Measure and Integration

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Module No. # 07

Lecture No. # 26

Computation of product Measure - I

Welcome to lecture number 26 on Measure and Integration. We have been studying the properties of the product measure spaces. We defined what the product of two measure spaces is and then, we started looking at the problem of how to compute the measure of an element E in the product sigma algebra. We will continue that study in today's lecture. So, in today's lecture the main aim is going to be computing the product measure.

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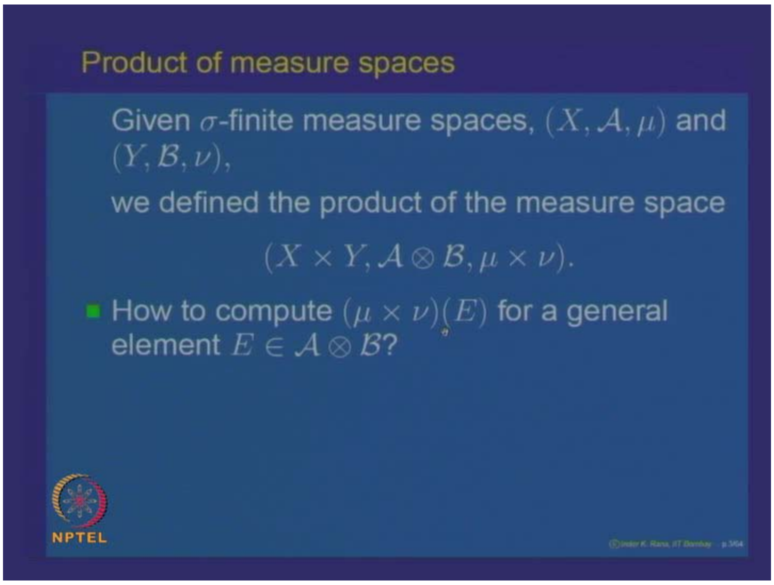

Let us just recall that given a sigma finite measure spaces X, A, mu and Y, B, nu; we had defined the product measure space namely, X cross Y and the sigma algebra on X cross Y is A times B, which is the sigma algebra generated by all measurable rectangles. That is, sets of the type E cross F where E belongs to A and F belongs to B and mu cross nu is the extension of measure which is defined on rectangles by the property that mu cross nu of a rectangle E cross F is mu E times nu of f then, via outer measures; we extend that to the sigma algebra A cross B.

So, the problem we wanted to analyze was that, given a set E in the product sigma algebra A times B, how do we compute the product measure mu cross nu of E, because at present we only know that mu cross nu is defined on the sigma algebra A times B via the extension theory. So, to do we had said that we try to look at what are called the sections of the set E; let us recall what is called the section.

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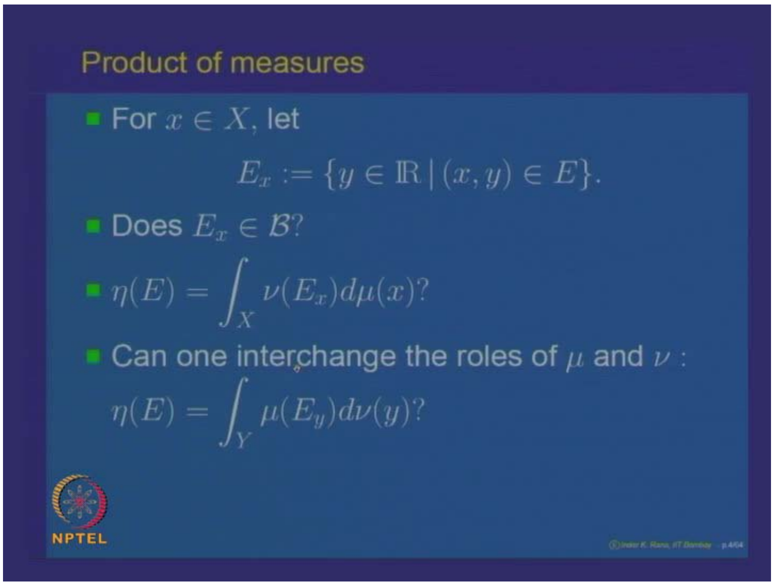

For a point element x in E - E lower X - this is the notation used for the set of all points y, this is not R this should be in y, so all the points y in y such that x comma y belongs to E. Similarly, we can define the section with respect to a point y in y. The main questions that we had formulated in the previous lecture where, can we say that this section is a element in sigma algebra B - this is a subset of y - is it an element in the sigma algebra B?

If yes, then we can define nu of E x which depends on x, so we get a function x going to nu of E x. The question is, is that a measurable function as a function of x? If it is measurable, we can define this integral with respect to mu and then, ask whether that is equal to the product measure eta of E. Similarly, can we interchange the role of x and y? These are the questions we want to analyze.

If it turns out to be true, then this gives us the way of computing the product measure of a set E by looking at the sections and taking the new measure and then adding up or integrating and similarly, the other way around. Let us start analyzing these problems one by one.

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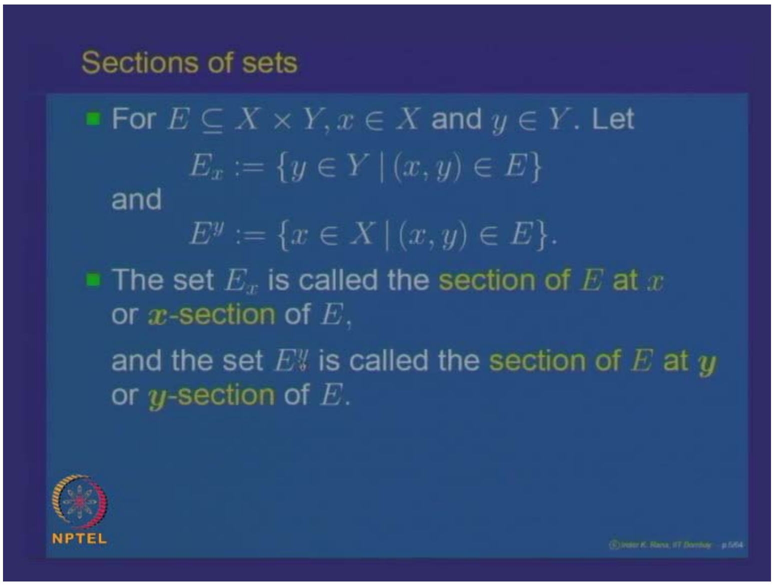

For the sake of clarity, let us note down E; for any element E, E a subset X cross Y, x an element in the set X and y an element Y. The section E lower x is defined as all points y in Y such that x comma y belongs to E. Similarly, the set E upper y whenever the point is coming from x, we will write it on the bottom E lower script x and whenever the point is coming from the set y, we will write as E superscript y on the right side, as all points x in X such that x comma y belongs to E.

So these, we will call as the sections, so E lower x is called the x-section of E at the point x and it is a subset of y. Similarly, the y section of E are the section of E at y is a subset of x that is E upper y.

