Measure and Integration

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Module No. # 07

Lecture No. # 28

Integration on Product Spaces

Welcome to lecture 28 on measure and integration. If you recall, we had started looking at the computation of product measure of a set E in the product sigma algebra and we had shown this can be computed via sections of the set E – integrating the sections and taking the measures. Let us recall this result and then we will continue to generalize this result for functions which are nonnegative and integrable functions.

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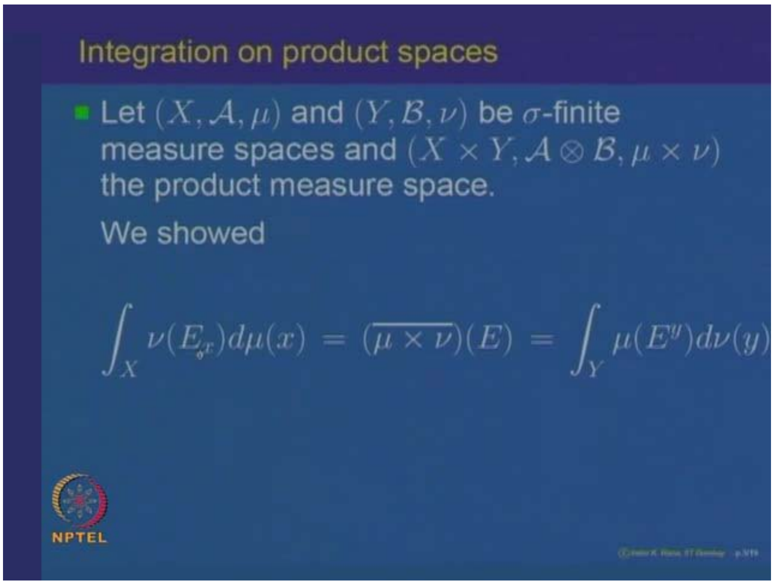

Let us recall the result that we had proved last time; namely, if X, A, mu and Y, B, mu are two sigma-finite measure spaces and the product sigma algebra a product space is X cross Y, A times B and mu cross nu is the product measure space, then we showed that for any set E in the product sigma algebra, the measure mu cross nu of E can be computed by either taking the section of the set E at a point x; that gives us a subset of the set Y and we showed that this is in the sigma algebra; you can take the measure of this section; this becomes the function of the variable x (Refer Slide Time: 01:39); then you can integrate out this function with respect to mu to get the product measure. Equivalently, you can take the y section of the set E; that gives a subset of x which is a measurable set; then you can take the mu measure of that; that gives a function of y; then integrate out that with respect to Y to get the measure of the set E. This result we want to reinterpret as follows.

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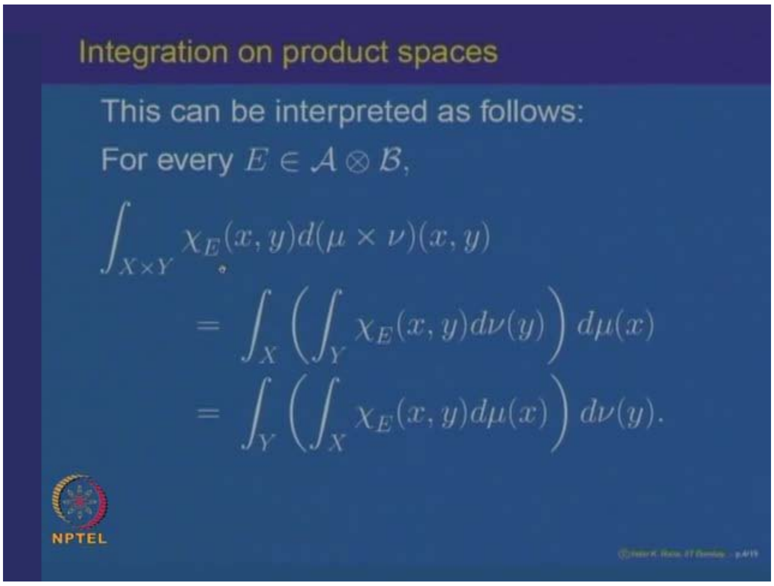

The measure of the set E_x can be written as the integral of the indicator function of the set E with respect to the product measure. So, mu cross nu of the set E is nothing but the integral of the indicator function of the set E on the one hand; on the other hand, if we look at the x section or the y section, they are nothing but the indicator functions of the set E again.

When you take the integration with respect to y, that means you are fixing x; you are looking at the section of E at x; so, nu of E_x is nothing but the integral over Y of the indictor function of E with respect to the variable y and similarly the other variable. As we had mentioned, the importance of this result lies in the fact that the indicator function of a set E is a function of two variables; to find its integral with respect to the product measure, what we can do is we can fix one of the variables, say, x; this becomes a function of one variable y and one shows that this is integrable with respect to nu.

When you integrate out with respect to nu the variable y, this is a function of x which again can be integrated with respect to mu and this integration gives you the integral of the indicator function of E.

The important thing is that here when you are integrating with respect to y, the variable x is fixed; so, this is only a function of the variable y. For every x fixed, you treat it as a function of the variable y, integrate that out and then integrate out that integral with respect to the other variable. Similarly, you can interchange x and y and get the result. What we want to show today is that this result is true for all nonnegative measurable functions on the product space X cross Y.

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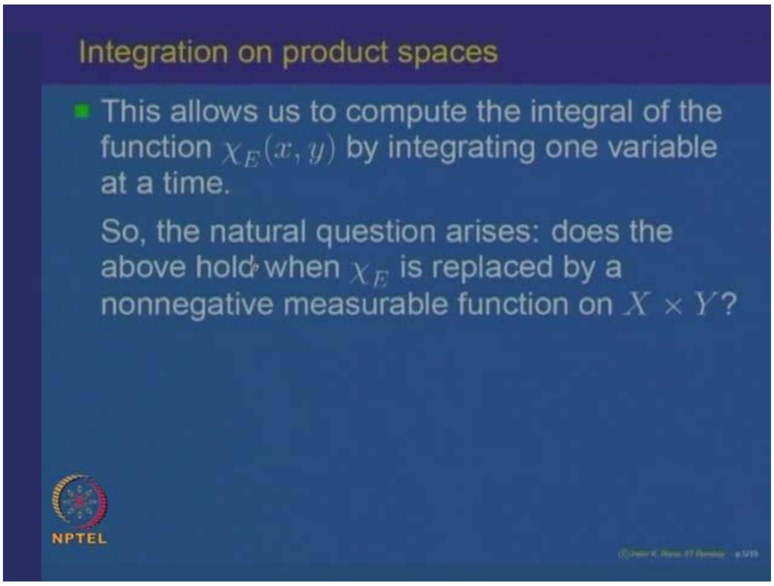

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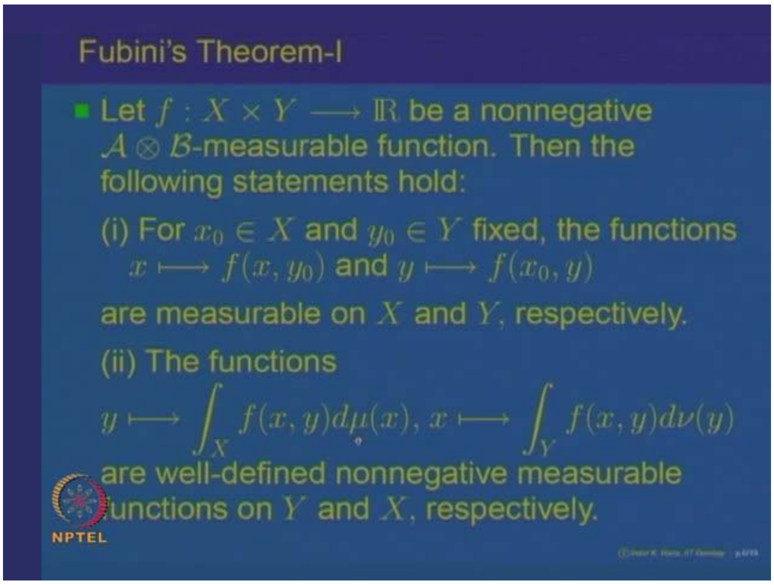

The theorem we want to prove is the following: if f is a function on the product space X cross Y and is a nonnegative function – f is a nonnegative measurable function on X cross Y and it is measurable with respect to the product sigma algebra, then the following statements hold; namely, if we fix one of the variables, say, x – if x_0 is fixed, then consider the function f of x comma y_0 and similarly y goes to f of x_0, y. For the function f of x, y, either you fix the variable y at y_0 or you fix the variable x at x_0 and treat it as a function of one variable only; then, these functions are measurable on X and Y respectively.

