Measure and Integration

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Module No. # 07

Lecture No. # 27

Computation of Product Measure – II

Welcome to lecture number 27 on measure and integration. In the previous lecture, we had started looking at how to compute product measure of a set in the product sigma algebra. We had shown part of theorem and we will continue looking at the proof of that theorem in this lecture.

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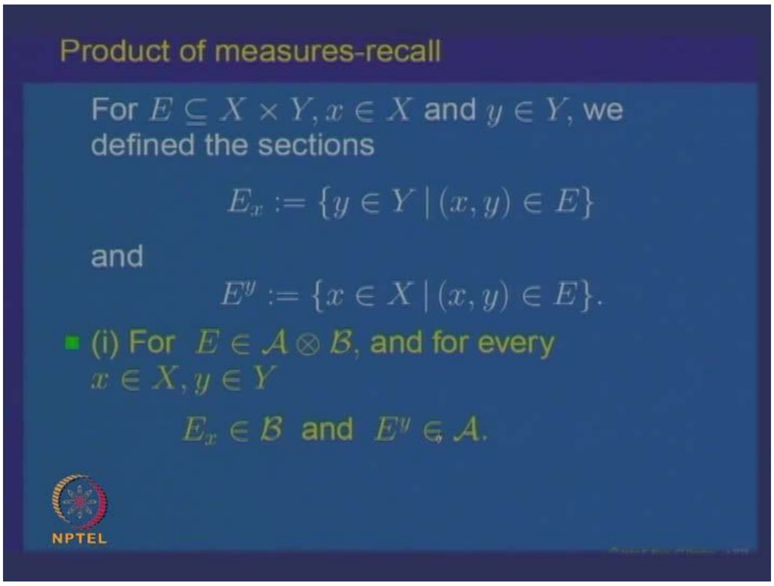

Let us just recall, what we have been doing. We were looking at computing the product measure; we will continue that study today. Let us just recall, the settings we have a set E contained in the product set X cross Y and for any element x in X and y in Y we defined what is called the x section E x and E y in the previous lectures.

Then, we claimed that for every set E in the product sigma algebra, set sections E x is a element of the sigma B and the section at y is an element in the sigma algebra A.

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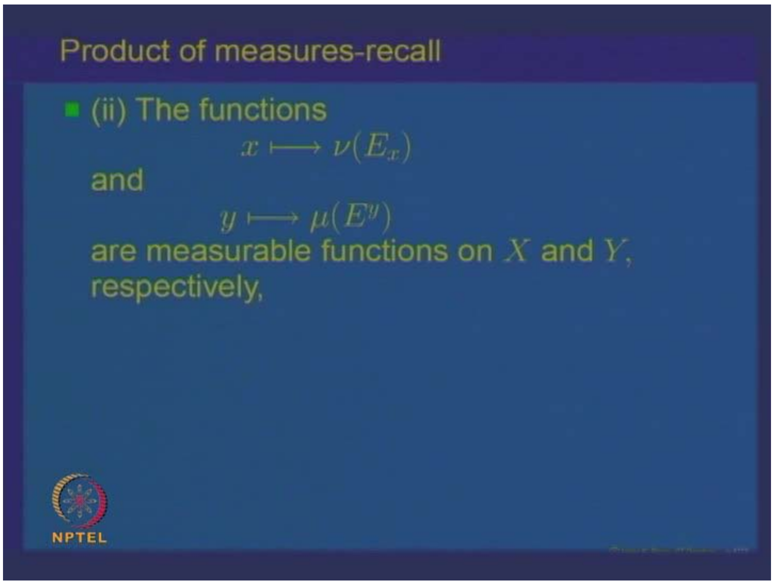

This, we had proved and I am just recalling them. Then, we proved that the function x going to nu measure of E x, E x is a sub set of is an element in the sigma algebra B and nu is a measure defined there.

We can compute, what is nu of E x and the claim is that the functions for every x, the image being nu of E x; this is a function defined on x and the claim is it is a measurable. Similarly, function y going to the measure of the y section is a measurable function on the set y with respect to sigma algebra B. These two we had proved and we wanted to prove finally the third one. If we integrate these functions with respect to mu and with respect to nu, these are non-negative measurable functions and we can integrate them. So, the claim is that the integral nu E x and d mu x is same as the product measure mu cross nu of E and it is same as the integral of the y section with respect to y.

This is the step we were trying to prove in the previous lecture. To prove this, what we said let us look at the class of those sub sets E in the product sigma algebra for which this is true.

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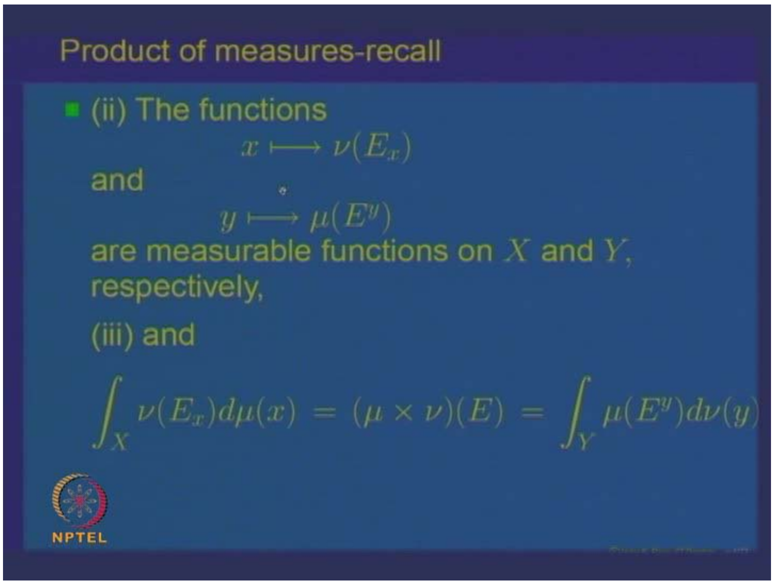

We constructed the class P. All those sub sets in the product sigma algebra such that the previous to claims, this claim 2 and claim 3 are both hold x going to nu E x and y going to mu E y are measurable functions.

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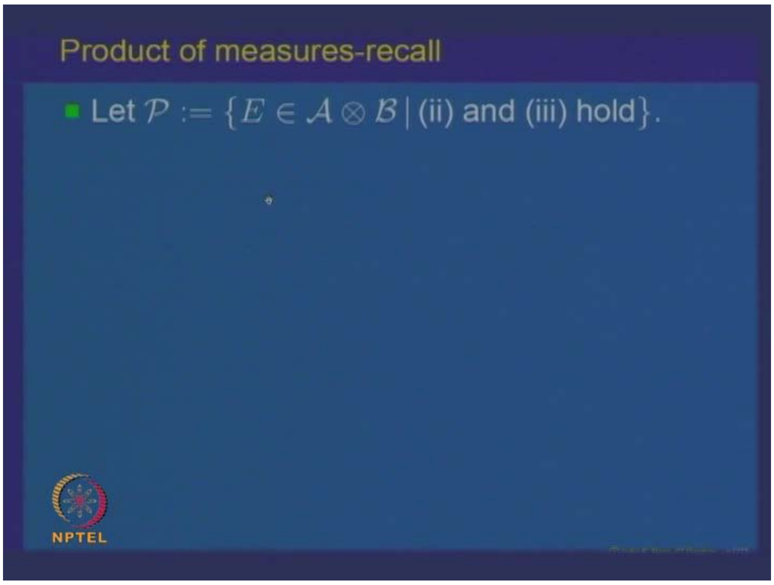

P is the family of all sub sets A cross B such that the property 2 and 3 hold. Let us just recall, what the properties 2 and 3 are.

Property 2 is that x going to nu E x and y going to mu E y, these are non-negative measurable functions.

The property 3 says that, the integrals of nu E x with respect to mu is same as the integral of mu E x with respect to nu and both are equal to the product measure of E. So, both these properties holds for a set E then that set is in the collection P.

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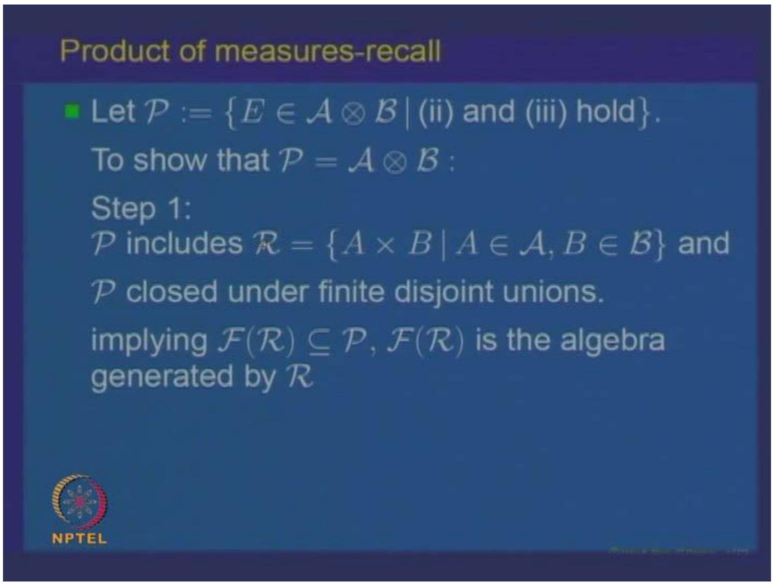

Our aim is to prove that P is equal to the product sigma algebra A cross B. We already observed in the previous lecture to show this. The first step is to prove that this class P includes the rectangles. So, that is 1 and that we had proved. Also, we had proved that this class P is closed under finite disjoint unions. Once this class P is closed under finite disjoints unions and includes the rectangles, the rectangles form semi algebra. So, the algebra generated by it looks like the class of sets, which are finite disjoint union of rectangles and P being closed under such operations, we will get that. As a consequence of this, the algebra F R generated by these rectangles is also inside P.

