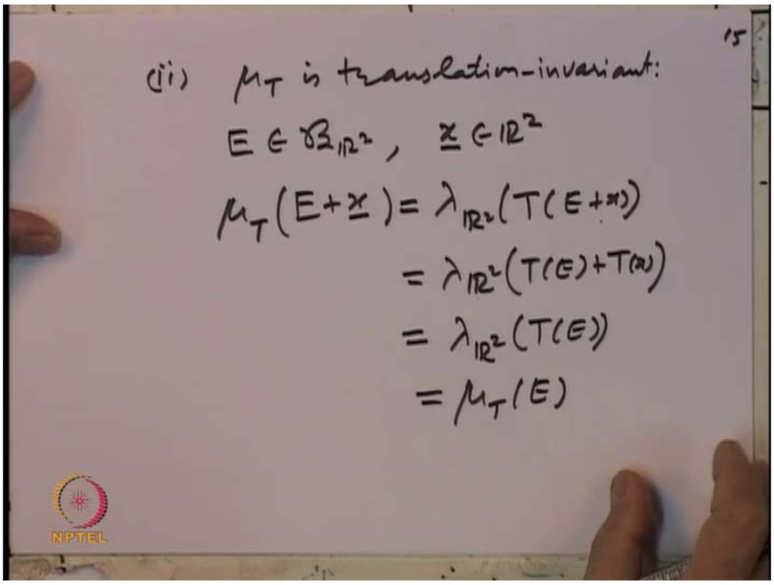

The second property is that mu_T is translation invariant on R 2. That means what? Let us take a Borel set in R 2. Let us take a vector x in R 2 and look at the set E plus x; that is a translated set. mu_T of this set is equal to by definition $lambda_{R\ 2}$ of T of E plus x, but T is a linear transformation; that implies T of E plus x is just T of E plus T of x; that is a consequence of the fact that T is linear; Lebesgue measure being translation invariant says this is $lambda_{R\ 2}$ of T of E; that is same as mu_T of E. That proves that mu_T is a translation-invariant measure.

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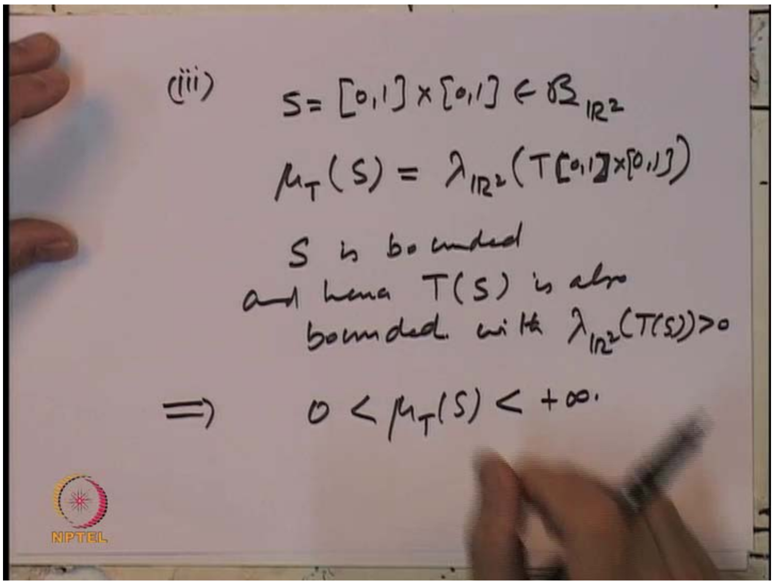

Another fact about this measure let us write as a third property. Let us take the set 0, 1 cross 0, 1 which is a Borel set. Let us call this set as S, the square in the plane; that is, of course, a closed set cross a closed set; that is a closed set; it is a Borel subset in R 2. mu$_T$ of this set S is equal to lambda$_{R\ T}$ of the Lebesgue measure of T of 0, 1 cross 0, 1. Now, let us observe T of 0, 1 cross 0, 1.

The observation is that S is bounded and hence T of S is also bounded; it is also a bounded set and, of course, its Lebesgue measure is positive – with Lebesgue measure of R 2 T S being positive. Being bounded, it has to be finite; that implies that the measure mu$_T$ of S is positive, is bigger than 0, and less than infinity. These are three properties of the measure mu$_T$; let us look at what are the three properties of the measure mu$_T$ that we have proved.

(Refer Slide Time: 45:54) (i): we defined the measure mu$_T$ of E to be equal to lambda$_{R\ 2}$ of T of E; we said first of all it is a measure; that is one property that we proved. The second property we proved it is translation invariant (Refer Slide Time: 46:08). The third property we proved that there is a set of finite positive measure with respect to mu of T.