What we are saying is that for a function of two variables if it is measurable with respect to the product sigma algebra, then fixing either of the variables gives you a function of one variable which is measurable on the corresponding ((.)) with respect to the corresponding sigma algebras. These are nonnegative functions; so, you can integrate them out. If you integrate this function x going to f of x, y_0 with respect to mu, then that gives you a function which depends on y – the function y going to integral over X of f of x, y d mu of x.

Integrate out the variable x; this gives you a function of y; similarly, you integrate out f of x, y with respect to the variable y; you get a function with respect to x. The claim is these two are well-defined nonnegative measurable functions on the respective spaces.

Finally, these are nonnegative measurable; so, you can integrate them out with respect to the corresponding variables.

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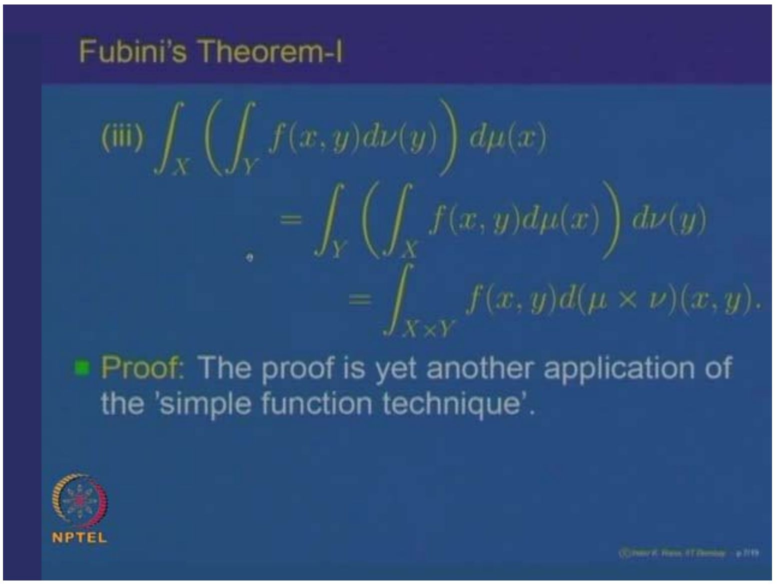

If you integrate out, first integrate with respect to Y and then with respect to X; that is same as integrating first with respect to X and then with respect to Y; both are equal to the integral of the function f of x, y with respect to the product sigma algebra. This gives an extension of the earlier result. It says that for nonnegative measurable functions if you want to integrate with respect to the product measure, then you can do it one variable at a time.

These two integrals are called the iterated integrals. The claim is that for a nonnegative measurable function, the integral with respect to the product measure is equal to iterated integrals, once again the importance being you are integrating one variable at a time. Let us prove this result. This proof is going to be built up step by step and this is what I call as the simple function technique.

The idea is that when f is the indicator function of a set, this result is true by the earlier result on product measures; everything involves integrals and integration being a linear operation, we will get that this result is true for nonnegative simple measurable functions. Once it is true for nonnegative simple measurable functions, application of monotone convergence theorem will give us that the result is true for all nonnegative measurable functions on the product space.

That is the approach basically we are going to follow and this is what I call as the simple function technique. When you want to prove something for nonnegative measurable functions on the measure space, verify it for the indicator functions, verify it for the nonnegative simple measurable functions and then verify it for the limits of nonnegative simple measurable functions. Let us prove this.

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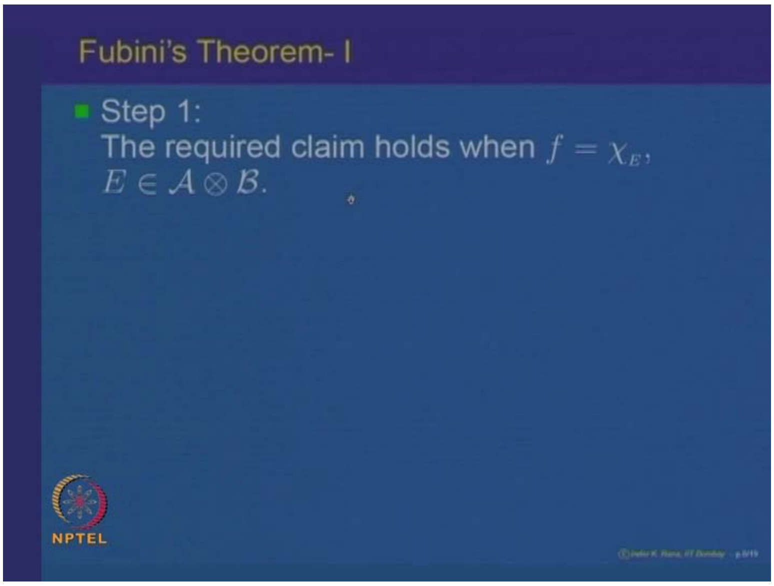

The step 1 is that the required claim holds when f is the indicator function of a set E in the product sigma algebra.

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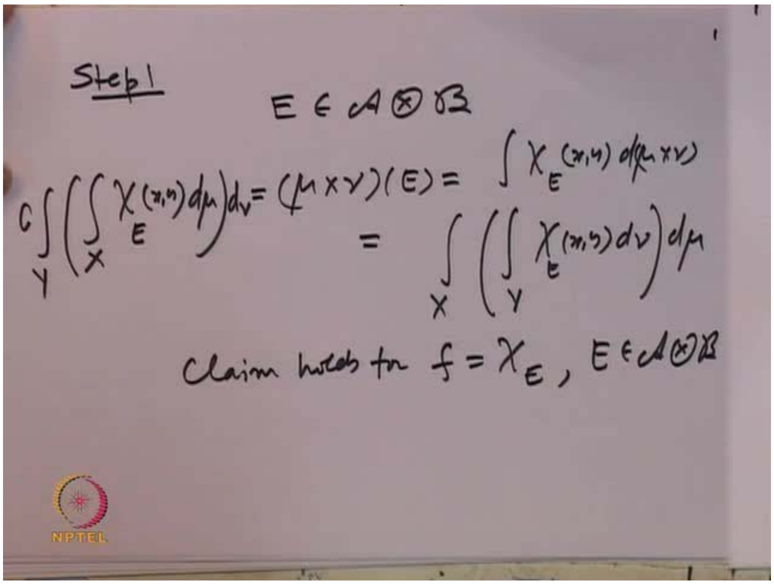

Let us look at that; that was what we have already shown. Step 1: when E is an element in the product sigma algebra, we had already shown that the product measure mu cross nu of E on one hand is equal to you take the indicator function of E and integrate out with respect to X d mu and then integrate out that with respect to the variable Y and so d nu (Refer Slide Time: 09:02).

That is same as you first integrate out the indicator function of x, y with respect to d nu; that means keeping the variable X fixed, you are integrating with respect to Y and then compute the integral of that with respect to the variable X. These two are equal and the middle thing if you recall, we said it is equal to the integral of the indicator function of E with respect to the product sigma algebra (Refer Slide Time: 09:30). This is precisely saying that the claim holds for f equal to the indicator function of a set E, E belonging to the product sigma algebra. This is step 1. From here, we want to go to step 2; let us take a function.

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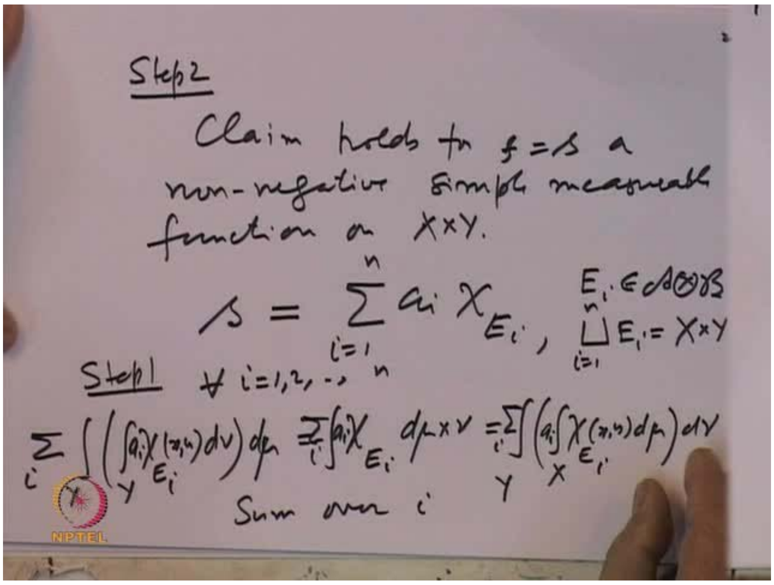

Step 2: the required claim holds for nonnegative simple measurable functions for f equal to s, a nonnegative simple measurable function on X cross Y. What does the function look like? A nonnegative simple function s on the product space looks like sigma a_i indicator function of some sets E_i where i is 1 to n. These E_is are sets in the product sigma algebra A times B; the union of E_is are pairwise disjoint and their union is equal to X cross Y.

