Measure and Integration

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Module No. # 07

Lecture No. # 29

Fubini's Theorems

Welcome to lecture 29 on measure and integration. In the past few lectures, we had been looking at measure and integration of product spaces. Let us just continue doing that in this lecture also. We will be studying, what are called Fubini's theorems. We had proved some versions of Fubini's theorems in the previous lectures, so let us recall what are those theorems that we have proved.

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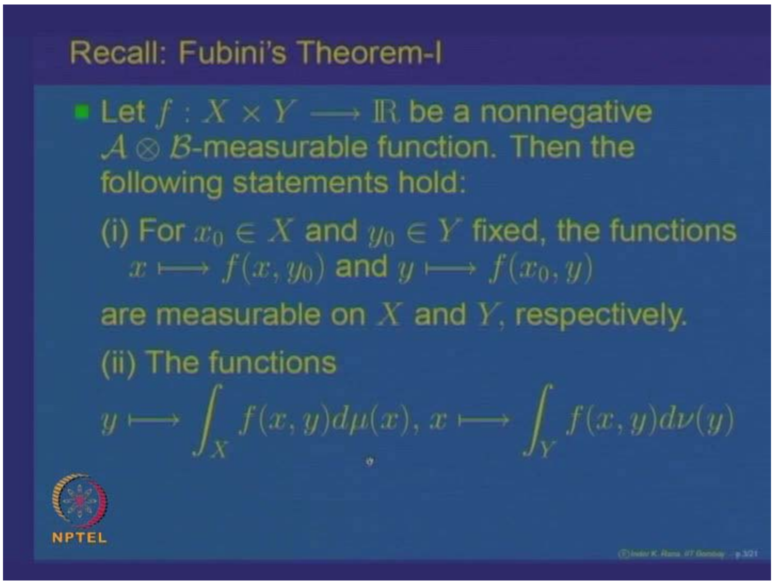

In that we first proved what is called Fubini's theorem I, which said that: suppose f is a function defined on the product space X cross Y taking values in the real line and suppose this function f is non-negative and it is measurable with respect to the product sigma algebra A times B.

So, f is a non-negative measurable function then we claim the following statements hold namely, if you fix one of the variables for this functions say x naught belonging to X or fix the other variable y naught belonging to Y, then with respect to the other variable this functions becomes non-negative measurable with respect to the corresponding sigma algebras.

For example, if x 0 in X is fixed then, the function y going to f of x 0 y is a function of the variable y on the space y, so it become measurable with respect to the sigma algebra B. Similarly, for every fixed x 0 the function for every fix y 0 in Y, the function x going to f of x y 0 is a measurable function on x with respect to the sigma algebra A. For every one of the variables fixed in the other variable, it becomes a non-negative measurable function.

Once, it is non-negative measurable you can integrate it out, look at the integral of f x y d mu x the variable y is fixed, so we are integrating with respect to X the integral depends on y. This gives us a function y going to integral of X of f x y with respect to the variable x. Similarly, if you integrate out the other variable y, then you get a function x namely x going to integral over Y of f x y d mu y. The second claim is that these functions are again non-negative measurable functions of Y and X respectively.

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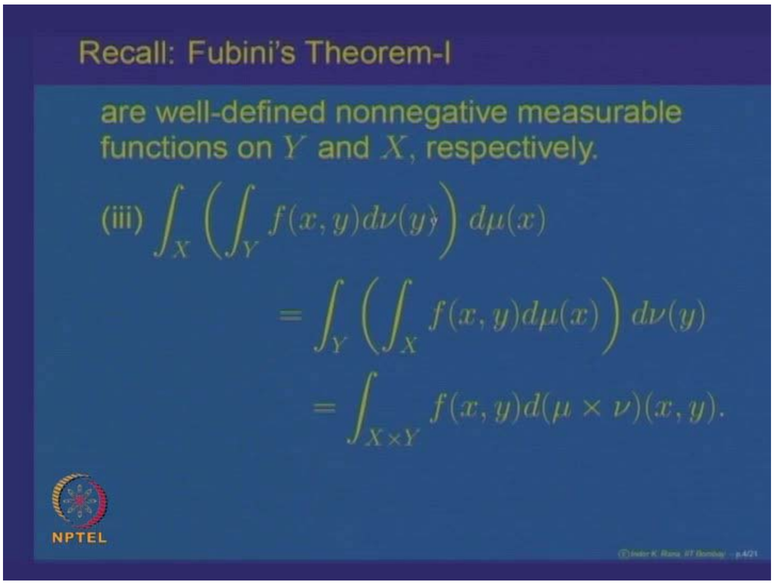

Once these are non-negative measurable functions you can integrate out the other variable now. So, integrate with respect to the variable x, you get one of the iterated integrals namely, integral with respect to X and then integral with respect to Y of f x y d nu y is equal to the other iterated integral namely, first integrate with respect to X and then with respect to Y; these two iterated integrals are equal and the claim is this is equal to the integral of the given non-negative function f x y with respect to the product sigma algebra.

As we had stress that importance of this theorem lies in the fact that to integrate a function of two variables f x y, we can fix either of the variables, first integrate it out and integrate out the other variable. So, you can do integration with respect to one variable at a time. This was called Fubini's theorem I for functions, which are non-negative measurable f x y.

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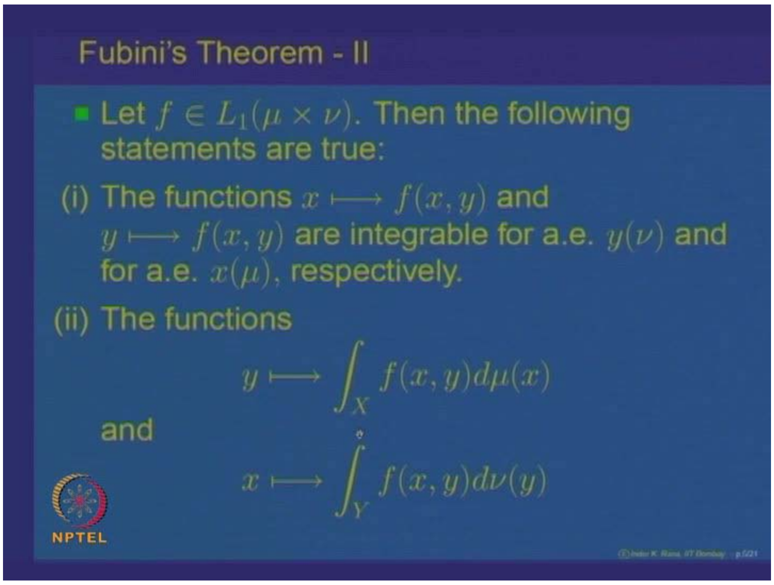

Then we extended this theorem to functions which are integrable. So that we called as Fubini's theorem II and stated that let f be an integrable function on the product sigma algebra product space.

So, f is L 1 of mu cross nu then the following statements hold namely, if you fix one of the variables say x or y then as a function of the other variable these functions are integrable, but not for almost all variable fix for almost all points, whichever are fixed. For example, if you are fixing y then it says for almost all y the function x going to f of x y is an integrable function.

Similarly, for almost all x fix y going to f of x y is a integrable function. When you fix one variable at a time for almost all such fixing the function of other variable are integrable, so you can integrate out. You get the function y going to integral over X of f x y d mu x. Similarly, the other function is x going to the integral over Y with respect to y of f x y d nu y.

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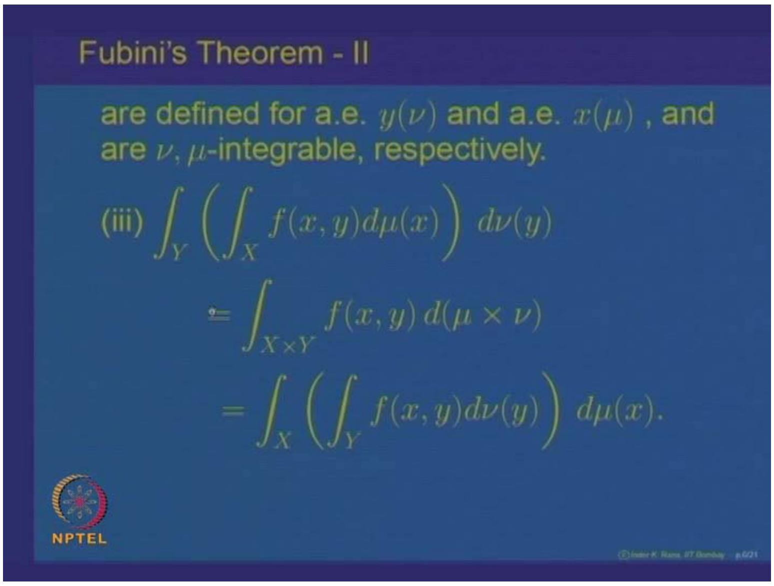

The claim is that these two functions are again integrable with respect to the corresponding measures nu and mu, so you can integrate them out. What you get is that the iterated integrals once again, in this case are equal to the integral of the function with respect to the product measure.