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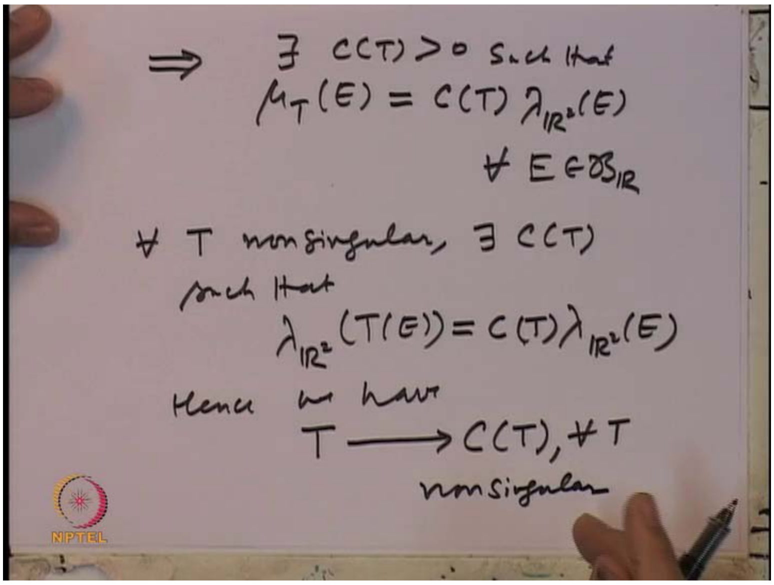

All these three properties by the uniqueness of the Lebesgue measure in R 2 imply that mu$_T$ of every set E has to be equal to a constant multiple, that is C of T, of the Lebesgue measure of E for every… It implies there exists a constant C of T bigger than 0 such that this holds for every set E which is a Borel set. This is where we are using that the

Lebesgue measure is essentially the only translation-invariant measure on the plane; any other translation-invariant measure has to be a constant multiple of the Lebesgue measure.

What we have gotten is for every T nonsingular, there exists a constant C of T such that Lebesgue measure mu_T which is nothing but the Lebesgue measure of the transformed set T of E is equal to the constant multiple C times T of the Lebesgue measure of the set E. This is the property that we have established: for every linear transformation, the transformed set T of E will be a Borel set and its Lebesgue measure will be a constant multiple of this. That means this gives us a map. Hence, we have for every nonsingular linear transformation we have got a constant C of T for every T nonsingular.

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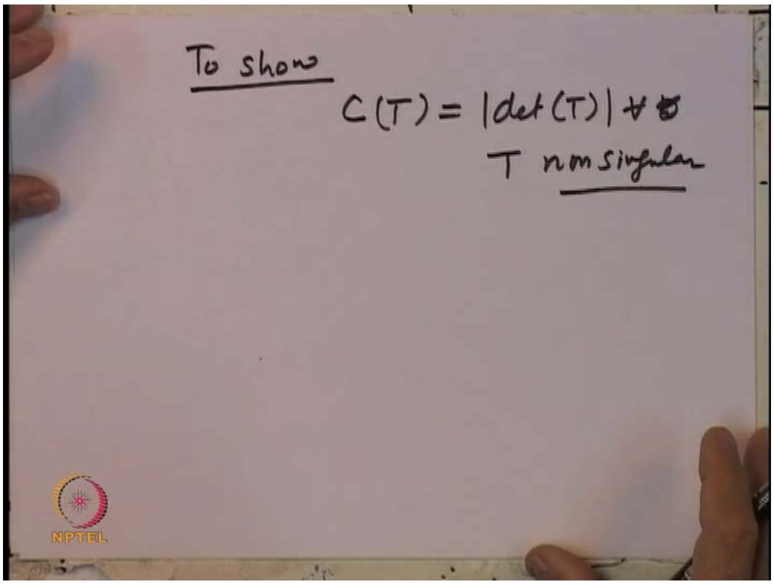

What we want to do is to show that C of T is equal to absolute value of determinant of T for every T nonsingular; that is what we want to show. Once we do that, we will be through with our construction because C of T being determinant that will prove that it is nonsingular. At this stage to prove that, I need some more facts about linear algebra; to prove that, we need some more facts about linear algebra; we will not be able to complete the proof in the remaining part of today's lecture.

I will continue the proof next time; we will start from here saying that we have got for every nonsingular linear transformation T a constant C of T; that means there is a map T going to C of T and we want to show that this map actually is nothing but the determinant of T. We will prove this next time. Thank you.