As a consequence of step 1, we get that the algebra generated by the rectangles in inside the class P.

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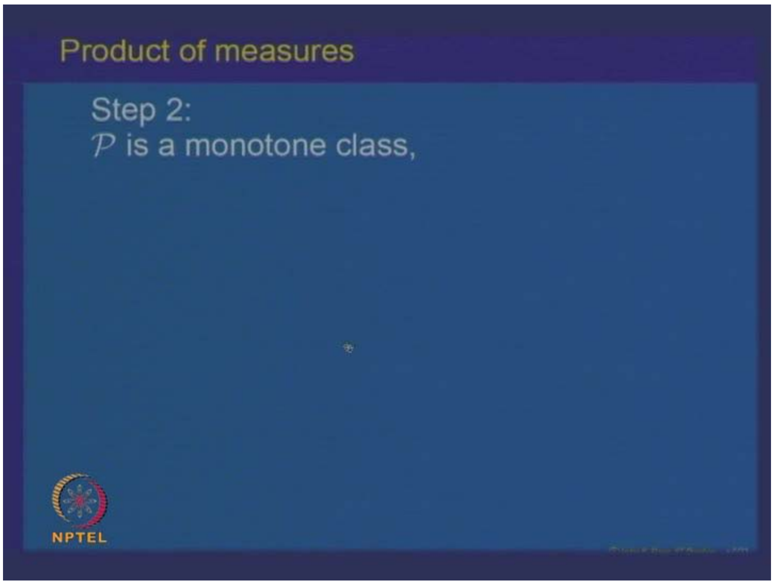

The second step, we wanted to prove that this class P is a monotone class. The reason for prove that it is a monotone class, it is directly difficult to show that it is a sigma algebra because if you could show directly that P is a sigma algebra. It includes algebra generated by a rectangle. Then, it will include the sigma algebra generated by it that direct route is not possible.

We follow the monotone class result. If, we are able to show that P is a monotone class and F R being inside it the monotone class generated by F R will be inside P and F R being algebra the monotone class generated by algebra is same as the sigma algebra generated by that class.

We will get that the sigma algebra generated by a rectangles will be inside P and that is precisely what we want to show and that is A times B because the sigma algebra generated by rectangles is the product sigma algebra A cross B.

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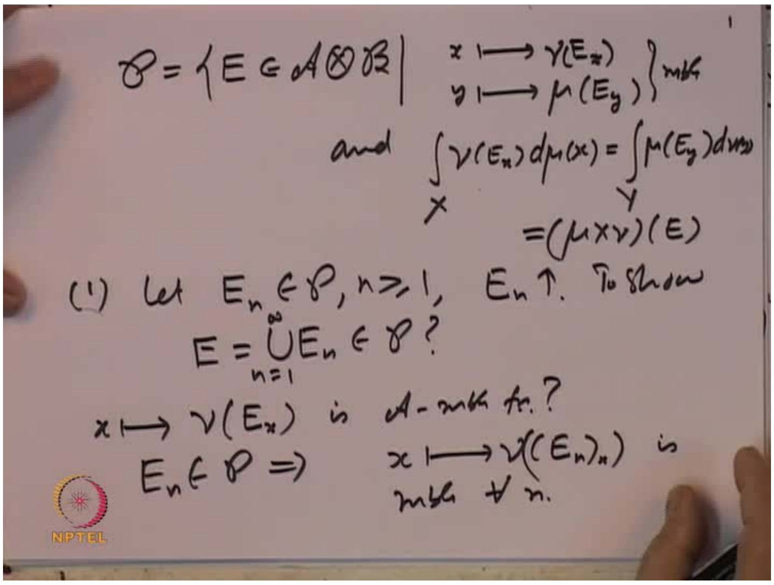

To complete that proof, we have to only show that the class P is a monotone class. Let us start proving that P is a monotone class.

So, P is the class of all those sub sets E belonging to the product sigma algebra A times B such that if we look at set x going to take the E take it sections x that is the sub set of the set y in the sigma algebra B so nu of that make sense so we get this function so this is measurable and the function y going to mu of E y that is E is measurable. So, both these functions are measurable and the property that if you integrate nu of E x with respect to mu; we are integrating over X this is same as the integral over Y of the second function mu of E y with respect to d nu E y and both of them are equal to the product sigma algebra mu cross nu of E.

This is the collection of all the sets E in the product sigma algebra this holds and we want to show that P is a monotone class. Let us look at the first property, let E n belong to P and E n be a collection of sets in the class P say that E n is increasing. To show that the set E, which is equal union of E n's also belongs to P.

This is what we have to show first. The P is a monotone class; we have to show it is closed under increasing unions and decreasing intersections.

That is the two properties we have to check. Let us take a sequence E n in P, which is increasing and let us say E is the union of this E n's. So, the claim is that E belongs to E n.

What we have to do? We have to look at the corresponding. What is the first property? We have to check. To check that E belongs, we have to look at nu of E x. So, the first thing we have to show is that this is measurable and A is measurable function.

To do that, let us observe the following. This is what we have to show. Each E n belongs to P, which implies that x going to nu of E n, it section at x is measurable for every n. So, this is what is given to us and we want to come to nu of E n.

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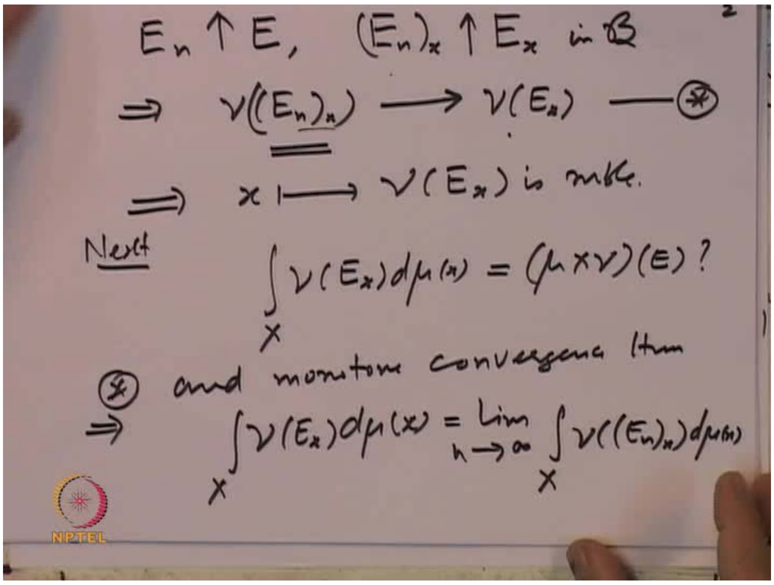

But for that, let us observe that as E n is increasing to E, the sections E n x is increasing sequence of sets increasing to E of x. This is a sequence of sets in the sigma algebra B.

So, that we have already seen, if a is a sub set of b then the section of a is the sub set of section of b. So, that will prove that the sections are increasing and increase to the union. So, union of the sections, union of E n at x is same as union of each E n and hence, this is increasing to x. This is a simple observation using the properties of the sections.

E n x is increasing and now you recall that nu being a measure, if a sequence of sets increases to another set, which implies that nu of E n x, the sections that will increase that will converge to nu of E x. So, that proves nu of E n x increases.

Each one of them is a measurable function. So, nu of E x is a limit of measurable functions, which implies that x going to nu of E x is measurable.

Basically, what we are saying is because nu of E x, the function x going to nu of E x is a limit of the functions nu of E n of x. That comes from the fact because E n is increasing to E, the sections E n x is increased to the section E x. That means, in the sigma algebra B and nu being a measure and nu of E n x must converge to nu of E x and each one of them being measurable because it is in the collection P.

Each is a measurable function. So, limit of measurable functions is measurable. That proves one part that x going to nu of E x is measurable.

Next, what we have to check is the following. We have to check that integral of nu of E x d mu x over X is equal to mu cross nu of E. This is what we want to check.

Once again let us go back to the earlier fact. We saw that nu of E n x the sections is measurable functions; these are actually non negative measurable functions and they are converging to the function nu of E x. That is an increasing sequence of measurable functions. This is nu of E n x is an increasing sequence of non-negative measurable functions converging to a measurable function nu of E x. We can apply our monotone convergence theorem that says so once again this property star star and monotone convergence theorem apply and apply and they give us as a consequence that integral of nu of E x d mu x over X because nu of E x is a limit of increasing sequence of non-negative measurable functions.

Integral of nu of E x must be equal to limit n going to infinity of the integrals of the corresponding sequence of non-negative measurable functions. They are nu of E n section at x d mu x. This is an application of monotone convergence theorem.