Basically, what we are saying is that the two iterated integrals are equal to the integral of the function with respect to the product measure, whenever f is integrable function. These two theorems we had proved and we want to give a one more version of this theorem, so to do that we need to prove a proposition about integrable function on the product space.

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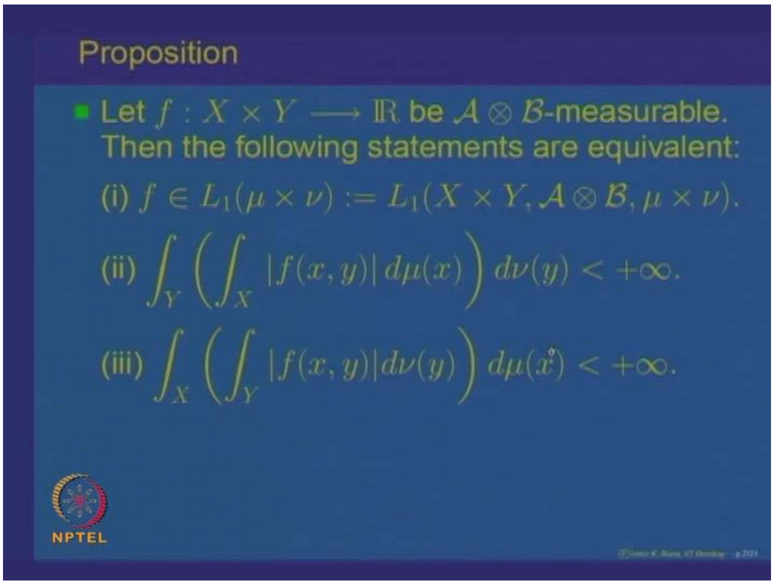

Let us take a function f on the product space f X Y on the product space A cross B, then the claim is that the following statements are equivalent, for the function is L 1 as a function of two variables the function is integrable, so that is f belongs to L 1. Secondly, if you look at the absolute value of the function f and look at its iterated integral with respect to X or Y.

Look at the absolute value of the function f that is a non-negative measurable function, look at its iterated integral first integral with respect to X and then with respect to Y that is finite, that is the second condition.

The third is you can interchange this, so look at the other iterated integral that first integrate with respect to Y and then with respect to X. Then, this is a second iterated integral that is finite and fourthly, these three conditions we want to show that these three conditions are equivalent.

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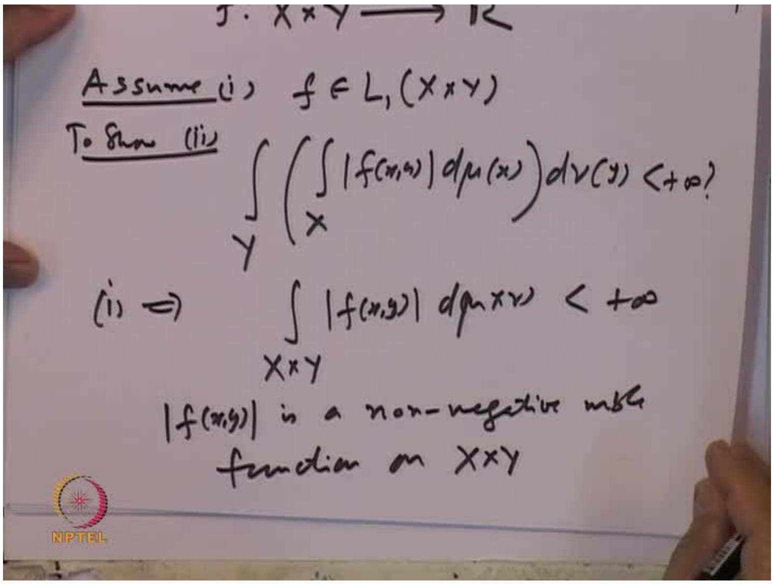

Let us prove them; Let us first assume f is a function given on the product space X cross X to R. Let us assume, 1 that is namely saying that f belongs to L 1 of X cross Y; we want to show 2 namely that the iterated integral of mod f x y d mu of x and then integrate that with respect to Y, so that is d nu y is finite, this is what we want to show.

Since, f is integrable; the condition that f is integrable so 1 implies that integral of the function mod of f x y with respect to the product sigma algebra, product measure mu cross nu is finite over the product space. Now, let us note that mod f x y is a non-negative measurable function on X cross Y. So, it is non-negative measurable functions on X cross Y, Fubini’s theorem 1 is applicable.

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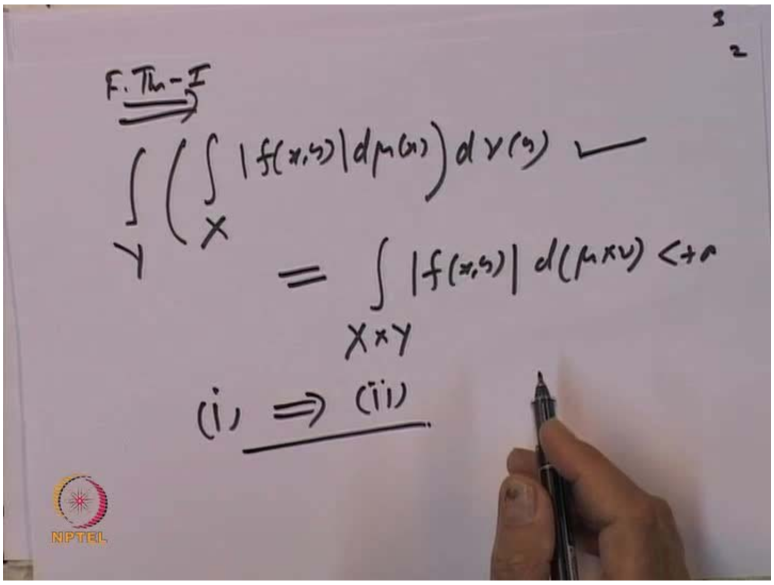

By Fubini’s theorem 1, we get implies by Fubini’s theorem 1, which was for non-negative, so for Fubini’s theorem 1 for non-negative functions, what we get is that the iterated integral of - so mod f x y the iterated integral with respect to X and then with respect to Y d mu x d nu y must be equal to the integral over the product space X cross Y of mod f x y d of mu cross nu, which is given to be finite.

So that implies this iterated integral is finite by the second condition. What we have shown is hence 1 implies 2.

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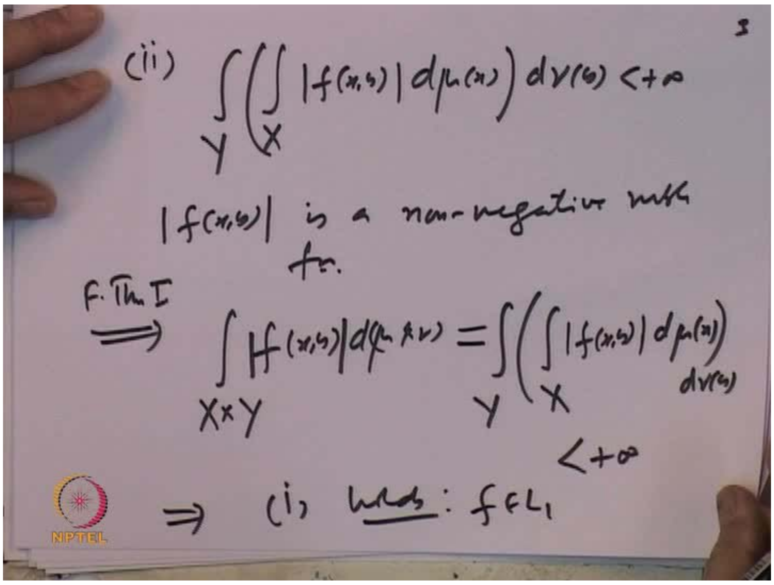

Let us try to show that 2 imply 1. What is 2? The condition 2 says that integral over X of mod f x y d mu x that integrated with respect to Y d nu y is finite; that is given to be finite. Once again, we observe that mod of f x y is a non-negative measurable function. Implies by Fubini's theorem 1 that the double integral, now we can revert that so double integral that means integral of the product space of f x y d mu cross nu of absolute value of f x y must be equal to the iterated integral.