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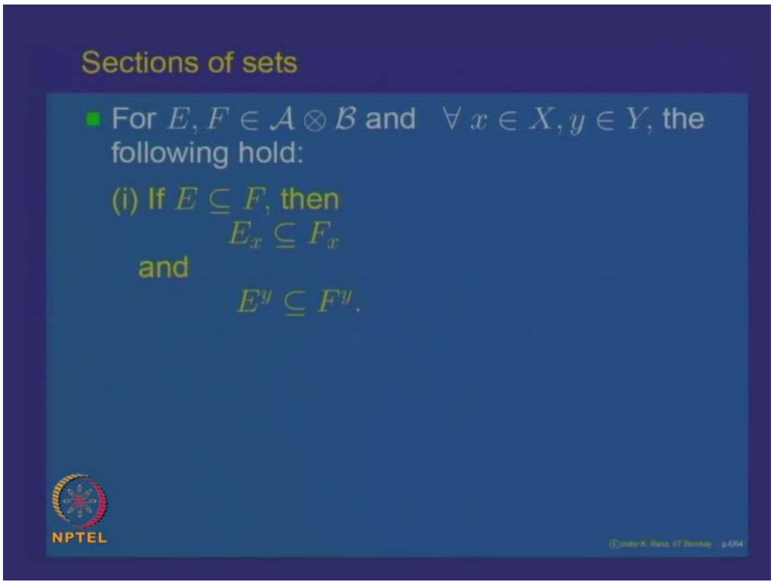

The questions we will like to analyze are the following and we want to claim that the following holds. Let us look at the some general properties of these sections; the first is, if E and F are subsets such that E is a subset of F then, the section of E at x is a subset of section of F at x and the section of E at y is a subset of the section of F at y. Let us prove these properties before going further.

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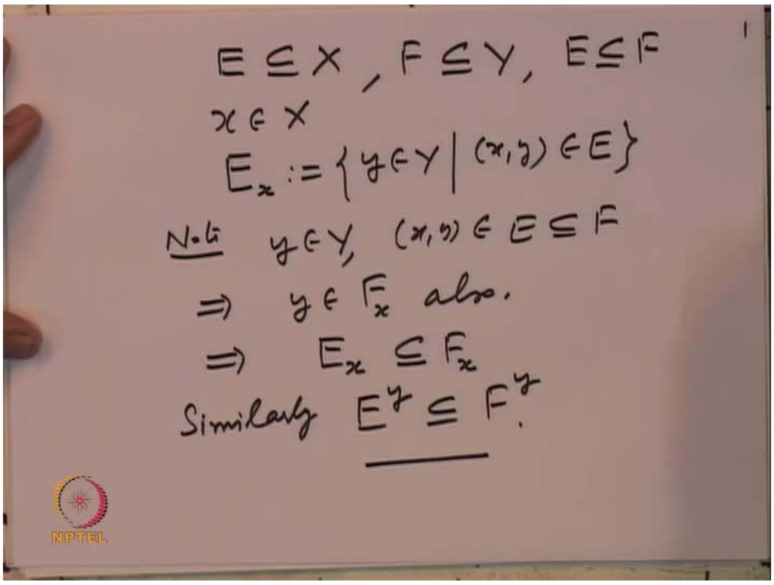

Let us verify namely, we are given E is a subset of X and F is a subset of Y. So for x belonging to X, let us look at what is E x and we are also given that E is a subset of F. So for any point x in X, let us look at the section E lower X that is, by definition all points y in Y such that x comma y belongs to E. So, if x comma y belongs to E and E is the subset of F that means from here, so note - what we are saying is note - y belonging to Y we have x comma y belongs to E that is the property here and E is the subset of F, so x comma y belongs to F that implies, y belongs to F of x also. So, this is the definition of E F lower x namely, y in Y such that x comma y belongs to F.

This implies that E lower x is a subset of F lower x. The other property similarly, we will have that E section at y is a subset of the section of F at y; these two properties hold; so, that is property one. Here, we do not use any effect that E cross F the subsets are in the sigma algebra A cross B, this is true for any subsets.

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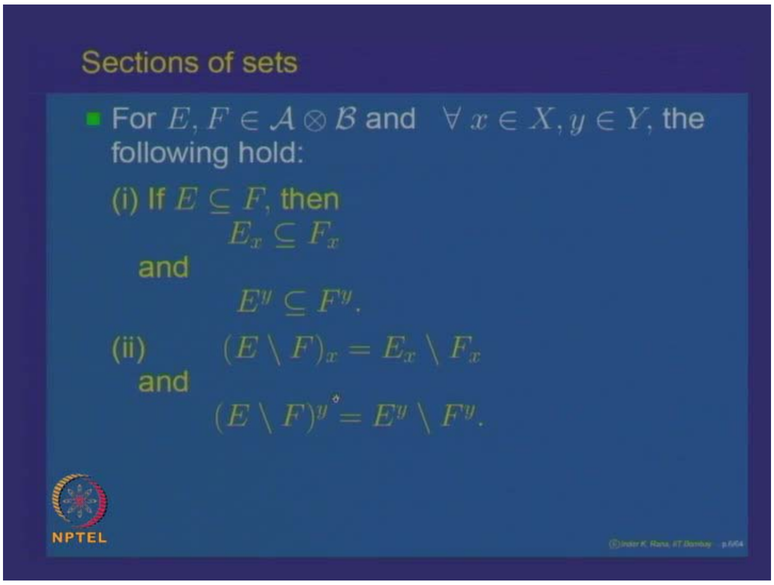

The next property says that if we look at the difference E and difference F and take its section, it is same as first taking the sections and then taking the differences whether at a point x or at a point y.

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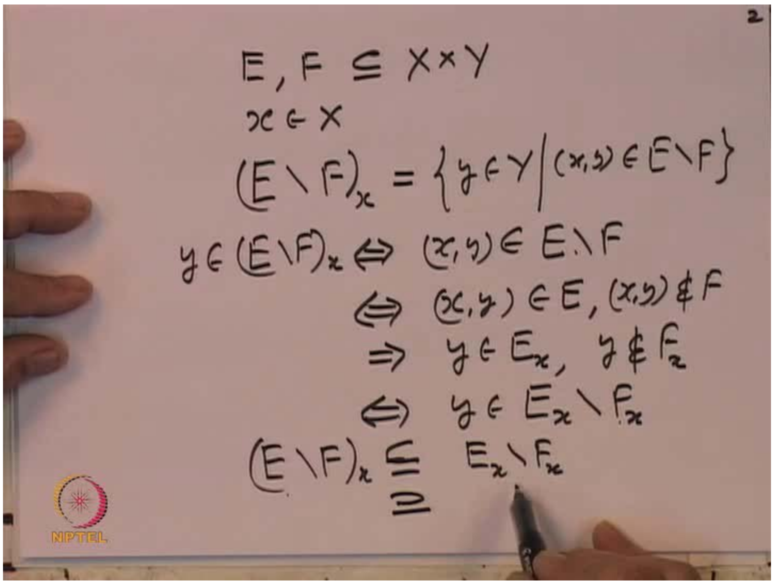

Let us analyze this, if E and F are subsets of X cross Y, we are going to look at the difference and the sections x is a point in X. Let us look at the section of E minus F at the point x. By definition, this is all points y belonging to Y such that x comma y belongs to E minus F.

So, what does that mean? That means, y belonging to E minus F section implies that x comma y belongs to E minus F, but what is the meaning of saying that x comma y belongs to E is difference F that is same as saying that x comma y belongs to E, but x comma y does not belong to F that is the meaning of this. That is same as saying that x comma y belongs to E that means, y belongs to E x and x comma y does not belong to F that means, y cannot belong to the section of F at x.

That implies, y belongs to the section of E at x but does not belong to the section of F at x that means, y belongs to E x difference of F of x. So that says, E difference F section x is a subset of E lower x difference F lower x, but notice in this all the arguments are reversible. Suppose, if y belongs here that is same as implying the earlier statement that y belongs to E x and y does not belong to F section at x, but that is same as saying x comma y belongs to E and x comma y does not belong to F and that is same as saying that the earlier statement that is the meaning, that is implied. All the arguments are reversible the other way round inequality also holds, so these two sets are equal.

