Measure and Integration

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Module No. # 08

Lecture No. # 30

Lebesgue Measure and Integral on R^2

Welcome to lecture 30 on measure and integration. In the previous lectures we had defined what is called the product measure on product space. In this lecture, we will specialize that construction on the set R 2, which is the Cartesian product of real line with itself and the sigma algebra being that of either Borel sets or Lebesgue measurable sets and the measure being the Lebesgue measure.

So the topic for today's discussion is going to be Lebesgue measure and integral on the space R 2.

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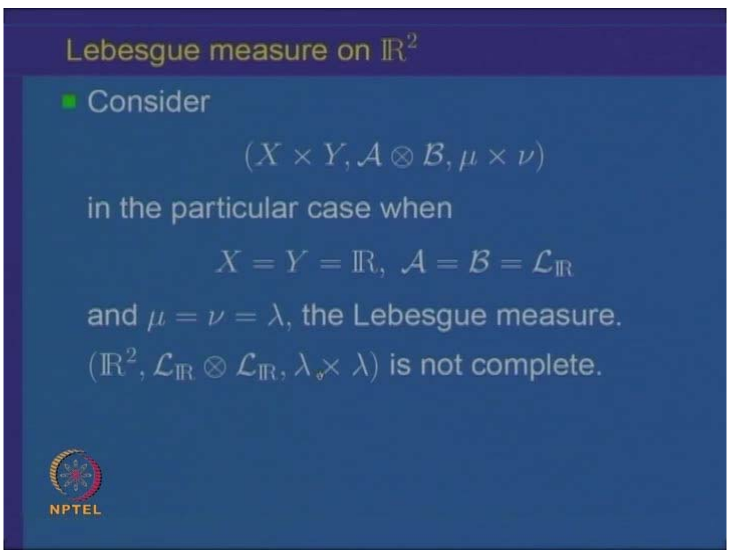

Let us just recall - we had defined the product measure space. Given measure space is X, A and mu and Y, B and nu, we defined the product sigma algebra A times B on the product space X cross Y and the product measure mu cross nu.

Today, we will start looking at the particular case when X is equal to Y equal to the real line and the sigma algebra A is same as the sigma algebra B is same as the sigma algebra of Lebesgue measureable sets on the real line.

mu is same as nu which is same as the Lebesgue measure. So, we are looking at a copy of the real line, the sigma algebra of Lebesgue measureable sets and lamda the Lebesgue measure and taking its product with itself. That will give rise to the product measure space R 2, the Lebesgue measureable sets times the Lebesgue measureable sets the sigma algebra and the product measure lamda cross lamda.

If you recall, we had mentioned that even if the original measure spaces are complete, the product measure space need not be complete. So, this product measure space R 2, lamda R cross lamda R and lamda cross lamda is not complete.

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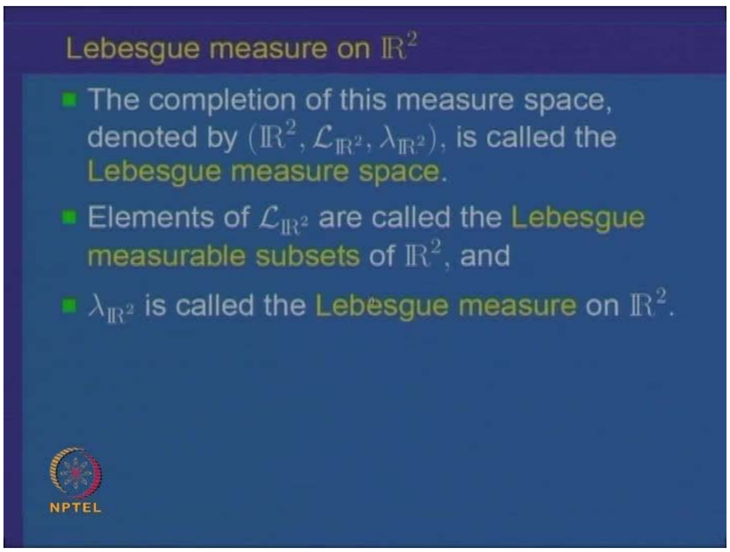

We can always complete it and thus a completion is denoted by R 2, Lebesgue measureable subsets of R 2 and lamda of R 2. This is called the Lebesgue measure space.

Lebesgue measure space is obtained from the sigma algebra Lebesgue measurable sets times Lebesgue measureable sets completed with respect to the product Lebesgue measure on R 2. So, this is normally called the product called the Lebesgue measure space on R 2.

The sets say in the sigma algebra lamda of R 2 are called Lebesgue measureable sets in R 2 and the measure, lamda of R 2 defined on this completed space is called the Lebesgue measure on R 2.

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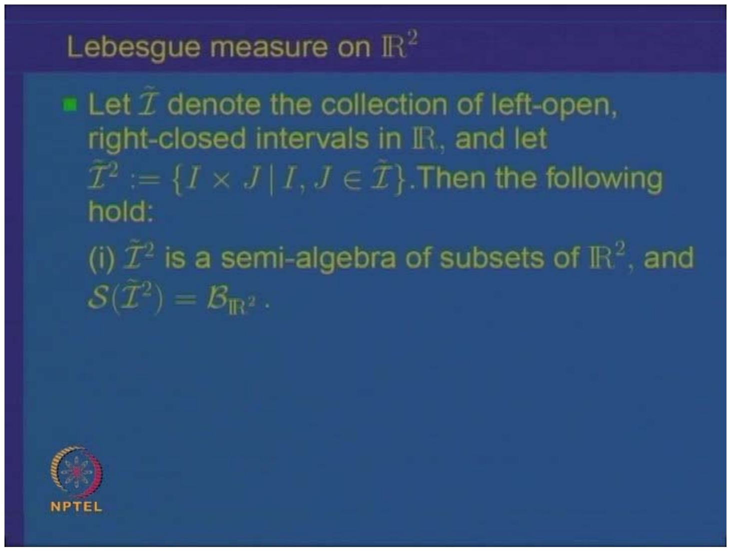

So, whenever one refers to the Lebesgue measure space, it is the complete measure space obtained via completing the product measure on the product sigma algebra. Today, we will start looking at properties of Lebesgue measureable sets and Lebesgue measure.

Let us denote by I tilde, as we have done for the real line, the collection of all left open right closed intervals in real line. Let us look at the rectangles obtained by such intervals so that we denote it by I tilde 2 - upper superscript 2, as I cross J rectangles whose sides are left open right closed intervals.

Then we claim that this I cross J is a semi-algebra of subsets of R 2 and the sigma algebra generated by this is equal to the Borel sigma algebra of R 2.

To prove this we already know that I tilde, the left open right closed intervals form a semi-algebra of subsets of real line and we have already shown that if you take rectangles consisting of elements of the semi-algebra, then itself form a semi-algebra namely the product of semi algebras is always a semi algebra.

So, that general construction will tell that the space the set of all rectangles with left open right closed intervals is a semi-algebra. To show that this sigma algebra generated by the rectangles I tilde 2 is the Borel sigma algebra, we observe few things.

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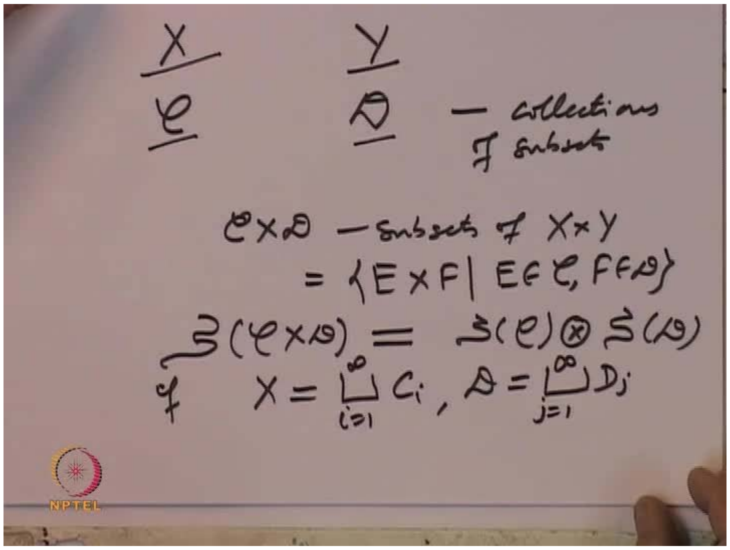

First of all if we recall, we had shown so Let me just recall a fact that we have shown in the beginning of defining product sigma algebras namely, if we take a set X and take a set Y and here we have got a collection of subset C and we have got a collection of subset; so, these are collections of subsets.

Then we can form C cross D, that is a collection of subsets of X cross Y. This is equal to all sets of the type E cross F, where E belongs to C and F belongs to D.

Now, one can generate a sigma algebra out of this collection C cross D. On the other hand, we can generate a sigma algebra by the collection C, we can also generate a sigma algebra by the collection D of subsets of Y and take the sigma algebra generated by the rectangles of this type; let us call it as S of C cross S of D. Then the claim is that these two are equal, but not always. We showed that these two are equal, if X can be represented as a union of sets partition 1 to infinity and D can be written as a union of some sets D js in the collection D of 1 to infinity.