Now, by step 1, what does step 1 say? Step 1 says for each E_i, the claim holds. Step 1 says for every i equal to 1, 2 and so on up to n, the integral of the indicator function of E i with respect to d mu cross nu on one hand is equal to the integral over X integral over Y of indicator function E of x, y d nu and d mu and also equal to the integral over Y integral over X of the indicator function of E d mu and then d nu; this is what we know (Refer Slide Time: 11:45).

Now, let us just observe because this is true for every i and integration is linear; that means if I multiply, I can multiply throughout by a_i. If I multiply by a_i, I can multiply here by a_i; this is E_i. If I multiply, this is a_i; I can multiply by a_i and I can multiply here by a_i and then take the summation (Refer Slide Time: 12:15). So, sum over i; summation over i, summation over i and here will be summation over i. This summation integration being linear, I can take the summation inside. When I take this summation inside the integral and then again take it inside, I will get summation of a_i times indicator function of E_i integral with respect to nu and then integral with respect to mu is equal to this I take it inside and that will be summation of a_i.

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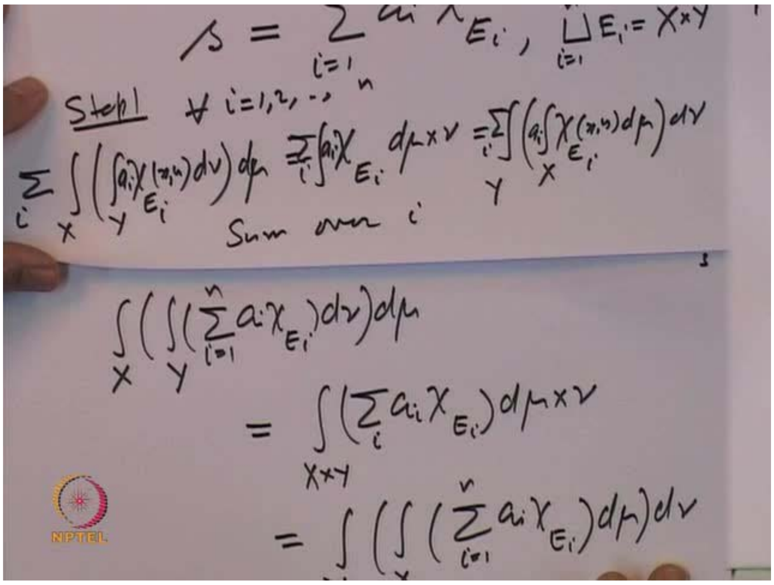

Let us just write that; this is more of writing than understanding. When I do this summation and take the summations inside, I will get integral over X integral over Y of summation i equal to 1 to n a_i indicator function of E_i d nu d mu is equal to the summation inside will give me integral over X cross Y of summation a_i indicator function of E_i d mu cross nu.

The last one will give me integral over Y integral over X of summation i equal to 1 to n a_i indicator function of E_i d mu d nu. This is just using the property that for every indicator function of a set, the result is true; integration is linear and so the result is true for finite linear combinations of these things also.

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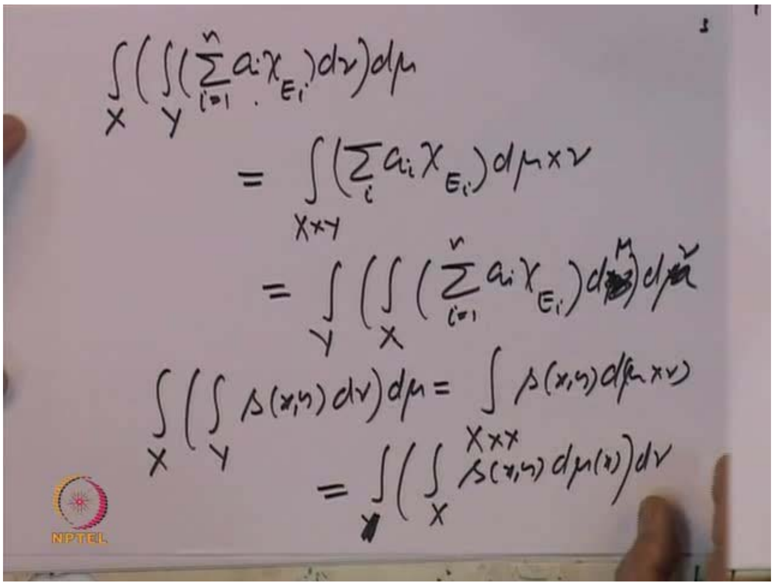

This is precisely my function s. This says integral over X integral over Y of s of x, y d nu d mu is equal to the integral over the product space; this is the function s; so, s of x, y d mu cross nu; that is also equal to the other iterated integral – integral over Y integral over X of s of x, y d mu of x d nu over we have already done it this is over Y; that should be nu actually and this should be mu; sorry (Refer Slide Time: 14:50).

This was over X and so that should be… no; that was okay; that was d mu and this is d nu (Refer Slide Time: 14:59). This is over X; so, d mu and d nu; that is okay. This says that the result holds; this proves the second step; namely, the claim holds for f a nonnegative simple measurable function (Refer Slide Time: 15:14).

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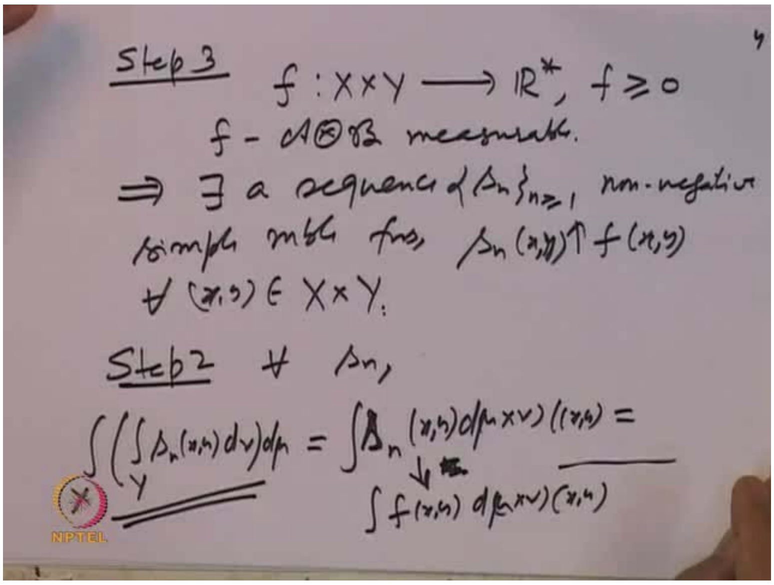

From here to go to general nonnegative functions, step 3 says we should be able to prove the result when f is… Let us take f on X cross Y to R star; f is nonnegative; f is A times measurable with respect to the product sigma algebra. Now, we look at the characterization of nonnegative measureable functions. f being nonnegative measurable implies that there exists a sequence s_n of nonnegative simple measurable functions, s_n of x, y increasing to f of x, y for every x, y belonging to the product set X cross Y.

This is by the fact that for every nonnegative measurable function, there is a sequence of nonnegative simple measurable functions converging to it. By step 2, we know that for every s_n the corresponding result holds. What does that mean? That means for every s_n, the integral of the nonnegative function s_n of x, y d mu cross nu of x, y is equal to the iterated integral; let us write, for example, integral over X integral over Y s_n of x, y d nu d mu and similarly, the other iterated integral.

Now, what we are going to do is observe the fact that for every s_n this result is true and s_n is a sequence of nonnegative simple measurable functions increasing to f. By the definition of the integral, this one converges to the integral of f of x, y d mu cross nu of x, y. This is by the fact of monotone convergence; this is not really monotone convergence theorem; this is actually the definition of the integral. If f is a nonnegative measurable function, then its integral is defined as the limit of any sequence of nonnegative simple measurable functions increasing to it; that is by the definition. On the other hand, we will compute this integral and show it is the corresponding iterated integral of the function f.

