Measure and Integration

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Module No. # 01

Lecture No. # 03

Sigma algebra Generated by a Class

Welcome to lecture 3 on measure and integration. We had started looking at the concept of algebra of subsets of a set X; we will look at some more properties of that today.

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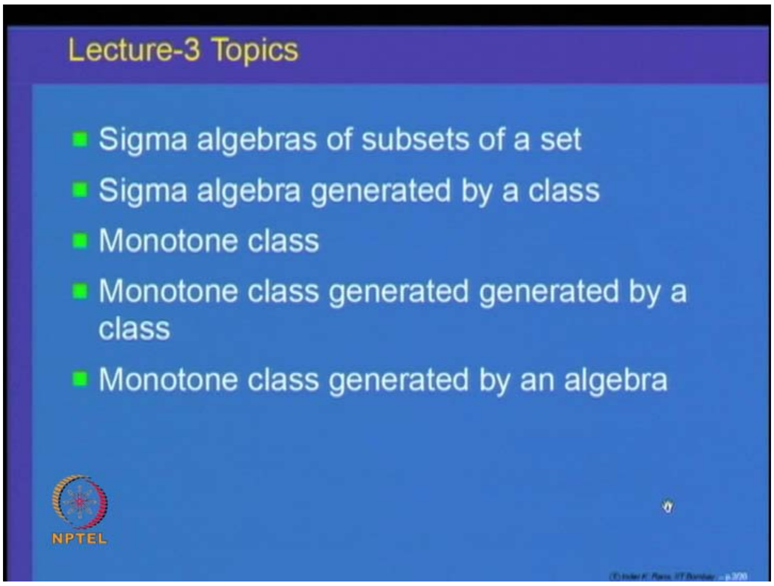

After that, we will start looking at what are called sigma algebras of subsets of a set and then come to sigma algebra generated by a class of subsets of a set x; we then go on to look at what is called a monotone class, the monotone class generated by a class and then look at a monotone class generated by an algebra.

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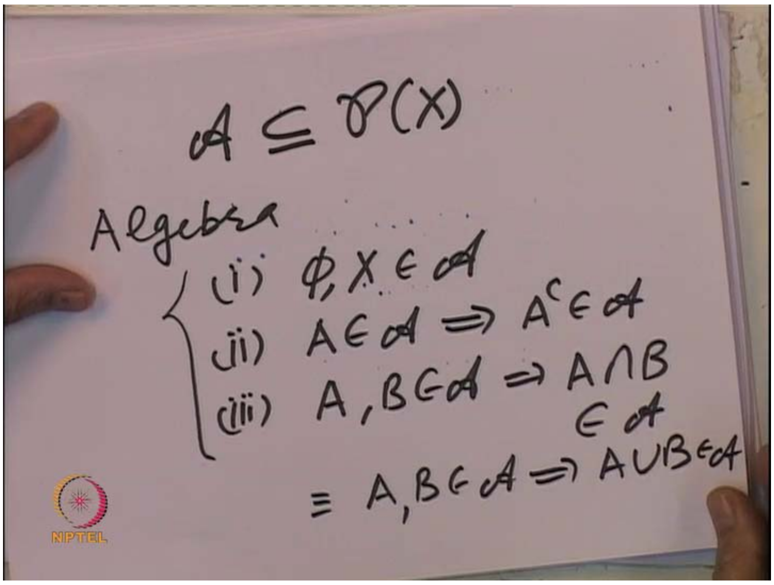

Let us just recall what we had started looking at, namely, the algebra. We said an algebra of subsets. A class of subsets A contained in P of X, that is the power set of X, is called an algebra if it had these properties: (i) – the empty set and the whole space is a member of A; secondly, whenever A belongs to A, it implies its complement is also inside the class A; that is, the class A is closed under the operation of complements; the third property was whenever A and B belong to A, that implies A intersection B belongs to the algebra.

These are the three properties that define a class A to B – an algebra. Keep in mind this property because of complements (Refer Slide Time: 01:53). This can be equivalently stated as A and B belonging to the class A implies A union B also belongs to the class A. This is what we had defined as a collection of subsets of a set X to be an algebra.

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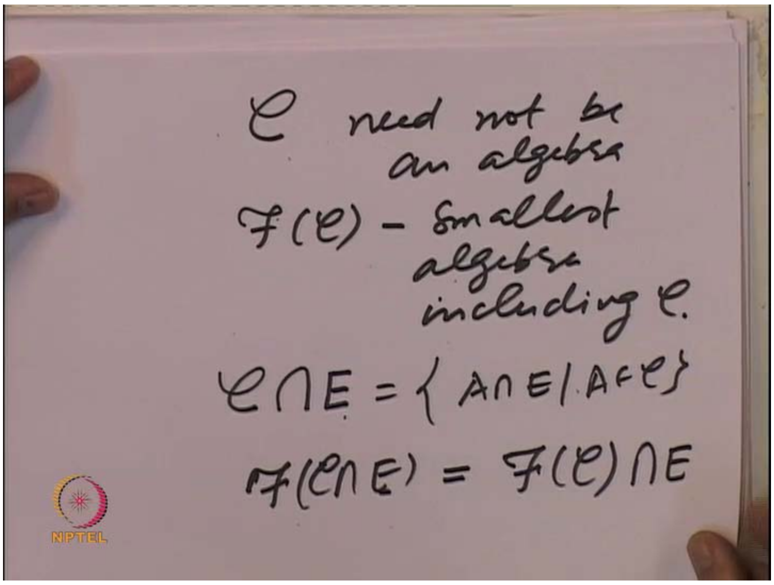

Then, we looked at various properties of algebras. For example, we proved one thing – that a class C need not be an algebra, but you can generate an algebra out of it; so, this is the smallest algebra including the given collection C. We then went on to prove that if you take a collection C and restrict its elements to a set E which is defined as all elements of the type A intersection E where A belongs to C, then if you generate an algebra out of this collection C intersection E which is the algebra of subsets of E generated by this collection C intersection E, we showed this is also equal to the algebra generated by C restricted to E. These are the various ways of generating more algebras out of the given algebra.

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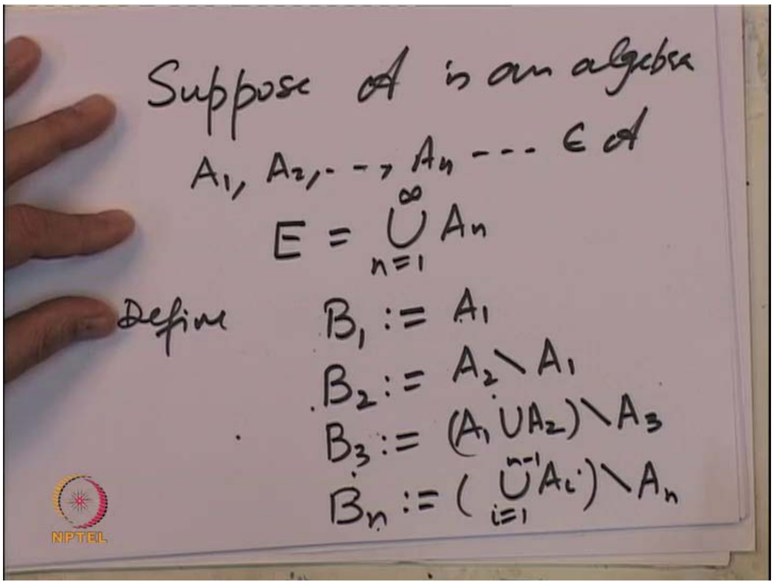

What is the advantage of having an algebra is the following. Let us suppose that we have got A is an algebra and then let us take a sequence A_1, A_2, up to A_n inside A, a collection of elements of A; let us take their union E equal to union of A_ns, n equal to 1 to infinity; of course, this E need not belong to the algebra because the algebra is only closed under finite unions.

However, there is something nice one can do. Let us define B_1 to be equal to A_1 itself; let us define B_2 to be equal to A_2 and remove from it the set elements which are in A_1. Similarly, let us define B_3 to be A_1 union A_2 and remove from it the elements which are in A_3 and so on. You will define B_n in general to be equal to union A_i, i equal to 1 to n and remove from it the elements which are n minus…. So, B_n is defined as the union of elements up to n minus 1 and remove from it the elements which are in A_n.

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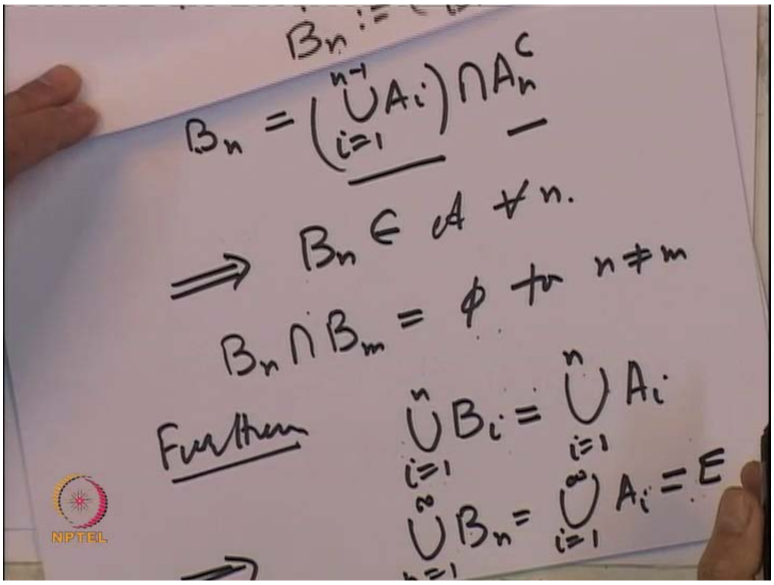

We have generated a new sequence out of the given sequence. Let us observe that B_n which is equal to union of these A_is minus A_n – what does this look like? It is union A_i, i equal to 1 to n minus 1 intersection A_n complement because removing ((.)) is the same as taking its intersection with A_n complement. Now, observe that each A_i is an element in the algebra; this is a finite union of elements in the algebra (Refer Slide Time: 05:47).

A_n complement is in the algebra because A_n is in the algebra; the algebra is closed under complements. This implies that each set B_n is an element of the algebra A for every n; that is one observation. Secondly, let us observe that B_n intersection B_m is empty for n not equal to m because what we are doing is B_1 is A_1; B_2 is from A_2 remove A_1; so B_1 and B_2 are going to be disjoint; B_3 is A_1 union A_2 minus A_3; we have removed what is in A_3; so, this B_3 is going to be disjoint from both B_2 and B_1.

In general, it is quite obvious that B_ns are ((.)); this is disjoint (Refer Slide Time: 06:39); they are elements in A. Further, here is an important consequence: the way we have constructed, if i take union of B_is, i equal to 1 to n, what is that equal to? B_1 is A_1 and B_2 is A_2 minus A_1. What is B_1 union B_2? That is the same as A_1 union A_2. Similarly, B_1 union B_2 union B_3 is the same as A_1 union A_2 and union A_3; that is the same as B_1 union B_2 union B_3. So, for every n, union of B_is, i equal to 1 to n is the same as union of B_is, i equal to 1 to n. As a consequence, this implies that union of B_n, n equal to 1 to infinity is the same as union i equal to 1 to infinity of A_i, which was our set E.