So, whenever this X can be represented as a disjoint union of sets from C and Y can be represented as a disjoint union of elements of D then whether you take the rectangles first and generate the sigma algebra or generate the sigma algebras and then take rectangles and generate the sigma algebra, both will be equal to same.

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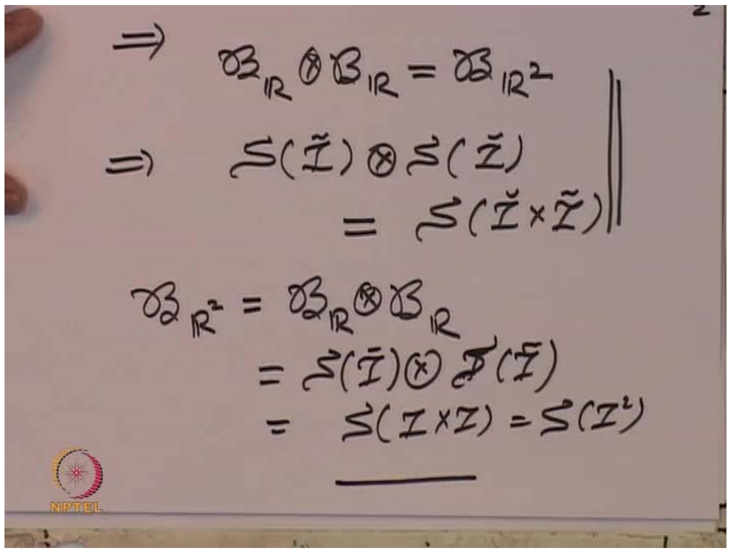

This result we had proved in the beginning of the topic. As a consequence of this, we obtained So, this implied one observation that if you take the Borel sigma algebra cross the Borel sigma algebra of R that is equal to Borel sigma algebra of the space R 2.

That is one observation because the real line can be represented as a countable union of say, open sets or intervals.

Similarly, the same argument also implies that if I take the sigma algebra generated by this left open right closed intervals cross the sigma algebra generated by left open right closed intervals and then look at the product sigma algebra then that will be same as the product sigma algebra of left open right closed intervals cross left open right closed intervals.

This is because the whole real line can be written as a countable union of left open right closed intervals. These two facts follow from our earlier construction. So, we will keep that in mind. Now, what we want to show is that the Borel sigma algebra of R 2 so that we know it is borel sigma algebra of real line times the product borel sigma algebra of real line.

Borel sigma algebra we know from our construction of real number is same as the sigma algebra generated by left open right closed intervals. So, left open Borel sigma algebra is generated by the sigma algebra of left open right closed intervals.

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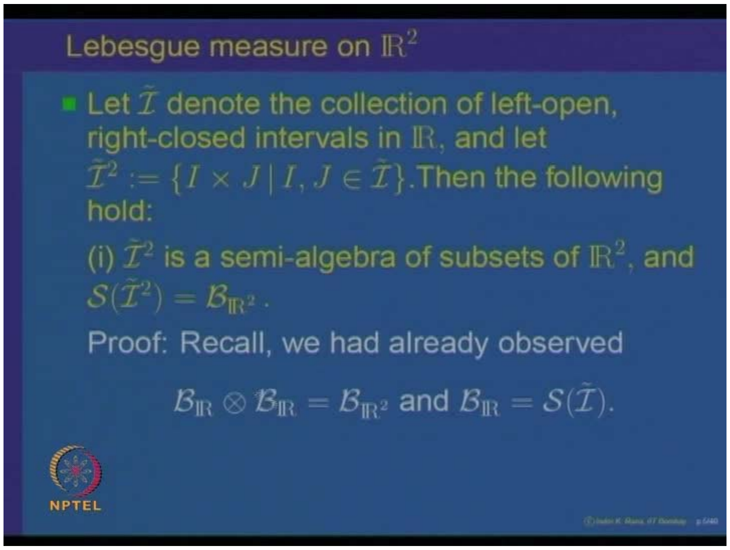

This just now we observed is the sigma algebra generated by I cross I and that is same as the sigma algebra generated by I 2. So, that completes the proof of the fact that the sigma algebra generated by rectangles which are left open right closed intervals is same as the Borel sigma algebra of R 2.

That is one observation that is very much similar to the result in the real line where the left open right closed intervals generated the sigma algebra of Borel subsets. Same result is true, if you replace intervals by rectangles which are left open right closed.

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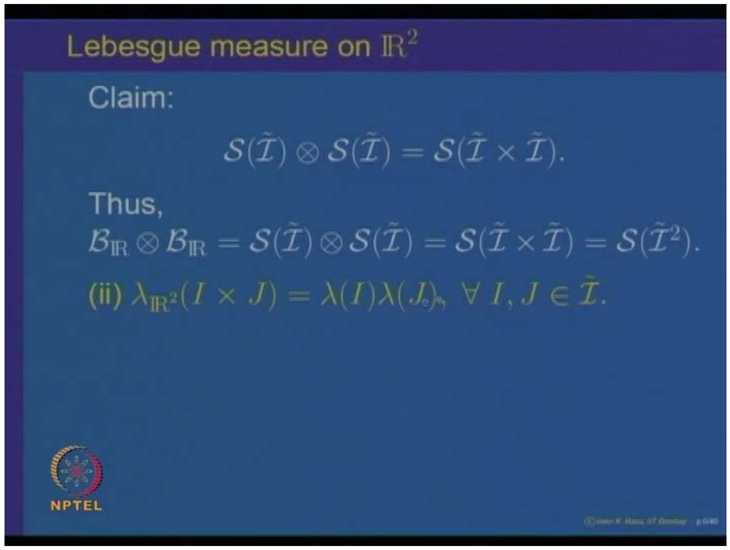

That is the proof we have just now said. S I is equal to S I cross I. So, Borel sigma algebra B R cross B R is the sigma algebra generated by intervals left open right closed cross left open right closed intervals which is same as rectangles.

Now, let us look at the next property that the Lebesgue measure that we have defined for a rectangle is lamda of I cross lamda of J is same as lamda of I into lamda of J. That is obvious because we obtained the product measure extension of the measure on the rectangles.

What we are saying is, the Lebesgue measure on R 2 is the natural extension of the notion of area in the plane. so this is properties obvious built in the definition of the product measure

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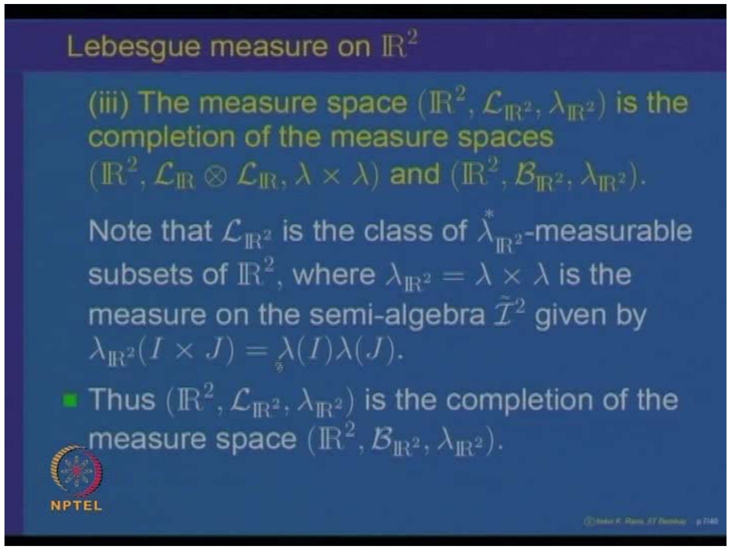

And third observation is so recall we just now said that the Lebesgue measure space R 2 Lebesgue measures subsets of R 2 and Lebesgue measure the Lebesgue measureable subsets so this space which is the space Lebesgue measures space

On one hand, we define it as a completion of the Lebesgue measureable sets cross Lebesgue measureable sets and this is also the completion of the measure space of real line with Borel subsets of R 2. That is once again by the fact that the Borel subsets of R 2 are inside this and the Borel set’s subsets of R 2 and the Lebesgue measureable sets that differ only by sets of measure 0. So, that is also the completion.

One way of looking at is look at the Lebesgue measureable subsets R 2, it being the completion. So, it is the class of all outer Lebesgue measureable subsets in R 2 with respect to the product measure and on the semi-algebra I 2 of rectangles, it is given by the product. This is obviously the completion of the measure space R 2.