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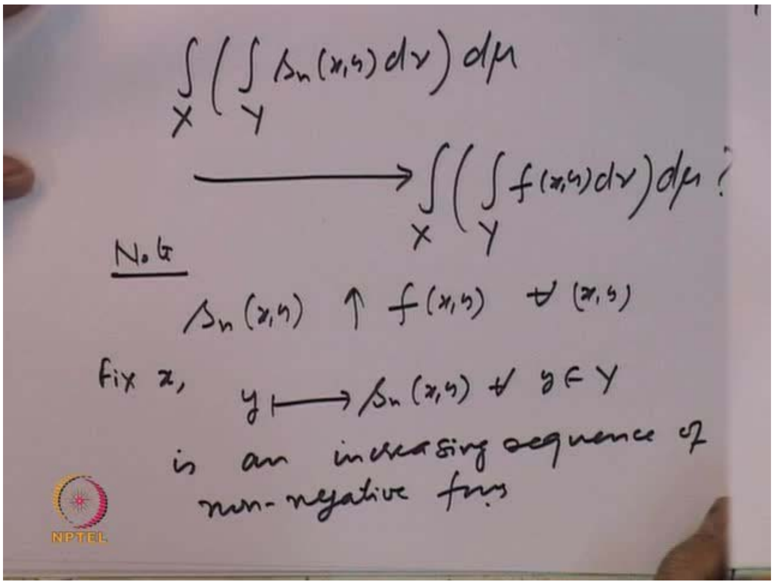

Let us look at this integral – integral over X integral over Y s_n of x, y d nu d mu. We want to show, claim, that this converges to integral over X integral over Y of f of x, y d nu d mu; this is what we want to show. Once that is shown, on one hand this converges to this iterated integral (Refer Slide Time: 19:15); on the other hand, this converges to this integral of f. These two will be equal and we will be through.

Let us try to prove that this iterated integral converges to this iterated integral. Here are the steps for proving this. First of all, let us try to prove that integral over Y with respect to nu converges to the corresponding integral. For that, let us note that s_n of x, y is increasing to f of x, y for every x and y. If we fix x, then we get a function y going to s_n of x, y for every y belonging to Y and because s_n itself is increasing, this sequence of functions we already know are measurable functions (Refer Slide Time: 20:18).

These are measurable functions; for every simple function, we had seen that if you fix one of the variables the other one is a measurable function; for simple functions, that is true. First of all, this is clear that for fixed x, this is an increasing sequence of nonnegative functions. This is an increasing sequence of nonnegative functions and each one of them is a measurable function on Y.

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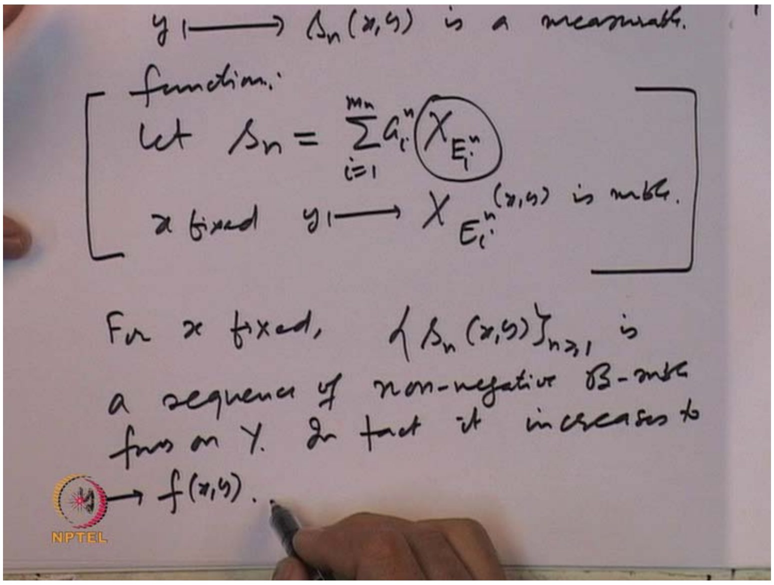

The second observation is that each one of them – the function y going to s_n of x, y – is a measurable function. That is in a sense obvious because… To see this, we note that for this… observation is…. Let us say let s_n be equal to something; let us say sigma a_i n into indicator function of E_i n for some i equal to 1 some indexing set 1 to m_n. Then for every x fixed and y going to the indicator function of E_i n of x, y is measurable; we have already observed that that is a measurable function; while computing the product measure we saw that.

Each one of them is measurable with respect to the product sigma algebra because it is an indicator function of a set. For every fixed x, this will be a measurable function on y (Refer Slide Time: 22:30). That will be the sections; that will be measurable, but scalar multiple of a measurable function is measurable and the sum of measurable functions is measurable.

This is an obvious fact that for a simple measurable function s_n if we fix one of the variables, then the other variable becomes a measurable function. This is a measurable function. That means what we have gotten is for x fixed, the sequence s_n of x, y over n is a sequence of nonnegative B-measurable functions on Y and it is also increasing. In fact, it increases because s_n increases to f; when we fix one of the variables x, this is going to increase to the function f of x, y, x fixed as a function of y. So, it increases to the function y going to f of x, y. It is a perfect setting for the application of monotone convergence theorem; by monotone convergence theorem, we get the following.

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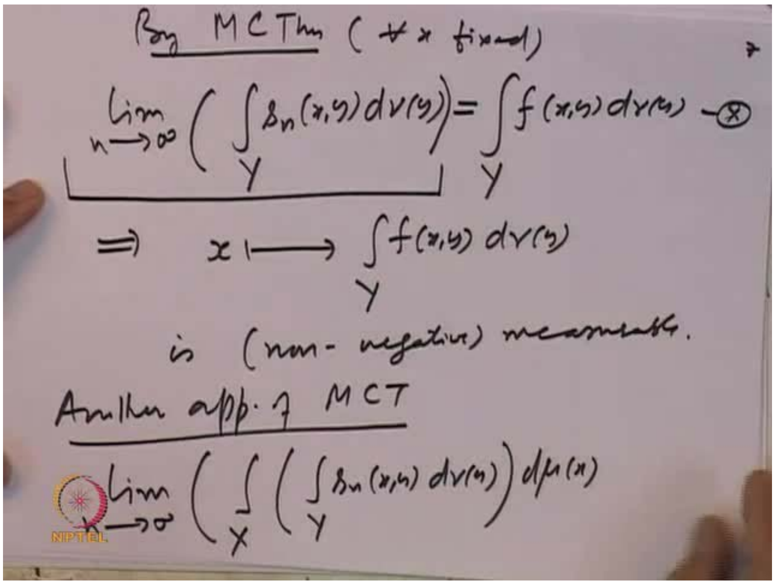

To this, apply monotone convergence theorem. By monotone convergence theorem, the integrals s_n of x, y d nu of y over Y limit of that must be equal to integral of f of x, y with respect to d nu of Y; this is what we get ((.)), of course, for every x fixed; for every x fixed, we get that this limit must be equal to this. That means what? This is a function of x.

That implies that if I look at x going to Y f of x, y d nu of y and if you treat this as a function of x, then it is a limit of these functions– limit of integrals of nonnegative simple functions (Refer Slide Time: 25:15). That means that this function is measurable; this implies that this function is nonnegative measurable; it is a nonnegative measurable function. This is a nonnegative measurable function and it is a limit of this sequence of nonnegative measurable functions.

By star, I can apply... Again, it is another application of monotone convergence theorem. On the left-hand side, the limit of integrals of s_ns with respect to nu is also a limit of measurable functions; this itself is a measurable function with respect to x; s_ns are increasing and these integrals are also increasing (Refer Slide Time: 26:27). So, this limit is equal to this.

Now, what we are saying is another application of monotone convergence theorem to the fact that if you look at the sequence of measurable functions, this is a measurable function (Refer Slide Time: 26:50) and so integral of that. The function x going to this is a nonnegative measurable function; what is left to be proved is we want to integrate this with respect to mu.

We are saying that to this, we apply monotone convergence theorem (Refer Slide Time: 27:07). We will get that limit of this function is this function and so integral limit of the integrals… This says limit n going to infinity integrals of this function so integral over X of these functions; these functions are s_n of x, y d nu of y d mu x. Limit of this must be equal to integral of this with respect to X, again by monotone convergence theorem.