What have we shown? We have shown that if we start with any countable union of elements in the algebra (Refer Slide Time: 07:50), E need not be an algebra but E can be represented as a disjoint union of sets B_n and each B_n is an algebra. We have proved a theorem which is going to be quite useful and that is the advantage of being inside an algebra.

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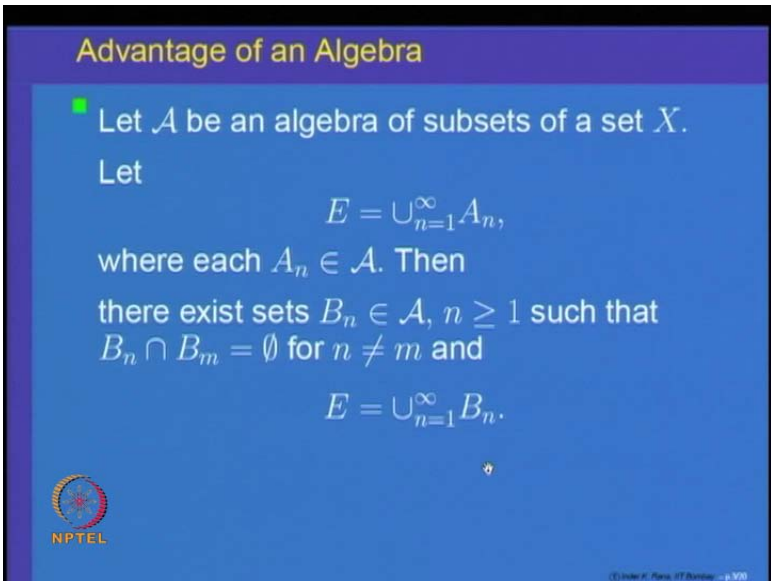

Let us recall once again: if A is algebra of subsets of a set X and a set E is union of A_ns, n equal to 1 to infinity where each A_n belongs to A, then there exist disjoint sets so there exists sets B_ns belonging to the algebra which are pairwise disjoint and their union is equal to B_ns. So, any countable union in algebra can be represented as a countable disjoint union; that is the advantage of being in a class which is an algebra. That is nice; we will see applications of this next time.

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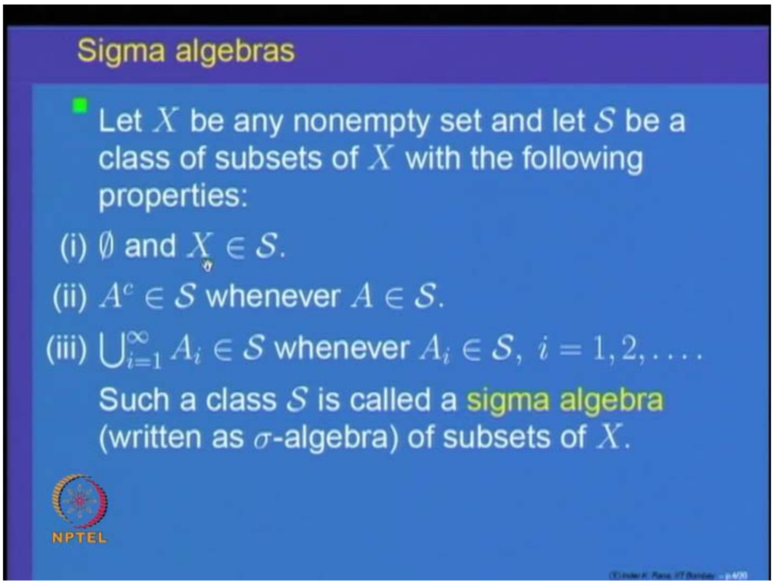

Let us now start with a class which is slightly stronger than an algebra; that is called sigma algebra. Let us start with a collection X is a nonempty set and let S be a class of subsets of the set X with the following properties: (i) – the empty set and the whole space are elements of it, like in a semi-algebra and like in an algebra; A complements belong to S whenever the set A is in S; that means the collection S is closed under taking complements, as in the case of an algebra; these two properties are the same as were the case for an algebra.

The third property is the one which distinguishes it from an algebra; we want that whenever sets A_is are in S, i equal to 1, 2, 3 and so on, that means whenever you take a countable collection of sets in S, their union i equal to 1 to infinity A_i also belongs to S; that means the collection S is closed under taking countable unions also. Such a collection we are going to call as a sigma algebra, sigma indicating that it is closed under a sequence of unions.

Let us just emphasize once again: a sigma algebra of subsets of a set X is a collection which includes the empty set and the whole space; it is closed under taking complements – if A belongs to S, A complement belongs to S; and whenever you take a sequence A_i of elements of S, their union is in S; that means S is also closed under taking countable unions; such a class is called a sigma algebra.

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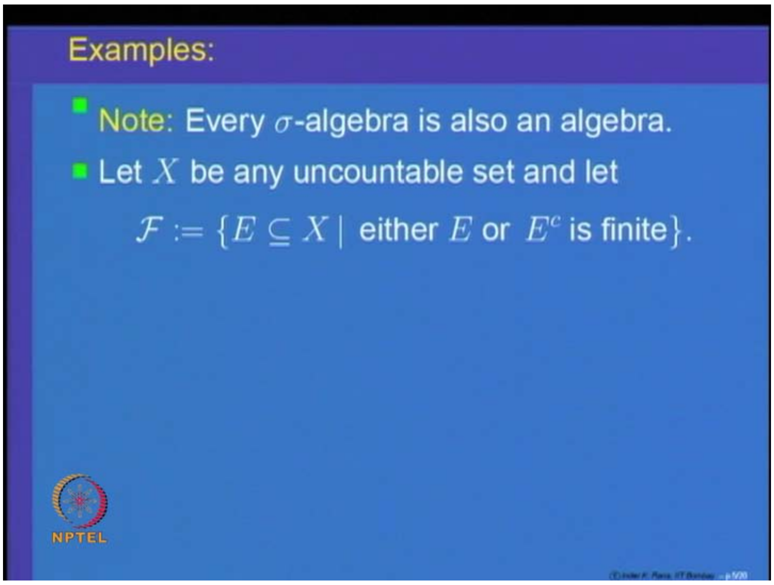

One obvious example is that every sigma algebra is also an algebra because sigma algebra means it is a collection which is closed under countable unions and an algebra only requires finite unions. Of course, both algebra and sigma algebra are closed under taking complements and the empty set and the whole space are always members of both of them; so, every sigma algebra is also an algebra.

Let us look at an example of X, an uncountable set. Let us look at the collection of all those subsets of X such that either the set is finite or its complement is finite. An element E is in this collection F if either the set is finite or its complement is finite. We had already shown that this collection F is an algebra.

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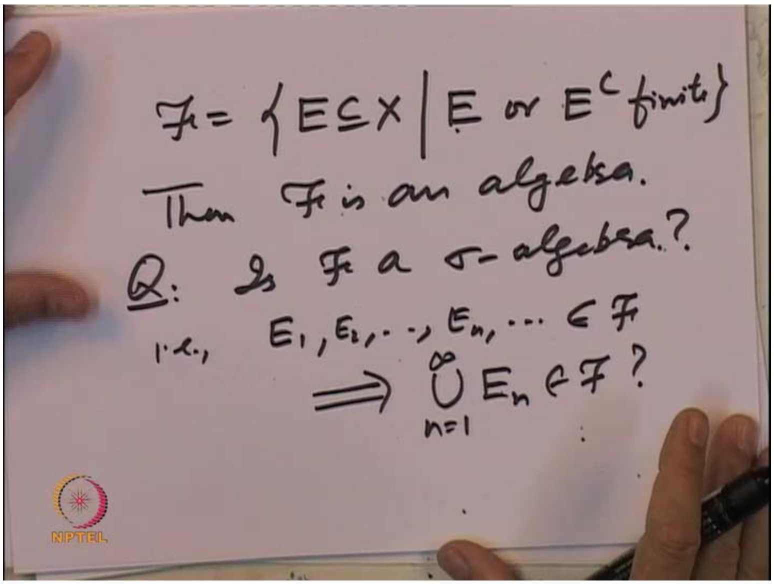

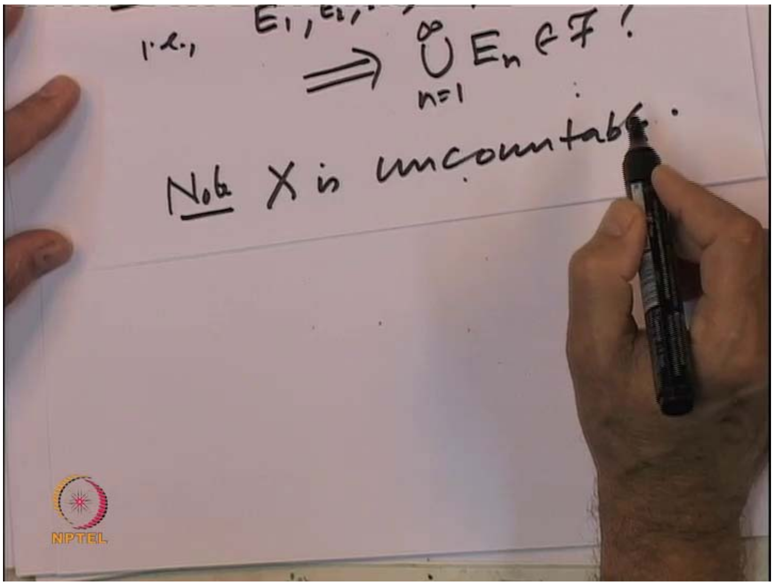

Let us recall what we have already shown that if I take this collection F of subsets E of X such that E or E complement finite, then we already observed that F is an algebra. The question is: is F a sigma algebra? Does it have the property that E_1, E_2 up to E_n belonging to F imply always that union of E_ns, n equal to 1 to infinity also belongs to F? That is not true for the following reason. Note that X is uncountable.

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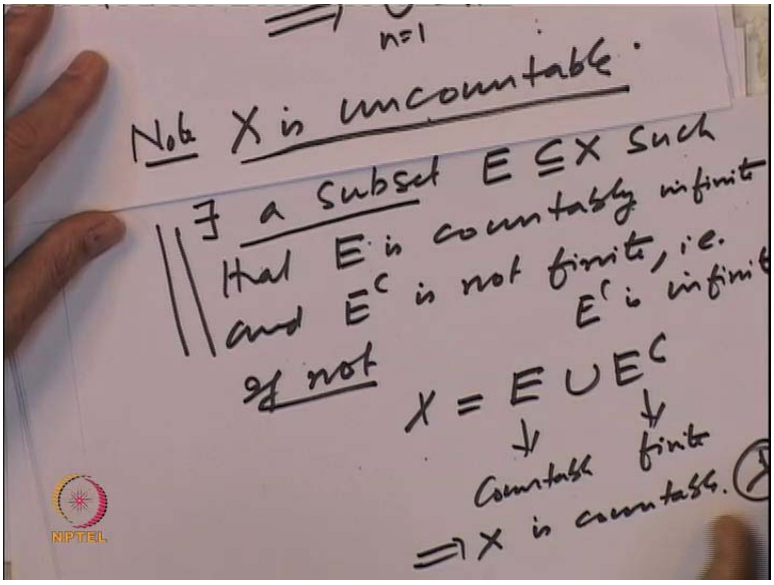

As a consequence of this, there exists a subset E contained in X such that E is countably infinite and E complement is not finite. If this is not true, then what will happen? We will have X which is equal to E union E complement. This is countable and this is finite; that will imply X is countable, which is not true – a contradiction. Whenever you have got a set X which is uncountable, there always exists a subset of it such that E is infinite and its complement is not finite; that is, E complement is infinite. We have got a set E which is countably infinite and its complement is not finite. That means what?