It says that if you take the difference of E with F and then take the section at x that is same as taking the sections first and then taking the difference, so that is at point x. A similar proof will work if you take the difference of E with F and take the section at y. The corresponding property says, it is first taking the section and then taking the difference for the corresponding section at y.

Basically we are saying, the properties of subsets are preserved undertaking sections that is a part one and the properties of taking sections is preserved also undertaking differences of sets. So, whether you first take the sections and then take the difference, it is same as first taking the difference and then the sections. So, these are two elementary properties of the sections. Let us look at some more general properties of the sections. Once again these are true for any sets not necessarily sets in A cross B, but we will be using them only for A cross B but they are true.

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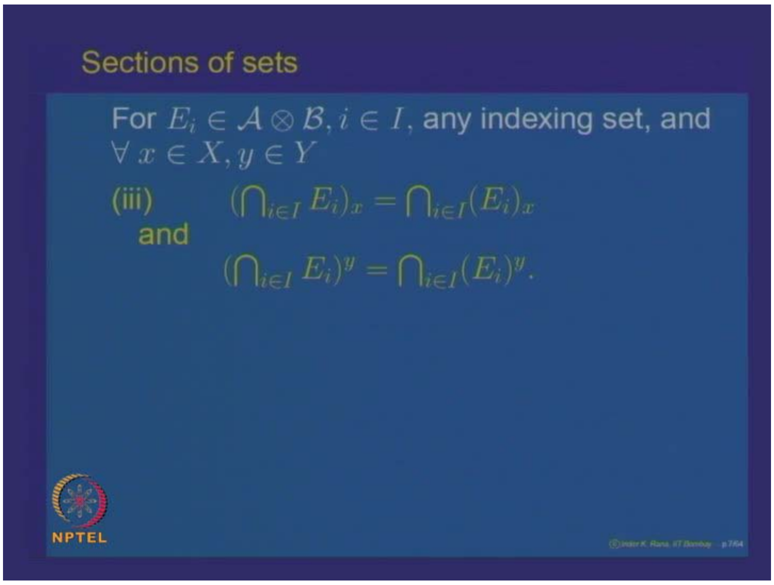

Let us look at a sequence of sets, not a sequence. Actually, a arbitrary family of sets E i’s in subsets of X cross Y, where i is any indexing set. Then, if we look at the intersections of the sets E i’s and then take the section, the claim is that it is same as taking the sections first and then taking the intersections and similarly at the point y, let us take a section whether at x or at y.

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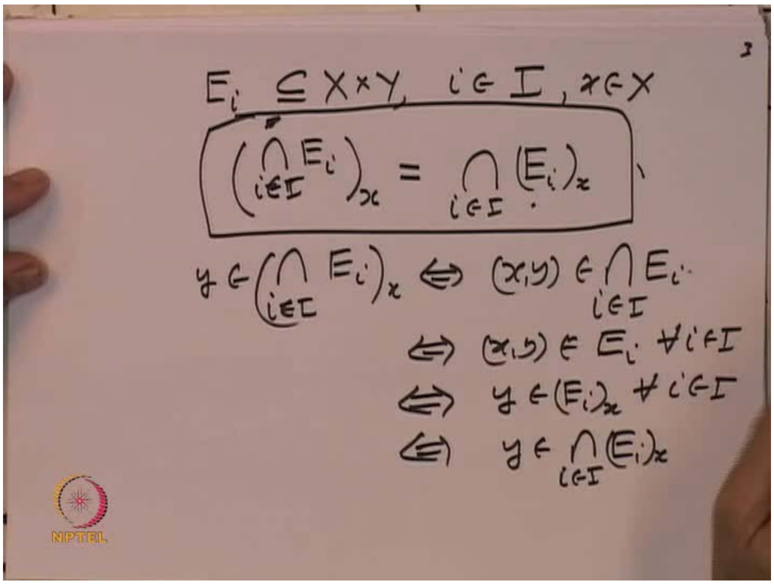

Let us look at this property, so E i's as subsets of X cross Y, i belonging to some indexing set I. We want to look at and take the intersections of the sets E I, i belonging to I, look at the intersections of this family E i's and then take its section at a point x. Let us take a point x belonging to X, we want to compute and show that this is same as first take the section of every set E i at x and then take the intersection i belonging to I…

To show this, let us take a point y belonging to intersection i in I of E i at x. Let us take a point y in this set on the left hand side that is, if and only if the definition says that means, x comma y belongs to intersection of i in I of the sets E i. If a point belongs to the intersection that means, the point belongs to each one of the sets, so it belongs to E i for every i belonging to I. Once that is true, saying that x comma y belongs to E i that is same as saying y belongs to E i x for every i belonging to I.

So, y belonging to the intersection section is same as y belonging to intersections. For every I, that means y belongs to intersections of the sections of E i at x and i belonging to I that proves that this property is true. So, what we have shown is that this property is true. For any arbitrary family of subsets of X cross Y, if you take the intersections of this sets and then take the section at a point x, it is same as first taking the sections and then taking the intersection of those sections.

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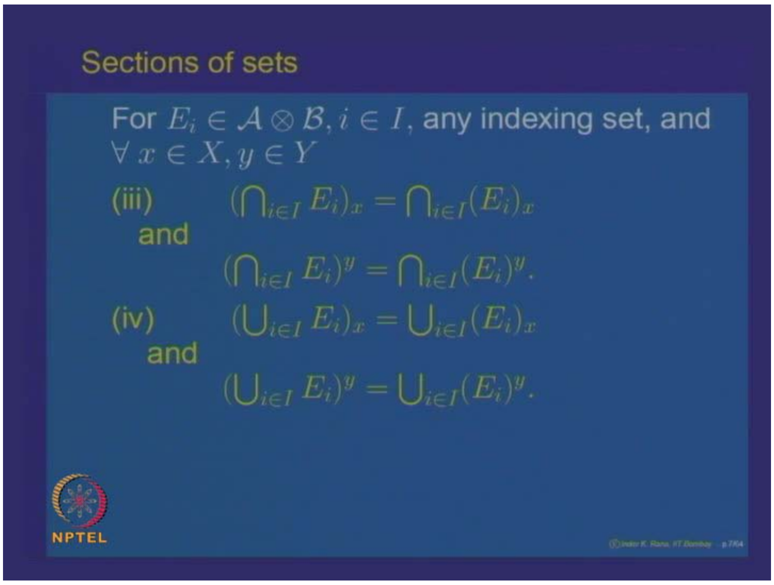

At a point x in x and a similar proof for the sections at y namely, for every y in y the section of the intersection is same as intersection of the sections. The corresponding result also is true for the unions, so let us prove that also. Once again the proofs are similar in all these cases; this is pure set theory actually.