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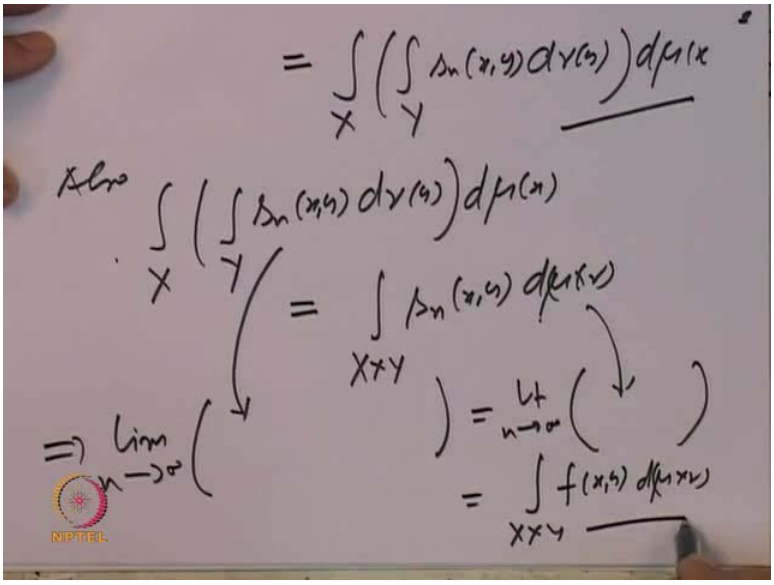

Let us write that. This is equal to integral over X integral over Y of s_n of x, y d nu of y d mu of x. This limit is equal to this (Refer Slide Time: 27:59). On the other hand, we had seen that this iterated integral for s_ns is equal to the double integral; this is one thing upon observation. Also, let us look at the other fact. What is the other fact? We have that integral over X integral over Y s_n of x, y d nu of y d mu of x… For nonnegative simple measurable functions, the claim holds. That means this is equal to the double integral – the integral over X cross Y of s_n of x, y with respect to the product measure mu cross nu.

This is because we have already proved in step 2 that the results holds for nonnegative simple measurable functions. This result is equal to this; so, limit of this must be equal to limit of that (Refer Slide Time: 29:14). It implies the limit n going to infinity of this left-hand side must be equal to limit n going to infinity of the right-hand side – this one; that is coming here, but limit of the left hand side we have already seen is equal to this. What is the limit of the right-hand side? s_n is a sequence; we have already seen that s_n is a sequence of nonnegative simple functions increasing to f; so, this must be equal to integral X cross Y of f of x, y d mu cross nu. That proves that this must be equal to this (Refer Slide Time: 30:02).

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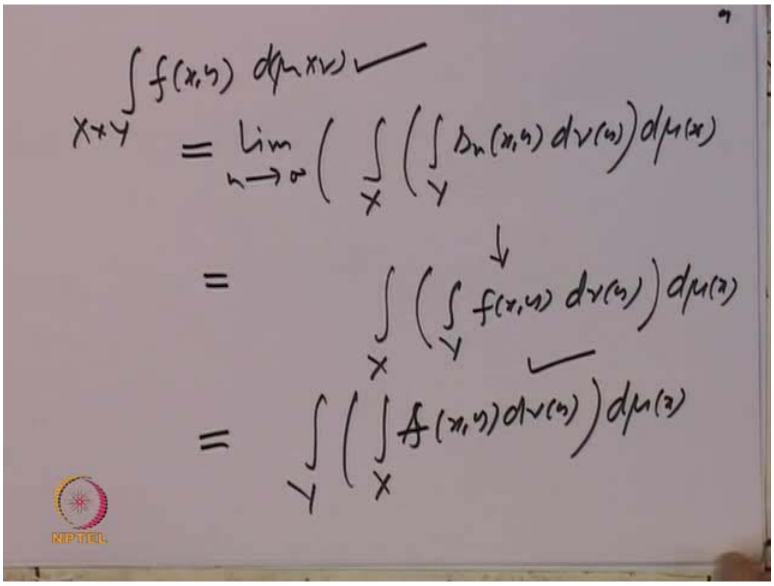

The step 3 proves that that the integral of f of x, y with respect to integral of... Let us just ((.)) prove that. What we have shown is this limit must be equal to this (Refer Slide Time: 30:25). What was that limit of that quantity? What we have shown is that integral of d mu cross nu over X cross Y is equal to limit n going to infinity of integral over X integral over Y of s_n of x, y d nu of y d mu of x. This is what we have proved just now – this limit on one hand side was this and other side was this; so, limit of these two quantities must be equal.

This is what we have proved, but this quantity let us see what it is. Note that s_n for every y fixed was increasing. Look at the sequence for every x fixed; that is an increasing sequence on nonnegative measurable functions increasing to the function f of x, y. So, monotone convergence theorem says this inner integral converges to integral of Y f of x, y d nu of y; that is what we had already observed.

Again, this is a sequence of nonnegative measurable functions (Refer Slide Time: 31:44). The application of monotone convergence theorem gives us that integral of this limit of that must be equal to d mu of x. That says that the double integral of the nonnegative simple function is equal to the iterated integral of the nonnegative measurable function ((.)) first with respect to nu and then with respect to mu. We can interchange X and Y; same arguments will imply. That will say that this is also equal to integral over Y integral over X of f of x, y d nu of y d mu of x.

Let us just go through the ideas in the proof. Basically, this proof is an application of the fact that integral for a nonnegative simple function is built from the limits of integrals of nonnegative simple measurable functions; that fact is used very effectively because we know that the corresponding result is true for indicator functions and integration is linear. That allows us to say that from the indicator functions you can go to nonnegative simple measurable functions by just taking scalar multiplications and additions of characteristic functions.

That will give us that the result is true for nonnegative simple measurable functions and then just an application, some suitable applications, of monotone convergence theorem will give us that the integral of a nonnegative measurable function on the product space can be computed via the iterated integrals. Let us just go through this proof through the slides once again so that we have a clear idea of what we are doing.

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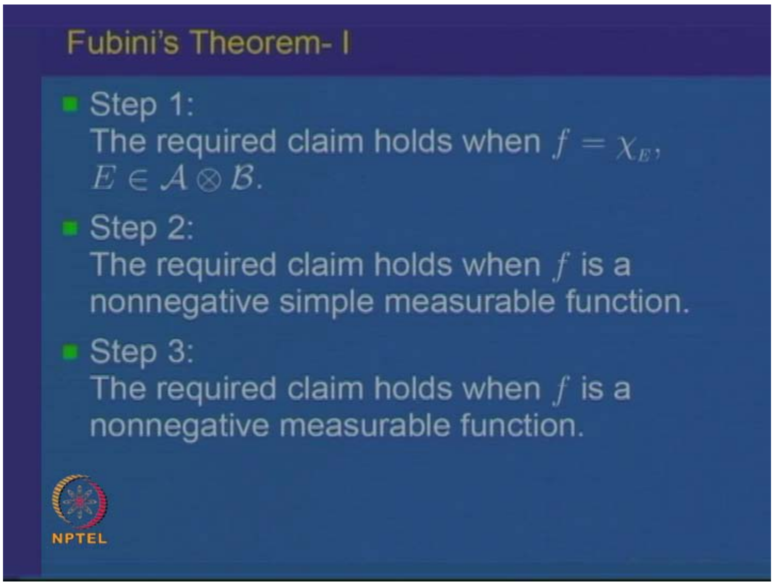

Step 1: the required claim holds when f is an indicator function of E; that is the previous theorem that we have proved. Step 2: the required claim holds when f is a nonnegative simple measurable function. From step 1 to step 2, one goes via the fact that integrals are linear operations; then one goes to step 3 that the required claim holds when f is a nonnegative measurable function; that requires applications of monotone convergence theorem.

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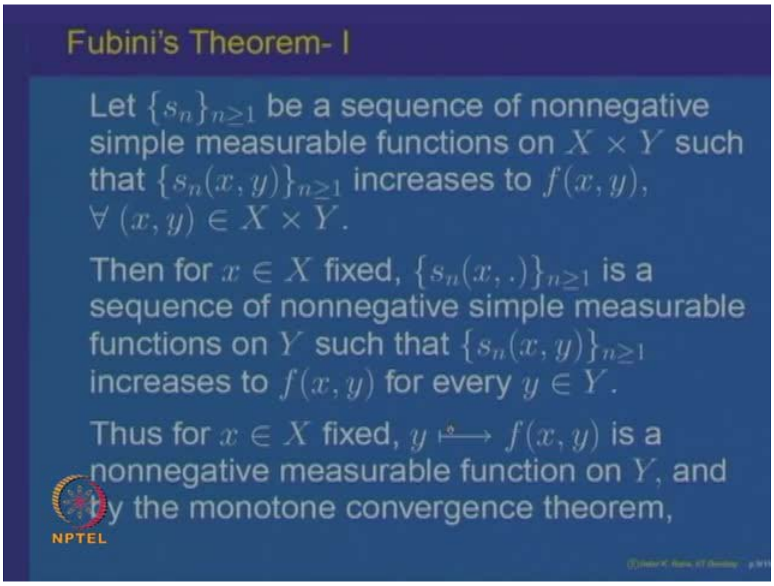

Step 3 is the crucial one where a lot of applications of monotone convergence theorem are used; let us just go through that again. Let s_n be a sequence of nonnegative simple measurable functions such that s_n increases to f; that is by the fact that f is a nonnegative measurable function. Now, let us fix x; then, the sequence s_n of x, x is fixed so in the variable y is a sequence of nonnegative simple measurable functions on Y and it increases to the function f of x, y for x fixed.