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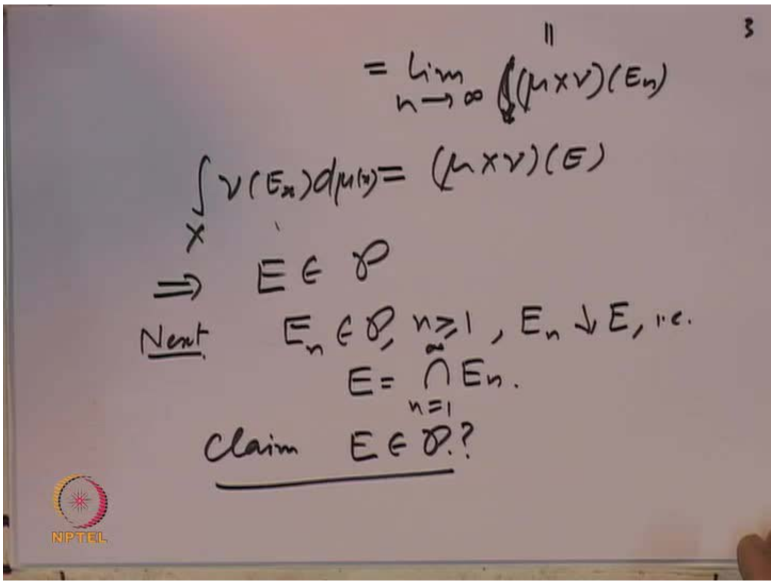

Let us observe that E n belongs to the class P. The property 2 says that if I integrate nu of E n section with respect to mu, this integral is equal to the product measure mu cross nu of E n.

That is because E n belongs to class P. So, by the third property of that collections of set in P that means nu of mu cross nu of the product measure of E n is the integral of the sections with respect to x. We can say that this integral is equal to limit n going to infinity of mu cross nu of E n.

Once that is true, we want to look at this limit. Let us observe that E n is an increasing sequence of sets in the sigma algebra A cross B and mu cross nu is a measure. Once again, using the property of measure that if a sequence of sets is increasing then the measure of limit of the measure of the sequence is equal to measure of the limit that is equal to mu cross nu of E.

Once again, we use the effect if E n is increasing to E and mu cross nu is a measure, this limit must be equal to mu cross nu of E.

What we get is that this limit is equal to this. That means, we get that nu of integral over X nu of E x d mu x is equal to mu cross nu of E.

We had proved that if E n is increasing, this implies that E belongs to the class P because we showed that E n is increasing to E then both the property holds for this.

Now, we want do the similar thing for decreasing. Next, let us consider E n belonging to P n bigger than or equal to 1 and E n's decrease to E that is E is equal to intersection E n's n equal to 1 to infinity.

We want to claim that E belongs to P. This is what we want to check. So, we can try to copy the proof for the increasing case. Let us go back to the proof of the increasing case and let us see, we can carry over the proof by saying similarly.

Now, we have got E n's decreasing because E n's belong, what we said first thing was that because E n's belong to P. This is a measurable function that is a property of the set E n being in the class P increasing or decreasing is not coming into picture. This step will carry over and then if E n is increasing to E, we have got E n is decreasing to E. This thing will change. If E n's are decreasing then it is true that the sections E n x will be decreasing to the section of the set E at x. So, the sections E n x will decrease to the set E x so that that step also will be ok.

Now, we want to say that when E n's decrease to E, we want to say that here we use the property that for the increasing case. We said whenever a sequence increasing nu of E n's converge the corresponding result we know is not true for decreasing sequences.

Here, the proof trying to copy the proof for the increasing thing will fell down because these steps will not this equation star will not hold.

To make this star hold, we have to put an extra condition that the measures are finite because if measures are finite then E n decreasing to E will imply measures converge.

So, if E n mu and nu are finite, for example if mu is finite then E n sections of each E n that is a decreasing sequence. So, nu of E n will converge.

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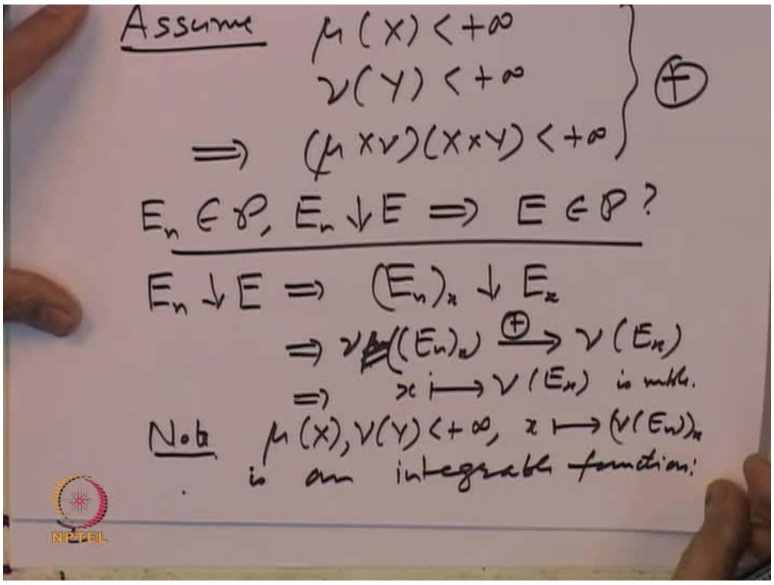

To carry over the proof; in the similar case, we have to put extra conditions. So for this claim to hold, we have to assume that mu and nu are finite. Let us assume that mu of X is finite and nu of Y is finite that implies mu cross nu of X cross Y is finite.

Under these conditions, we want to show that if E n belongs to P E n's decrease to E that implies E belongs to P.

To show that, we can repeat the steps. So, E n's let me just go through the proof again but the decreasing case also to emphasize we are exactly will be using the finiteness condition. So, E n's decrease to E so that implies that the sections E n x decrease to E of x

That implies nu of E n x, because E n's x is a subset of B. This converges to nu of E x. This is the stage, where will be using this condition. Under this condition, mu and nu are finite. Now, each E n belongs to P. So, each one of them is a measurable functions that will imply x going to nu of E x is measurable.

This is a measurable function and we have got nu of E n x decreases to nu E of x. Earlier, we use monotone convergence theorem to conclude that nu of E x integral of nu of E x must be limit but here it is a decreasing sequence. So, we cannot use monotone convergence theorem here.

But let us note here that because mu of X is finite, nu of Y is finite. So, this function mu x going to each of the functions nu of E n x is an integrable function.

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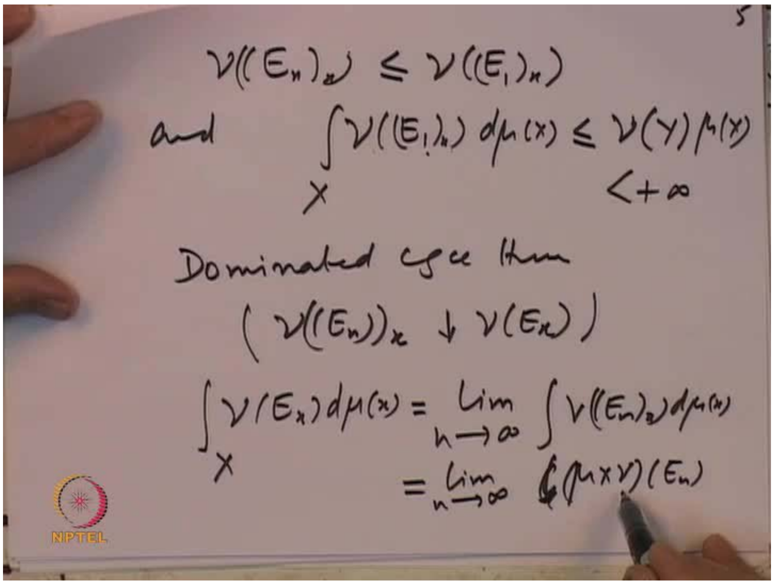

Why is that? It is a decreasing sequence. Let us observe, nu of E n x for every n if I look at this non negative function, it is less than or equal to nu of E 1 of x and nu of the section E 1 x integral over X d mu x is less than or equal to nu of E 1, E 1 x is less or equal to nu of Y and so this is integrant is less than nu of Y. So, this integral of 1 d mu x is less than mu of X, which is finite.

So, nu of E 1 x is a integrable function on the measure space Y B nu and each nu of E n x is less than or equal and each nu E n of x is integrable.

We can apply Dominated convergence theorem. Dominated convergence theorem applies to the fact that nu of E n x, which is a sequence of non-negative integrable functions and they are decreasing to the integrable to the function nu of E x.

This is also integrable, which implies by dominated convergence theorem. This observation that the function nu of E x d mu x over X, this function is integrable. Its integral is nothing but the limit n going to infinity of integrals nu of E n x d mu x.

For the decreasing sequence, the proof differs in both the steps; first of all when we want to say that E n's are decreasing the sections decrease. So, the finiteness condition allows us to say that nu of E x is a limit of this functions and that implies that this is a measurable function.

Finiteness say this function is measurable because of this fact, the finiteness condition says this is a sequence of integrable function decreasing to the function. So dominated convergence theorem can be applied and that gives us so this limit is equal to nu of E x integration with respect to mu is limit of and now the proof is as before this E n being in the collection P. So, this integral is nothing but measure of mu cross nu of the set E n.