That is integral over Y integral over X of mod f x y d of d mu x and d nu of y and that is given to be finite by 2. So, implies that 1 holds namely, integral of mod f with respect to the product measure is finite that is that f belongs to L 1. What we have shown is that the condition 1 is equivalent to condition 2 and a similar proof implies that condition 2 is also equivalent to condition 1.

Saying that the function is integrable is equivalent to saying that the either of the iterated integrals of mod f are finite, so all these three are equivalent conditions. These can be put into the statement of Fubini's theorem. Combining Fubini's theorem 1 2 and then this proposition gives us what is called as the combined Fubini's theorem.

Basically, combined Fubini's theorem gives you conditions under which you can say that the iterated integrals of a function of two variables are equal and equal to the integral of the function over the product measure space.

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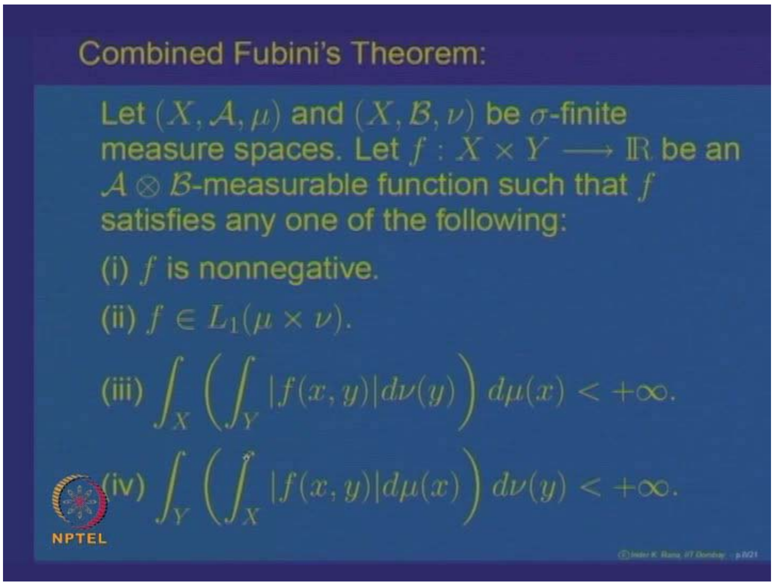

Let us look at the combined Fubini's theorem, which says let X, A, mu and Y, B, nu be sigma finite measure spaces. We have been working under sigma finiteness, because the product measure is defined only for sigma finite measures. If two sigma finite measure spaces are given and you are given a function f on the product set X cross Y, which is measurable with respect to the product sigma algebra A times B.

Then anyone of the following conditions namely f is non-negative and secondly, f is integrable. Third condition that the iterated integral of mod f with respect to y and then with respect to x is finite or the other one, the iterated integral of mod f with respect to mu and then with respect to nu is finite.

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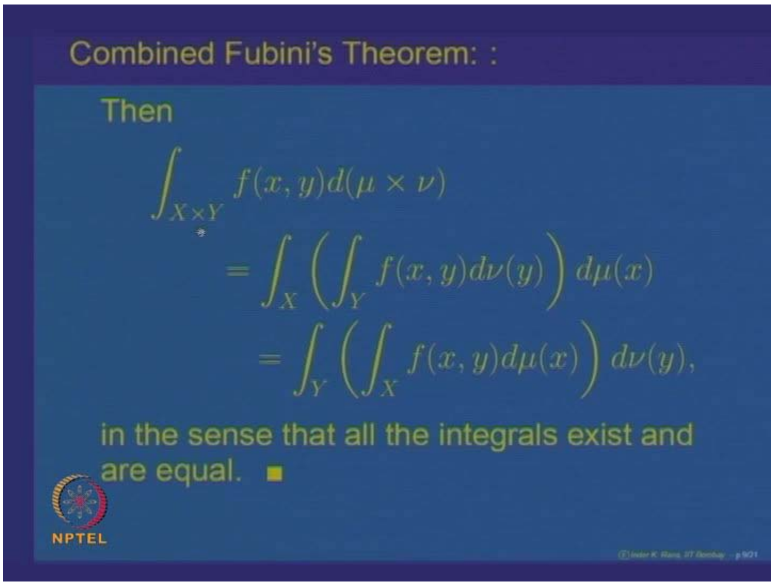

So, if any one of these four condition is satisfied then we can say that the integral of the function f over X cross Y with respect to the product sigma algebra is equal to both of the iterated integrals namely, either integrating with respect to Y first and then with respect to X or integrating with respect to X first and then with respect to Y. Basically, these three integrals are equal in the sense that if either of them exist then all the three will exist and are equal.

Basically, Fubini's theorem says that the two iterated integrals are equal and equal to the integral with respect to the product measure, whenever either of these things either of these integrals are define; for example, if f is non-negative then these are all defined and hence there should be equal by Fubini's theorem 1.

If f is L 1 then the integral of f x y over X cross Y that exist, all three must be equal and other iterated integrals of mod f they are equivalent to say f is L 1, so they all will be equal. This is called combined Fubini's theorem and it is off importance, we will see lot of applications of this soon.

Let us look at some examples, so I want to stress this point namely that in the statement of the theorem, we have assumed the condition that mu and nu are sigma finite measures. These conditions are going to be important, so let us look at some example to illustrate this.

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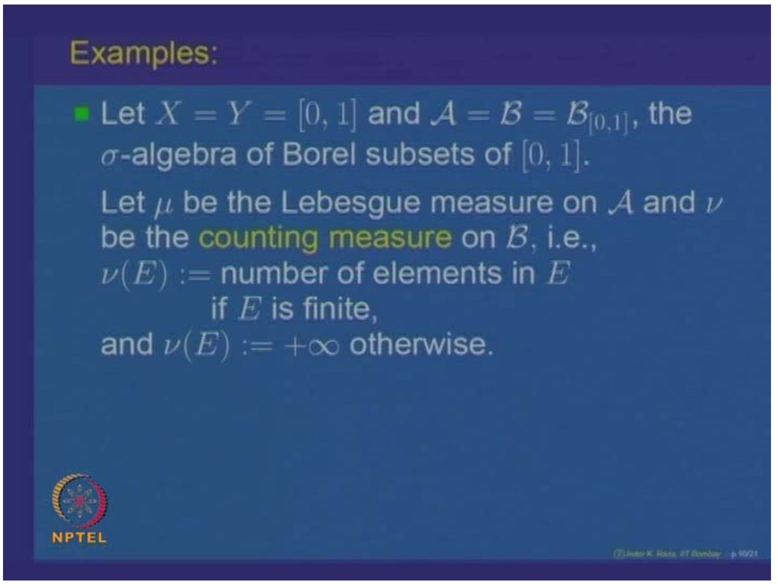

Let us look at the space when X is equal to Y is equal to 0 1, so my under lying space is X is same as Y as the interval 0 1. The two sigma algebras: sigma algebra A is same as the sigma algebra B is equal to Borel sigma algebra; the sigma algebra Borel subsets of 0 1.

Let us define the measure mu to be the Lebesgue measure on A and the measure nu on when treat it as Y 0 1 as treat it as other space, we defined it to be the counting measure. What is the counting measure? Counting measure is the number of elements in a set E, if the set E is finite; otherwise it is equal to plus infinity.