Pointwise, s_n of x, dot ((.)) x fixed s_n x as a function of y increases to the function f of x as a function of y. This is increasing (Refer Slide Time: 35:21). An application of monotone convergence is not required here. This is a limit of increasing sequence of measurable functions; that says that the function y going to f of x, y is a nonnegative measurable function because this function is a limit of measurable function. The first fact being used is that limit of measurable functions is a measurable function. Now, we can also apply monotone convergence theorem to conclude that the iterated integral of s_n must converge to the iterated integral of f with respect to y.

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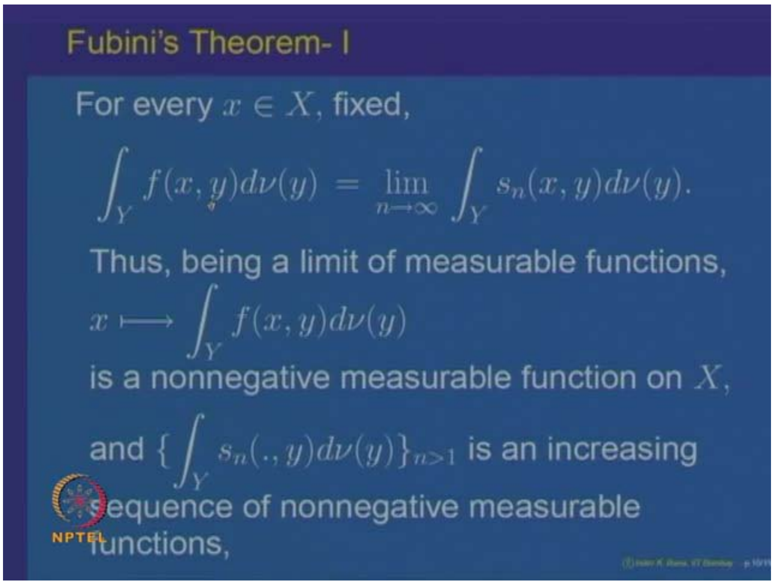

That is the first application of monotone convergence theorem; for the nonnegative measurable function f for the variable x fixed, its integral with respect to the variable y is well defined because this is a nonnegative measurable function. It is equal to limit with respect to n of the iterated integral of the nonnegative simple measurable function s_n with respect to y.

This result also says, this equality also says, that the right hand-side – treat it as a function of x; that means that converges to this integral and by the fact that the required result holds for nonnegative simple measurable functions, this function integral of s_n with respect to Y is a measurable function of x; here we are using the step 2. This is a sequence of measurable functions converging to a function; that means this integral must be a measurable function (Refer Slide Time: 37:10).

Again, limits of measurable functions are measurable. That gives you x going to integral over Y f of x, y d y. The iterated integral of f with respect to Y is a measurable function with respect to x and it is nonnegative. Once again, this is a nonnegative function and it is a limit of these measurable functions. Another monotone convergence theorem application gives that integral of s_n with respect to Y and its integral with respect to X must come to the corresponding integral of f with respect to X.

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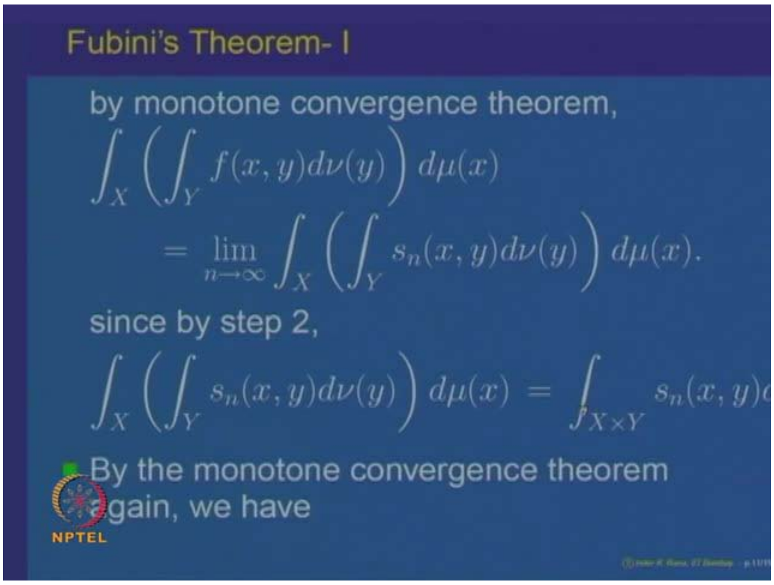

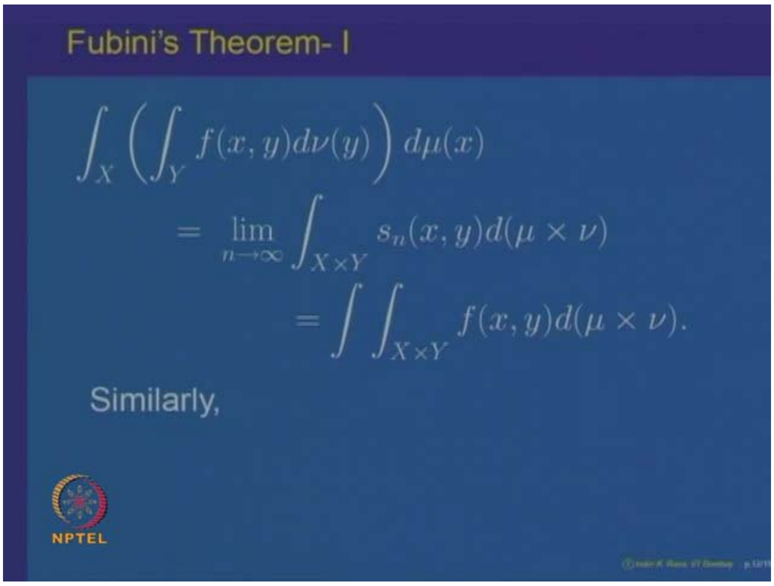

Here, we are applying monotone convergence theorem that the integral over X of the integral of f with respect to Y must be limit of the corresponding integrals with respect to the nonnegative simple functions. Now, come back; for nonnegative simple functions, we know the result is true; so, this iterated integral must be equal to the double integral; so, this is equal to the double integral (Refer Slide Time: 38:24).

s_n is a sequence of nonnegative measurable functions on the product space; again by either you can say application of monotone convergence theorem or just by the definition, this limit must be… so this is equal to this and the limit of that must be equal to the integral of the function f over X cross Y. That says the corresponding result holds;

so, this iterated integral of f is equal to the double integral of f with respect to mu cross nu.

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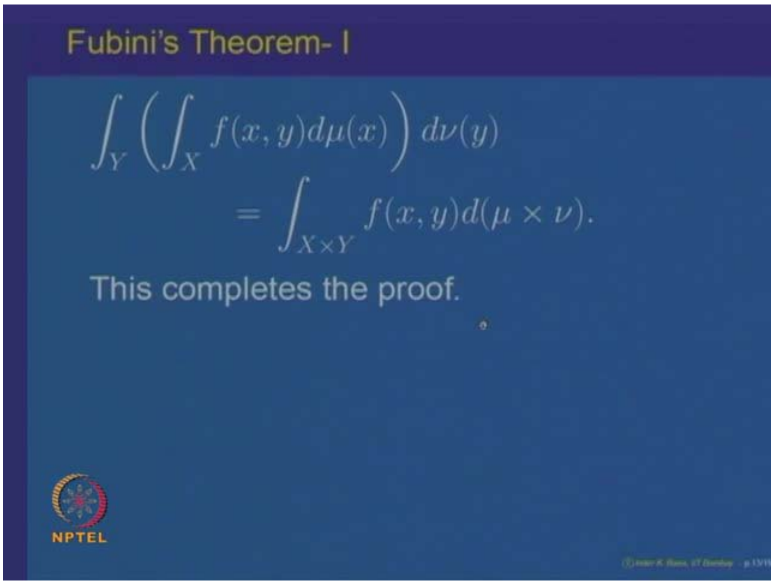

Similarly, the other thing can be proved; you can interchange the variables X and Y; so this result is true. This is a result which is called Fubini's Theorem I; this the first Fubini's Theorem which says that for a nonnegative measurable function on the product space if you want to integrate – find its integral with respect to the product measure, you can do it by integrating one variable at a time.