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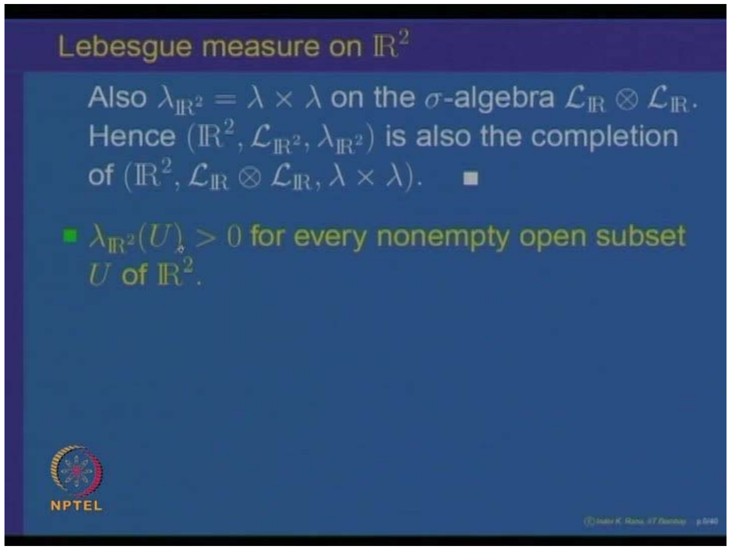

These are obvious facts. We should keep in, which are very much similar to that of the real line. They play a role later on when we want to look at null sets in R 2.

So, basically the sets which are going to be of importance are going to be the Lebesgue measureable sets R Lebesgue measureable sets cross Lebesgue measureable sets in R 2 or Borel subsets in R 2.

Here is another useful fact about Lebesgue measure in R 2 which connects it with the topologically nice sets namely the Lebesgue measure of R 2 of any non-empty open set is always bigger than 0.

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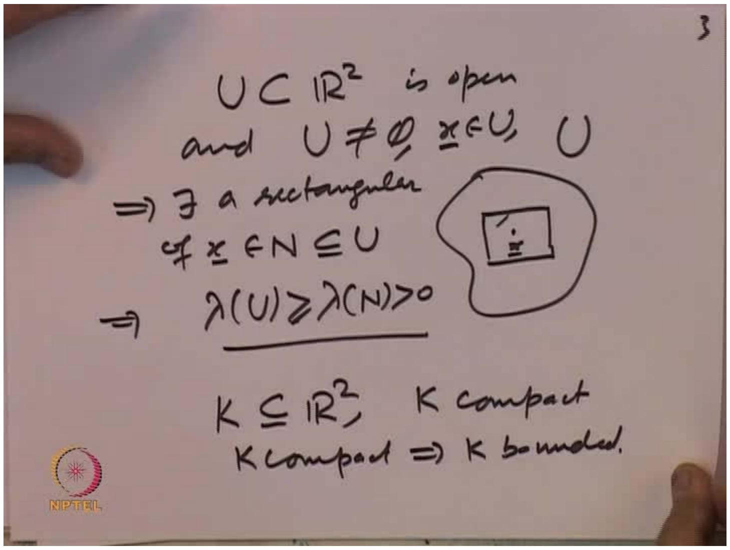

That follows from the fact that if U is contained in R 2 is open and U is not equal to empty set then Here is a set U. There is always a left open right closed rectangle inside it. There is a rectangular neighborhood. It implies there exists a rectangular so nonempty. There is a point x belonging to U. So, there is a rectangular neighbourhood of x and let us call that neighbourhood as N.

So this is the rectangle N which is contained in U. That means, the Lebesgue measure of U will be bigger than Lebesgue measure of N which is always going to be bigger than 0 because it is a non-empty neighbourhood. So, for every non-empty open set, the Lebesgue measure is always positive, if the set is non-empty.

The second important thing is supposing you take a set k which is a compact subset of R 2. Let us look at a compact subset of R 2. So, k is contained in R 2 and k is compact. If a set is compact that implies it must be bounded. So, k compact implies k bounded. and that means so saying the set is bounded implies that

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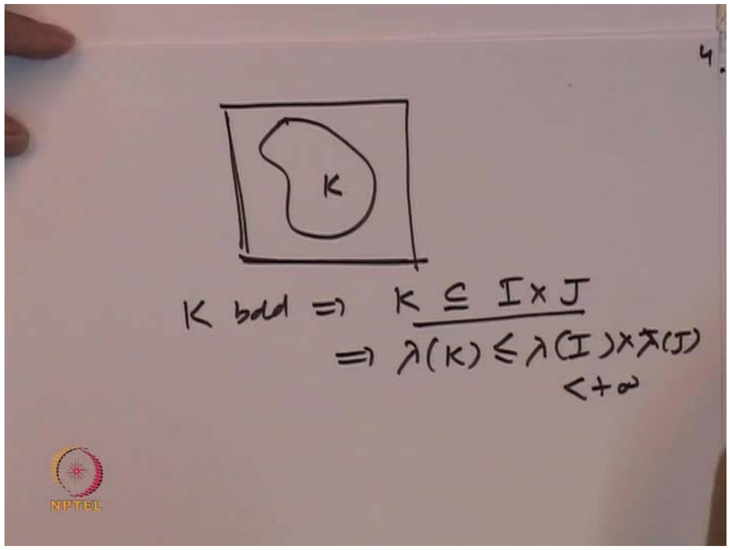

This is the set k which is compact. That means, if this is bounded, it must be inside a rectangle. So, k bounded implies k is inside some I cross J. It implies lamda R 2 of k will be less than lamda of I cross lamda of J which is finite. So, finite intervals, compact implies bounded. So, there is a finite rectangle including it and that means it is finite.

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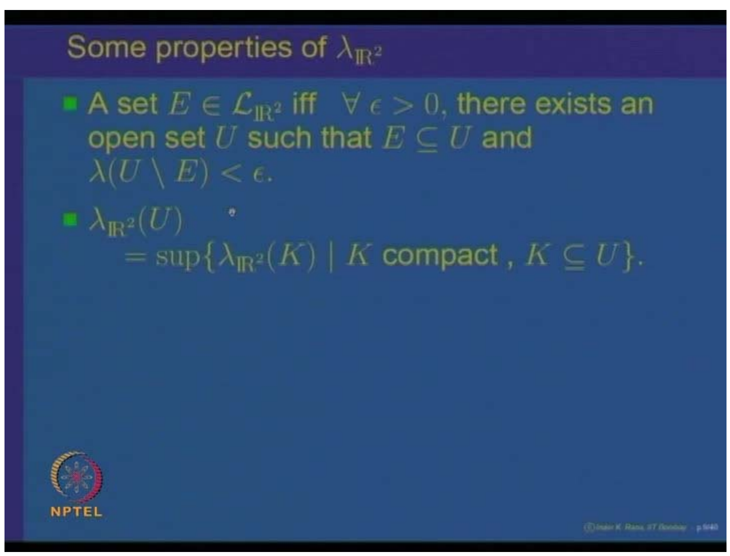

These are two relations about open sets and compact sets. There are more relations and like in the real line one can prove a result. For example, a set E is Lebesgue measureable if and only if for every epsilon, you can find an open set which includes it and the difference as measure small. So, that is very much similar to the real line and the proof is also very much similar to the real line.

We will not prove this result. Somebody who is interested should try to copy the proof of the real line and extend that proof to the case of R 2.

This will give us that Another result is that for the Lebesgue measure of R 2, you can approximate it from inside by compact sets, as supremum of lamda of R 2 where k is compact.

These results basically are of importance; these are called regulatory conditions for the Lebesgue measure in R 2. We will not prove these results. Just for the sake of knowledge, I am mentioning these results here so that later on if you come across, you can look at proofs of these results.

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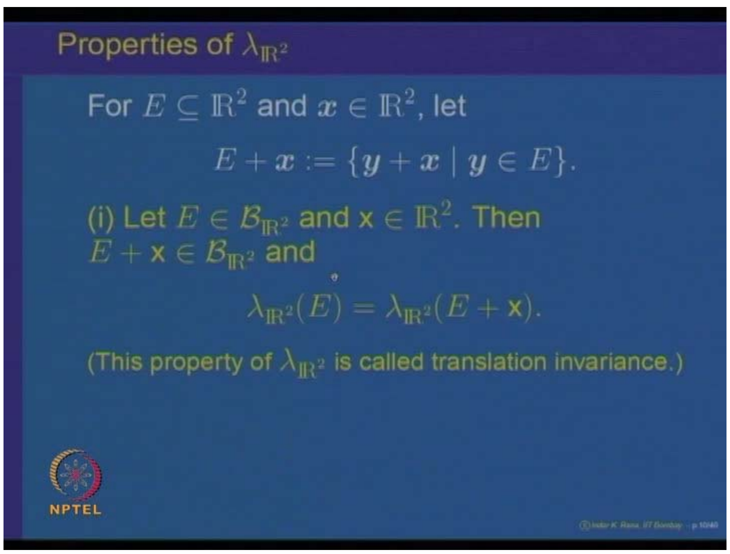

The next result we want to look at is how are the Lebesgue measurable sets related with the group structure of the space R 2. Let us take a subset E of R 2 and let us look at a vector x in R 2. We will define the translate of the set E by x to be, as in real line, y plus x where y belongs to E.