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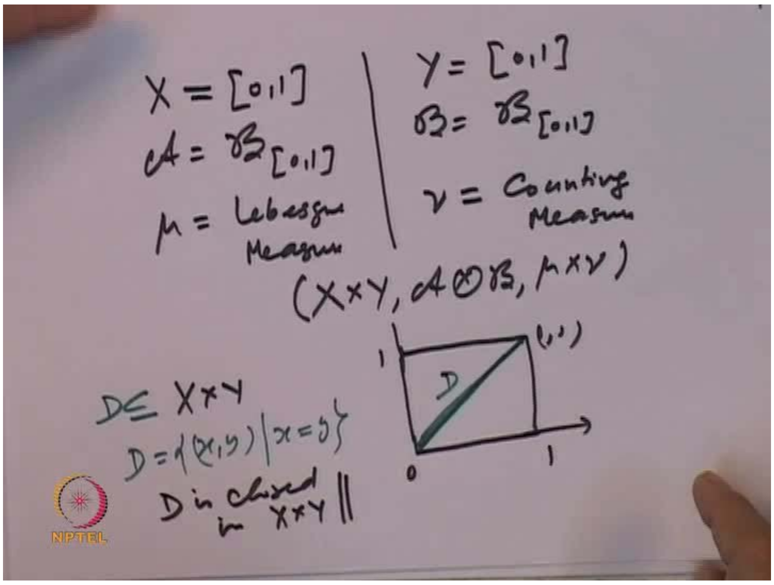

We have got two measure spaces, let us look at the example that we have gotten, so we have got X which is equal to 0 1. The sigma algebra A is Borel sigma algebra on 0 1 and mu is Lebesgue measure. On the other side, we have got Y which is again 0 1 and the sigma algebra B is again the sigma algebra of Borel subsets of 0 1 and we have got nu which is equal to the counting measure.

Counting measure as defined, if the set E is finite then the number of elements other it is equal to plus infinity. Now, let us look at the product space X cross Y, A times B and mu cross nu, so let us look at the product space.

Let us look at the set; so we are going to look at the set D which is the set X is equal to Y. So, note that X cross Y this is a subset of X cross Y, the claim is that this is a close subset of X cross Y and hence is an element in A cross B. What is our space X cross Y? X cross Y is, if you picturize it is just 0 1 cross 0 1, so that is our space; this is 1 1 (Refer Slide Time: 18:08).

What is D? We are looking at the set D, which is contained in this, so that D is equal to x comma y x is equal to y, so that is this line is my D. So, the claim is that this D is a Borel set, one way of looking at it is that D is closed in X cross Y.

What is the meaning of that this set is close? One way of showing that this is a close subset of 0 1 is to show that it contains all its limit point that is one way of showing it.

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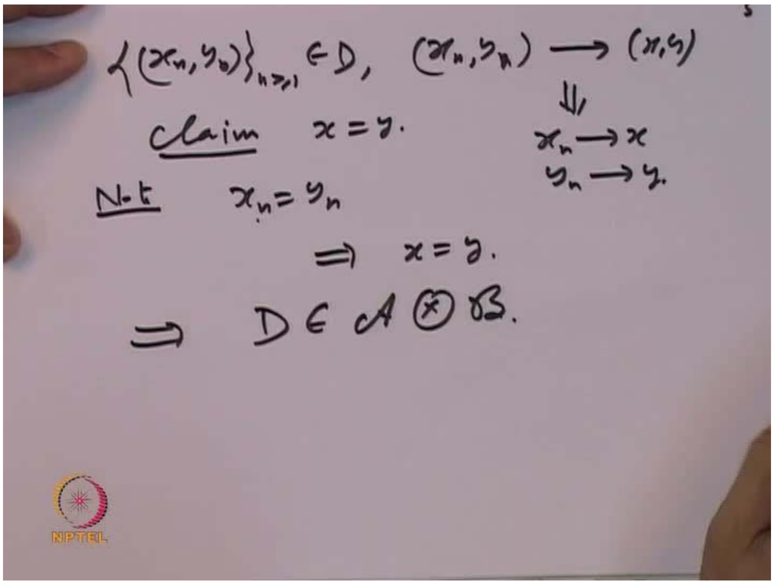

If we take a sequence x n comma y n belonging to D and x n comma y n converges to x y then claim, you should be able to show that x is equal to y. Let us note, x n y n belonging to D that implies; so note that x n is equal to y n because x n comma y n belongs to D is a diagonal and x n y n converging to x y, so this condition implies that x n converges to x and y n converges to y.

So, x n is equal to y n; x n converges to x, y n converges to y, so all that will imply that x is equal to y. The set D is a close subset of 0 1, implying that D belongs to the product sigma algebra A times B, so the D is close subset of 0 1.

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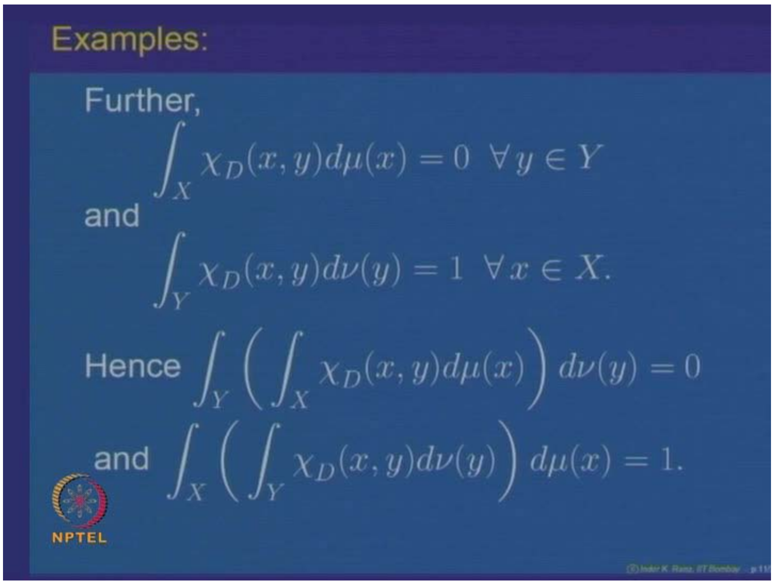

Next, we want to compute the iterated integrals of the indicator function of the set D of this diagonal with respect to mu and the claim is that if I fix one of the variable say y and integrate the indicator function of D with respect to X then, this is equal to 0 for every Y.

The other integral when you fix x integrated with respect to y the counting measure that is equal to 1 for every x. So that will imply that the two iterated integrals are not equal, one of them is equal to 0 and the other one is equal to 1, let us verify this first.

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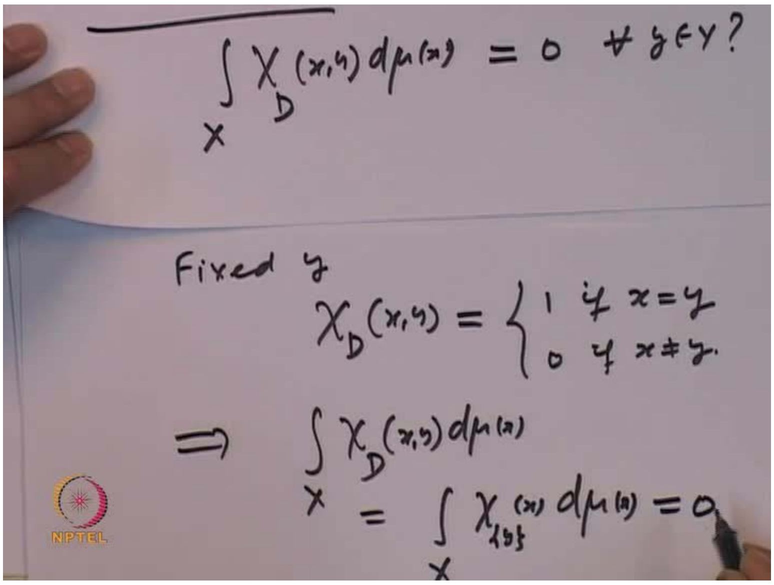

Let us verify the condition that if I take the indicator function of the set D x y and integrate with respect to mu over x. The first claim is that this is equal to 0 for every y belonging to Y. Let us see, why is this integral equal to 0? To show that let us observe, for fix y we want to compute chi the indicator function D at x y.

What is that equal to? So this is equal to 1 if x comma y belongs to D that means x is equal to y and it is 0 if x is not equal to y. This function, indicator function of the diagonal for y fix takes only two values 1 and 0 and only at one point, it takes a value y it takes a value 1, when x is equal to y at all other points it is just 0. So, it is an indicator function of a singleton set.