Either you can fix x, integrate out with respect to Y and then integrate with respect to X or interchange; the choice is yours; you can first integrate with respect to X and then with respect to Y. The two iterated integrals for a function of two variables is equal to the double integral for nonnegative measurable functions. This is called Fubini's first theorem which helps one to integrate functions of two variables. Next, we want to show that this result also holds for functions which are integrable; we want to prove that for an integrable function the corresponding result holds.

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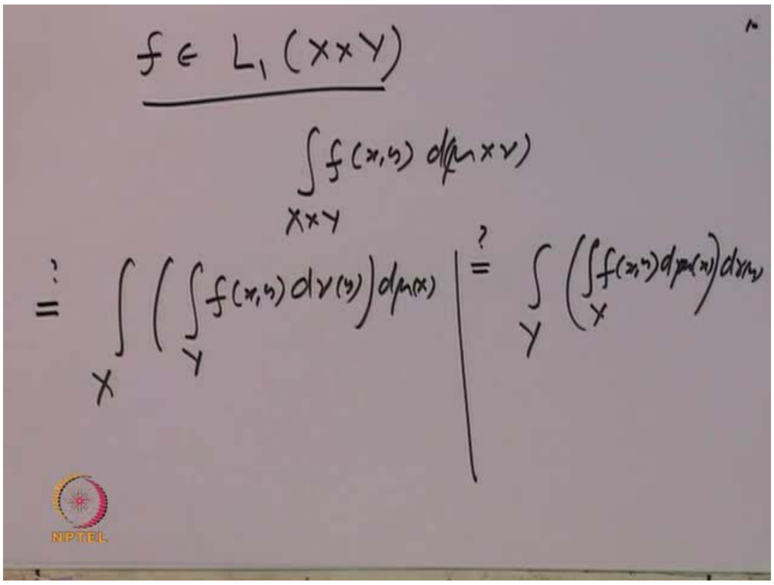

Let us look at the proof of that; let us take a function f which is L_1 on X cross Y; it is integrable with respect to X cross Y and we want to say that the integral of f of x, y over X cross Y d mu cross nu on one hand is equal to you can integrate first x, y with respect to nu, we want to claim this, with respect to y over Y and then integrate out that with respect to X so d mu x (Refer Slide Time: 41:02).

Or one should be able to say that this is also equal to you take the function f of x, y and integrate out the variable with respect to mu of x and then integrate out with respect to Y d nu of y; we want to say that these two, these results, hold. If these equations are to hold where f is not necessarily nonnegative, that means what? First of all, the inner integral, for example, integral of f of x y with respect to y must exist; that means we should be able to say for a function of two variables which is integrable when I fix the variable x as a function of y that is integrable.

That is integrable; then that gives us a function of x; then we should be able to say that is integrable with respect to x. Finally, these two are equal; similarly, the other result must hold. The theorem which we want to prove is the following; that is called Fubini's theorem II.

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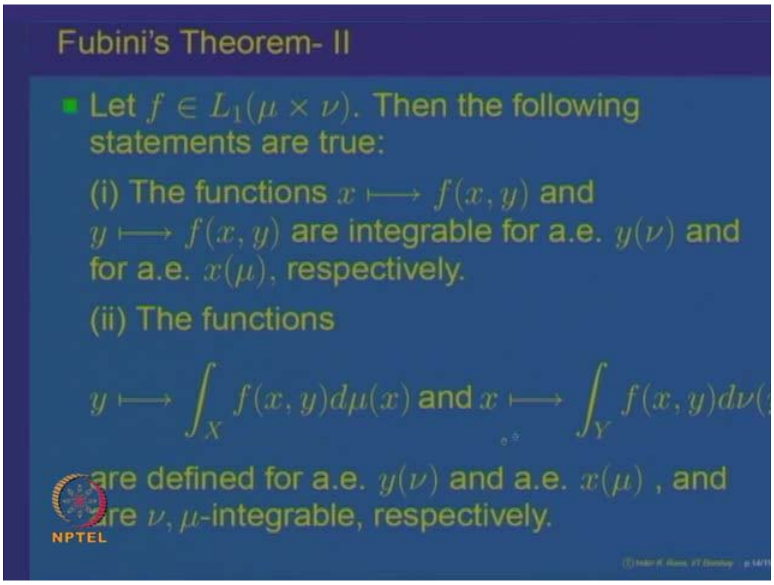

If f is an integrable function, f is integrable, we want to prove the following. If f is an integrable function, then the following statements are true. (i): for the function of two variables if I fix either of the variables, then with respect to other variable it is integrable; not for all, but we are able to say that the function x going to f of x, y and y going to f of x, y for the other variables are integrable for almost all y and for almost all x.

For almost all fixing of coordinate, the other variable becomes a function which is integrable with respect to the other one; that is (i). Secondly, once these are integrable you can integrate out. It says the function y going to integral of f over X with respect to mu and similarly the integral of f with respect to Y – these two – are defined almost everywhere; of course, they are defined almost everywhere and are integrable.

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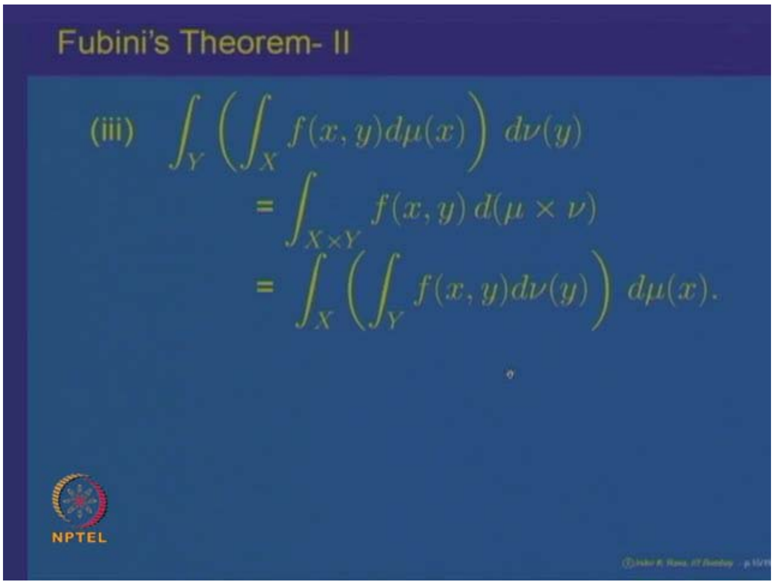

Hence, the third step says they are integrable and indeed the iterated integrals are equal to the double integral. We would like to prove this theorem. To prove this, let us proceed as follows.

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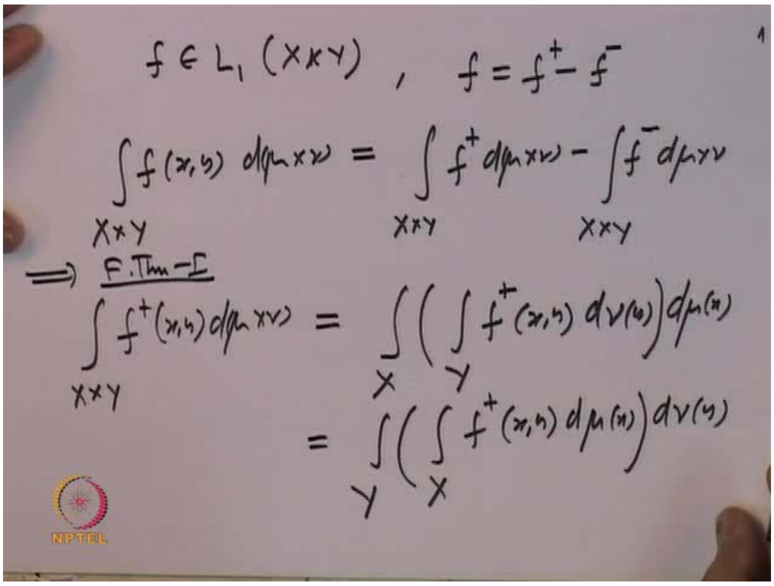

We are given that the function f belongs to L_1 of X cross Y. Let us write the positive and the negative parts of the function; f is equal to f plus, the positive part, minus the negative part. The integral of f of x, y with respect to the product measure mu cross nu is equal to the double integral of f plus with respect to the product measure minus the integral of the negative part d mu cross nu over X cross Y. That is the definition of the integral.