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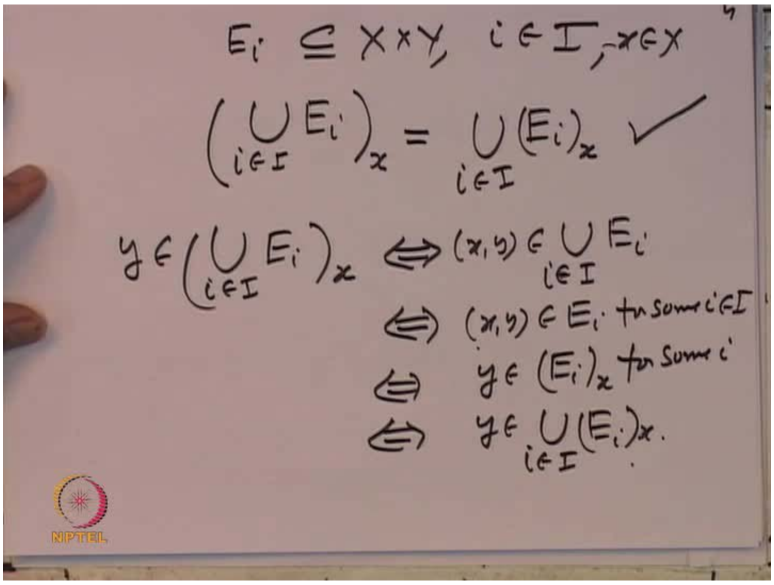

So, what we want to do is E i’s are subsets of X cross Y where i belongs to I, what we want to do? We want to look at the union of these sets E I, i belonging to I and then take its section at a point x, so let us take a point x in X. We want to show this is same as this, for each one of the E i's take the section x at a point x that X we have fixed and then take its union over i belonging to I.

So, section of the unions is equal to union of the sections that is what we want to prove. To prove, note that y belong to the left hand side of the set that is i belonging to I union E i and its section at x, but y belonging to this section at a point x is same as saying that the point x y belongs to the union of E i's i belonging to I, but the definition of saying that a points belongs to the union means at least it should belong to one of them.

So, x comma y belongs to E i for some i belonging to I but that is same as saying x y belongs to E i that means, y belongs to the section E i at x for some i that is same as saying it belongs to the union. So, it belongs to at least one of the sections of E i's that means, y belongs to union i belonging to I of E i at the point x.

So, y belonging to the section of the union, if and only if, y belongs to union of the sections that proves that this property is true namely, section of the unions is union of the sections at a point x in x and a similar property holds section of the unions at a point y in y.

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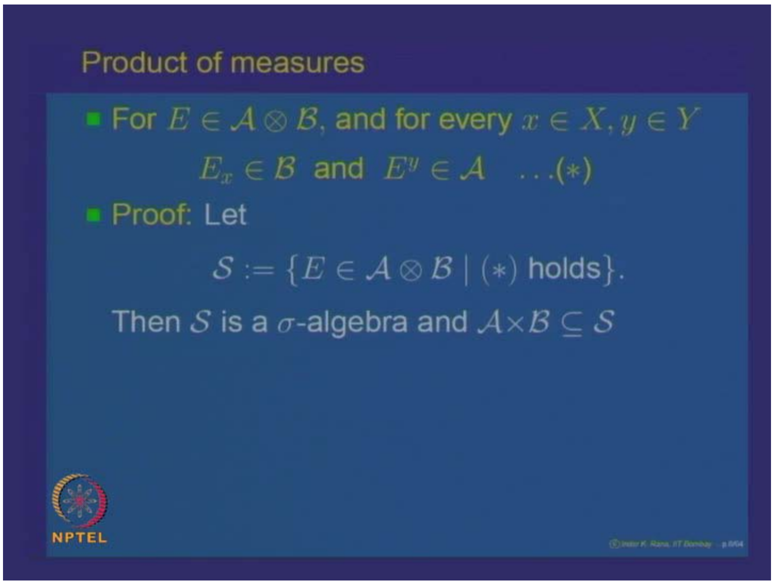

Basically, what we are saying is all the set theoretic operations behaves nicely with respect to the taking sections and this is true for all subsets E i's of X cross Y. Now using these properties, we will prove if E is a set in the product sigma algebra A times B, x and y are elements where x is in X and y is in Y then, the claim is that the section E x belongs to B and the section E y belongs to A.

That means, for every x in x look at the subset of y which is the section of E at a point x that belongs to the sigma algebra on B whenever E is element in the product sigma algebra. Similarly, the section at y is a subset of x and our claim is that this belongs to the sigma algebra A.

So, these are the two properties that we want to check for every set E belonging to A cross B. Now, here is the technique of proving all these results in the product sigma algebra. Basically, we will apply the monotone class sigma algebra techniques. Whenever we want to show a property holds for elements in A cross B, we will collect together all subsets for which this property is true and try to show - that we will collect sets for which this property is true in A times B - that the collection includes rectangles and this collection is sigma algebra. Once this collection is sigma algebra and includes rectangles, it will include the product sigma algebra A times B.

That is what I called as the sigma algebra technique; we will apply that technique here. Let us define the collection S to be all subsets in A times B such that this property which we are calling as star, so E x the section at x belongs to the sigma algebra B and the section at y belongs to the sigma algebra A. What we want to prove? We want to prove that S is equal to A times B. To prove that S is equal to A times B, we will prove two things namely, S is a sigma algebra and it includes rectangles and that will prove that it actually is equal to A times B.

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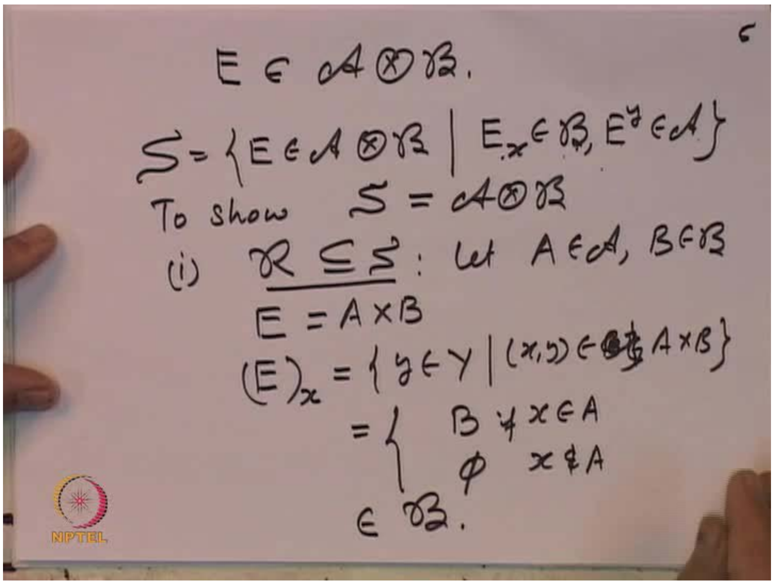

Let us prove these properties, so what we are given? We are given that the set E belongs to the sigma algebra A times B. So, S is the collection of all the subsets E belonging to A cross B such that the section E x belongs to B and the section E y belongs to the sigma algebra A. To show, S is equal to A product sigma algebra B; note it is already a subset of A cross B.

We will follow two things; let us check the properties of this first is, the rectangles are inside S. To check this property, let A belong to A and B belong to B and let us take the rectangle E which is equal to A cross B. If you recall, we had calculated what the section of E at x that is, all y belonging to Y such that x comma y belongs to B sorry A cross B.

Now, x comma y can belong to A cross B only when x belongs to A and in that case, y should belong to B. So if x belongs to A, for all x belonging to A this set is equal to B, the section is equal to B, if x belongs to A. If x does not belong to A then in no way x comma y is going to belong to this is empty set if x does not belong to A.

For a rectangle - we have already seen it, I am repeating the steps which we have done earlier - for a rectangle A cross B, the x section is either B or empty set, in either case this belongs to the sigma algebra B. The property that E x belongs to B is true, a similar argument will show that E y also belongs to A; this proves that the rectangles are inside the sigma algebra S.