Hence, this implies that integral over X chi D x y d mu x is equal to the indicator function of the singleton set namely y x d mu x and that is single point and measure the mu is the Lebesgue measure, so this is equal to 0. That says a simple observation that the indicator function of the diagonal x y is equal to 1, if x y is fixed, x is equal to y otherwise it is 0, so that proves that this is equal to 0 (Refer Slide Time: 23:00).

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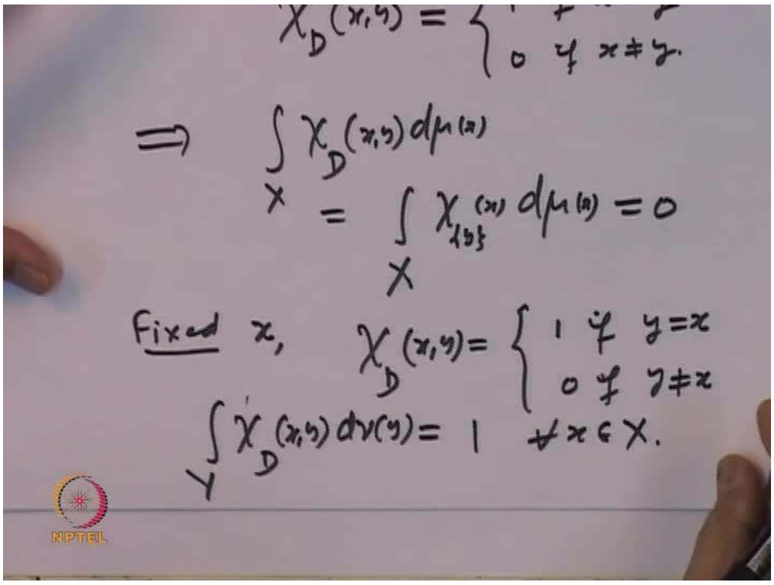

Similarly, let us compute the other one for fix x, let us look at the indicator function of x y equal to - what is that equal to? So that is again equal to 1 if y is equal to x and 0 if y is not equal to x. That is again the same function but now, let us observe that if I integrate with respect to Y of the indicator function of D x y with respect to D nu y and nu is the counting measure, so for one point when y is equal to x the value is 1, the value is one for 1 point and the measure of the single point is equal to 1, so that is equal to 1 for every x belonging to X.

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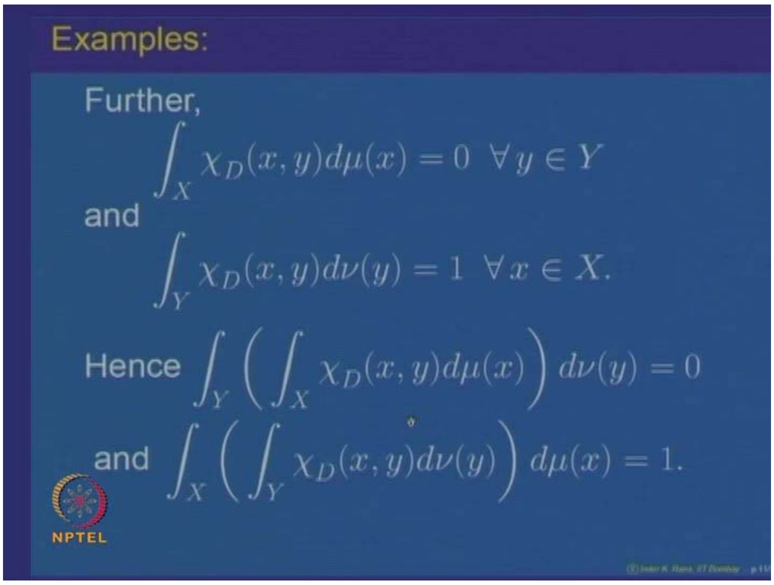

That proves the required claim namely, the integral of D; indicator function of D for every y fix is 0 and for every x fix is 1. One of the iterated integral is equal to 0, while the other iterated integral is equal to 1. So that seems to contradict Fubini's theorem because the indicator function is non-negative function that the two iterated integrals are not equal.

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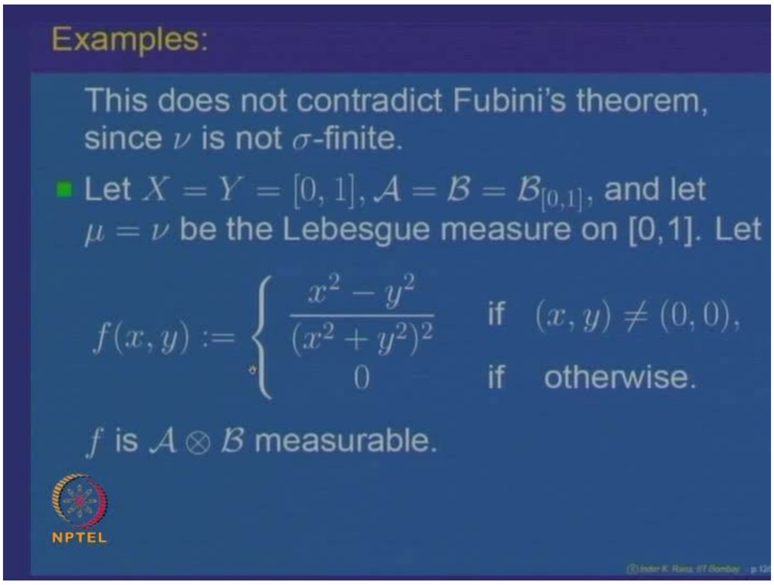

But that is not the case, because the measures involved not both of them are sigma, Lebesgue measure is sigma finite but the counting measure is not sigma finite, so this does not contradict Fubini's theorem. Since, the counting measure is not sigma finite; the counting measure of the whole interval 0 1 is infinite and you cannot divided it into countable number of pieces each having finite because 0 1 is uncountable. The counting measure is not sigma finite, so that is why this does not contradict Fubini's theorem.

Let us look at another example; let us look at two sets once again, the underlying space is same X is equal to Y is equal to 0 1 and A the sigma algebra on both of them is equal to the Borel sigma algebra. Now, let us look at measures mu equal to nu to be the Lebesgue measure. Basically, what we are doing is, we are taking 0 1 the Borel sigma algebra and the Lebesgue measure and take a copy of it and take the product of that.

Let us define a function of two variables f x y to be equal to x square minus y square divided by x square plus y square to the power 2, if x and y is not equal to 0 0 and it is equal to 0; if it is 0 otherwise. First of all, we want to claim that this function f of x y is the measurable function on the product sigma algebra.

For that one has to look at the basic properties of functions of two variables. One can show that this function is continues everywhere except at 0, it is the almost everywhere continues function of two variables. Hence, it is going to be measurable function with respect to the product sigma algebra that is the Borel sigma algebra on the square 0 1 cross 0 1.

Saying that f is a measurable function requires a proof and the proof - I am giving you the hint; the hint as follows, look at this function; this function as a function of two variables, let me just write down the steps so that you are able to verify yourself later on.

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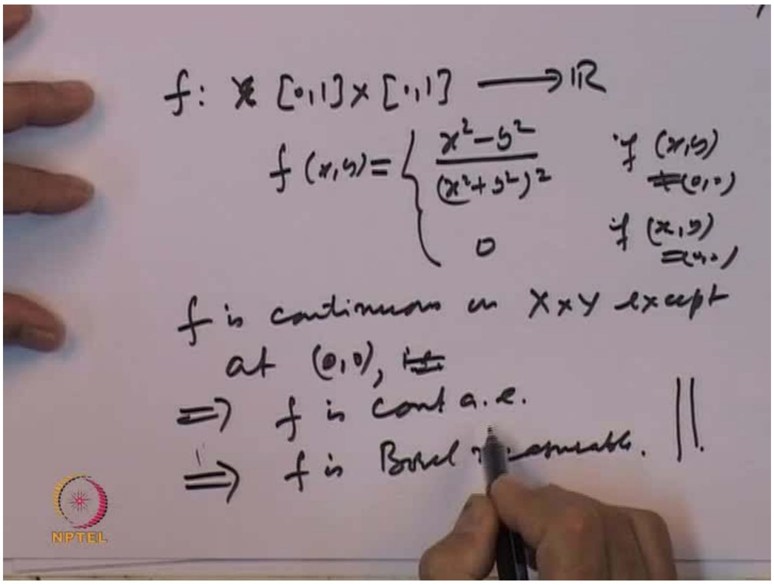

Let us look at the function f on X cross Y, so that is 0 1 cross 0 1 to R, this is a function which we are saying is f of x y is equal to x square minus y square divided by x square plus y square square, if x y not equal to 0 0 and it is equal to 0, if x y is equal to 0 0. The claim is this function f is continues on X cross Y that is 0 1 cross 0 1 except at the point 0 0.