Take the set E and shift every element of E by the vector x; so, it is y plus x. So, the first claim is that if E is a Borel set and the point x belongs to R 2 then E plus x also is a Borel set; that is one property and the Lebesgue measure of R 2 of the set E is same as the Lebesgue measure of the set E plus x.

That means the Lebesgue measure is, one says, it is translation invariant on the class of all Borel subsets of R 2.

The proof of this fact that for every set E, E plus x belongs to B R 2 and the fact that the Lebesgue measure of the translate set is equal to Lebesgue measure of the original set are standard applications of the techniques that we have been using namely the sigma algebra monotone class theorems.

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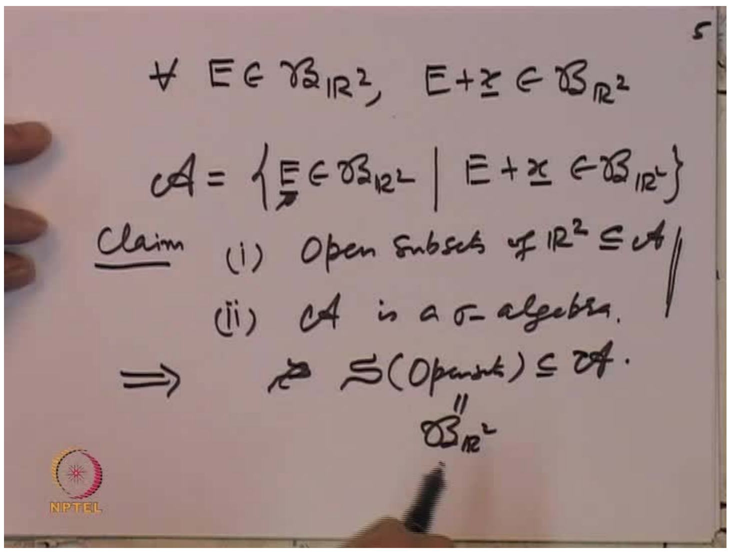

Let me illustrate this once again so that this idea of using monotone class convergence theorem, monotone class sigma algebra technique settles down in the mind. We first want to prove namely that we want to show that for every E, a Borel subset of R 2, if I look at E plus x that is also a borel subset of R 2.

The technique is s follows. Let us collect together all sets A. So, form the collection A of all those subsets E belonging to B R 2. All borel subsets say that the required property is true - E plus x belongs to B R 2.

So, look at all sets having this property. so claim That is the sigma algebra technique. Claim one: all open subsets of R 2 are inside this collection. We will prove two claims. One is this and secondly, that the class A is a sigma algebra.

So, if you prove these two facts about the class A then that will imply because it includes open subsets of R 2 so it will include the smallest sigma algebra generated by so these two facts will imply these two facts will imply that the sigma that the sigma algebra generated by sigma algebra generated by open sets will be inside the class A and that is equal to the Borel sigma algebra.

So, that will prove that the Borel sigma algebra is equal to A. Let us first show that the open subsets of R 2 are inside A.

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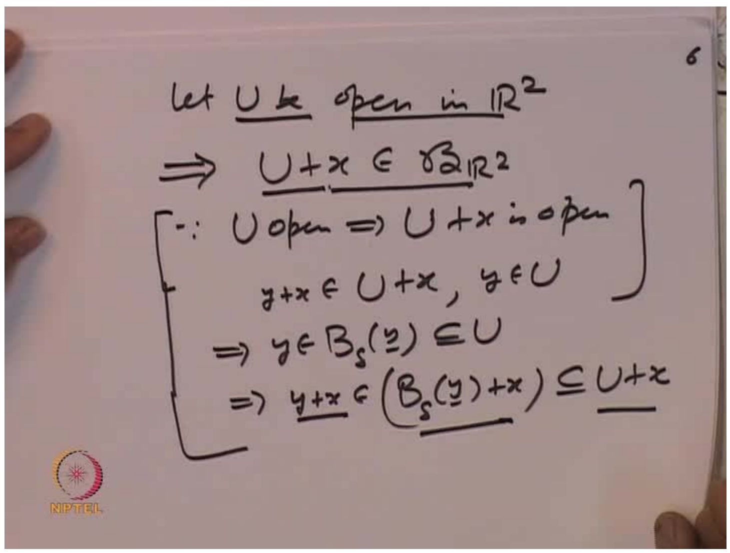

Let us take an open set. To prove the first fact, we have to show that if a set so to the show the first one, let U be open in R 2. We want to show that this implies U plus x belongs to B R 2 and this follows because if U is open, it implies that U plus x is open. That is a simple fact because how do you show that U plus x is open? Basically, saying that U is open.

Let us take a point y plus x belonging to U plus x. If y plus x belongs to U plus x where y belongs to U and U open implies there is a neighbourhood. Let us call it as B delta y. So, y belongs to a neighbourhood which is contained in U.

But then that implies that y plus x belongs to the translation of the neighbourhood that is contained in U plus x. That means, for every point y plus x, there is a neighbourhood. When you shift a ball that remains a ball in the plane.

That is a basic fact, we are using. If you translate a neighbourhood that remains a neighbourhood in U plus x. So, that implies if U is open in R 2 then U plus x is also an open set and hence belong to B R 2. So, that proves the first fact namely open subsets belong to A.

Let us now show that so this proves the first fact that open subsets belong to R 2 so To show that A is a sigma algebra, there is this very standard technique we have been using it very often. If a set E belongs to A that means E plus x belongs to B R 2. Let us write that. so if E belongs

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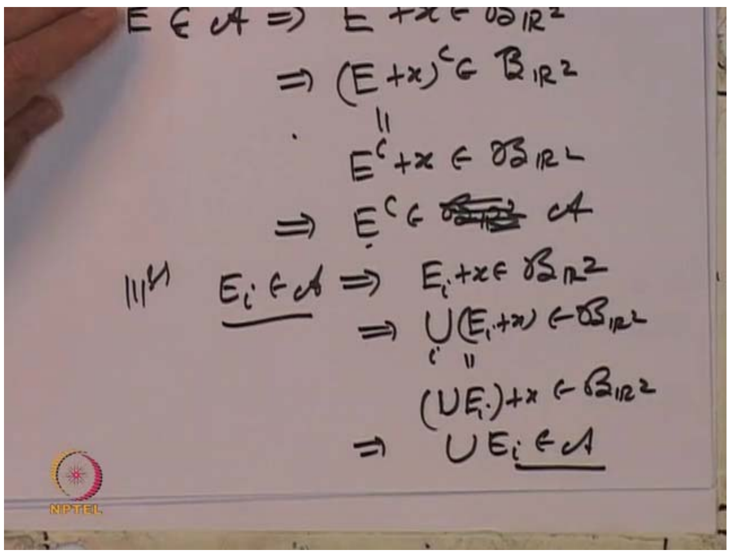

If a set E belongs to collection A that implies E plus x belongs to B R 2 and that implies because B R 2 is a sigma algebra, that will imply, its complement belongs to B R 2.

This is same as first taking complement and then taking translate. So, that belongs to B R 2 and that means E complement belongs to B R 2.

Whenever E belongs to collection A, its E complement plus x belongs to B R 2. That means E complement belongs to A. So, E is closed under complement. Similarly, E i s belonging to A will imply that union of E i plus x belongs to B R 2. That will imply that union of E i plus x belongs to B R 2 union over i.

But this is same as union of E i s plus x belongs to B R 2 and that will imply that the union of E i s belongs to A.

So, E is belong to union also belongs to A. (Refer Slide Time: 24:37) That will prove that A is an algebra so A is a sigma algebra including open sets and so, it includes everything.

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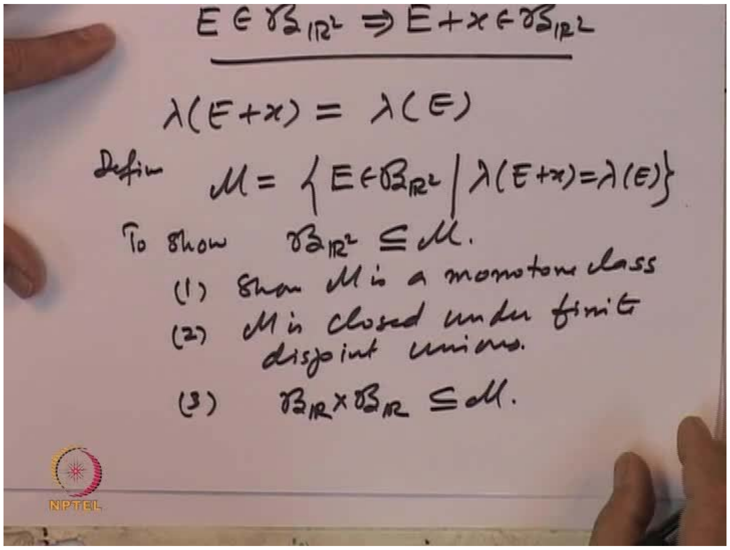

So that will prove that (Refer Slide Time: 24:56) This is the sigma algebra technique, I have been mentioning. The sigma algebra technique that we have mentioned implies that whenever E belongs to B R 2 implies E plus x belongs to B R 2 and that is what we have proved.