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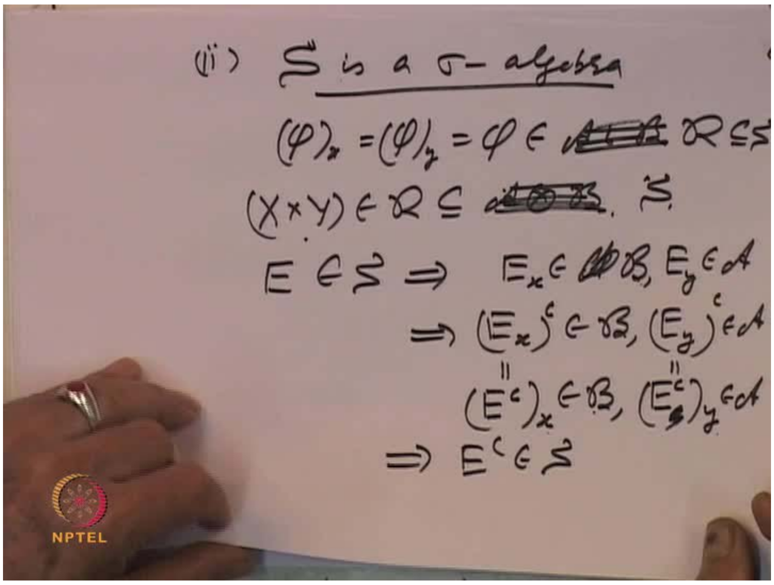

The next step we want to check is the following second step, what we want to check is that this collection S is sigma algebra; this is what we want to check. For that, the first property look at the empty set; the sections of the empty set either x section is same as the section y and that is empty set belongs to both A and B. Similarly, if I look at the whole space that is X cross Y that is actually a rectangle which is inside A times B and, sorry, already rectangles are inside S. So, both the whole space and empty set are inside S A and B, hence it is also a rectangle. Actually, we should say that this belongs to a rectangle and which is part of S. So, empty set and the whole space both belong to S.

The next property is, let us take a set E belonging to S and show that its compliment also belongs to A. E belongs to S implies the sections E x belongs to B and E y belongs to A that is the definition of S. Let us just recall what was the definition of the set S? The definition of the set S is all subsets A cross B say that E x belongs to B and E y belongs to A.

By the definition this is true, but E x belongs to B and B is a sigma algebra, E y belongs to A and A is a sigma algebra that implies, E x compliment belongs to B and E y compliment belongs to A, because of the properties of sigma algebras that A and B are both sigma algebra, so they must be closed under compliments.

On the other hand, this set taking section and the compliment just now we observed it is same as that I can take the compliment first and then take the section that should belong to B. Similarly, here the section y and then compliment is same as taking the compliment first and then taking the section that should belong to A, so this set is same as this set.

For every set E in S, if I look at the set E complement its section at x belongs to B and its section at y belongs to A that implies, E compliment also belongs to S. So, S is closed under taking compliments. Finally, to show it is a sigma algebra I have to show it is also closed under say countable unions.

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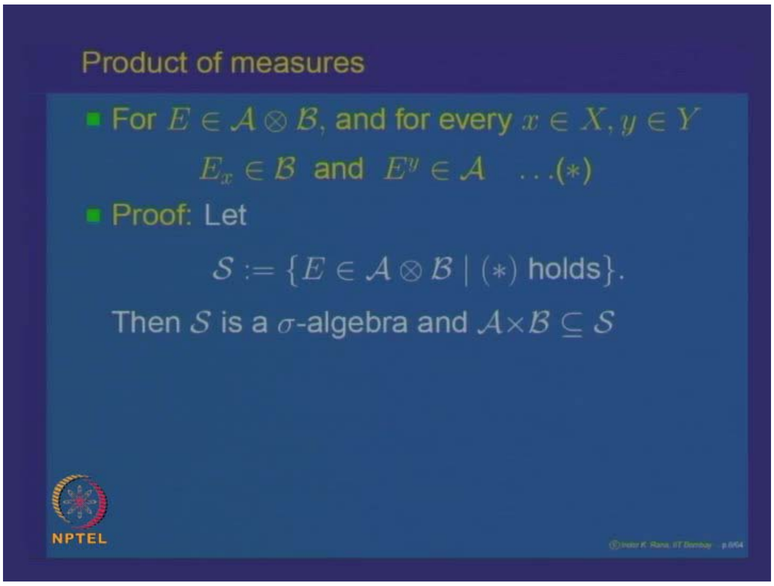

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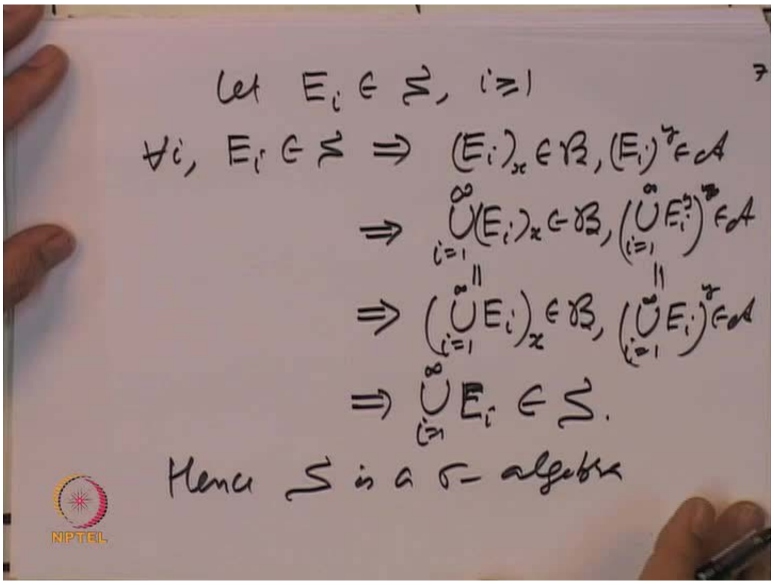

To show, let E i’s belong to S; S bigger than or equal to 1, but each E i belonging to S implies - for every i E i belongs to S implies - that E i section at x is in the sigma algebra B and E i section at y belongs to the sigma algebra A, this property is true. Once again, A and B both are sigma algebras that implies that the union of E i’s sections at x i equal to 1 to infinity belongs to B. Similarly, the corresponding one the union i equal to 1 to infinity of E i’s section at y belongs to A.

This is true but that implies by the fact that this set taking the sections and taking the union is same as first taking the unions and then taking the sections, just now we observed so that belongs to the sigma algebra B. Similarly, this set first taking, sorry, this was E i’s at y because unions belongs to A that is same as - now I can write as this is same as - union 1 to infinity of E i’s at section at y belongs to A. That implies, the set union of E i’s its section at x belongs to B and its section at y belongs to A that means, this belongs to the calculation S, hence S is a sigma algebra.