That is limit n going to infinity of measures of the sets E n and once again E n's are decreasing to E and mu cross nu is a finite measures. So that will imply so this is equal to mu cross nu of E, again using the fact that mu and nu are finite.

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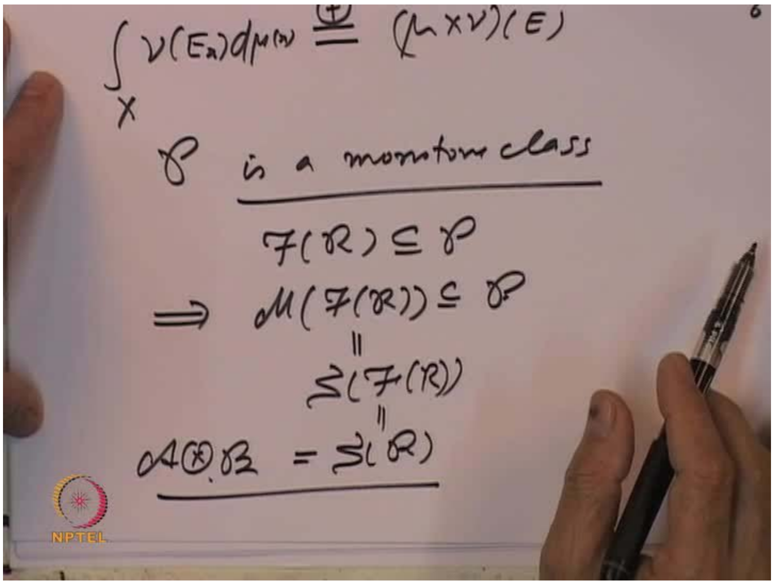

We get the conclusion that using finiteness condition the integral nu of E x d mu x over X is equal to; we have already shown it is the class P is closed under increasing sequences. Now, we have shown it is closed under decreasing sequences. So, P is a monotone class. That proves that P is a monotone class.

As a consequence of the fact that P is a monotone class, the consequence of this would be namely - we already have F of R is inside the class P and P is a monotone class, which will imply that the monotone class generated by F of R will also be inside P but this is nothing but the sigma algebra generated by the class R of rectangles are same as the sigma algebra generated by rectangles and that is same as the product sigma algebra.

So, this will prove that the product sigma algebra is equal to P namely that the required conditions hold for the corresponding that proves step 2 namely - P is a monotone class and that implies that the monotone class generated by F of R is inside P.

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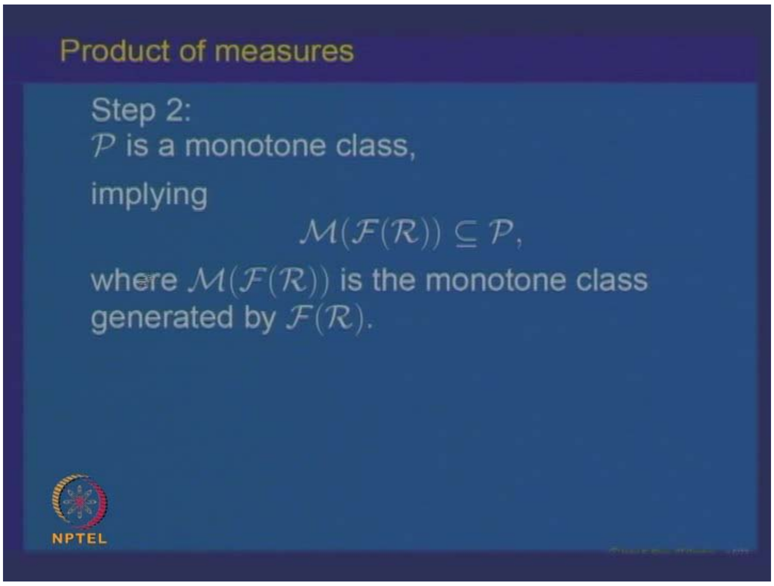

Hence this will prove that monotone class generated by F of R is algebra. So, the monotone class generated by algebra is precisely the sigma algebra generated by E and A cross B will be inside the class P. Hence, everything is inside so A cross B is equal to P.

This is a theorem, where we have used very sensibly the fact that when mu and nu are finite; in that case, we can extend that argument of the increasing to the case of decreasing also. This also illustrates the technique the monotone class sigma algebra technique.

We have approved the theorem required claim that P is a monotone class under the conditions mu and nu is finite.

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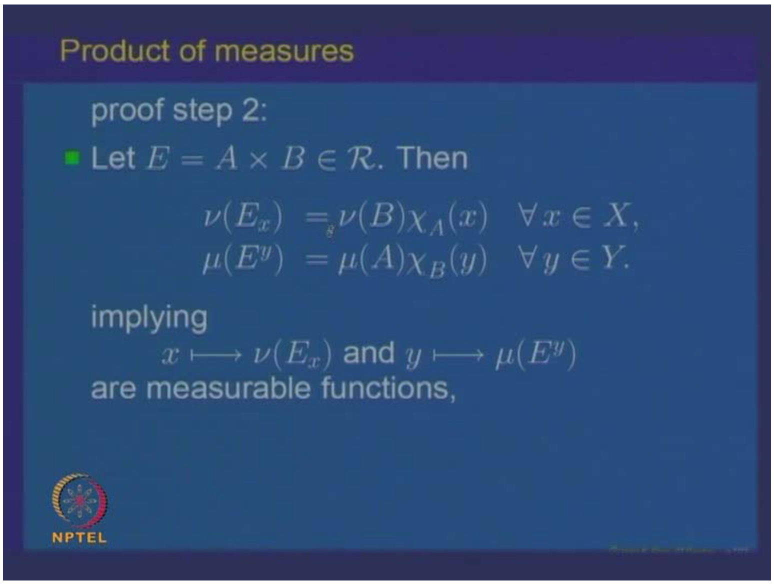

Now, with the usual arguments one can extend to the case when it is sigma finite. Let us see that, but before doing that let me just go through the proof of the step 2 again to illustrate the basic facts.

The first thing we looked at was, if E is a product set A cross B. So, I am just reversing the proof of step 2 to highlight the important points in the proof.

So, A cross B belongs to R is a rectangle then nu of E x was the first step showing the R is the class R includes rectangles. There we use the fact that if you take a set, which is a rectangle then it section is nothing but either the set A or the set B or the empty set according to the point x or y.

So, nu of E x is nothing but nu of B times the indicator function of x because if x does not belong to A then this is 0 and the section is just B. Similarly, mu of E y is mu of A times the indicator function of B.

These two facts prove that x going to nu of E x and y going to mu E y for rectangles is measurable functions.

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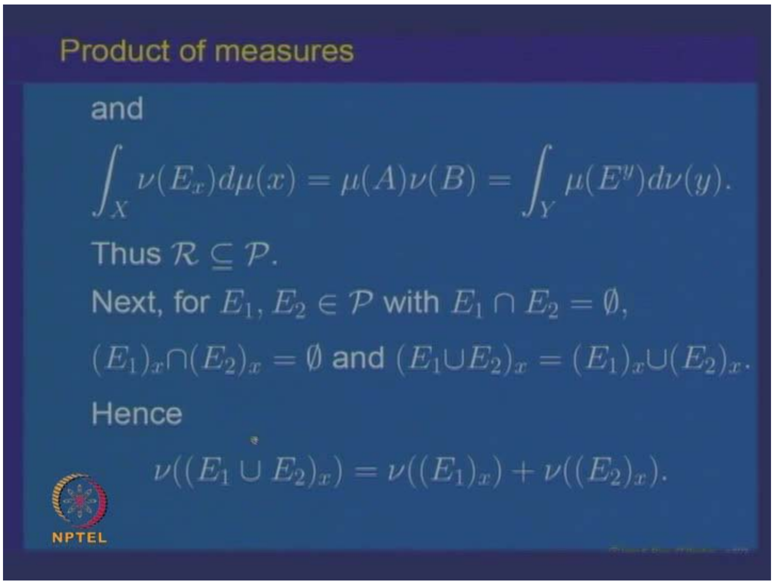

If we integrate this integral of nu will be equal to nu of B into mu of A. So this is the product measure of the product set A cross B. That says the rectangles are inside it and so that is the straight forward argument, which says rectangles comes inside P.

Showing that P is closed under finite disjoints unions, this also straight forward because that follows from the fact that E 1 and E 2 are 2 sets in the class P, which are disjoints.

Then, the sections are disjoint of these two sets and the sections of the union are equal to union of the sections. As a consequence of this, the nu of the section of the union section of union so E 1 union E 2 section at x mu of that is addition nu of E 1 x plus nu of E 2 x because the sections are disjoints and E 1 and E 2 both belong to P imply these two are measurable functions. Hence, the sum of measurable functions is measurable so this becomes measurable.