To show the other thing, to prove that lamda of E plus x is same as lamda of E, everything is in R 2, once again let us define M to be the collection of all those subsets E belonging to B R 2 for which this property is true - lamda of E plus x is equal to lamda of E.

We want to show that B R 2 is inside M because M is already a subset of B R 2. That will prove that M is equal to B R 2 and hence this property will hold for all subsets of B R 2.

To show this, the technique is the monotone class theorem. So, one: show M is a monotone class; two: M is closed under finite disjoint unions and third, the rectangles B R cross B R rectangles are inside M.

Once these three facts are proved, we will be through as follows because these rectangles are inside it and this is a monotone class. The idea is that step three will imply that the monotone class generated by B R cross B R is also inside M.

This cross is also closed under finite disjoint union. This collection, the sets which are inside M will also be closed under finite disjoint unions. So, it is a monotone class closed under finite disjoint unions. That will imply that so B R the rectangles are inside it so the algebra generated by the monotone class generated by finite disjoint unions also will be inside it

(Refer Slide Time: 28:05) Because this is inside and so, this is closed under finite disjoint unions. That means, the algebra generated by rectangles will also be inside it, but M is a monotone class which is closed under finite disjoint unions and that must be a sigma. So, monotone class generated by algebra is also sigma algebra. Sigma algebra generated will come inside it and hence will have everything is equal.

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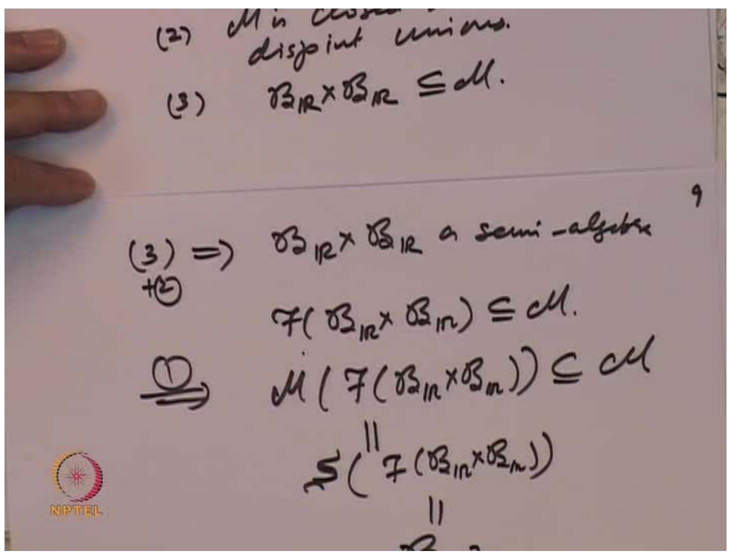

One should prove these three things. because after these three things are proved, What will the proof imply? 3 plus 2 will imply, this is a semi-algebra because B R cross B R is a semi-algebra; it is inside M and M is closed under finite disjoint unions and this will imply that the algebra generated by F of B R cross B R will be inside M.

So, the algebra generated by this come inside M. Now, 1 implies, M is a monotone class and it includes this algebra. So, the monotone class generated by this algebra is also inside M

But the monotone class generated by algebra is same as the sigma algebra. So, this is same as the sigma algebra generated by this algebra B R cross B R and that is equal to the Borel sigma algebra of R 2.

(Refer Slide Time: 29:57) So, this is the line of argument which will prove that B R 2 is a subset of m.

So, we have to verify these three things namely M is a monotone class, M is closed under finite disjoint unions and rectangles are Borel rectangles and are inside M.

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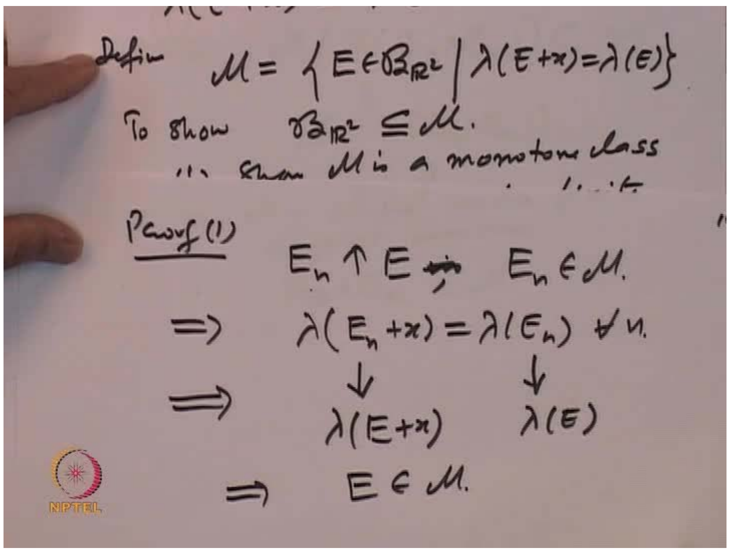

To show that, let us look at the first one that is, M is a monotone class. To show that M is a monotone class let us look at the proof; so, proof of 1. Let us look at a sequence E n which is increasing to E and let us say E ns belong to m.

That will imply that lamda of E n plus x is equal to lamda of E n for every n. Now, if E n is increasing then E n plus x is also increasing and lamda being a measure, this converges to lamda of E plus x and by the same thing, this converges to lamda of E.

So, that says lamda of E plus x is equal to lamda of E. If E ns increase to E then that will imply that these two are equal; so, E belongs to M. Similarly, for a decreasing sequence also similar property; if E ns are decreasing to E and say lamda of E 1 is finite then E which is the intersection will also belong to M.

So, that will prove the fact that M is a monotone class.

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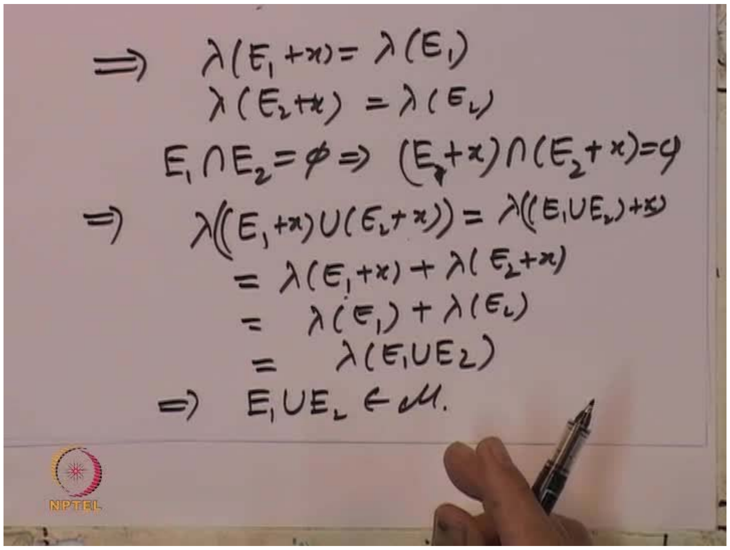

Now, let us show that M is closed under finite disjoint unions. Let E 1 and E 2 belong to M and E 1 intersection E 2 equal to empty set.

E 1 and E 2 belong to M and this fact implies lamda of E 1 plus x is equal to lamda of E 1. Similarly, lamda of E 2 plus x is also equal to lamda of E 2. E 1 and E 2 disjoint implies that the sets, translate of E 1 and translate of E 2 are also disjoint.

That is a simple thing to observe. So, that will imply that lamda of E 1 plus x union of E 2 plus x because these sets are disjoint, the Lebesgue measure of the union of in R 2 is same as the Lebesgue measure in R 2 of E 1 plus x plus lamda of E 2 plus x.

E 1 and E 2 belong to M; so, this is equal to lamda of E 1 plus lamda of E 2. E 1 and E 2 are disjoint; so, it is lamda of E 1 union of E 2.

So what we have shown is, if E 1 and E 2 belong to M and they are disjoint then lamda of E 1 union E 2 is same as lamda of E 1 x plus union of E 2 x, but a simple observation will tell you that this is also same as lamda of E 1 union E 2 plus x.

So, whether you take translates first and then take the union that is same as taking union and then translates. This will imply that E 1 union E 2 also belongs to M. Whenever E 1 and E 2 are disjoint that union also belongs to A.

So, that proves the second fact namely M is closed under finite disjoint unions. Finally, we prove the third fact namely the rectangles are inside M. That again is a set forward simple fact to prove.