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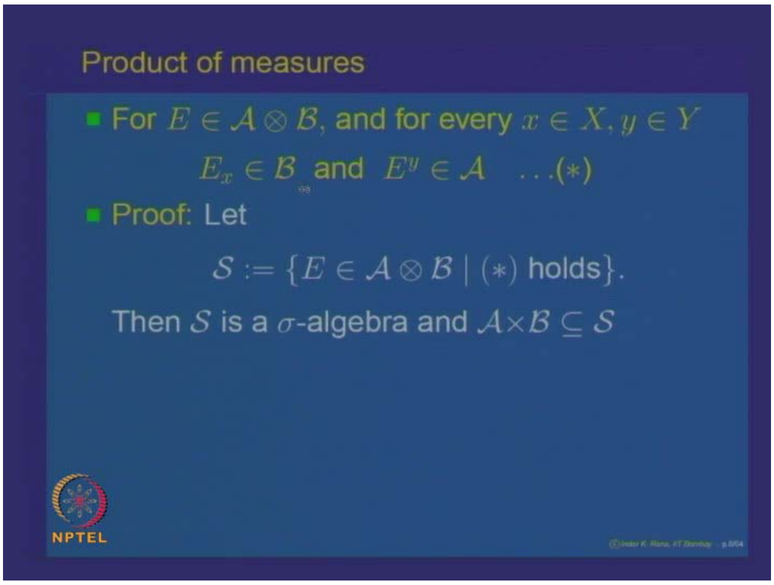

S is a sigma algebra and we know that rectangles are inside S that implies that A cross B is inside S; S is a subset of already A times B all these are equal that means, the property for every set in the product sigma algebra the x section belongs to B and the y section belongs to A is true. Let me once again emphasize the fact that we are looking at this proofs, which are nothing but application of the technique called the sigma algebra technique.

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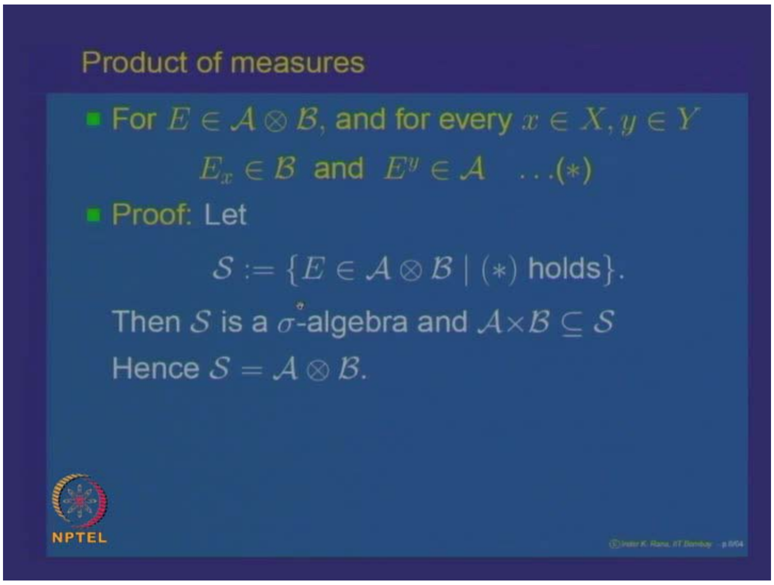

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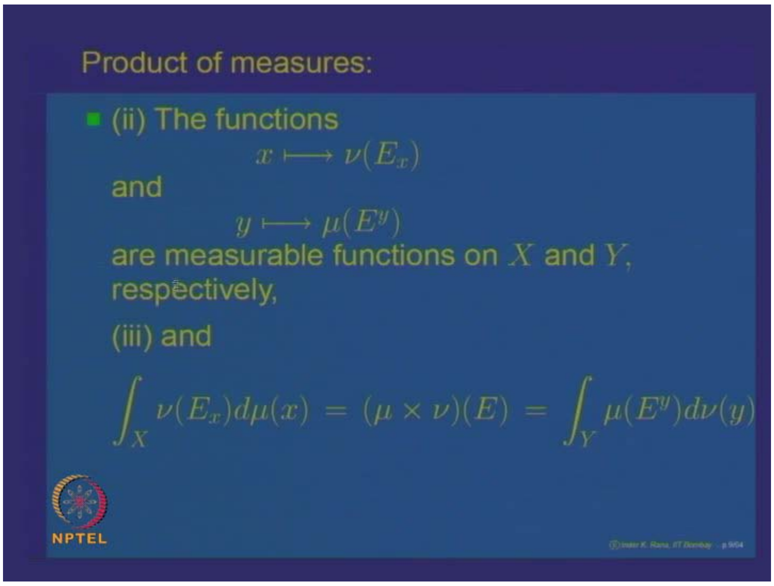

Now, let us go to the next property namely, we want to check the property. We already know that for every x, E x is a section of E at x, if E is in the product sigma algebra this set E lower x E section of E at x is in the sigma algebra B. So, nu of that set makes sense because nu is defined on the sigma algebra B. Similarly, the section of E at y is in the sigma algebra A, so mu of this section makes sense but both nu of E x depends on x and nu of E y depends on y. This gives us two functions x going to nu of E x and y going to nu of E y. The first one is a function on the set x and the second one is a function on the set y.

We want to prove that both of these are measurable functions and clearly these are non-negative functions, so they are non-negative measurable functions on x and y. Their integrals make sense with respect to - this is a function on x, its integral with respect to - mu makes sense and this is a non-negative measurable function with respect to y. So, x integral with respect to the measure nu make sense. We want to claim that the integral of the first function with respect to mu is same as the product measure of the set E and which is same as the integral of the second function with respect to mu.

That will give us a nice way of computing namely, the product measure of a set E can be computed either by taking its sections with respect to x, finding the size of those sections that is, the new measure of the sections with respect to x and then summing it up that is taking integrals with respect to mu or we can interchange the roles of x and y. We can take sections of E with respect to y. First take its measures with respect to mu and then add up, so take integrals with respect to nu. So, we want to prove that this property 2 and property 3 hold for every subset E of product sigma algebra A cross B.

Once again, this proof is going to be an application of the sigma algebra monotone class technique and you will see how effective these techniques are. So, what we will do? We will collect together all the subsets of A cross B for which these two properties are true and we will try to show rectangles are inside it and hence everything is inside it.

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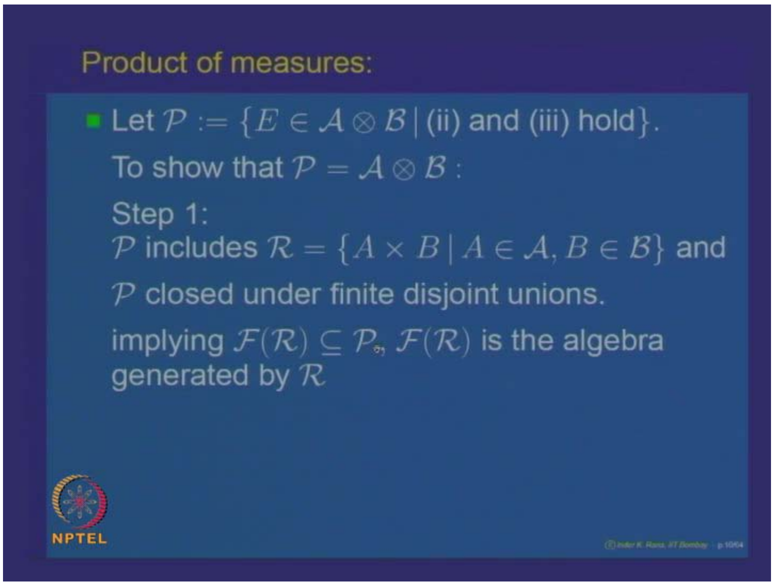

Let us look at the collection P of subsets of E cross elements in A cross B so that property 2 and 3 both hold. So, what is going to be our technique and what is our problem to be proved? The problem is to show that P is equal to A times B, to show that we will do the following. First we will show that rectangles are inside A cross B, so that is 1, that the set of all rectangles are inside the class this collection P. We will show the second step that this collection P is closed under finite disjoint unions.