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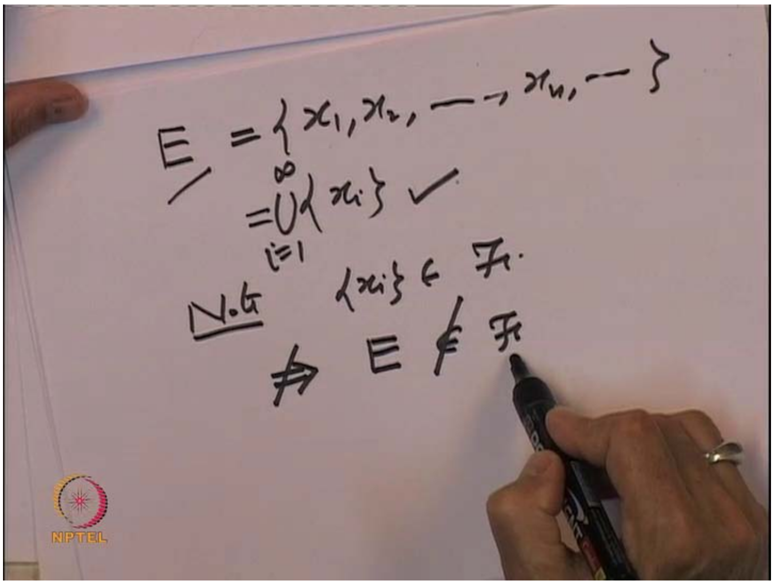

Since E is countably infinite, I can write E equal to x_1, x_2 up to x_n and so on; it is a countably infinite set; I can write it as a sequence; I can enumerate the elements of it. That is equal to singleton x_i union, i equal to 1 to infinity. Let us observe that singleton x_i is an element in the algebra F because singleton is a finite set; that does not imply E which is a union of these elements (Refer Slide Time: 15:19) belong to F; E does not belong to F.

Why does not E belong? It is because if E has to belong to this collection F, E should be either finite or E complement is finite; both of them are not true. Basically, if I take a set E which is countably infinite, then it is countable union of elements of F and it does not belong to F; so, F is not going to be a algebra.

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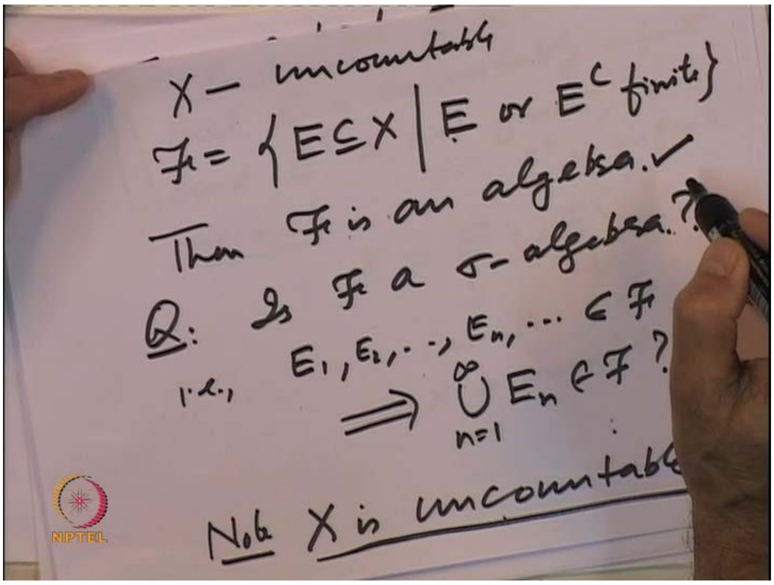

What we are saying is whenever X is uncountable and look at this collection of sets E contained in X so that E or E complement is finite, then it is an algebra and it is not a sigma algebra of subsets of X.

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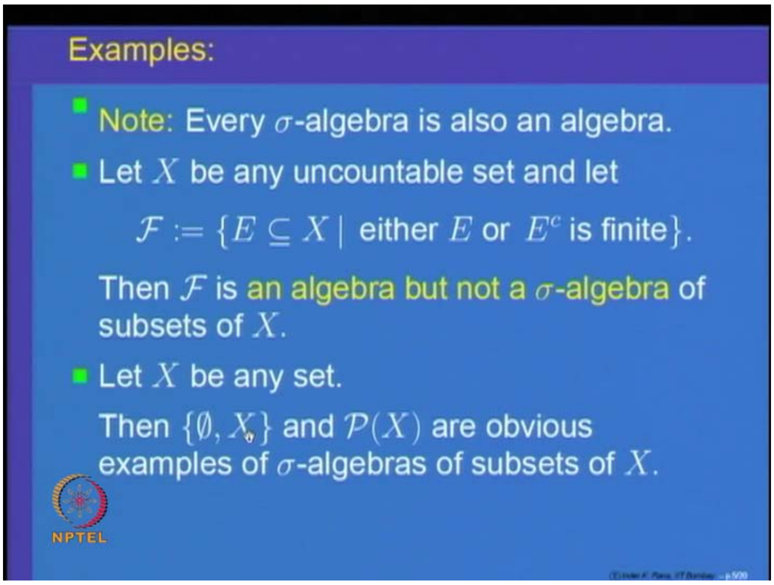

This collection that we have taken, we have proved that it is an algebra of subsets of X but it is not a sigma algebra of subsets of X. So, every sigma algebra is an algebra, but every algebra need not be a sigma algebra; that is the observation that we get from here (Refer Slide Time: 16:36). Let us look at some more examples of sigma algebras. Let X be any set; then, obviously, the empty set and the whole space put together (the two elements) – that collection is a sigma algebra because there are only two elements; their union belongs closed under complements and so on.

Of course, the collection of all subsets of X, the power set of X, also is a sigma algebra of subsets of X because it is closed under all kinds of operations. The empty set and the whole space put together is an example of a sigma algebra; power set is an example of a sigma algebra of subsets of any set X. These are obvious examples of sigma algebras of subsets of X.

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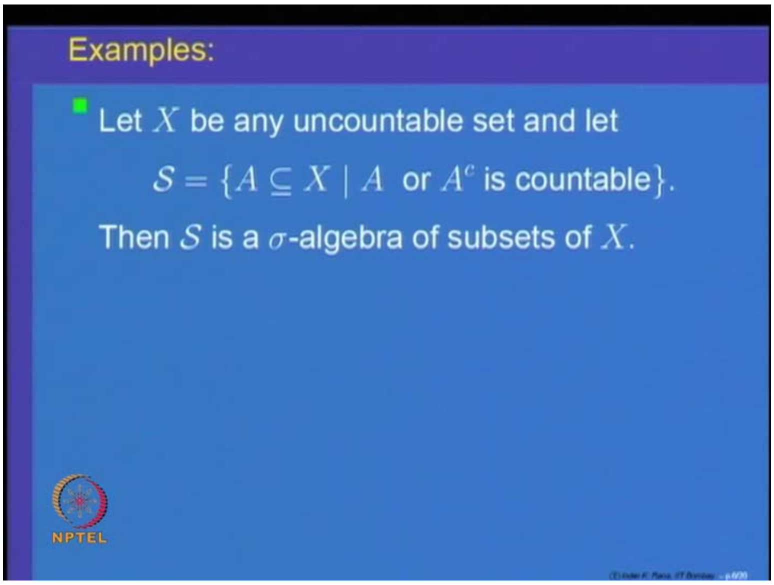

Let us look at some nontrivial examples of sigma algebras of subsets of X. Let us take X, an uncountable set; let us take S to be a subset, all those subsets of X, such that A or A complement is countable. The claim is S is an algebra of subsets of the set X. Let us try to prove that this collection S is a sigma algebra of subsets of X.

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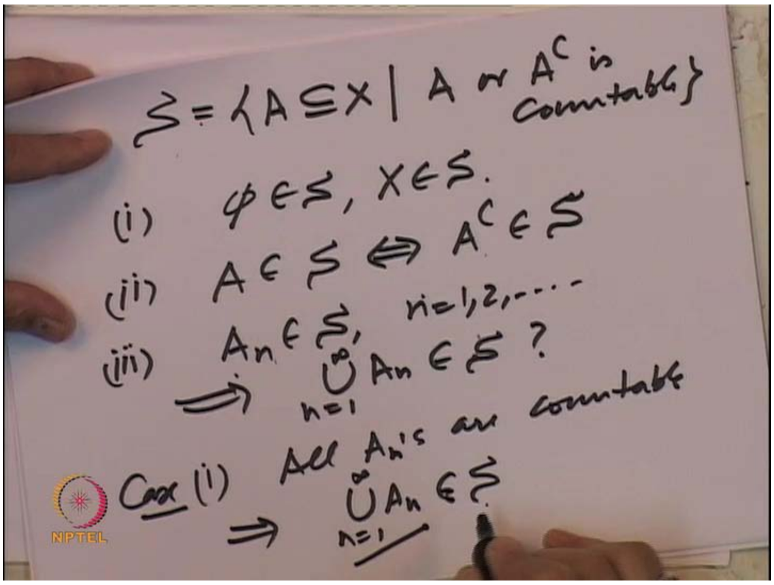

What is S? S is the collection of all those subsets A contained in X such that A or A complement is countable. The first observation is that the empty set belongs to S, because the empty set is taken to be a finite set and so it belongs to S. Does X belong to

S? Yes. X belongs to S because its complement is empty set and hence that belongs to S; the empty set and the whole space both belong to S. Clearly, if A belongs to S, then this implies ((.)) if and only if A complement belongs to S because our defining condition is symmetric with respect to A and A complement.

Let us check the third property that if A_n belongs to S, n equal to 1, 2, 3 and so on, then this implies union of A_ns, n equal to 1 to infinity also belongs to S; let us check that property. Obviously, like in the case of finite and complement finite, we have to divide it into cases. The first case is case (i): all A_is or all A_ns are countable, but that will imply that union of A_ns is also countable and hence belongs to S. Why is union of A_ns countable? It is because a countable union of countable sets is countable – that is the set theory property; this set is countable and so it belongs to S.