If f is integrable, then the integral of the function is nothing but the integral of the positive part minus the integral of the negative part of the function. Now, let us look at them separately. f plus of x, y and of course d mu cross nu over X cross Y. f plus is a nonnegative function; it is nonnegative measurable function; by the result Fubini's Theorem I, I can write this as integral over X integral over Y f plus of x, y integral over f of x, y d nu with respect to y and then d mu with respect to x.

It implies by Fubini's Theorem I, that is Fubini’s theorem for nonnegative measurable functions, that integral of a nonnegative measurable function can be computed by iterated integrals. Let us write the other one also; you can interchange; integral over X f plus of x, y d mu x and d nu of y. For f plus, we have used the Fubini's Theorem I. Now, let us observe; f being integrable, this quantity is finite; so, all these integrals are finite quantities.

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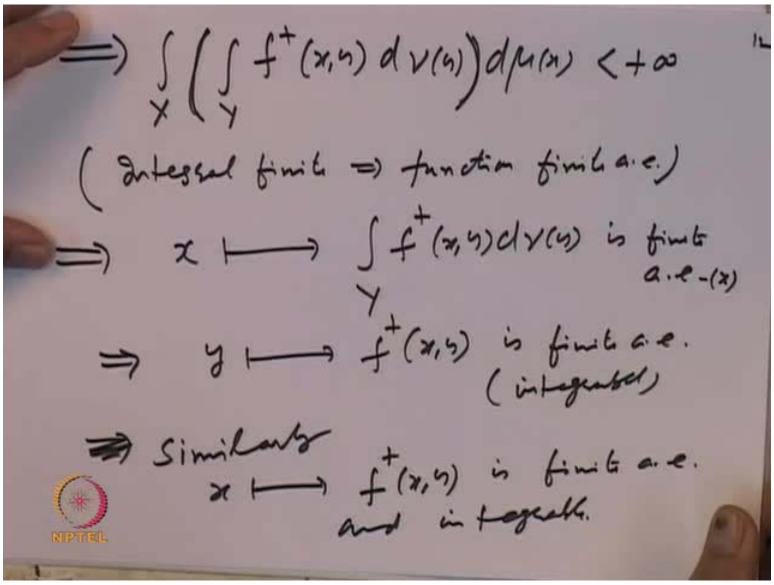

For example, the first one implies because of integrability that integral over X integral over Y f plus of x, y d nu of y d mu x is finite. Here is an important observation that we have earlier proved – if the integral of a function is finite, then the function must be finite. Here, we are using integral finite implies function finite almost everywhere; this we had already proved; this fact we are going to use now.

Look at this integral with respect to mu of this function is finite (Refer Slide Time: 47:25). That implies that the function x, x going to integral over Y of f plus of x, y d nu y is finite almost everywhere with respect to x. We have used the fact that integrable function implies that the function is finite almost everywhere. Once again, for almost all x this is finite; this also implies that the function y going to f plus of x, y is finite almost everywhere and, of course, integrable because this integral is finite almost everywhere; it is a function which is integrable and finite almost everywhere.

It implies I can integrate it out. This is a nonnegative function; it is integrable – a nonnegative integrable function. We have already seen that for the nonnegative measurable function, this is equal to this integral (Refer Slide Time: 48:55). Similarly, the function x going to f plus of x, y is finite almost everywhere and integrable. Similar results holds for f minus. That means what? All those four functions are finite and integrable; so, we can integrate them out. We have the results corresponding results; that is the first part.

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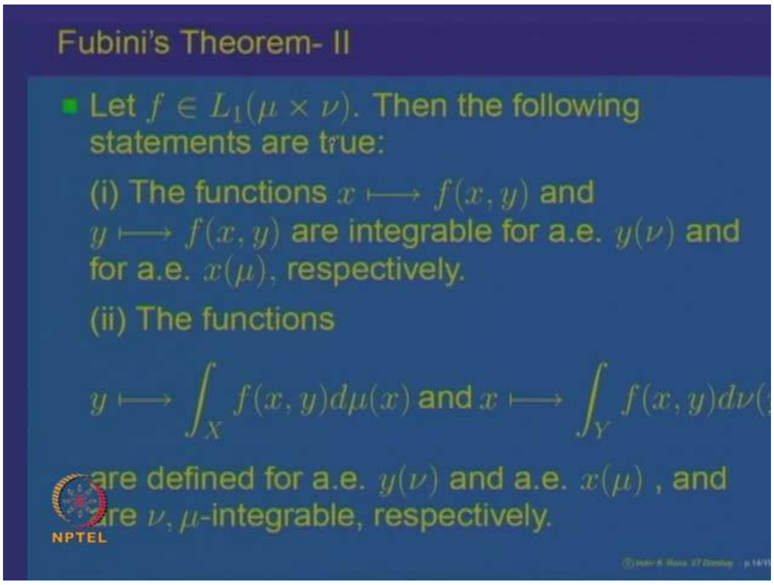

That is the first part that these functions are integrable almost everywhere and correspondingly almost everywhere with respect to x and y. These functions are also defined and are integrable (Refer Slide Time: 50:06). That means we have the following. They are all integrable and finite.

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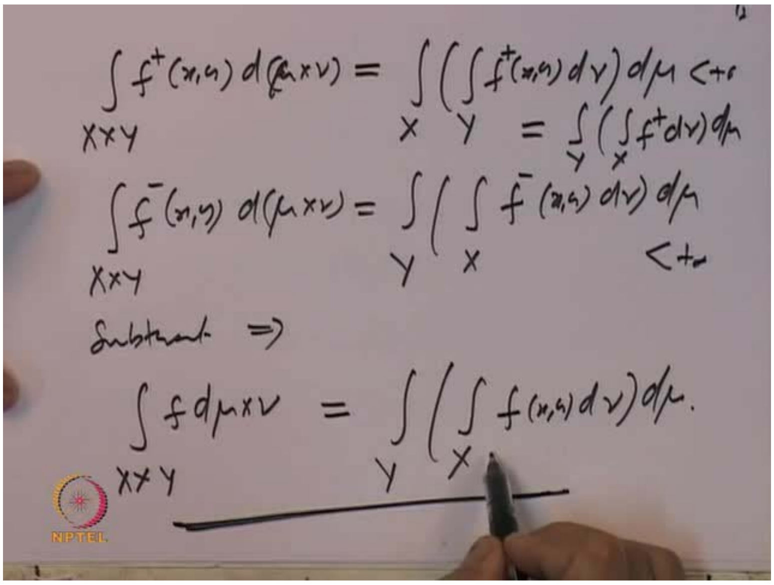

For f plus of x, y with respect to X cross Y d mu cross nu, we have got this is equal to the iterated integral with respect to X with respect to Y of f plus of x, y d nu d mu. Similarly, for the negative part, we have x, y d mu cross nu X cross Y is equal to integral over Y integral over X of f minus of x, y d nu d mu. All these are finite quantities because f plus and everything is integrable.

These are all finite quantities; this is finite and this is finite (Refer Slide Time: 51:04). I can take the difference of the two. Subtract second from the first and use the fact that integrals are linear; subtract implies subtraction and similarly… sorry… and also the corresponding identities for the other one – interchanged thing. This is also equal to integral over Y integral over X of f plus d nu d mu; for nonnegative, that is true. We will subtract this from this (Refer Slide Time: 51:47).

We will get integral of f d mu cross nu X cross Y because of integral f plus minus integral of f minus is integral of f is equal to the iterated integral of f plus with respect to X and Y minus iterated integral of f minus with respect to the same iterated integral. That will give you Y integral with respect to X of f plus minus f minus; that is f of x, y d nu d mu. That will prove that for the integrable function the double integral is equal to the iterated integral of one of them; the other proof is similar (Refer Slide Time: 52:35).

This result that for integrable functions the corresponding interchange of integrals holds is basically coming from the previous result that the corresponding result holds for nonnegative measurable functions (Refer Slide Time: 52:49). What we have proved is two Fubini's theorems – Fubini's Theorem I and Fubini's Theorem II. Fubini's Theorem I says that for nonnegative measurable functions, the double integral (the integral over the product space) can be computed by integrating one variable at a time; similarly, this can also be done for functions which are integrable. We will continue this Fubini's Theorem a bit more and then specialize it for integrals for R 2, R 3 and so on. Thank you.