So, what will that prove? You recall, we had shown that R is sigma algebra and if the collection P which includes R is closed under finite disjoint unions that means, finite disjoint unions of elements of the rectangles also will be inside P, but finite disjoint union of rectangles is nothing but the algebra generated by this semi algebra R. So that will prove this step will imply that the algebra generated by the rectangles is inside the class P.

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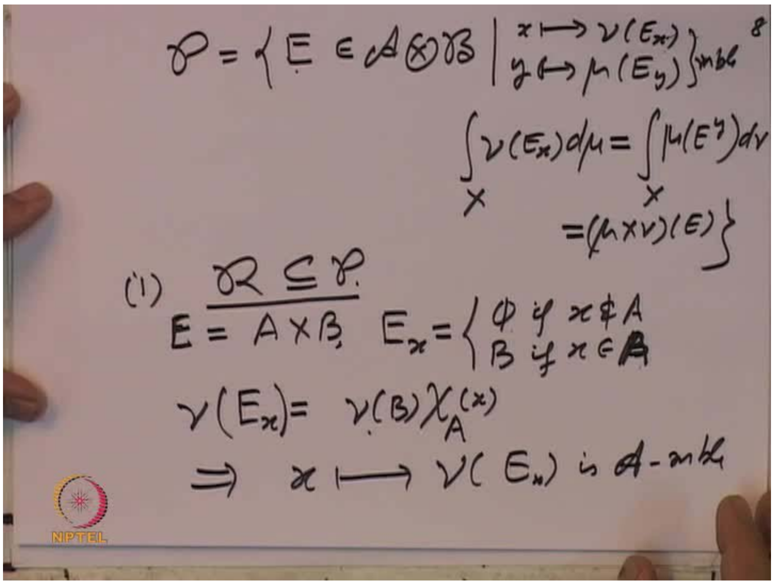

So, this first step is to conclude that the algebra generated by rectangles is inside P and the method is to show that R is inside it and F of R and it is closed under finite disjoint unions. Let us prove this step one first, so we have got the collection P; P is the collection of all subsets E belonging to A times B such that those two properties hold.

So, what are the two properties? The properties were that x going to nu of E x and y going to mu of E y, these two is measurable functions and that the integral of nu E x with respect to mu is same as integral of mu E y with respect to d nu. This is over x and this is over y and both of them are equal to the product sigma algebra namely, mu cross nu of E (Refer Slide Time: 33:55). So essentially what we are saying is, we are looking at the sets E in the product sigma algebra for which the required properties hold. The first thing is we want to show the rectangles are inside P.

To prove this, let us take a rectangle E; E is equal to A cross B where A belongs to the sigma algebra A and B belong to the sigma algebra B. Let us recall what were the sections? The section E x was equal to empty set if x does not belong to A and it is equal to B if x belongs to B. That means, this E x is nothing but when x does not belong to A, it is empty set. So, what is going to be nu of that? That is going to be 0, E x is going to be set B, so it is nu of B into the indicator function of A at x (Refer Slide Time: 35:10).

What is important that for a rectangle A cross B. We already computed the sections, x section was empty set if x does not belong to A and it is B if x belongs to B. So, nu of E x is going to be nu of empty set which is 0, if x does not belong to A and if x belongs to A then, it is nu of B and here nu of A is 1. This equality holds because if x belongs to A this value is 1 and indicator function of A at x, x does not belong to A is 0, so we have got this equation namely, nu of E x which we want to show is measurable is nothing but the constant times the indicator function of a set in the sigma algebra - A is in the sigma algebra - that implies that x going to nu of E x is A measurable, this is a measurable function. Similarly, if you take the corresponding sections with respect to y, let us write that also.

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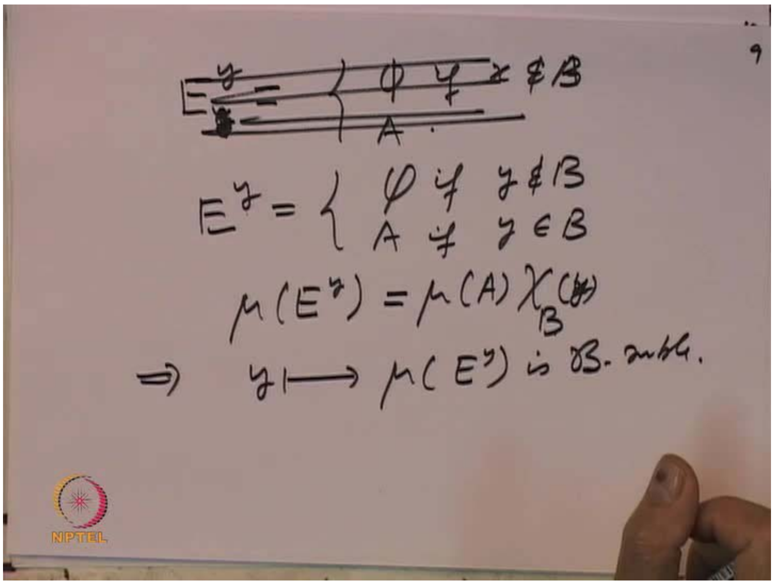

If we look at E of y so that is we are writing it up actually so let me write follow the same notation the section of E at y so E superscript y is equal to it is empty set if x does not belong to A and it is equal to A I am sorry let me write it properly.

The section E y is equal to empty set if y does not belong to B and it is A, if y belongs to the set B because, y is a point in x. That means, mu of E y is going to be equal to mu of the set A times the indicator function of the set B at the point y. As a function of y, it is just the indicator function of the set B at the point y multiplied by a constant, so that will imply y going to mu of E y is B measurable.

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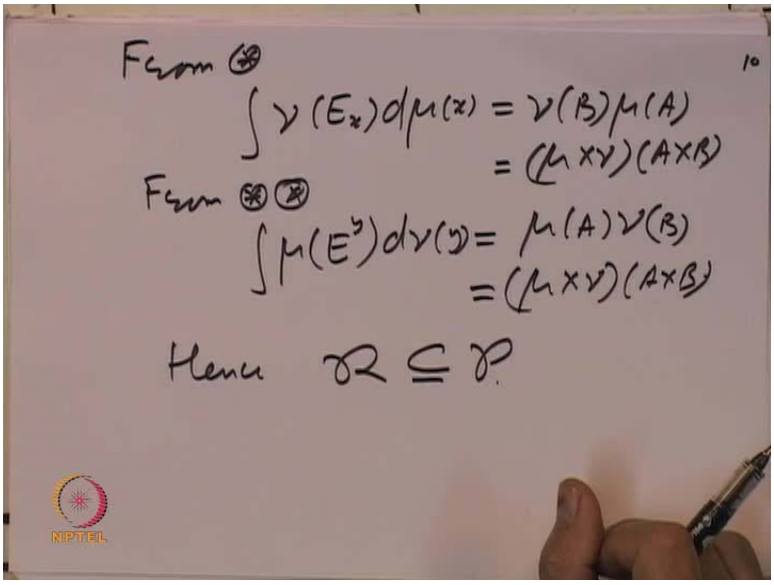

That proves the first thing namely, we wanted to show that rectangles are inside, so what we had shown here is for a rectangle the first property x going to nu of E x and y going to mu y are measurable with respect to the corresponding sigma algebras. Let us compute the integrals of these things. So nu of E x is this function, what is its integral with respect to mu? This is a constant and this is the indicator function, it is nu of B into mu of A (Refer Slide Time: 38:25).

From the equation star it follows, from star integral of nu of E x d of mu x is equal to - which is the integral of this quantity that is - nu of B times mu of A that was the property star. Similarly, let us look at the integral of the other function. So, we want to compute integral of the function mu of E y, but function mu of E y is equal to this quantity let us call it as double star.