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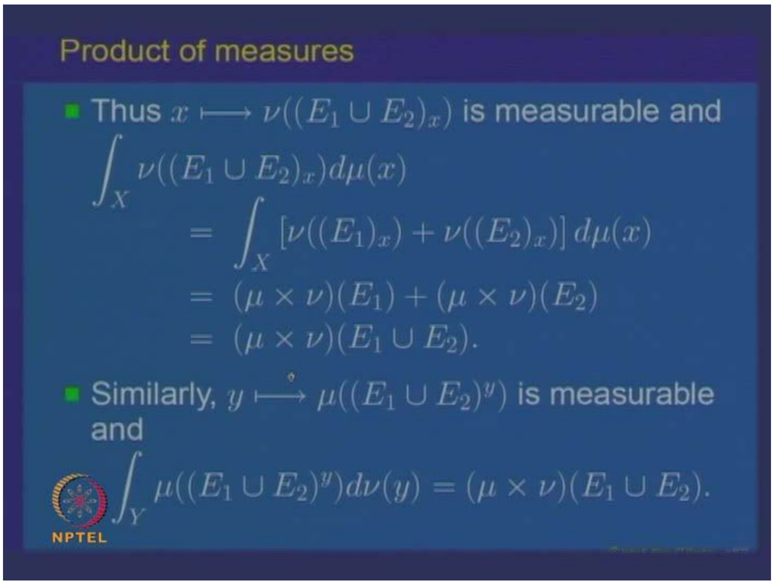

That is the straight forward proof of fact that, if E 1 and E 2 belong to P then E 1 intersection and there disjoint then the union also belongs to P. Finally, look at the integral, integral of nu of the section of the union because that splits into two parts. So, nu of E 1 union E 2 is nu of E 1 x plus nu of E 2 x with respect to mu.

The integral splits into two parts that is mu cross nu of E 1 because E 1 belongs to P and this is mu cross nu of E 2 because E 2 belongs to P.

Now, using the fact that mu cross nu is a measure that gives us this equal to mu cross nu of E 1 union E 2.

Similar thing will work for the y sections. So, proving that rectangles are inside the class P and P is closed under finite disjoint unions is there other straight forward computation.

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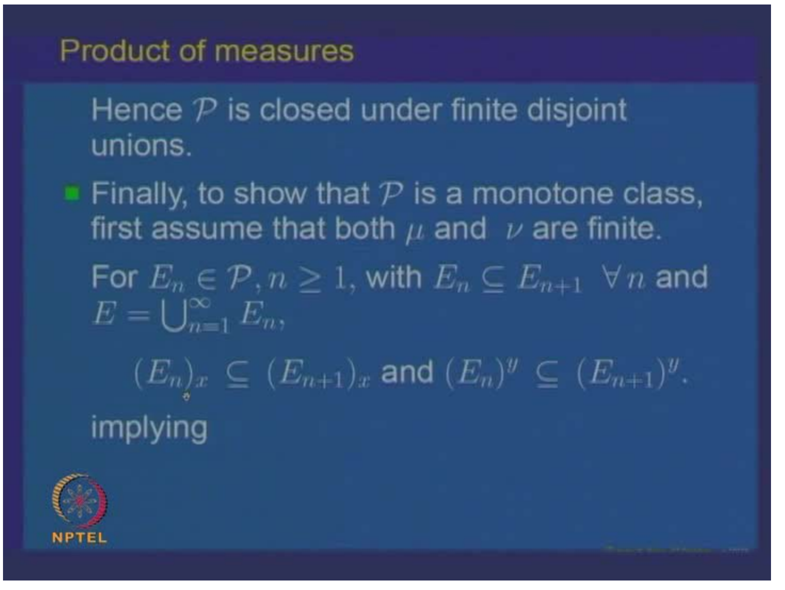

The problem arise is when we want to show that P is a monotone class. So, there we first assume that mu and nu are finite. Once mu and nu are finite, we want to show it is closed under increasing union and decreasing intersections. Take a sequence of sets E n's which is increasing.

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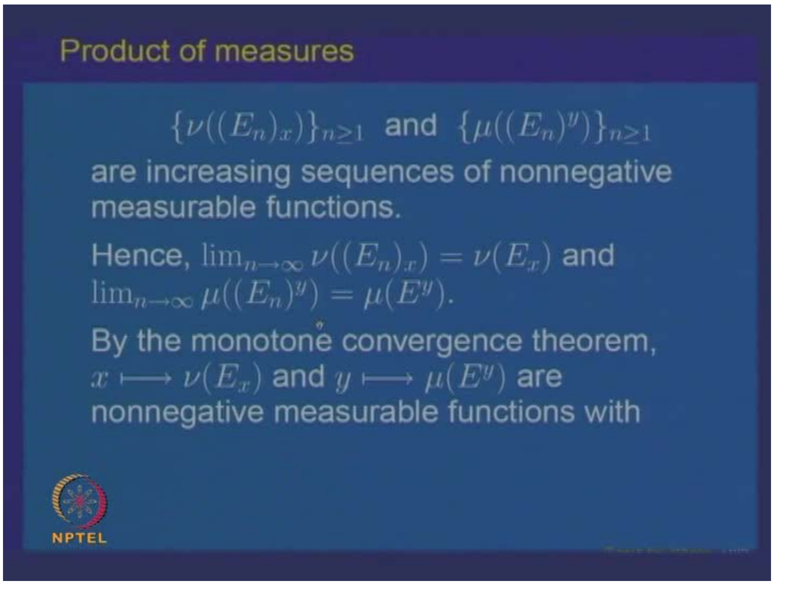

Simple fact that if E n's are increasing, the sections are increasing and mu and nu being measures imply mu of the sections E n's will converge to mu of E. So, mu of E x and nu of E y are limits of limits of measurable functions. So, they become measurable.

Till now, no finiteness condition has been used. So, this is true whenever mu and nu are any two measures.

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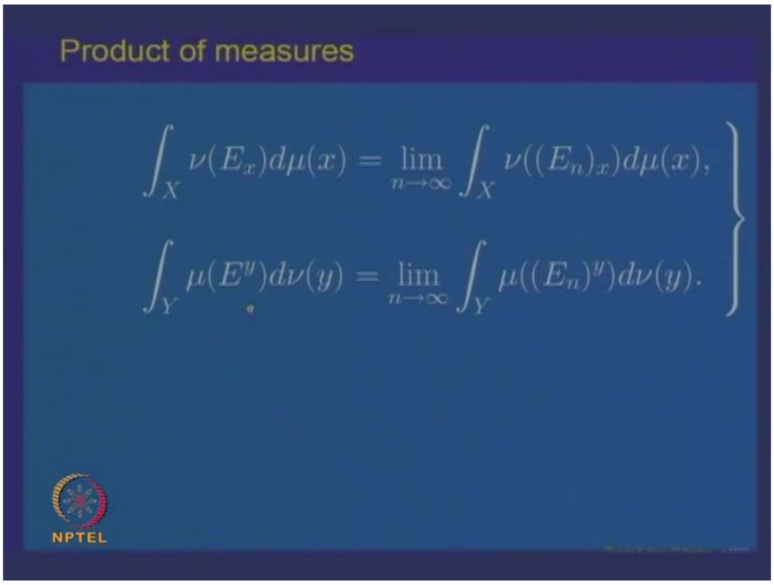

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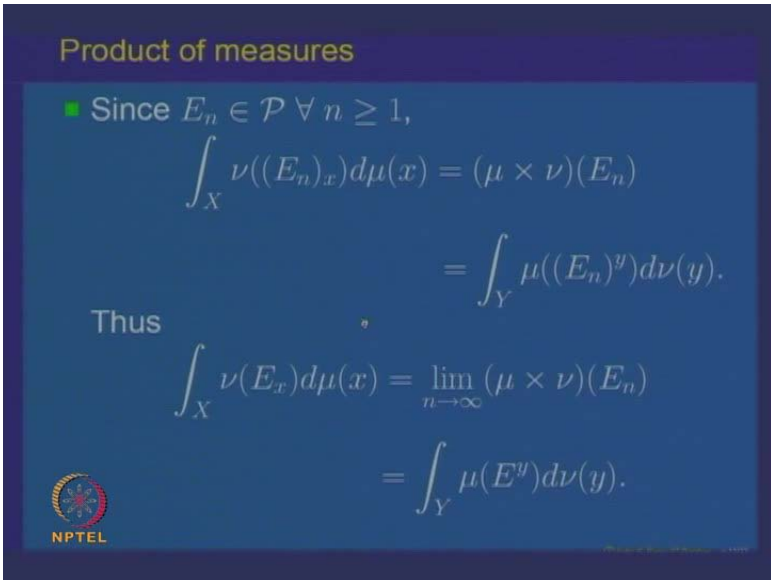

For the decreasing part, we will need the finiteness condition. So, for the increasing part everything goes straight monotone convergence theorem application gives you nu of E x is limit of that and that is equal to the product measure and everything is ok.