So at 0 0, if you can we can see that if I take y to be equal to 0 that is x square divided x to the power 4, so it looks like 1 over x square and x approaches 0 it is going to blow up. It is not continues at the point 0 0 but that does not matter, so that means f is continues almost everywhere because one point does not matter. So that implies f is continues almost everywhere and this continues almost everywhere implies f is Borel measurable, because continues means inverse images of open sets are open and hence there will be Borel sets.

To show that inverse images of Borel set is Borel, you apply that sigma algebra technique, so this we have shown that every continuous function is continues almost everywhere is the Borel measurable. Using A in this step, you will be using the sigma algebra technique, so look at all sets for inverse images are Borel open sets are inside and so on. This will prove that this is a measurable function.

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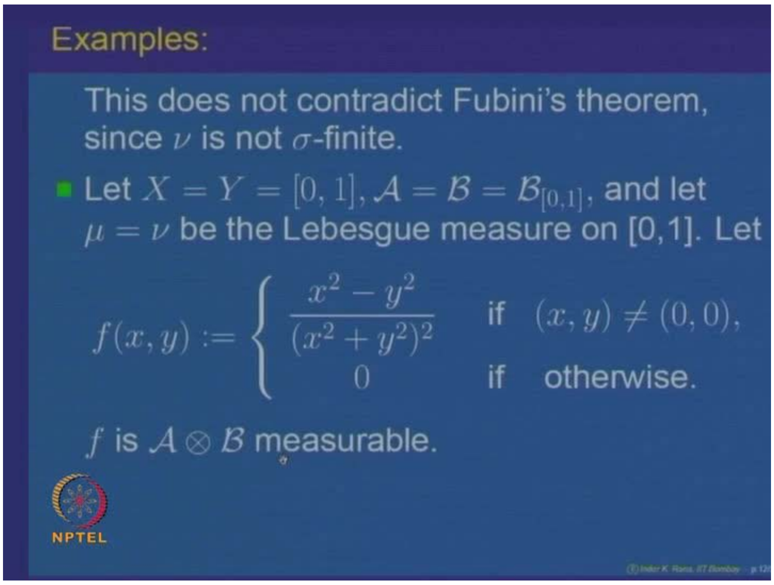

Now, let us compute the iterated integrals of this function with respect to the measures separately.

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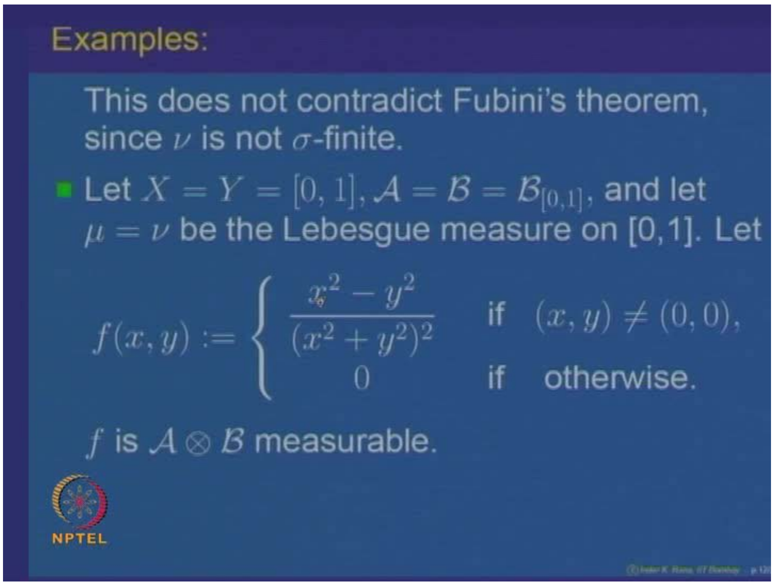

Let us first observe that for every fixed x; let us look at for every fixed x, if I fix x at any point then this is a function which looks like minus y square divided by x square plus y square over whole square. So that is function of y for every x fix continuous almost everywhere except at the point 0, it is going to be a Riemann integrable function.

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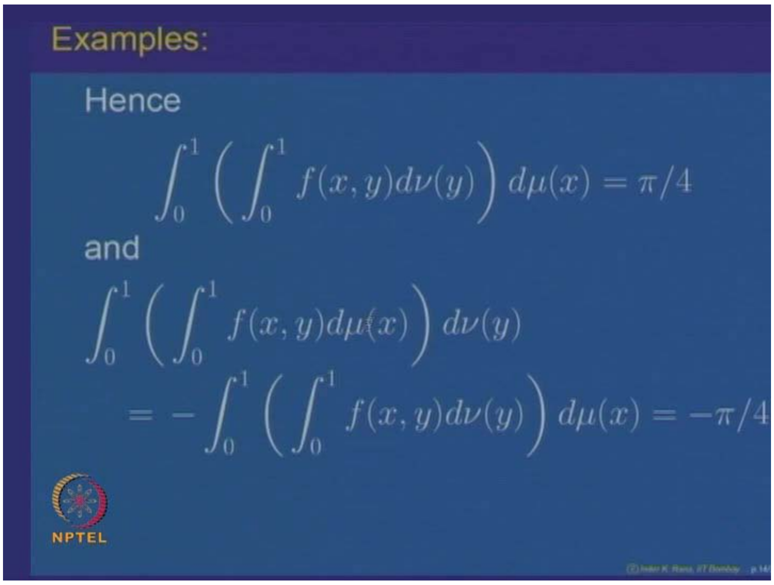

It is a function for every fix x it is function which is Riemann integrable function of y and if you look at the integrant it is the derivative of the function y divided by x square plus y square. So, partial derivative of the function y divided by x square plus y square is equal to f x y. I know the anti derivative of this function for every fix x, so I can integrate it out with respect to the variable y.

When I integrate f x y with respect to the variable y, because it is the Lebesgue measure, we are integrating and for Riemann integrable function Lebesgue integral is same as the Riemann integral. So that will give me that the integral 0 to 1 of f x y d nu y is equal to 1 over because y is 1, so that will give me integral of this is equal to 1 over 1 plus x square.

Now, that we want to integrate with respect to x; so integrate this with respect to x and we know how to integrate this function 1 over 1 plus x square by substitutions putting technomatric substitutions. I leave it for you to verify that the integral of 1 over 1 plus x square is between 0 to 1 is equal to pi by 4, one of the iterated integrals is equal to pi by 4.

But a simple observation in the function tells me that if I change x to y - let us look at the function - if I change x to y, interchange x and y then you get a negative sign outside because there is a negatives x square minus y square, so I do not have to compute the other iterated integral by using this property that if I interchange x and y that gives me a negative sign.

If you want to integrate first with respect to x and then with respect to y answer will be same as the earlier one with the negative sign, so the other integrated integral is going to be minus pi by 4. For this function of two variables we have got two iterated integrals: one of them equal to pi by 4 and the other is minus pi by 4. Both the measures here are sigma finite, so the question is what is going wrong? We have got two sigma finite measure spaces actually the same measure space 0 1 Borel sigma algebra and Lebesgue measure.

On this we have got a function f x y equal to x square minus y square divided by x square plus x square were the point is not 0 0 and this function has got iterated integrals which are different. So this does not contradict Fubini's theorem, the answer is no, this is simply because iterated integrals are different because the function is not integrable on the product space, so we cannot apply Fubini's theorem 2 to it.