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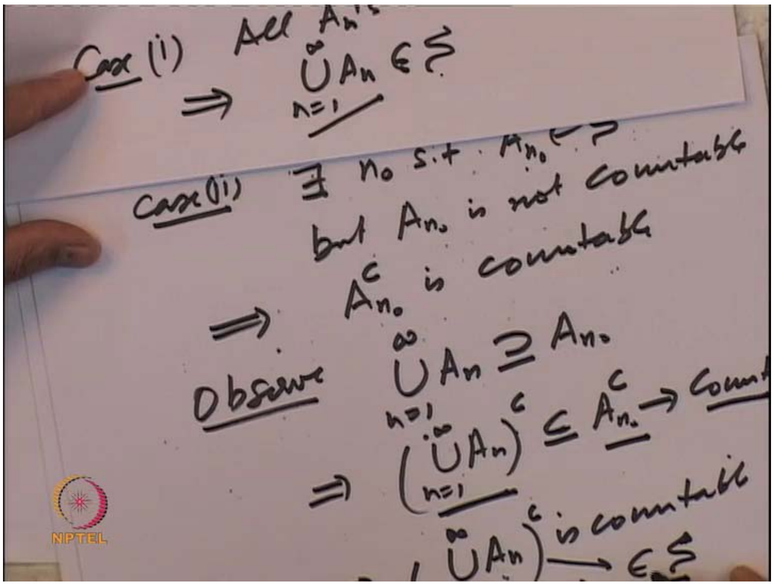

Let us look at the second possibility, the second case. What is the possibility? Not all A_ns are countable. That means there exists some n_0 such that A_{n0} belongs to S, but A_{n0} is not countable. That means what? A_n belongs to S not countable means that A_{n0} complement is countable by the very definition of S. Now, observe that union of A_ns, n equal to 1 to infinity includes the set A_{n0} because that is one of the members. That implies that union of A_ns, n equal to 1 to infinity complement is contained in A_{n0} complement and this is countable. So, this set's complement is a subset of a countable set.

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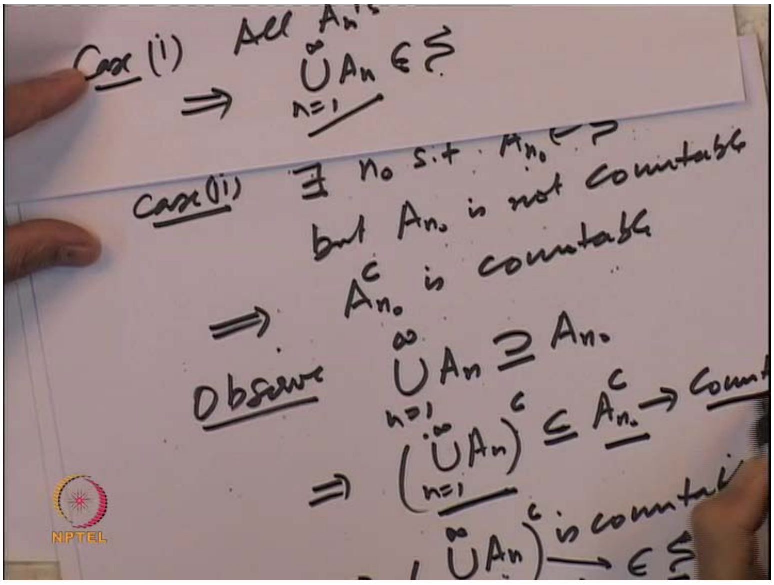

That implies union A_n, n equal to 1 to infinity is countable, implying that this belongs to the class S. This is a set whose complement is countable; that means this set must belong to S. This is contained in A ((.)) and this is countable; that means this set is countable (Refer Slide Time: 21:28) and because the complement of this set is countable, this belongs to S.

Let us look at the collection S of all subsets of A such that A or A complement is countable (Refer Slide Time: 21:44). Then, this collection is a sigma algebra of subsets of X. Let us observe that we have not used anywhere the fact that X, the underlying set is a countable set; this is true for any, actually. What we have shown is that if X is any set and let us take the collection of all those subsets of X which are either countable or their complements are countable, then that forms a sigma algebra of subsets of X (Refer Slide Time: 22:05).

Let us observe one thing: we have not used anywhere the fact that the set X is uncountable; this property even remains true when X is any set. Of course, the collection S still remains a sigma algebra, but its nature will change in the sense that, for example, if X is a countable set (you can try to prove yourself), then for this collection of those subsets of X we will say that A or A complement is countable. In fact, that will be all subsets of the set X. So, it is a nontrivial example only when X is an uncountable set. We have given an example of a collection S of subsets of set X which is a sigma algebra.

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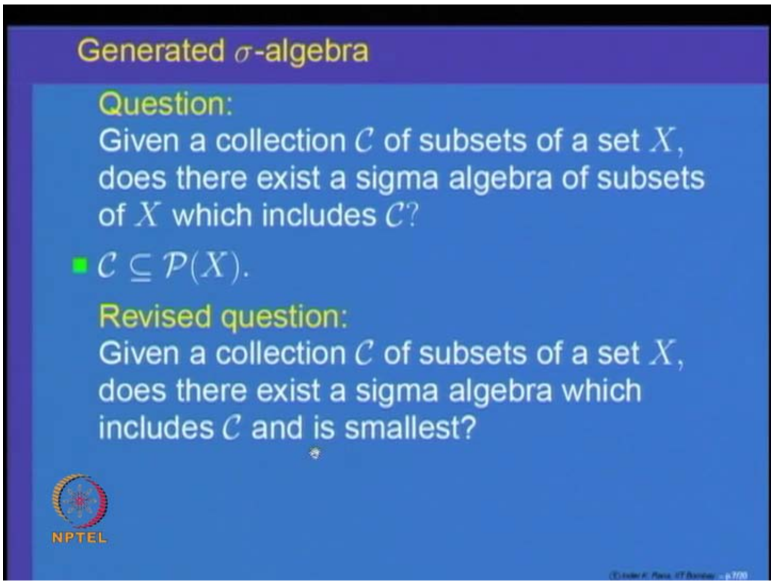

Next, let us ask a question. Given a collection C of subsets of a set X, it may not be an algebra and it may not be a sigma algebra, the question arises: can we say that we can find a sigma algebra of subsets of the set X which includes this collection C? In some sense, this collection C may not be closed under complements or may not be closed under taking countable unions; we would like to enlarge it so that it becomes a sigma algebra.

The obvious examples are if you take all subsets of the set X, then that itself is a sigma algebra. That includes C but that is a very trivial example of a sigma algebra which includes C. We would like to modify our question such that given a collection C of subsets of a set X, does there exist a sigma algebra of subsets of X which includes C and is smallest?

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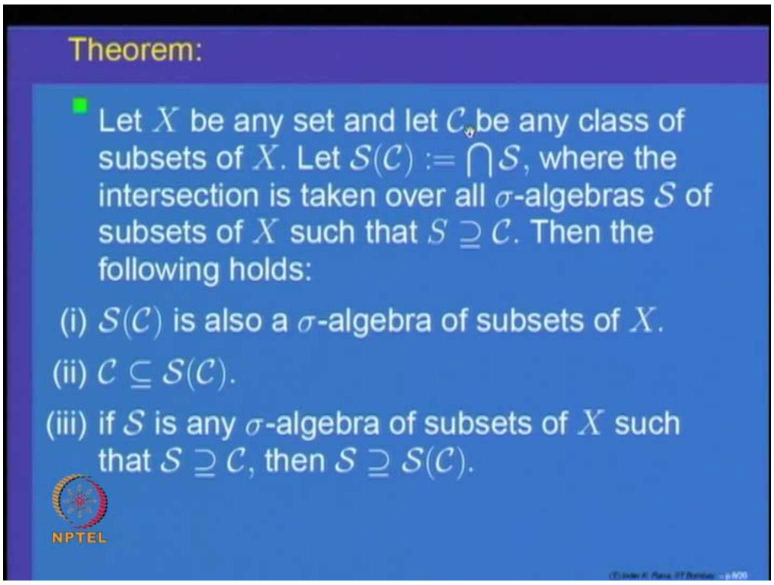

The answer is yes and it is something similar to what we have done for the case of algebras. Let X be any set and C be any class of subsets of set X. Let S of C denote the intersection of all the algebras S of subsets of X which includes C. Then, the claim is that this collection S of C is a sigma algebra of subsets of X; it includes the class C and it is the smallest. That will show that given a collection C of subsets of a set X, you can always find a sigma algebra of subsets of X which includes C and which is the smallest.

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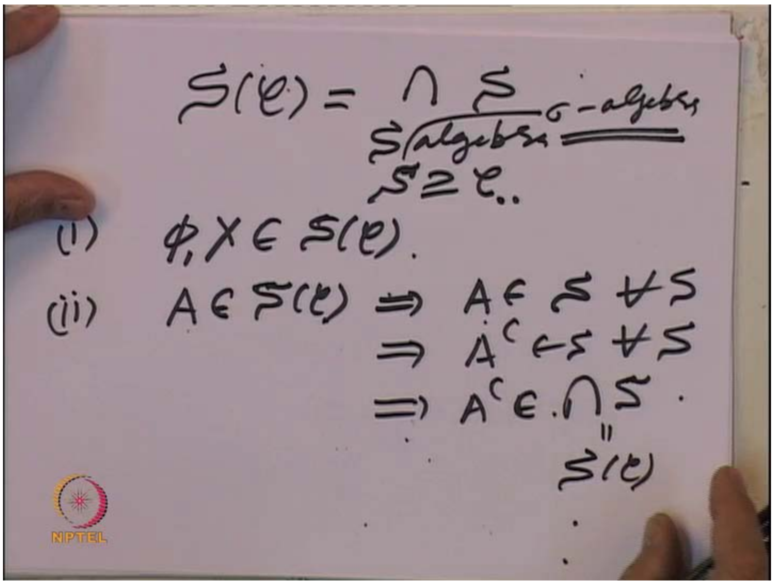

Let us check these three properties one by one. S of C is defined as the intersection of all the algebras S such that S algebra and S includes C. The first property we want to check is that the empty set and the whole space belong to S of C; that is obvious from the fact that the empty set and the whole space will belong to every algebra S which includes C because S is an algebra; the empty set and whole space belong to it – every element S in this collection whose intersection we are taking; so, the intersection also will have that property. That is the obvious property, as observed in the case of the algebra generated.

The second thing: let us take a set a which belongs to S of C; that implies that A belongs to S. Sorry, this is a sigma algebra; we are looking at the case of sigma algebras (Refer Slide Time: 26:14). We are taking the intersection of all sigma algebras which includes C. So, if A is inside the class S of C, then A belongs to S and S is a sigma algebra. That implies that A complement belongs to S for every S; that implies that A complement belongs to the intersection of all this collection. Hence, A complement is nothing but S of C.