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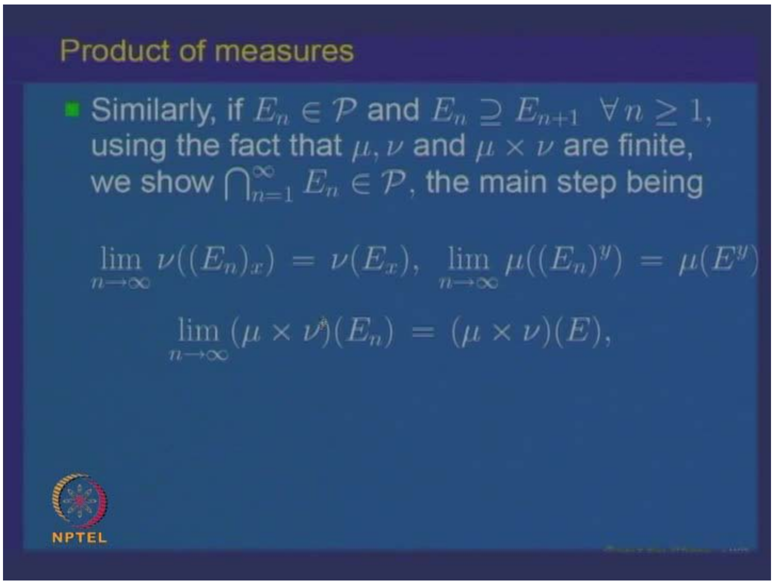

Let us look at the part, where we find difficulty arises. So, difficulty arises, when we want to show that E n belongs to P and E n’s are decreasing then the set E, which is a intersections of E n also belongs to P.

Here, the main step is to conclude that nu of E n x is equal to nu of E x.

For that we need finiteness condition because whenever a set a sequence of sets is decreasing to a set. Then measure of the sets need not converge to measure of the limiting set unless the measures are finite.

Finiteness conditions will give us that and then instead of monotone convergence theorem, we can apply the dominated convergence theorem to conclude that mu cross nu of E n is equal to corresponding integral.

So, that will prove that mu and nu being finite. P is a monotone class but still we are not concluded the proof for the general case. For the general case, one can apply the usual sigma finiteness criteria namely - whenever to measure a sigma finite, the whole space can be cut up into finite number countable disjoints pieces. Each of finite measure and on each the result holds and put them together to get the result holds for the whole space.

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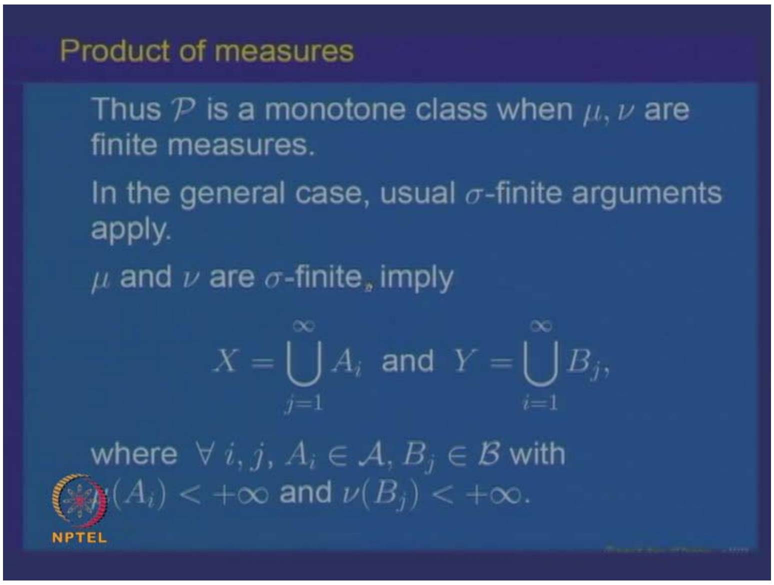

Let us see the argument, how it works? Because mu and nu are sigma finite, X can be decompose into a disjoint union of sets A i and Y can be decompose into a union of sets B j. Such that disjoint unions mu of each A i is finite and nu of each B j is finite.

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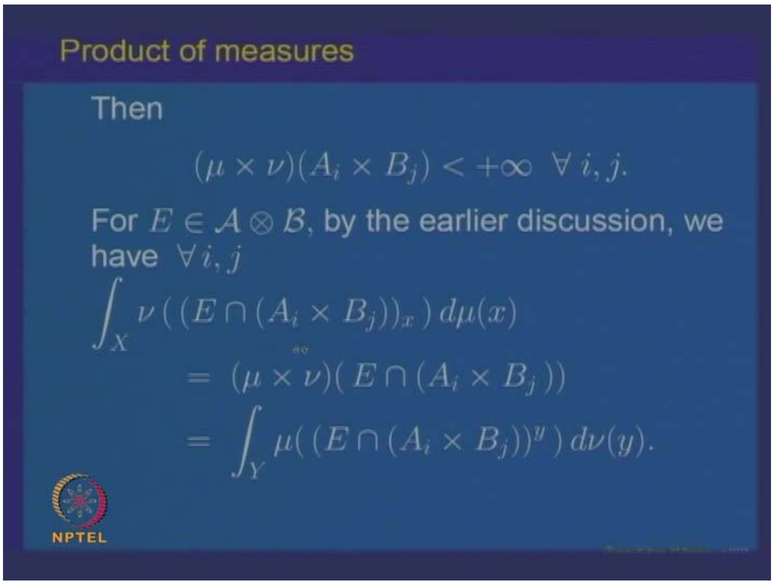

Using that, we can write down that mu cross nu of A i cross B j is finite because this is nothing but mu of A i times nu of B j. As a consequence, on each of these pieces our earlier results hold the P was a monotone class.

Let us see, how that is use to prove for a general set E in A cross B. For a set, in the sigma algebra A cross B note that the integral of the measure nu of E intersection A i cross B j x d mu x because each nu of each of these sets has got finite measure.

We are applying the earlier result on the piece A i times B j. For every i and j using the earlier case, we have that the integral over X of the x sections of E intersected A i cross B j is nothing but mu cross nu of E intersection A i cross B j and that is equal to the mu integral of the Y sections of the corresponding sets.

This step follows basically from the fact that mu cross nu of A i intersection B j is finite. For any set, E intersection this rectangle A i cross B j and on that rectangle mu and nu are finite. So, this earlier case gives us the result.

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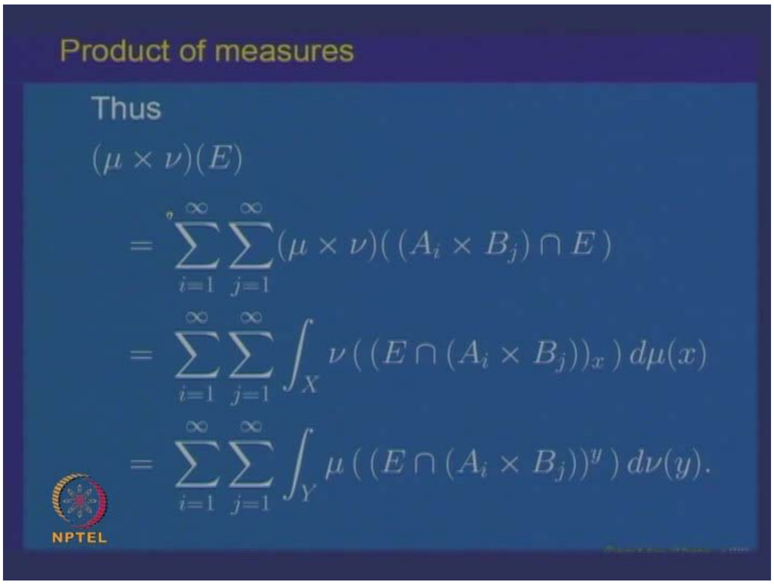

Now, we have to only sum both sides with respect to i and j. Let us look at mu cross nu of E which is equal to or the whole space is equal to union over i and j of the rectangles A i cross B j that is a partition.

So, mu cross nu of E can be written as using countable additivity of the measure mu cross nu. As summation over i, summation over j and mu cross nu of the pieces A i times B j. Now, for each one of this piece, we know the result holds. So, I can write this as integral of the x sections or as integrals of the y sections.

This term mu cross nu of A i cross B j intersection E is equal to this integral or this integral because of the fact that for the finite case the result holds.

Now, using the fact that if you look at the section E intersection A i cross B j of x. This section is nothing but nu of E x times A i cross.

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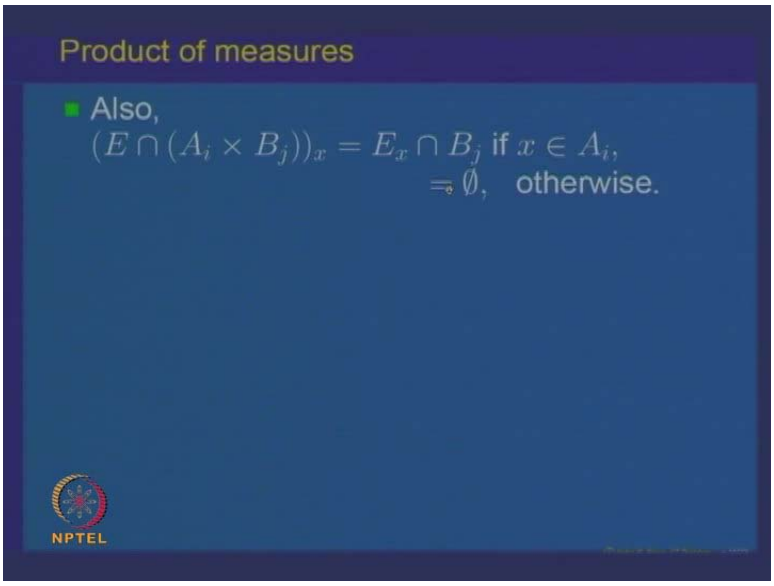

So, this is a small observation that you will look at set E and take its piece inside the rectangle A i cross B j and take it section. So, this section is going to be equal to the section of E intersection with B j. Of course, if x belongs to E j and x does not belong to E j then there is not going be any intersection. So, this is going to be empty set.