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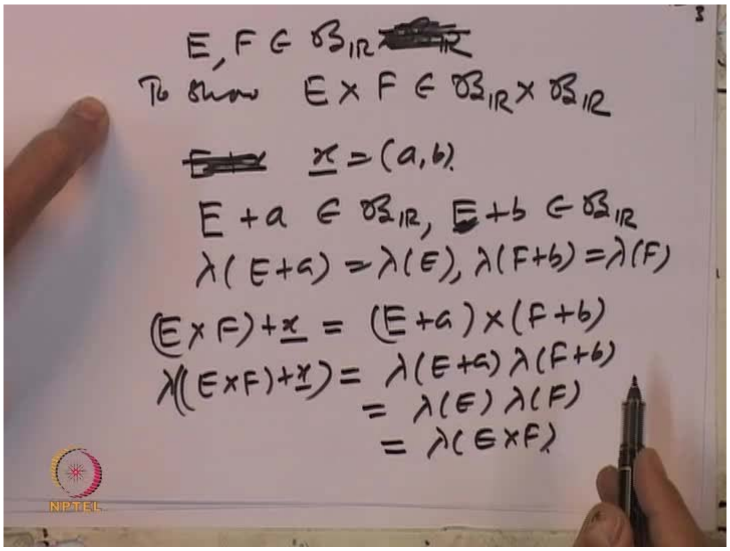

To prove the third thing, let us observe the following namely, let us take E, F belonging to B R cross B R. To show E cross F, E and F both belong to B R, we want to show that the cross product belongs to B R cross B R.

That is what we want to show. To show that let us observe E and F belong to B R. so we know that whenever E is in F so E plus x Let us take a vector x which is equal to a comma b.

Then what is E plus? We know that E plus a belongs to B R and also E plus b belongs to B R because E and F are subsets in B R and so, the translates belong. lamda of E plus a is same as lamda of E, and that was a set F and lamda of F plus b is same as lamda of F.

Now, look at the set E cross F translated by x; x is a b. What is that? That is equal to E plus a cross product with F plus b.

So, the Lebesgue measure of this set E cross F plus x will be equal to, this is a rectangle, Lebesgue measure of E plus a into Lebesgue measure of F plus b, but that is equal to Lebesgue measure of E into Lebesgue measure of F because Lebesgue measure on the real line is translation invariant; that is equal to Lebesgue measure of E cross F.

What we have shown is that if E cross F is a Borel rectangle then translate of the Borel rectangle has the same measure as the rectangle itself. That proves the third thing namely that the Borel sets cross the Borel sets is inside M.

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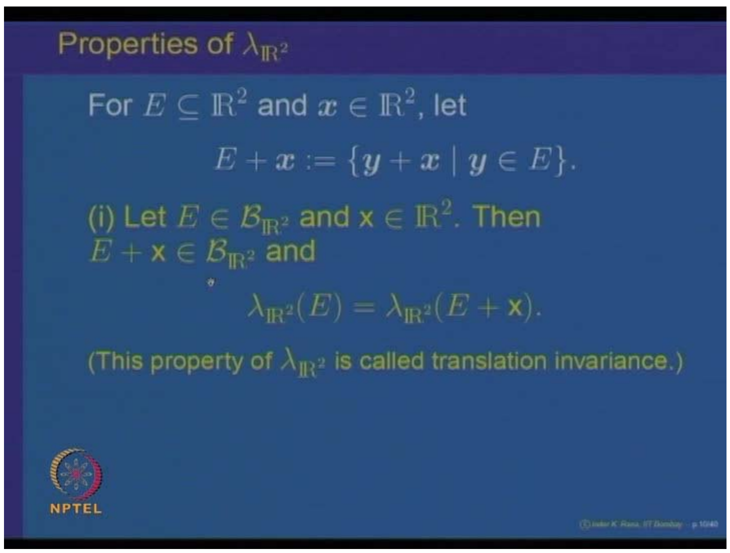

So, all the three facts are proved and that will imply that B R 2 is a subset of M. and hence for all. That is what, we have shown is that the Lebesgue measure is a measure on the plane which has the property that Lebesgue measure for every Borel set E, its translation is also a Borel set and the Lebesgue measure of the translated set is equal to Lebesgue measure of the original set.

This is called as the translation invariance properties of the Lebesgue measure on the plane. As in the case of real line, we showed that the Lebesgue measure on the line is a translation invariant measure. Similarly, we have shown that the product of that Lebesgue measure taken on R 2 is also a translation invariant measure.

Of course, the natural question arises that on the real line we have shown that essentially Lebesgue measure is the only translation invariant measure and we will show that Lebesgue measure on the plane also is essentially the unique translation invariant measure.

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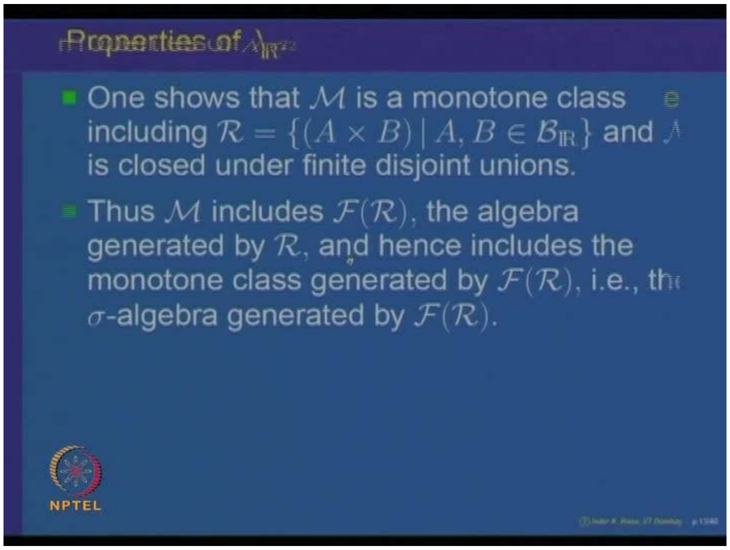

Unique in the sense that a scalar multiple is again translation invariant anyway. So, upto a multiplication by a scalar we will show that the Lebesgue measure in the plane is a unique translation invariant measure on the Borel sigma algebra, but before that let us prove a property about the integrals of functions on the plane.

The next property we want to analyze is the following. We have already gone through the sigma algebra monotone class technique. That sigma algebra monotone class technique, we have already explained. That is just shown here that shows that M includes F of R and hence it will include the sigma algebra generated by it and that will prove.

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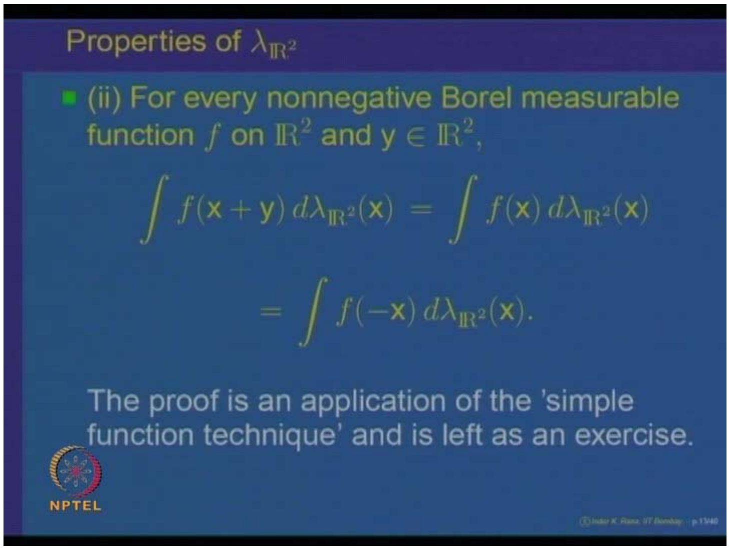

The next property I wanted to illustrate is the following namely, for every non-negative Borel measurable function f on R 2 and any vector y in R 2, the integral of the translated function, integral of f of x plus y with respect to the Lebesgue measure is same as the integral of the function itself and it is also same as integral of the negative of the function namely f of minus x.

That means, the Lebesgue integral for non-negative functions is invariant under translation and this is what is called deflection - x goes to minus x. Proof of this is basically applications of the simple function technique. Let me just illustrate 1 or 2 steps of this proof that this is true.

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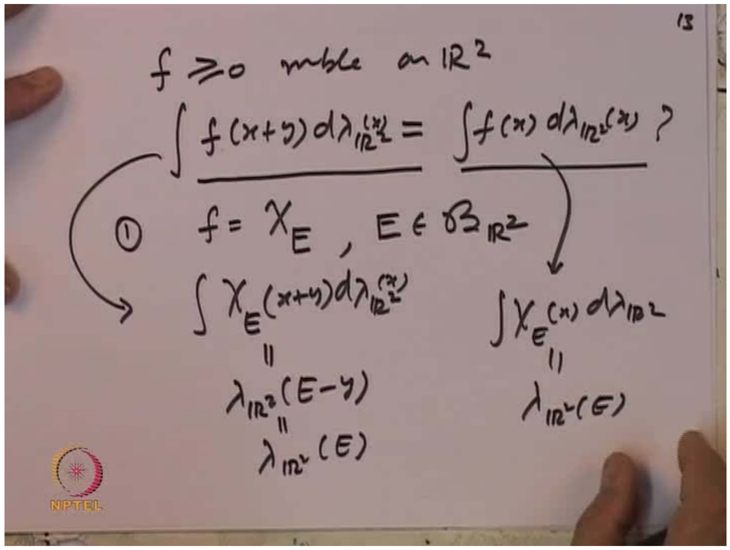

Let us look at the first one. Let us prove that if f is a non-negative measurable function on R 2 then we want to prove that the integral of f of x plus y d lamda R 2, this is over R 2, is equal to integral of f of x d lamda R 2 of x. This is what we want to prove.