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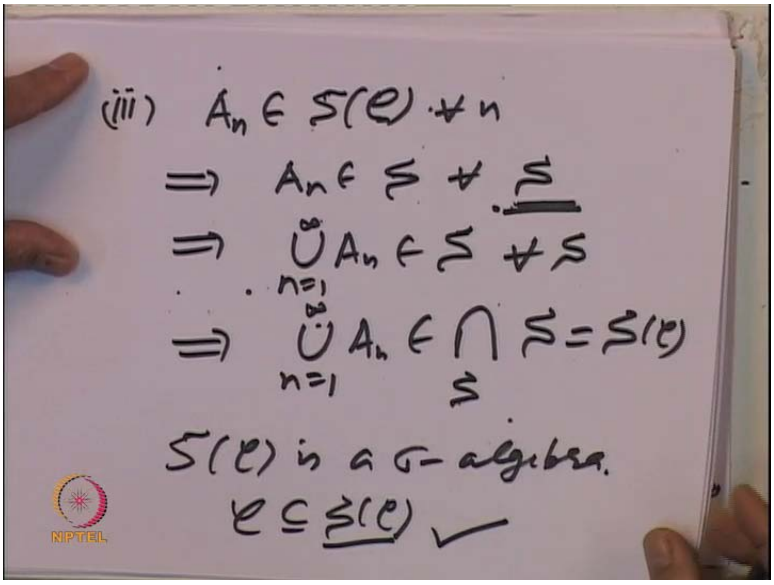

Similarly, let us take the third property; let us take a sequence A_n which belongs to S of C for every n. That implies A_n belongs to S for every S and that implies, because this is a sigma algebra, the union of A_ns n equal to 1 to infinity also belongs to S for every S. That implies that union n equal to 1 to infinity A_n also belongs to the intersection of all this algebras S over S; that is nothing but S of C.

What we have shown is that if you look at S of C (Refer Slide Time: 27:35) – the intersection of all sigma algebras which includes C, then they themselves form a sigma algebra. What we have shown is S of C is a sigma algebra and that C is contained in S of C; it is once again obvious because S of C is the intersection of all sigma algebras which includes C; so, the intersection also will include it; that also is an obvious property.

Why it is it the smallest? The smallest property also is true once again by the very fact that S of C is the intersection of all the sigma algebras which include C (Refer Slide Time: 28:22). S of C being the intersection is the smallest anyway; that proves the fact that S of C is a sigma algebra of subsets of X; C is inside S of C; if S is any other algebra of subsets which includes C, then S of C must be inside S, because S of C is the intersection of all (intersection is inside every element).

What we have shown is given a collection of subsets of the set X, C – a collection of subsets, it may not be an algebra, but you can put it inside a sigma algebra of subsets of X denoted by S of C and such a thing exists because of this construction. Such a thing is called the sigma algebra generated by the class C.

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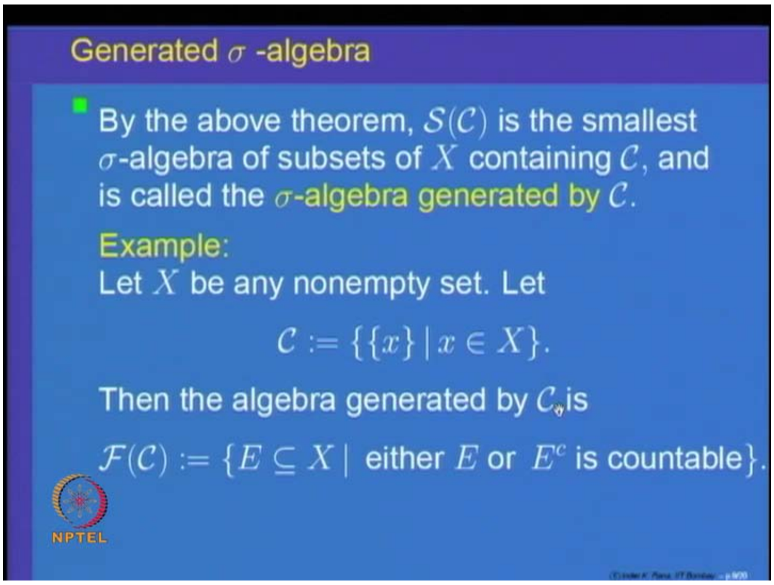

S of C we are going to call it as the sigma algebra generated by a class. Let us look at some examples of sigma algebras generated. Let us look at X, the collection of all… X is any nonempty set. Let us look at all singleton subsets of this set X. Let us call that collection as C; C is the collection of all singletons where singletons are elements of the set X. The claim: we want to find out what is the sigma algebra generated by it.

If we take a sequence of elements x_1, x_2, x_1 up to x_n in X and look at those singletons, then their union is going to be an element in C. That means all countable sets must be elements of C. Similarly, all sets whose complements are countable also must be elements of the set E. As a consequence, we expect that this answer is nothing but the algebra F of C, the sigma algebra generated by… This is not correct (Refer Slide Time: 30:34). What we should have is the sigma algebra generated by C; this is a correction here. Then, the sigma algebra generated by this must be equal to this collection F of C, where either E or E complement is countable.

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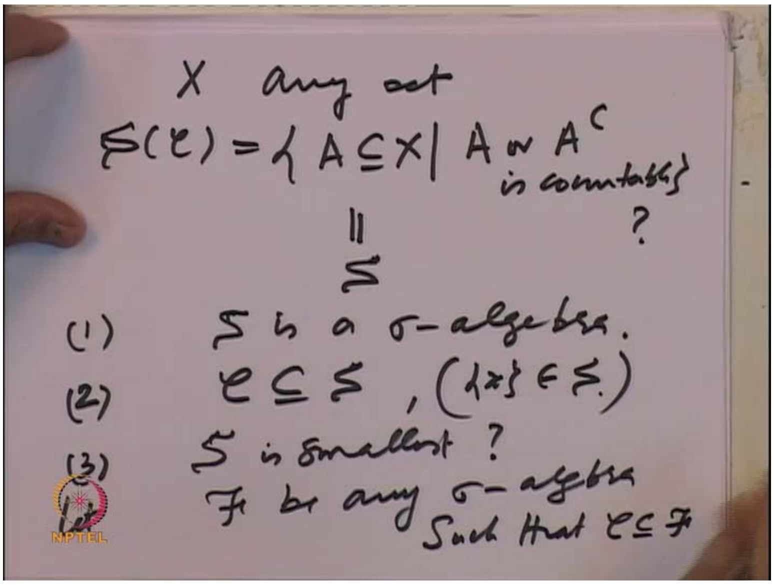

Let us prove this fact. X is any set. We are taking S of C to be equal to all those subsets A contained in X such that A or A complement is countable; this is what we want to prove. Let us call this collection S; sets which are countable or their complements are countable – that collection is called S. The claim is S of C, the sigma algebra generated by the singletons, is nothing but this collection.

Let us observe. The first observation is that S is a sigma algebra; we have just now proved that this collection S is a sigma algebra. The second is that C is inside S because what are the elements of C? They look like singletons; a singleton is countable and so this belongs to S. It is because of this reason that C is a subset of X. This is a sigma algebra which includes…. Third, we want to show it is a smallest – S is smallest. Let us take any other algebra; let F be any sigma algebra such that that C is a subset of F.

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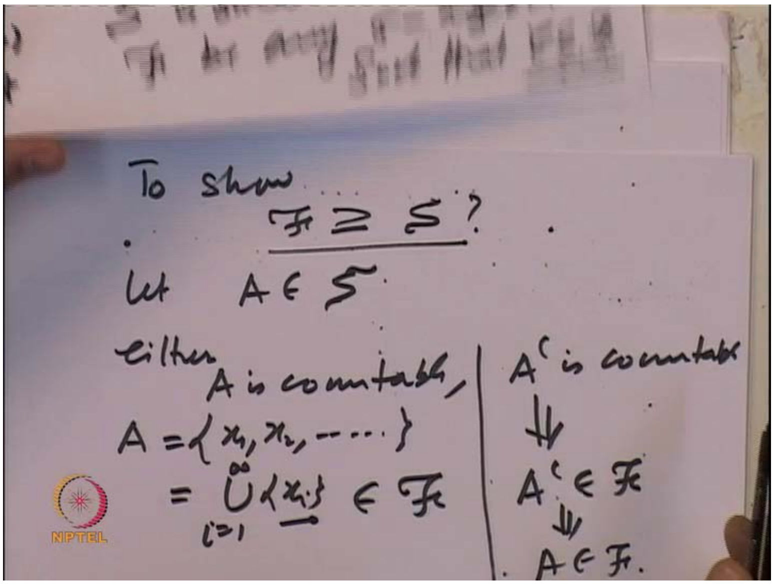

What do we want to show? We have to show that F includes this collection S. Let A belongs to S. Two cases arise: either A is countable. In that case, A can be written as x_1, x_2 and so on; that can be written as union of singletons x_i, i equal to 1 to infinity. Each x_i is an element in C and C is inside F (Refer Slide Time: 33:44). When C is inside F, every singleton belongs to F and F is algebra; so, this is a sigma algebra; this implies this belongs to F. If A is countable, then it belongs to F.

If not, what is the second possibility? A complement is countable. Then, by the same argument, this will imply that A complement belongs to F. F is a sigma algebra and so this implies that A belongs to F. In either case, we have shown that if F is any sigma algebra (Refer Slide Time: 34:23) which includes C, then this F must include S. That proves the fact that S of C, the sigma algebra generated by the singletons, is nothing but all those sets such that either the set is countable or its complement is countable.

We are able to give a description. As a consequence, we are able to give a description of the sigma algebra generated by a collection of subsets in this case when the collection C consists of singleton sets, but let us be very careful; in general, for a set X given a collection C of subsets of X, it is not always possible to describe the elements of S of C explicitly in terms of elements of C; it is not possible always; remember that was also the case when we looked at the algebra generated by a collection of subsets C.

Only when C was a semi-algebra were we able to describe the algebra generated by the semi-algebra; we were not able to give a general description of the algebra generated by a class C. Similarly, it is not possible to give a description of the sigma algebra generated by a collection of subsets of a set X, but such are the collection of sets which are going to play a role in our subject later on; so, we have to study them carefully in detail.

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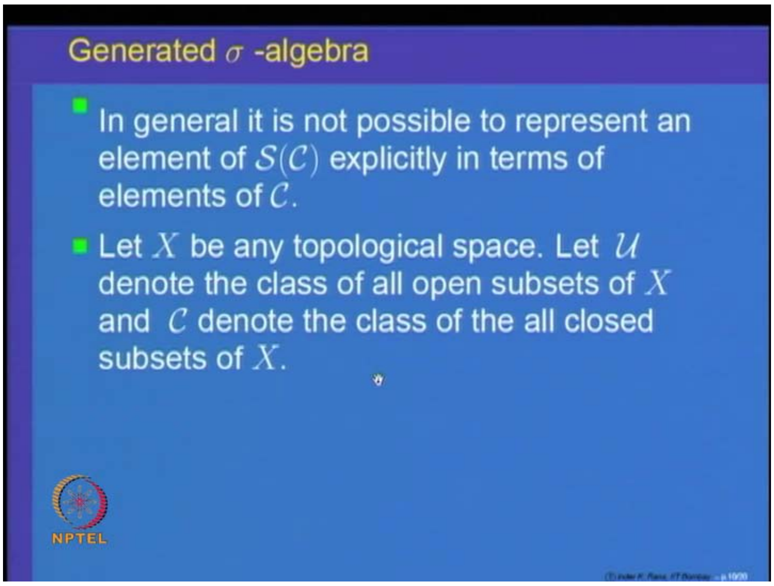

Let us look at some more properties of such kind of objects. Let us look at something called a topological space; I hope some of you are aware of what is a topological space.