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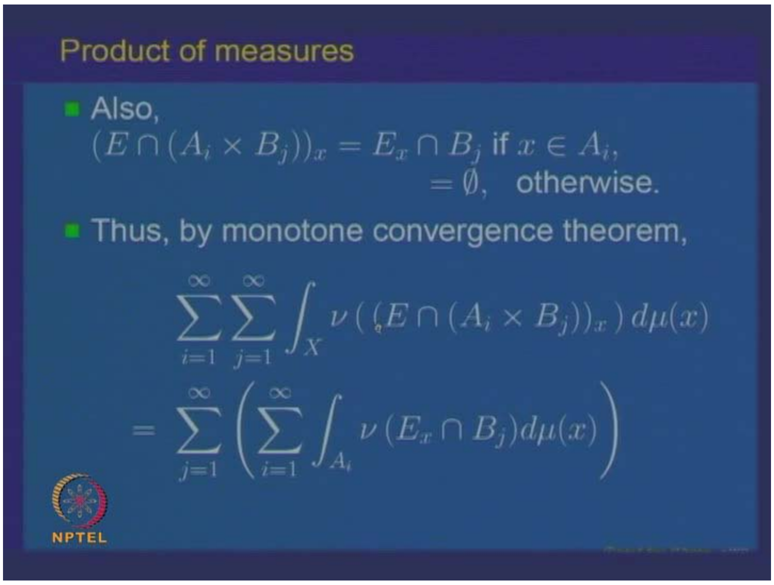

This is the observation and that observation can be used in this part that if x does not belong to A I, then this thing is going to be 0. Using that we can write that sum. So this sum which was integral over X of E intersection this can be written as so this set is nothing but nu of E x intersection B j because that is the only place, where the section appears.

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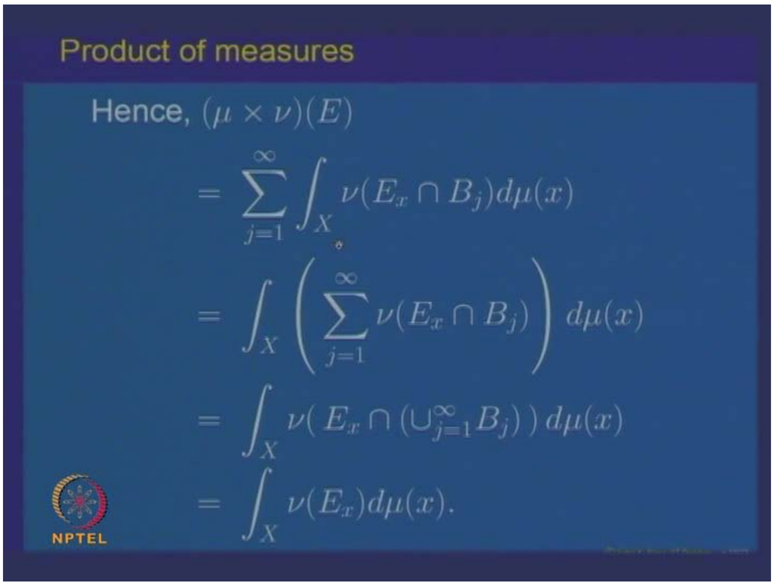

When x belongs to A I, this is integral over A i of nu E x intersection B j. So, this integral is equal to this because of this fact. Now, the summation over i means that this integral is over x. This summation you can transform into integral over X.

Now, you can interchange the two integral and the summation again you will be using fact here that this is an integral, which depends on j. So, you can push it out and take it inside basically you will be applying implicitly a monotone convergence theorem. To say that this is equal to, I can take the integral sign X and because this is the sequence of functions, which are non-negative measurable and so on.

Here, an application of monotone convergence theorem, which helps you to interchange summation and the integral sign. So, summation goes inside and now summation over B j B j is are disjoints that gives you over the whole space y so that is just E x.

We get that mu cross nu of E is equal to the integral of the section nu of E x d mu x. You see here, almost every step we are using sum theorem or the other to justify the facts. This is the case for the x sections and the similar results will hold for y sections.

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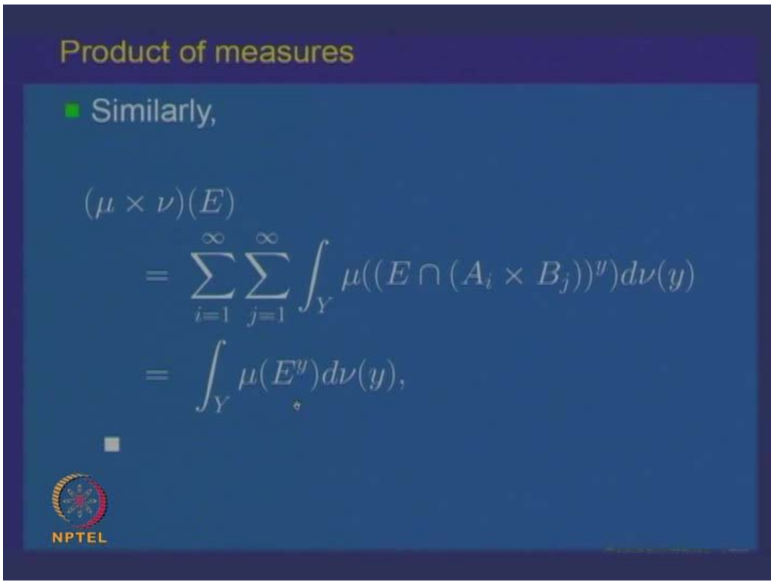

That will prove that mu cross nu is also equal to integral over the y sections and that will complete the proof of the fact that one can reduce the result in the case of sigma finite. So, from finite to sigma finite is almost straight forward in the sense that we can split the whole space into countable number of pieces of finite measure. So, on each piece we apply and then submit up to go back to the original piece.

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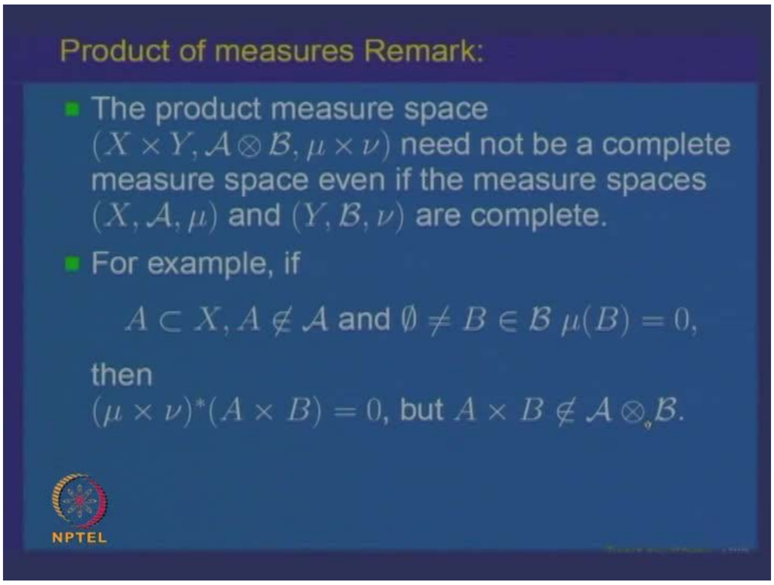

We have proved the theorem namely how to compute the measure of a product set.

Let us observe one thing here, even if we start with measure spaces X A mu and Y B nu to be complete, the product measure space which we are denoting by X cross Y A times B mu cross nu need not be complete.

Because how do we get this measure mu cross nu on A cross B, we looked at product mu cross nu on rectangles and extended it and defined the outer measure and then looked at measurable sets mu cross nu and that included the sigma algebra.

This A times B the product sigma algebra is not the sigma algebra with respect to which of all mu cross nu measurable sets. So, it may not be complete say for example, you can take any set A in X such that A does not belong to the algebra A.

Take any non-empty set B of measure 0, then the outer measure of mu cross nu will be equal to 0 because mu of B is equal to 0.

But the rectangle A cross B does not belong to product sigma algebra because A does not belong to A. In case, one wants to look at the completion of this, which is possible.