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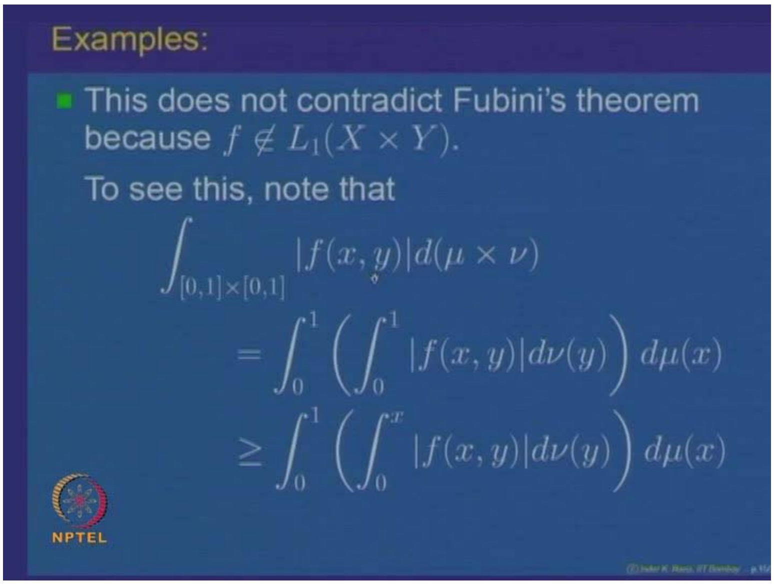

Let us verify that with respect to the product sigma algebra the function is not integrable and that is a very simple observation, so let us look at that. Look at the function for two variables; so let us look at mod of f x y, so this mod of f x y with respect to two variable these are non-negative function.

So, by Fubini's theorem 1 we know that this is equal to the iterated integral of mod f x y with respect to say variable y, because the integrant is non-negative and the inner integral the x is between 0 and 1, so we can write the iterated integral by 0 to x, so it will become bigger than or equal to. So the integral of absolute value of f x y with respect to the product measure is bigger than or equal to integral 0 to 1, integral 0 to x of mod f x y d nu y.

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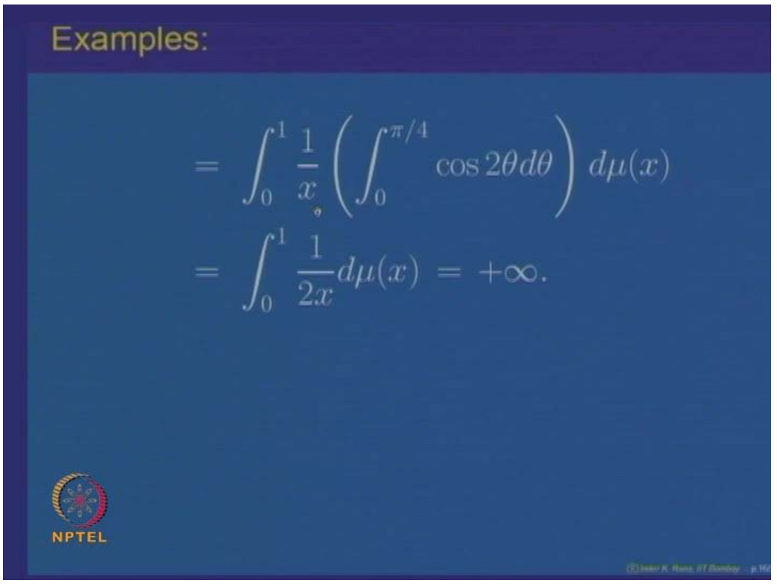

Mod of that is nothing but 1 over x and let us just look at mod of 0 to x here, x is fixed with respect to y. Once you compute that inner integral, so that is nothing but 1 over x 0 to pi by 2 cos 2 theta and those and that can be computed and that comes out to be 1 over 2 so it is 1 over 2 x d mu x, which is equal to plus infinity because 1 over x is not integrable. The simple computation shows that this function is not integrable.

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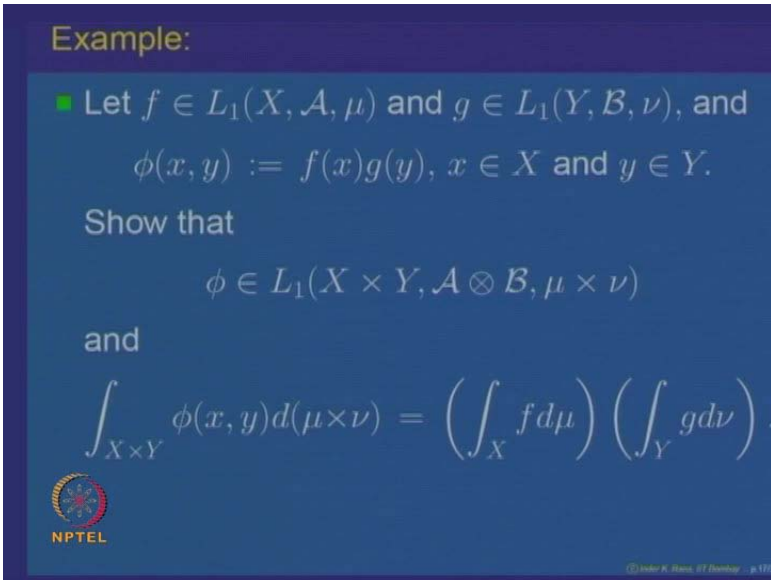

As a consequence again, the Fubini's theorem is not contradicted because the two iterated integrals are not equal because the function is not integrable and let us look at an application of Fubini's theorem, we want to prove that if f is a integrable function on X, A, mu and g is another integrable function on Y, B, nu then look at the product of the two functions f x and g y. So phi x y is equal to f x into g y. The claim is that this function is integrable and its integral is equal to the integral of f into integral of g.

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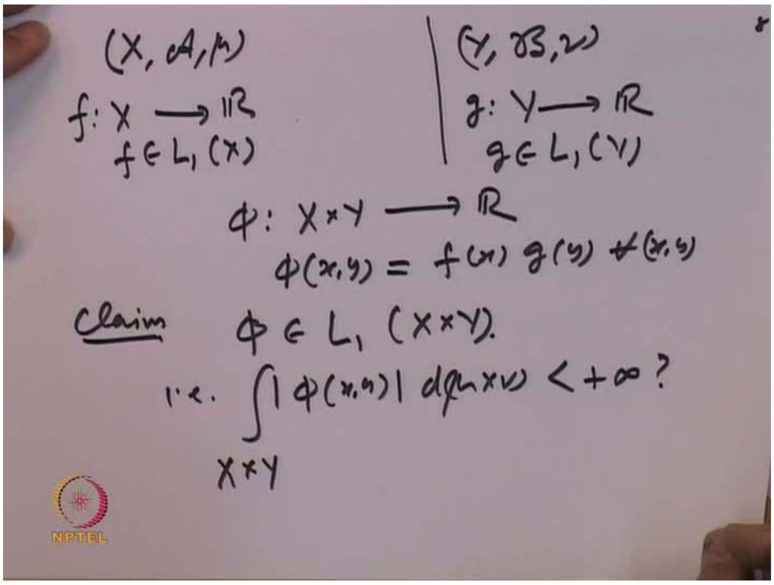

Let us show this as a simple application of Fubini's theorem, so we want to look at a function f, we have got a measure space X, A, mu of course, sigma finite and f is a function defined on X and f belongs to L 1 of X. On the other hand, for Y B mu we have got a function g Y to R and g belongs to L 1 of Y.

So, define phi; we are defining a function phi on X cross Y taking values in R and the function defined is phi of x y is equal to phi of x y is equal to f x g y for every x y. So, claim that phi belongs to L 1 of X cross Y, so that is what we want to show.

Let us see, how will the proof go? How the proof of this theorem goes? So phi belongs to L 1, so that is we want to show mod phi x y integral over X cross Y d mu cross nu is finite, so this is what we want to show.

To show that it is enough to show, let us observe to show this enough to show because of Fubini's theorem 1 or earlier, so to show this, so this is what we want to show.