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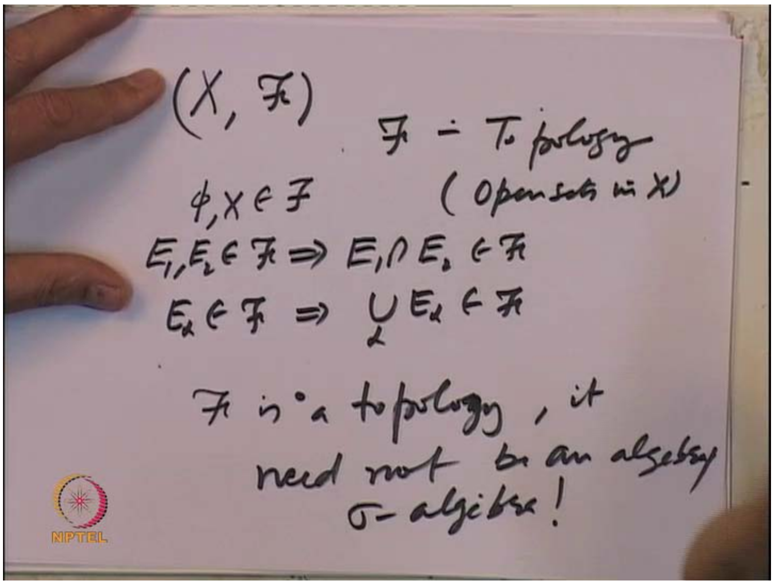

A topological space consists of a set X and a collection of subsets of X; tau is called a topology; it is a collection of open sets in X. This is a collection which has some properties: the empty set and the whole space belong to it; if two sets E_1 and E_2 belong to F, then that implies E_1 intersection E_2 belongs to F; and if E_{alpha} is a collection of sets in F, that implies union of E_{alpha}s belong to F.

A topology is a collection of subsets of a set X such that the empty set belongs to it; it is closed under finite intersections and closed under arbitrary unions; such collections of sets are called open sets. Obviously, if tau is a topology, it need not be an algebra or a sigma algebra; the reason is obvious because this collection need not be closed under complements, which is required for an algebra or a sigma algebra; there it lacks.

Actually, in a topological space, the sets whose complements are open are called closed sets. If you want the complement also to be in that collection, then those are the sets which are both open and closed; there are not many examples of such things. Let us look at a topological space X. F is a topology on X and this need not be a sigma algebra. The question is: can we generate the sigma algebra given by this topology (Refer Slide Time: 38:40)?

The answer is yes. Let us look at the collection of all open sets in this topological space and let us look at the collection of all closed subsets in the topological space. We can generate the sigma algebra given by all open sets and we can also generate a sigma algebra by all closed subsets of the topological space. The question arises: is there any relation between these two sigma algebras? Let us recall a set is closed if and only if its complement is open.

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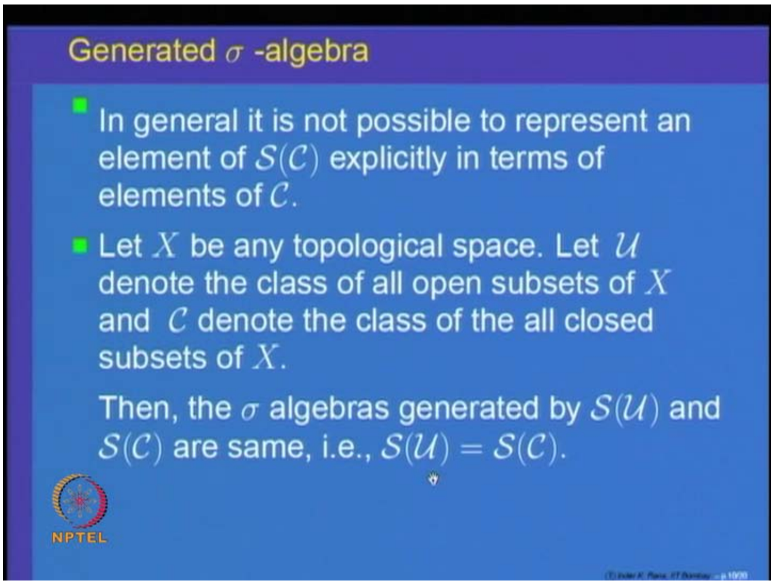

Using this fact, we will prove that collection of all the sigma algebra generated by open sets is equal to the sigma algebra generated by closed sets.

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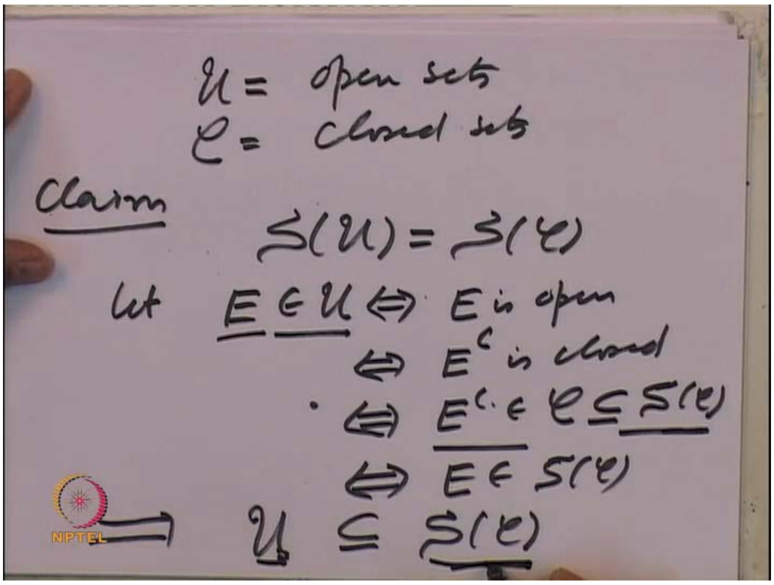

U is open sets and C is closed sets. The claim is that the sigma algebra generated by open sets is the same as the sigma algebra generated by the closed sets. To prove this, we will follow a technique which is going to be used again and again. Let us observe; let us take a set E which belongs to U; that is the same as saying that E is open; it is equivalent to saying E complement is closed.

That implies that E complement is in the collection C which is inside S of C. What we have shown is if E is an open set. then E complement belongs to S of C. As a consequence of this, E belongs to S of C because S of C is a sigma algebra. This implies that all open sets are inside the sigma algebra generated by all closed sets. Now, this is a sigma algebra which includes open sets; so, this sigma algebra must include the smallest sigma algebra containing u. That means once U is inside the sigma algebra S of C, the sigma algebra generated by it must also come inside S of C, by the very definition.

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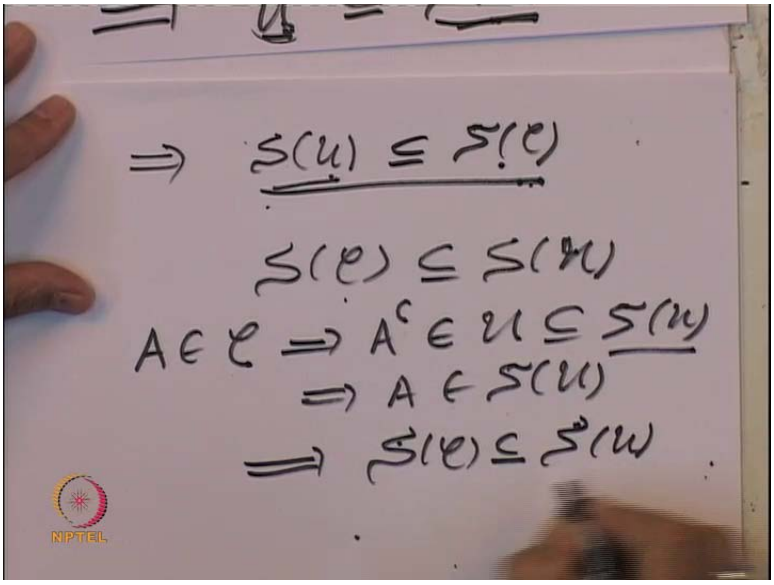

This implies that S of U, the sigma algebra generated by open sets, comes inside S of C. This is a technique which is used very ((.)). To prove S of U is inside S of C, what we have done is we have shown that U is inside S of C and hence S of U is inside S of C. This technique is going to be used very often in our course of lectures. When we want to show certain collection of sets has a required property, we show that that collection of sets includes a set of generators and hence will include the sigma algebra generated by it, provided that collection forms a sigma algebra.

By this technique, we have shown S of U is inside S of C; by the same technique, you can show that S of C is also inside S of U because if I take a set A which is in C, that means A is closed; that means A complement is open and so it belongs to U which is in S of u; that will imply that A belongs to S of U because that is a sigma algebra. Hence, this implies that the sigma algebra generated by C must come inside S of U; so, the collection S of C is same as S of U (Refer Slide Time: 42:48). This is a very important collection of subsets of a topological space.

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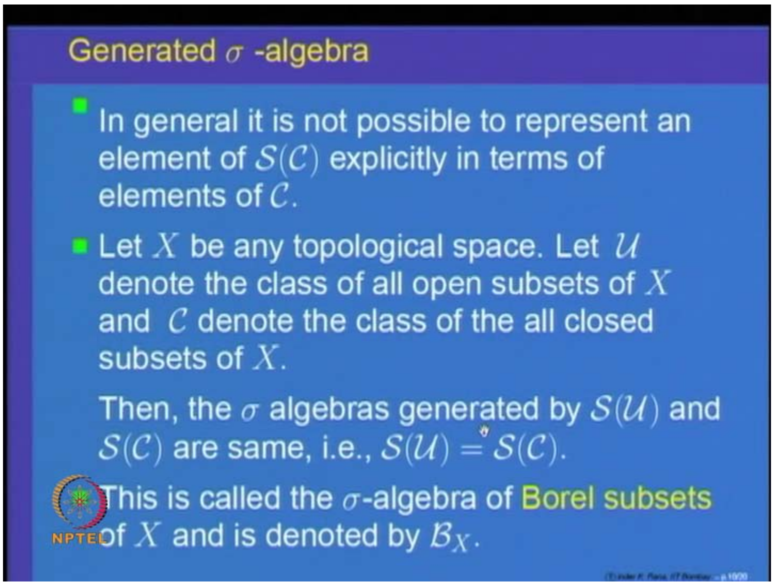

This is given a name; this collection is called the Borel subsets of the set X. The sigma algebra generated by all open sets or by all closed subsets of a set X is called the Borel sigma algebra of subsets of the set X.