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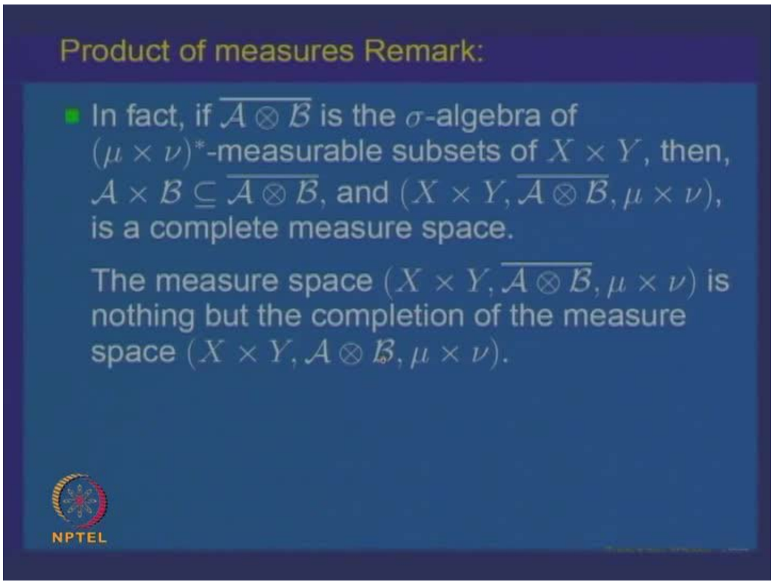

If we look at the sigma algebra A times B bar and denote that to be the sigma algebra of mu cross nu are measurable product sub sets. The product space, then of course that the product sigma algebra is inside it and that will be a complete measure space. We can say that X cross Y and mu cross nu measurable sets as before is the completion of product measure space X cross Y A times B mu cross nu.

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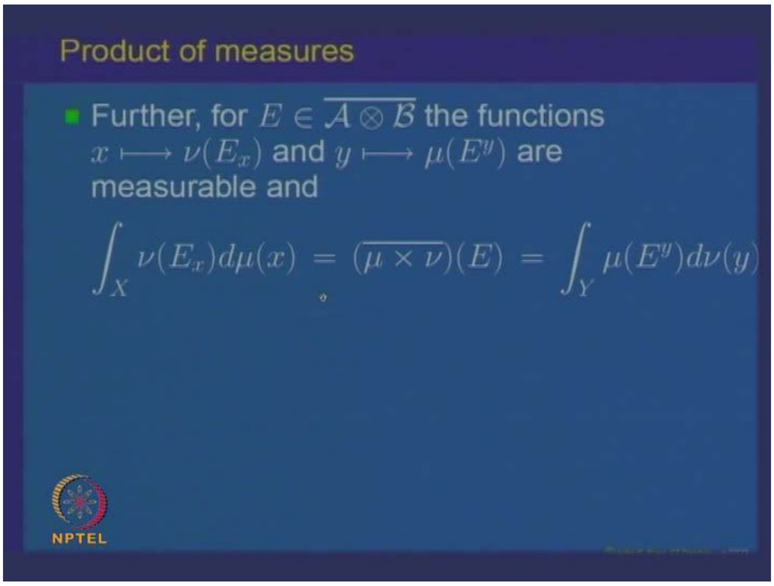

This is a just a small observation, which we should keep in mind that the product sigma algebra, which is a sigma algebra generated by the rectangles need not be giving you a complete measures space.

However, one can always complete it and the corresponding result holds for sets in A times B that is a small technical results, which we can proved. We had proved this result for sets in the product sigma algebra; you can integrate the sections and get back the product measure. This also applies to any set E in the product sigma algebra that means in the completions space also the corresponding result holds.

This is the way; we can compute the product measure of a set in the sigma algebra. I want to go over to an interpretation of this result, which leads to a very important result in integration of product spaces.

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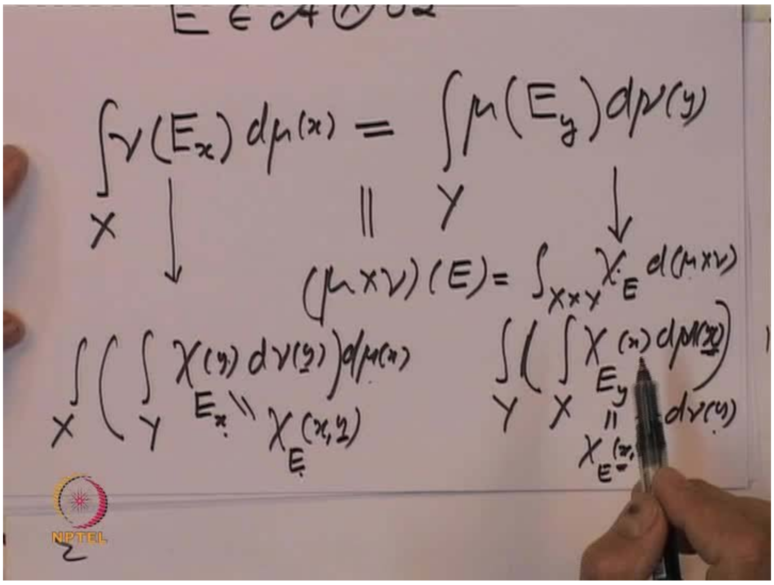

What we had was the result, what we have shown is for every set E in the product sigma algebra A times B. We can take it section with respect to every point x that gives us a set in the sigma algebra B. We can define nu of that and that becomes, we show it is a non negative measurable function.

So, I can integrate this over X with respect to mu. On the other hand, I can also take the section of E with respect to every point y and then take its measure. We showed that the section belong to A. Take its measure mu of E y. We should that that is a non negative measurable function and I can integrate it over Y d nu of y.

We showed that these two are equal and in fact both of them are equal to the product mu cross nu of E. But a simple observation that the measure of a set is the integral of the indicator functions.

What is this? I can write it as integral over X, this nu of E x I can write it as integral over Y of the indicator function of E x y d nu y. Similarly, this thing I can write it as integral over Y mu of E y. I can write integral of over X of the indicator function of E y x d nu of y.

Then, we should have d mu of x. This E y and this is d nu y. So, this E y there should be d mu of x and then d nu of y. This product thing, I can write it as integral over X cross Y of the indicator function of E d the product measure mu cross nu.

We get an integral representation of this result that I can take the indicator function of the set E but note that the indicator function of E x y is nothing but see this is non zero when y belongs to E x that means x comma y belongs to E.

So this is just the indicator function of E x comma y. Similarly, this also the indicator function of E x y. Everywhere, it is an indicator function of E. What we are saying is look at the indicator function of the set E and integrated with respect to y. Keep x fix and integrate with respect to y that depends on x integrated with respect to x or take the indicator function of E then integrated with respect to x. Keep y fix, that integral depends on y and integrate over y. So, that is another number that you will get and it says both of them are equal to integral of the indicator function of the set E with respect to the product measure mu cross nu.

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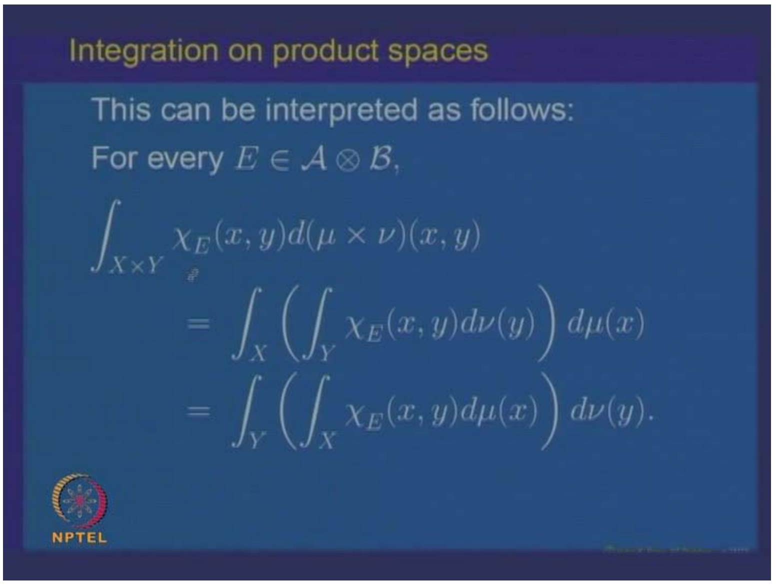

Let me just rewrite and show it to you in the form of in the slide. What we are saying is the result that we proved just now for every set E in the product sigma algebra A cross B, I can rewrite the result in the form of integrals that it is same as saying that the integral of the indicator function of E with respect to the product measure mu cross nu is same. Look at the indicator function it is a function of two variables.

For this function of two variables, I can fix in x. If I fix in x and vary only y then this indicator function becomes a function of one variable y.

It says let me integrate this function indicator function of E for a fixed x with respect to y. This integral can be computed and this integral depends on x. It says that, it is a measurable function and its integral can be taken with respect to x with respect to the measure mu and that is same as that integral.

Similarly, instead of fixing the first variable x, I can fix the second variable as y. I can fix this as y, and then this becomes a function of x. I can integrate it to with respect to x. I get a number which depends upon y and that function is integrable with respect to y and that integral is also equal to the original one.

So, the result of computation of product measure of a set E in the set A cross B can be written in terms of the integrals of indicator function over the product set. Basically, this illustrates that to integrate the indicator function, which is a function of two variables. I can integrate it as one variable at a time.

This is an important result, which leads to important results in integration that given a function of two variables, if you want to integrate it with respect to the product measure then, this gives the hint that possible what we can do is fix one variable of the two variable function. So, it becomes the function of one variable, integrate it out the one variable. Then, it becomes the functions of the other variable, integrate out that variable also you get the integral with respect to the product measure. We will prove this in the next lecture. This result can be extended to non negative measurable functions on product spaces and eventually it can be extended to integrable functions.

That leads to important theorems in the theory of integration on product space is called Fubini’s theorem. So, we will continue looking at that in the next lecture. Thank you.