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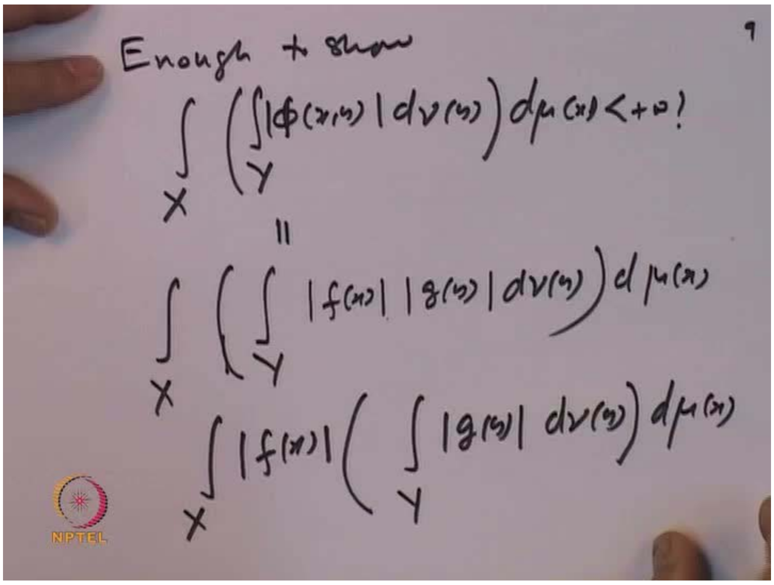

So, enough to show say for example, integral with respect to Y of phi x y integral with respect to X this is d nu of y d mu of x is finite, enough to prove this that this is finite. Let us compute what is this quantity? This quantity is equal to the inner one integral over Y phi x y is f x g y, so it is mod of f x mod of g y d nu of y integral over X of d mu x.

This we want to show is finite is equal to this (Refer Slide Time: 39:08). Now, this is independent of integral x is fix, so this is independent; it is integral over X mod of f x, inside is integral over Y of mod of g y d nu y d mu of x. Now, g is integrable, so this quantity is finite. What is that quantity equal to? Let us write this quantity equal to; this is a constant, it is finite so it comes out, you can write this quantity is equal to integral over X mod f x d mu x integral over Y mod g y d nu of y and that is finite because of both of them are finite.

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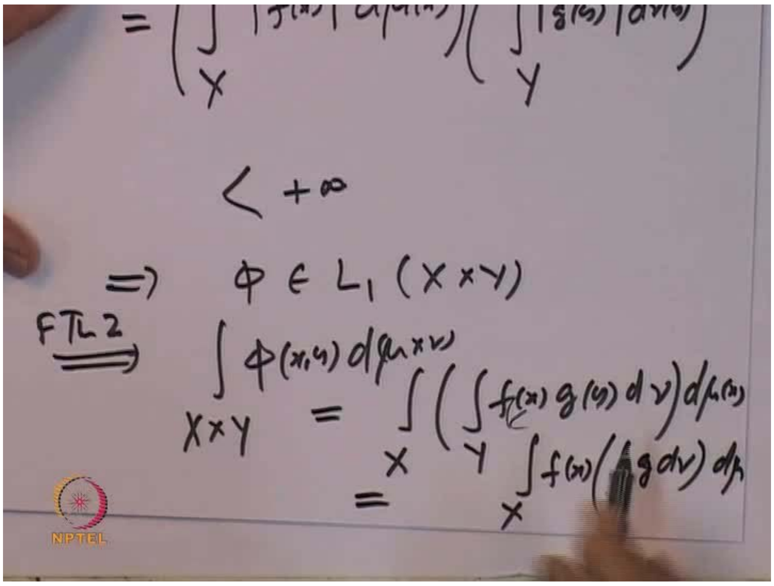

What we have shown is that integral of mod of f x y the iterated integral is finite, so that just now we observed that is equivalent to saying that the function mod f x y is less than finite.

This will imply that phi belongs to L 1; phi is L 1 function, let us just say this last thing that we proved implies that phi is L 1, because of product space l X cross Y, because it is in L 1 so Fubini's theorem is applicable, so implies by Fubini's theorem 2 that the integral of X cross Y of phi x y d mu cross nu is equal to the iterated integral either one we can write, so let us write it over X integral over Y of phi x y - what that is f x g y d nu y and d mu x and that has just now we have observed this independent of this integrant is independent of y, so this you can take it out.

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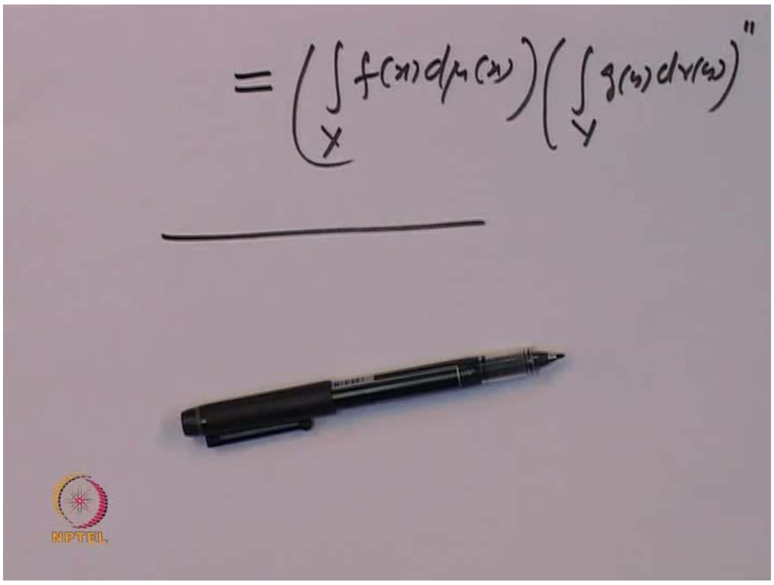

This is equal to integral over X f x integral over Y integral over g d nu d mu and this integral that we have written is precisely equal to integral over X of f x d mu x the first one and the second one is integral over Y g y d nu y. So that is how Fubin's theorem is applied.

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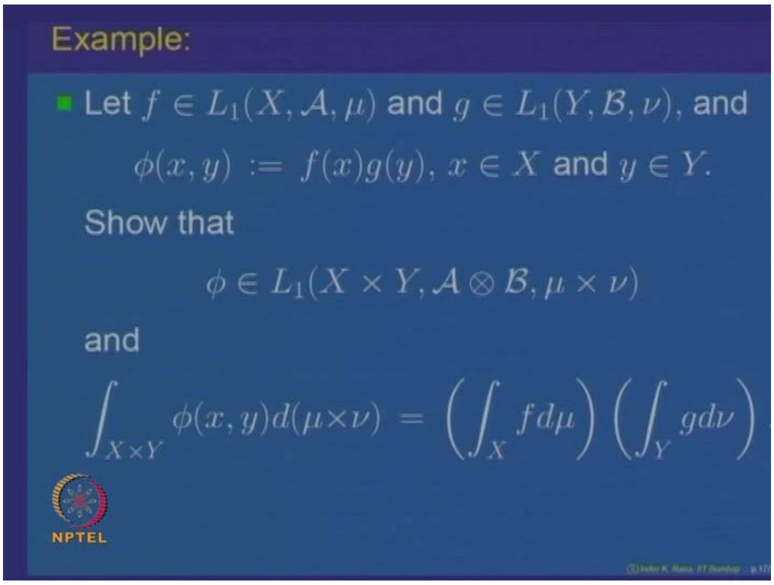

You have seen that two application of Fubini's theorem to prove this result namely, that if f is L 1 function then and g is a L 1 function on y then the product is a L 1 function on the product space and integral is equal to the product of the two.

So, this is how Fubini's theorems are going to be applied. Let me just recollect or revise what we have done till now given product measure X A mu and Y B nu, which are both sigma finite. We constructed the product sigma algebra A times B the product measure mu cross nu, so we got the product measure space X cross Y A times B and mu cross nu.

For this product measure, the first thing we did was how to compute the measure of a set in the product sigma algebra. We said you can go via sections, so that gave us the product measure mu cross nu of a set E is same as you look at the x section e x; look at the measure of that nu of e x and integrate with respect to x or similarly, do with respect to y. So that gave us ways of computing the product measure of a set in the product sigma algebra.

That result when interpreted as an integral, gave us the first Fubini's theorem namely, for non-negative measurable functions you can fix one variable at a time and integrate it out and then we extended this to functions of which are not necessarily non-negative but integrable. So, those gave us the Fubini's theorems.

In the next lecture, what we will do is, we will now specialize when x is the real line, y is the real line, A is the Borel sigma algebra, B is the Borel sigma algebra or Lebesgue sigma algebras and look at the product of the Lebesgue measure on x that is real line and product measure and Lebesgue measure on y is again real line.

We will look at the Lebesgue measure space on the real line and take its products with itself to come to a notion of a measure Lebesgue measure on the plane, which will extend the notion of area in the plane. So, we will continue this study in our next lecture, thank you.