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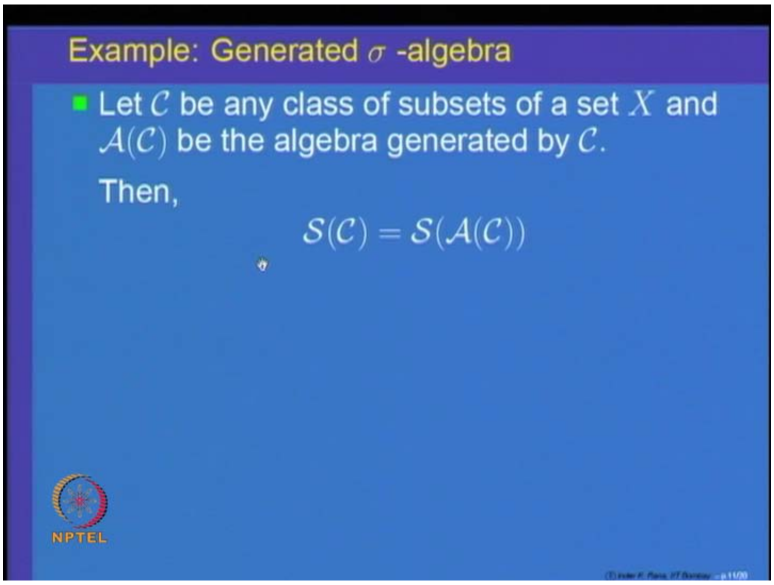

Let us observe a few more things. We recall we started with a collection C of subsets of a set X. We said that given a collection of subsets of set X, we can generate an algebra out of it. An algebra has properties which the class C may not have. Now, given this collection C, we can also generate a sigma algebra out of it; we can also generate the sigma algebra by the algebra generated by that class. The question comes: what is the relation between these three things? The observation we are going to prove is that given a collection C, you can directly generate the sigma algebra by this collection or you can generate first the algebra and then the sigma algebra by that algebra; both processes are the same.

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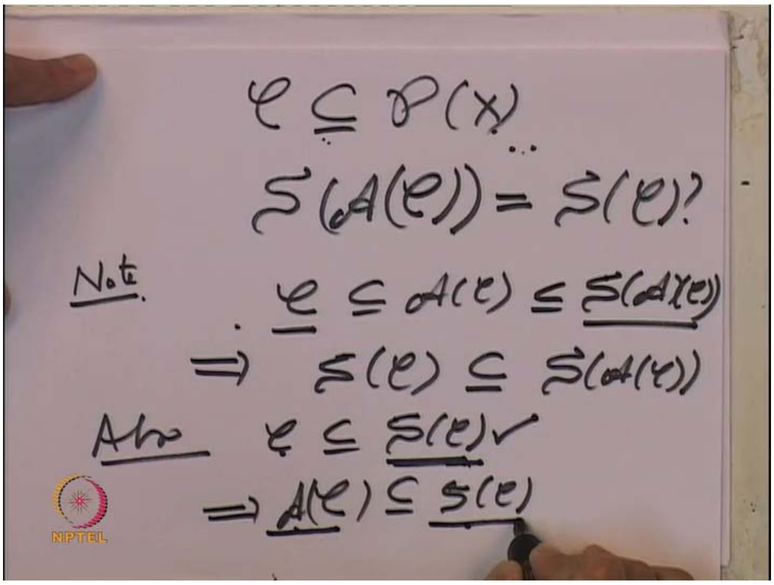

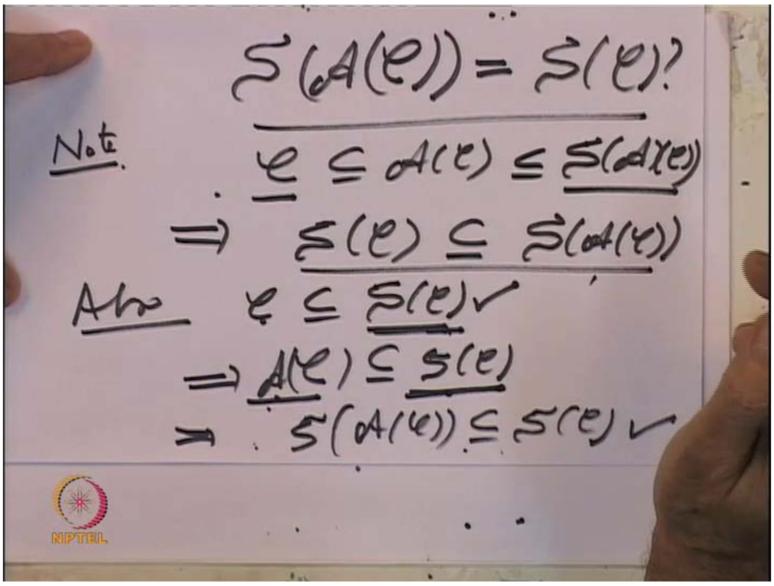

Let us give a proof of this obvious fact, because such kind of proofs are going to be useful. Let us look at the proof. If C is any collection of subsets of a set X and I take the collection C, I generate the algebra by this collection and then generate the sigma algebra by this collection; that is the same as the sigma algebra generated by C. This is what we want to prove. Let us observe. Note that C is contained in A of C which is contained in S of A of C by the very definition.

That implies C is the collection which is contained in this sigma algebra; that means the sigma algebra generated by C is contained in the sigma algebra generated by the algebra generated by C; so, that proves one way. To prove the other way round, what do we have to show? Let us also observe that C is contained in S of C and S of C is a sigma algebra; hence, it is also an algebra; this implies that A of C must be inside S of C.

Here, we have used the fact that every sigma algebra is also an algebra. This is an algebra including C and so the smallest one must be inside it. Now, this collection is inside S of C; this is a sigma algebra; that means the smallest one and so the sigma algebra generated by A of C must come inside S of C (Refer Slide Time: 46:02). That proves the other way round inequality and this is already proved.

These two together imply that the sigma algebra generated by any collection C is the same as first generating the algebra and then generating the sigma algebra; it does not matter; both are same. The proof illustrates the use of this technique again and again. If C is inside something and that something is algebra, then the algebra generated comes inside and so on. These are going to be techniques which are going to be used again and again in our course of lectures. This is one observation: the sigma algebra generated by any collection C is also the sigma algebra generated by the algebra generated by that collection (Refer Slide Time: 46:49).

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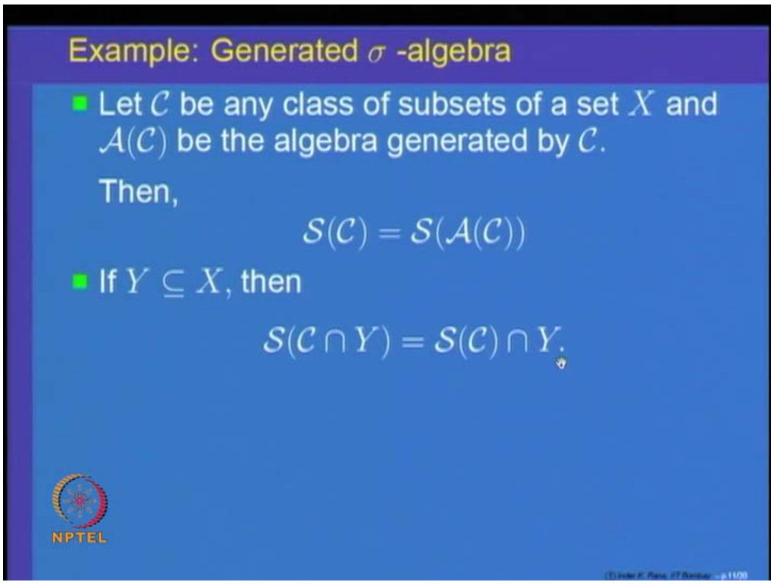

Let us observe another thing, another way of generating more examples of sigma algebras. If you take any collection C and take a subset Y of it, then that gives us C intersection Y is the collection of subsets of Y which are inside C. Then, we want to show that the sigma algebra generated by this... We are restricting C to Y and then generating the sigma algebra by it. The claim is it is the same as the sigma algebra generated by C first and then restricting it to Y.

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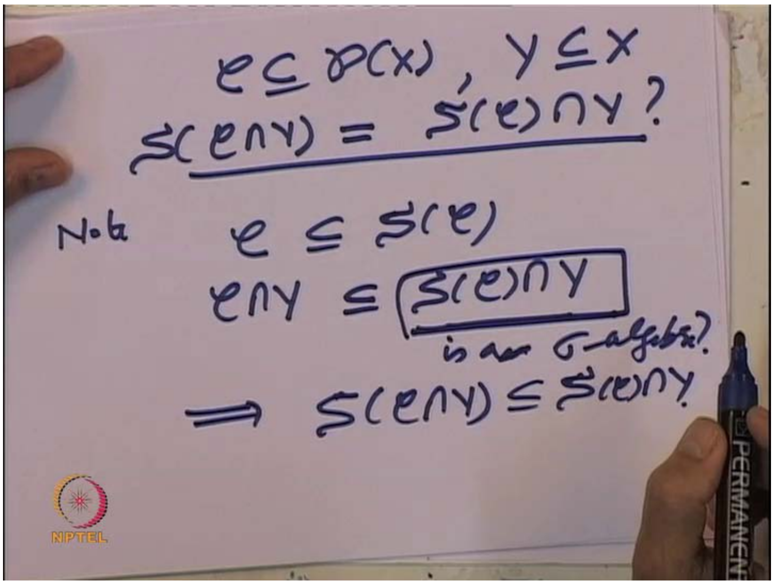

Let us give a proof of this fact. X is any set; C is a collection of subsets of P of X; Y is a subset of X. We want to show that the sigma algebra generated by C intersection Y is equal to the sigma algebra generated by C restricted to Y; this is what we want to prove. Let us observe; note that C is contained in S of C; by the very definition, C is inside. If I look at C intersection Y, look at the intersection of these sets, that is going to be inside S of C intersection Y. If I can show that this is a sigma algebra, then I would have that S of C intersection Y will be inside S of C intersection Y; one should try to show that this is a sigma algebra.

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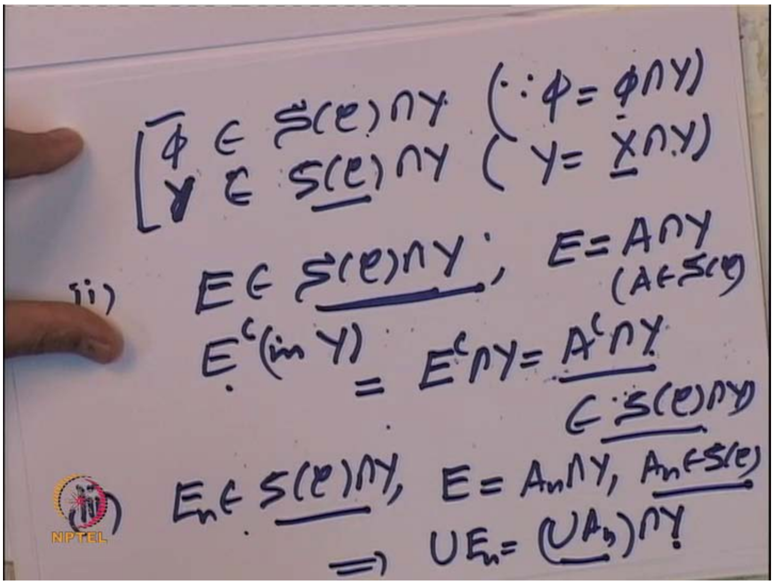

Let us try to show that that is a sigma algebra. Does the empty set belong to S of C intersection Y? Yes, because the empty set can be written as empty set intersection Y and empty set belongs to S of C. Similarly, the whole space. What is the whole space? The whole space here is Y; ((.)) S of C intersection Y; the claim is that this belongs to S of C intersection Y.