The simple function technique as we recall, is the following first step. Let us take f to be the indicator function of a set E, where E is a Borel subset of R 2.

In that case, this left hand side is integral of the indicator function of E x plus y d lamda R 2. Here integrating with respect to x, that is same as x plus y belonging to E means it is x belonging to E minus y. This is integral of the indictor function of E minus y; so, it is lamda R 2 of the set E minus y.

That is same by the translation invariant property. It is lamda of the set E and this thing f is an indicator function. So, indicator function of x d lamda R 2 is same as lamda R 2 of E.

So what we are saying is that as a first step, the required claim namely integral of f of x plus y is integral f of x holds, whenever f is the indicator function of a set E.

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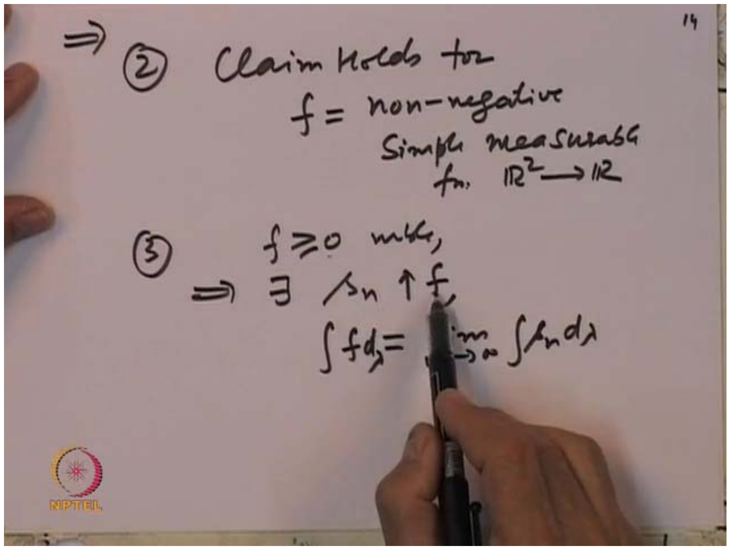

Now both sides being integrals, step 1 implies step 2 namely, required claim holds for f equal to non-negative simple measurable function R 2 to R.

So this claim will hold because any non-negative simple measurable function is a finite linear combination of characteristic functions or the indicator function. For each indictor function we have shown this. so that will imply that the required claim holds for non-negative simple functions.

The third step, if f is non-negative measurable then it implies that there exists a sequence s n of non-negative simple measurable functions s n increasing to f and integral of f to be equal to limit n going to infinity integral of s n d lamda.

So, saying that f is non-negative measurable means that f is limit of non-negative simple measurable functions and the integral of f can be defined as the limit of the integrals of non-negative simple measurable functions.

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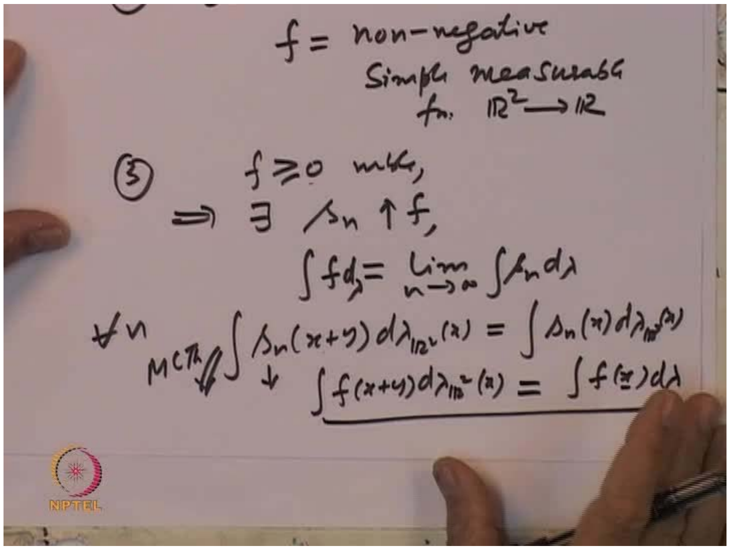

For non-negative simple measurable function, each s n and for every n, we know that the required claim holds by step 2. By step 2, we know that s n of x plus y d lamda R 2 of x is equal to integral of s n of x d lamda R 2 of x; that is by step 2.

Now as s n is increasing to f, so clearly the translates this will increase to the translate of the function f. This implies in the limit by monotone convergence theorem An application of monotone convergence theorem will say that as n goes to infinity this will converge to integral of f of x plus y d lamda of R 2 of x.

On the other hand, we know this converges to integral of f x d lamda of R 2. These must be equal. That means, for a non-negative measurable function, this required conclusion holds.

So, that is how one proves the claim namely, f of x plus y, f of integral of the translate is equal to the integral of the original function. Basically, this is what we call as the simple function technique applied to it.

So, similar argument will show that integral of f of x is same as integral of f of minus x.

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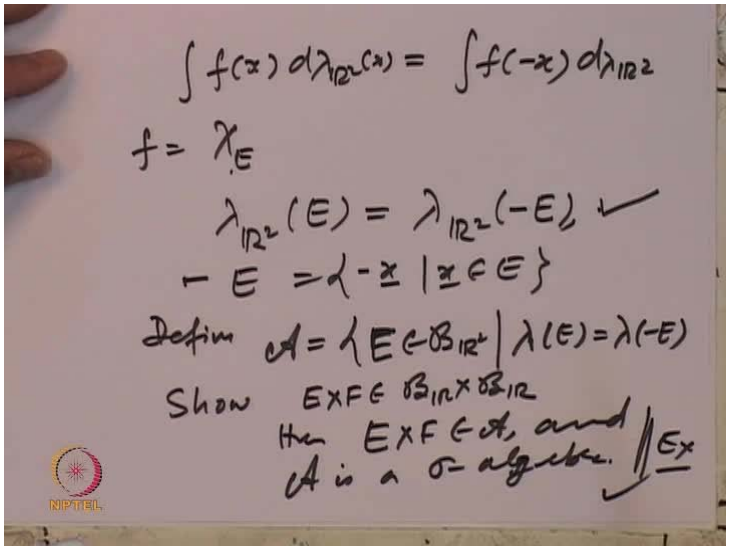

For that, one has to use the fact that the Lebesgue measure of a set E in R 2 is same as the Lebesgue measure of Let me just indicate what we need for step 2 to show that integral of f of x d lamda R 2 of x is equal to integral of f of minus x d lamda R 2, when f is equal to indicator function of the set E. That means, we need the fact that lamda R 2 of a set E is equal to lamda R 2 of minus of E.

What is minus of E? Minus of E is the set minus the vector x, here x belongs to E. To prove that this is so, once again one has to go to the sigma algebra technique.

Define A to be the collection of all those sets E belonging to B R 2, for which you can say that lamda of E is equal to lamda of minus E.

Look at all the collections of these sets. So, one will show rectangles are inside it that means, if I take set E cross F belonging to B R cross B R then E cross F belongs to A and A is a sigma algebra.

Once again, if these two steps are proved, that will prove that this claim holds for every Borel subset also and hence for the indicator function of a set E. We leave it as an exercise. Once again is a straight forward verification; so, do that.

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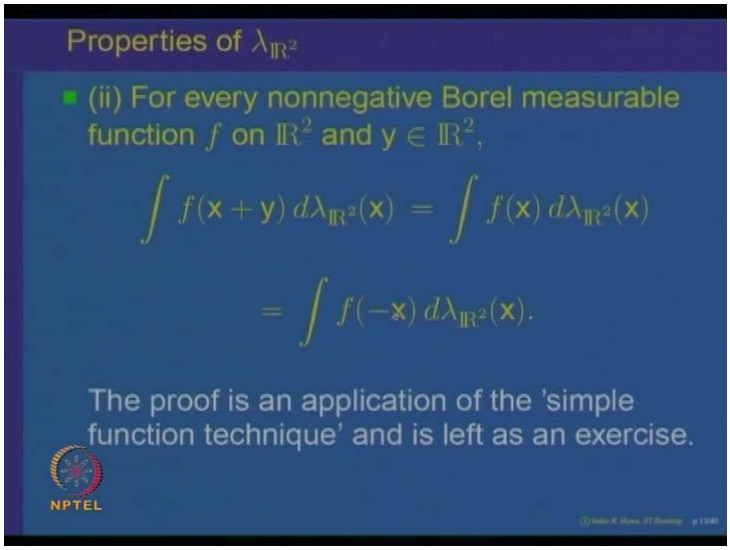

Once that is done, that will prove the second equality also. Now, in this proof one more observation we want to make here is the following.

If I replace lamda R 2 by any See in the proofs of these two things, we have not used anywhere the fact that the lamda is especially the Lebesgue measure. Essentially, we use the fact that this measure lamda of R 2 is translation invariant.