That is true because Y can be written as X intersection Y and X is in the sigma algebra S of C; so, both these things belong. The second thing: let us look at a set E belonging to S of C intersection Y. I want to look at the complement of this, but what is the complement of this? If this set belongs to S of C intersection Y, that means this set E must be equal to some element A intersection Y where A belongs to S of C; that is by the definition.

What is E complement? Keep in mind that we are looking at subsets of Y. What is E complement in Y? That is the same as E complement in Y. That means E intersection Y and that is the same as A complement intersection Y. That again belongs to S of C intersection Y. This collection S of C intersection Y is closed under complements. Finally, let us show it is closed under countable unions. If E_ns belongs to S of C intersection Y, let us assume E_n because it belongs here, it will be some A_n intersection Y where A_ns belong to S of C; that will imply union E_ns is union A_n intersection Y. This belongs to S of C (Refer Slide Time: 50:55); so, it is intersection Y; it belongs to this.

That implies that union also is an element; if E_ns belong to this, then the union also is an element of this.

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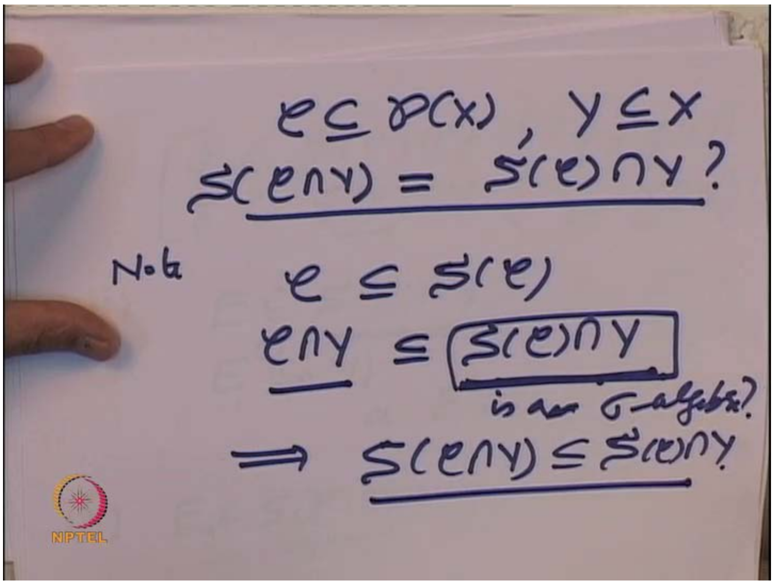

We have proved that this collection is a sigma algebra; this collection is inside this and so this will prove one-way inequality. We have to prove the other way round inequality.

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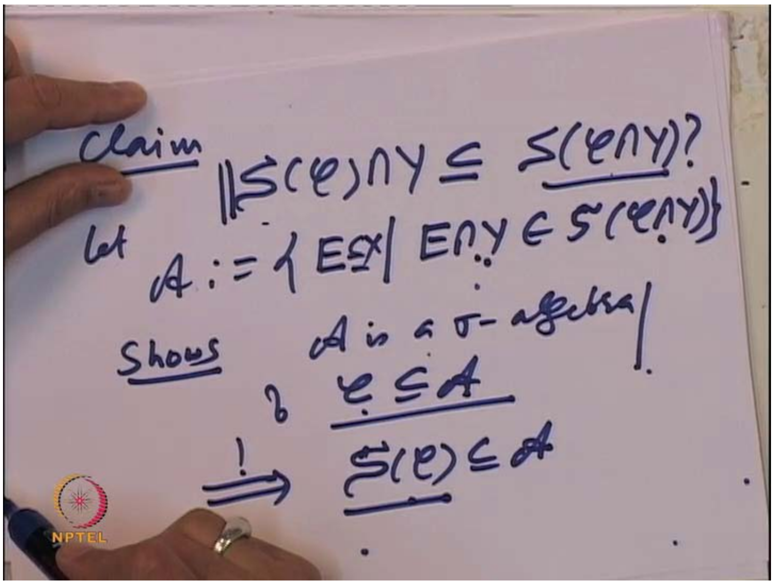

Let us try to prove the other way round inequality; namely, the claim that S of C intersection Y is also a subset of S of C intersection Y; this is what is to be proved. The proof once again uses a similar technique. Let us write A; look at all those subsets E such that E intersection Y is an element in S of C intersection Y. Look at all those subsets in X such that their intersection in this Y is inside this sigma algebra.

One shows that A is a sigma algebra and C is inside A. That means what? This will imply that because C is inside A, it is a sigma algebra; that will imply that S of C is inside A; that means for all elements in S of C if I take as intersection, that is going to be inside it; that will prove this required inequality (Refer Slide Time: 52:42). This claim is equivalent to proving these two things; namely, A is a sigma algebra and C is inside A. Let us observe why it is a sigma algebra. That is once again by similar properties. Let us show that this is a sigma algebra; let us try to show that E is a sigma algebra.

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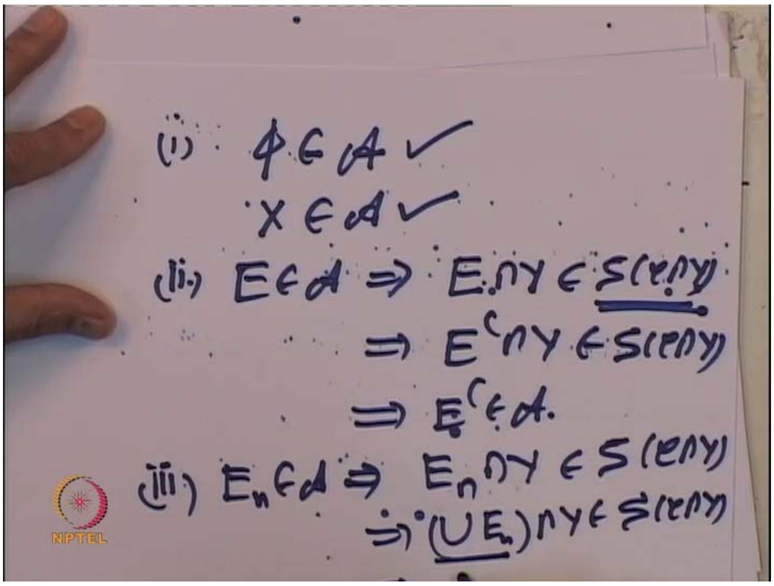

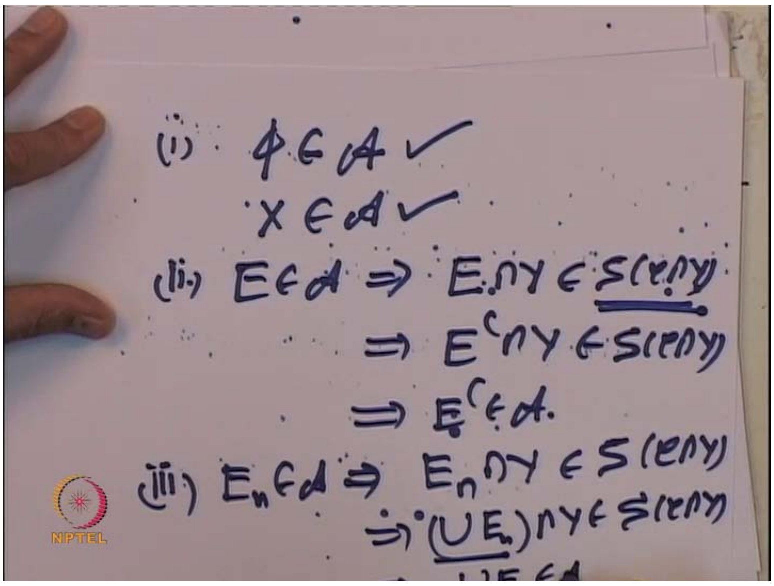

(i): does empty set belong to A? Yes, because empty set intersection Y is empty set which belongs to this because this is a sigma algebra (Refer Slide Time: 53:23); so, that is okay (Refer Slide Time: 53:25). Similarly, the whole space belongs to A; that will be okay. Second property: let us take a set E which belongs to A. What does that imply? E intersection Y belongs to S of C intersection Y, but this is a sigma algebra (Refer Slide Time: 53:244); so, it must be closed under complements and complements in Y.

That means E complement intersection Y also belongs to S of C intersection Y. That implies that E complement belongs to A; the collection A is closed under complements. Similar arguments will show that it is closed under unions also; so, E_ns belonging to A will imply E_n intersection Y belongs to this sigma algebra and hence implies union of E_ns intersection Y also belongs to the sigma algebra. Hence, this set union E_n belongs to A (Refer Slide Time: 54:31).

This proves clearly that A is a sigma algebra (Refer Slide Time: 54:35). Clearly, C is inside A because E intersection Y in that case will belong to this. This will prove the required fact that we were trying to show (Refer Slide Time: 54:53); if I take a set Y inside X and restrict the collection C to subsets of Y and generate the sigma algebra, that sigma algebra is the same as first generating the sigma algebra and then restricting it to Y.

This is another way of generating more sigma algebras out of a given sigma algebra for a given collection. This is a kind of technique which you are going to use later on in our subject. For example, X will be the real line; we will look at Y; it will be an interval. We will look at the open sets in the whole space of real line and generate the sigma algebra; that is a Borel sigma algebra of subsets of real line; then, we can restrict them to the interval; it is the same as looking at the open sets in the interval and generating the sigma algebra out of it.

There are various ways of generating sigma algebras. What we have done today is the following. We started with looking at algebras and described a special property of algebras. We said that in an algebra, any countable union can be represented as a countable disjoint union – a very important aspect of an algebra. Then, we moved on to looking at restricting the algebra to a set. We said if E is a subset of the set X and C is a collection, then you can restrict the class C to E and generate the algebra out of it.

That is the same as first generating the algebra and then restricting ((.)) to it; that is another way of restriction of algebras. Then, we defined what is called sigma algebra subsets of it. It is a collection of sets which is closed under complements; it includes empty set and the whole space and is closed under countable unions, unlike algebra – algebra is closed under complements and finite unions only; every sigma algebra is also an algebra.

We gave examples of a collection of subsets of a set X which is an algebra, but is not a sigma algebra; so, sigma algebra is something stronger than being an algebra. Then, we finally looked at how we can generate a sigma algebra out of a given collection of sets and that is called a sigma algebra generated. The process is similar to the algebra generated: take the intersection of all sigma algebras that include that collection and that will be the sigma algebra generated by it.

The sigma algebra generated by the open subsets of a topological space is an important sigma algebra called the sigma algebra ((.)) Borel subsets of that topological space. That is same as the sigma algebra generated by all closed subsets of the topological space; that is an important thing. Finally, we proved that you can take any collection restricted to a set and generate the sigma algebra; that is same as generating a sigma algebra first and then restricting it to the set. Next time in the next lecture, we will continue the study of classes of subsets of the set X and we will look at another important class called the monotone class of subsets of a set X. Thank you.