If you replace this Lebesgue measure on R 2 by any translation invariant measure then this result that f of x plus y is equal to integral of f of x will remain true for lamda of R 2 replaced by any translation invariant measure. This is an observation you should keep in mind for the future reference.

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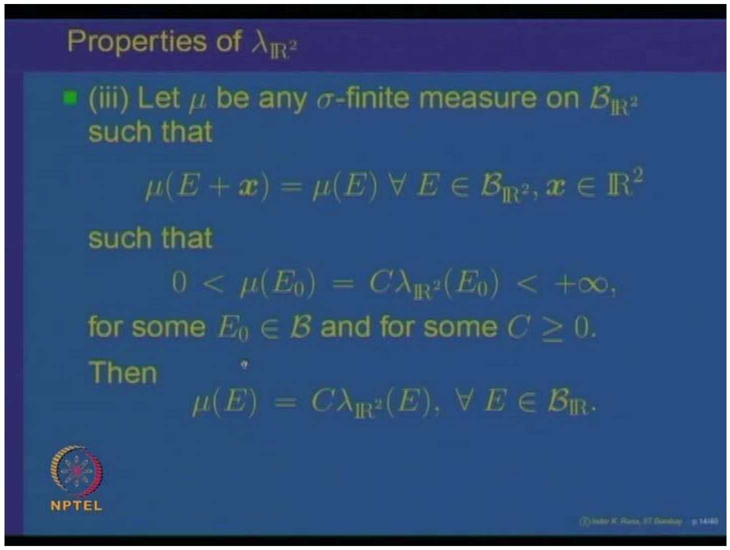

Finally, we want to prove the fact that the translation invariance is a unique property for the Lebesgue measure.

Let us take any measure mu which is sigma finite on the Borel subsets of R 2 and assume it is translation invariant. Let us assume that there is some particular set E naught such that the measure of the set E naught is positive and the measure mu of E naught is C times a constant multiple of Lebesgue measure of the set E naught and it is finite.

So, there is a set of finite positive Lebesgue measure such that mu of E naught is a constant C times Lebesgue measure of E naught for some particular set, E naught.

Then the claim is that this property holds for every subset of Borel subset. That means, mu of E is constant multiple of the Lebesgue measure. That will prove the uniqueness of the Lebesgue measure with respect to translation invariance.

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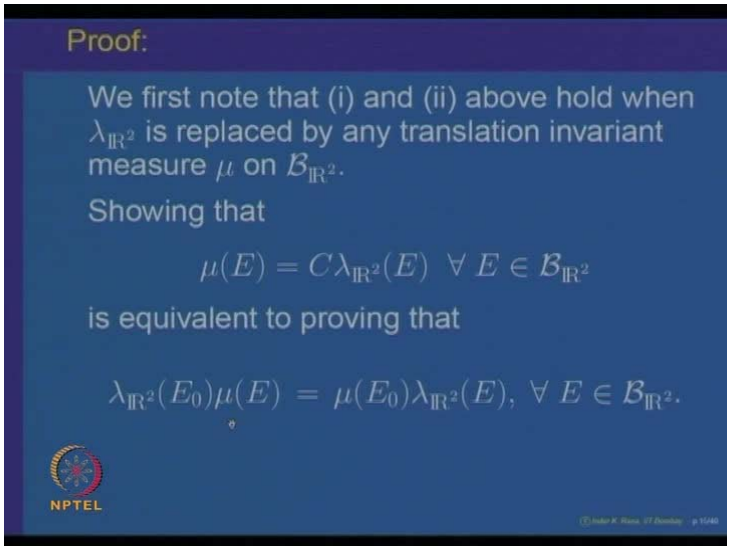

Let us prove this. As I observed that the integral of the translate of a function is equal to integral of the function remains true for any translation invariant measure, in particular for mu; that property we will be using.

So, we want to show that mu of E is constant multiple of Lebesgue measure of E for every set E. What is C? I can compute from here, C is equal to mu of E 0 divided by lamda of E 0.

To show that mu of E is equal to C times lamda R 2 of E, it is equivalent to showing that lamda of E 0 Lebesgue measure of E 0 into measure of E is same as measure of mu of E 0 into Lebesgue measure of E for every subset. This equality we should show for every subset E of R 2.

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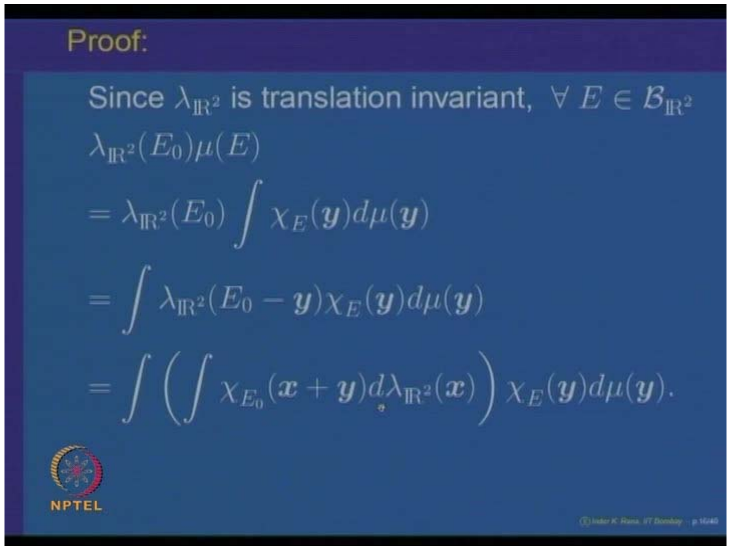

We will show it as an application of Fubini's theorem. Let us take the left hand side. So, lamda of E 0 mu of E is equal to lamda of E 0 and mu of E is integral of the indicator function with respect to y d mu y.

Now, take this lamda R 2 inside and the use the fact that it is translation invariant. So, lamda R 2 of E 0 is same as lamda R 2 of E 0 minus y and I put it under the integral sign.

The required quantity is equal to integral of Lebesgue measure of E 0 minus y into indicator function of E. Now, this Lebesgue measure, I will write it in the form of integral. I get Lebesgue measure of E 0 minus y is integral of the indicator function of E 0 minus y. It is same as the integral of the indicator function of E 0 of x plus y d lamda R 2.

Here, we have got double integral, iterated integral and the function involved are non-negative. So, by Fubini's theorem, the first part for non-negative functions, I can interchange the order of integration.

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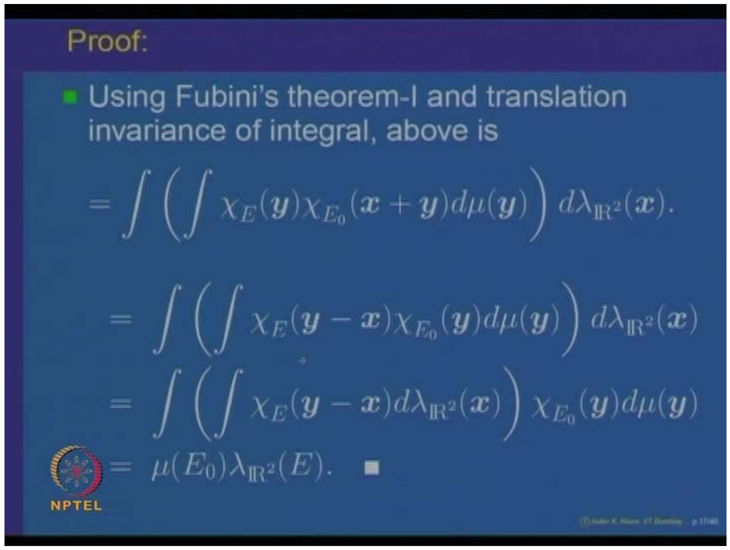

Let us interchange. Earlier, the inner integral was with respect to lamda and outer with respect to mu. When we interchange, mu comes inside and lamda goes outside and so, that is the integral. Now, once again mu is translation invariant. That means, in this integral if I shift y to y minus x then the integral will remain the same.

Let us do the shifting; shift this to y minus x. So, indicator function of y minus x, indictor function of E 0 x plus y, that becomes y d mu y. Now, once again we apply Fubini's theorem and go back. When I apply, mu goes out and lamda R 2 comes inside. That is indicator function of E y minus x lamda of R 2 of x, but that is same as Lebesgue measure of the set E and this is mu of E naught. So, that is equal to this.

So, twice an application of the Fubini's theorem for non-negative functions and the earlier property gives us the required fact namely the Lebesgue measure is the unique translation invariant measure on.

Today, we have looked at the properties of Lebesgue measure with respect to the topologically nice sets namely open sets, compact sets and with respect to the group operation of translation. On the plane, there is another transformation possible namely you can take a set E and rotate it.

Not only you can translate, you can also rotate it or magnify a set. So, next lecture we will analyze how Lebesgue measure changes with respect to what are called linear transformations in the plane and which include rotations and magnifications. Thank you.