Once you integrate this what we will get is, we will get integral of mu of E y with respect to y d nu y is equal to mu of A into nu of B. From equation double star, we will have integral mu of E y d nu of y is equal to mu of A into nu of B. In either case these integrals which is nothing but the product that says, this is equal to the product measure mu cross nu of the rectangle A cross B. Similarly, this is the product mu cross nu of A times B. Hence, what we have shown that rectangles are inside the collection P of the sets for which we wanted to prove the required claim holds.

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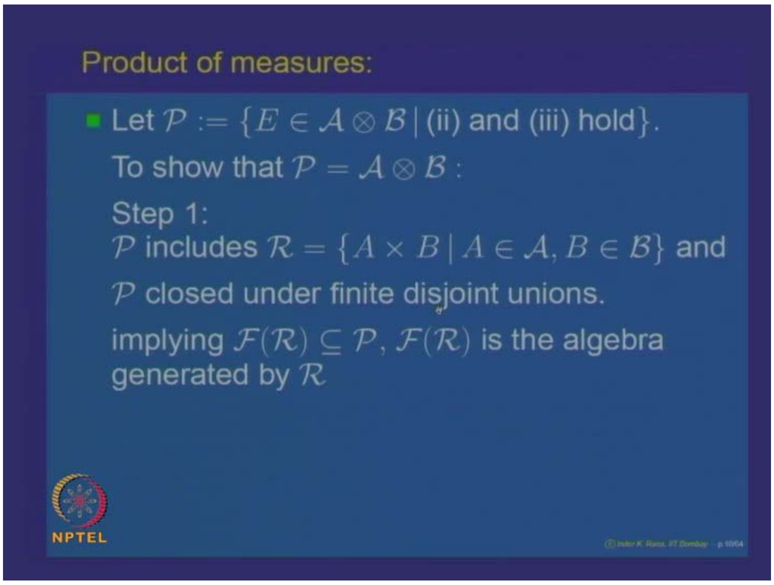

So, what was the second step we wanted to prove? We want to prove that this collection r and p claim the second collection - here is the second thing in the step 1. We have proved part of the step 1; we have proved that P includes R; P includes rectangles.

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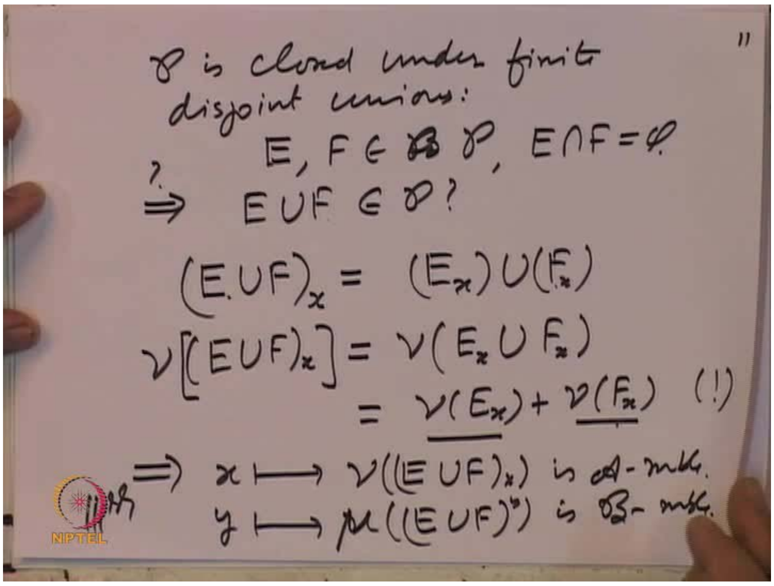

The next part of the proof requires showing that P is closed under finite disjoint unions. Let us prove that P is closed under finite disjoint unions. For that, let us take two sets E and F which belong to the collection P that means what? That implies, for E and F the corresponding results are true; E and F are disjoint that is also given to us intersection is equal to empty. We want to show that E union F belongs to P that is what we want to show.

Let us start with looking at the sections of E union F its section at a point x, by the definition the properties of the sections, the section of the union is union of the sections, so it is union of E x, union of F x. These are the sections, so what is going to be nu of the union E union F section, what is the nu of that? That is, as E and F are disjoint these sections are going to be a disjoint.

So, it is nu of the disjoint union of the sections E x union F of x these being disjoint that means, this is equal to nu of E x plus nu of F x that is nu of E x plus F of x. Here is something for you to think and confirm that if E and F are disjoint sets then, their corresponding sections are also disjoint and hence this property is true.

Now, E and F both belong to P that means, this is a measurable function of x and this is also a measurable functions of x and we have proved that sum of measurable functions is measurable. This will imply that x going to nu of E union F section at x is A measurable, this is a measurable function. Similarly, y going to mu of E union F section y is B measurable. To check that P is closed under finite disjoint unions, we have checked the first property namely, if E and F are two disjoint sets in P then, x going to nu of E union F section and y going to mu of the section E union F at y are both respectively measurable functions.

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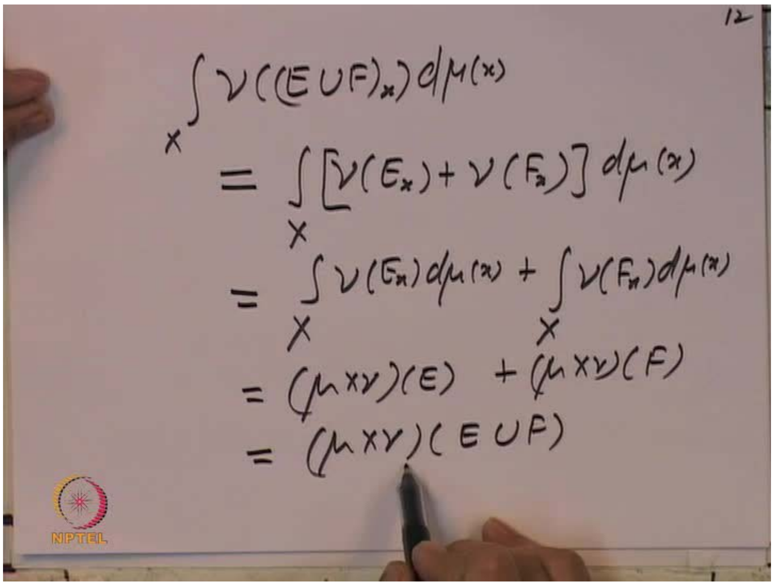

Now, let us check the next property namely, the integral property is true for the union. For that what we want to do is the following, we want to integrate. Let us integrate nu of E union F section, this is a measurable function with respect to mu we can integrate this. This over the set x of course, this we know by the just now we proved that nu of E union F at x is nu of E x plus nu of F of x, so let us use that property, this we can write it as x. The integrant nu of E union F of x is equal to nu of E x plus nu of F of x d mu x.

Using the properties of the integral, we can split it as integral over x of nu E x d mu x plus integral over x of nu of F of x with respect to d mu x, because E and F both are inside the class inside the collection P for which this property integral of the section nu E x d mu x is nothing but mu cross nu of E and the second integral is nothing but mu cross nu of F.

By the fact that E and F are disjoint and mu cross nu is a measure, this is nothing but mu cross nu of E union F. So, what we have shown is that if I integrate nu of E union F section with respect to mu that is the product measure of the set E union F, a corresponding result will also hold when it takes y section namely, we can show that integral of mu E union f at y; let me just write the argument that the corresponding result will be similar.

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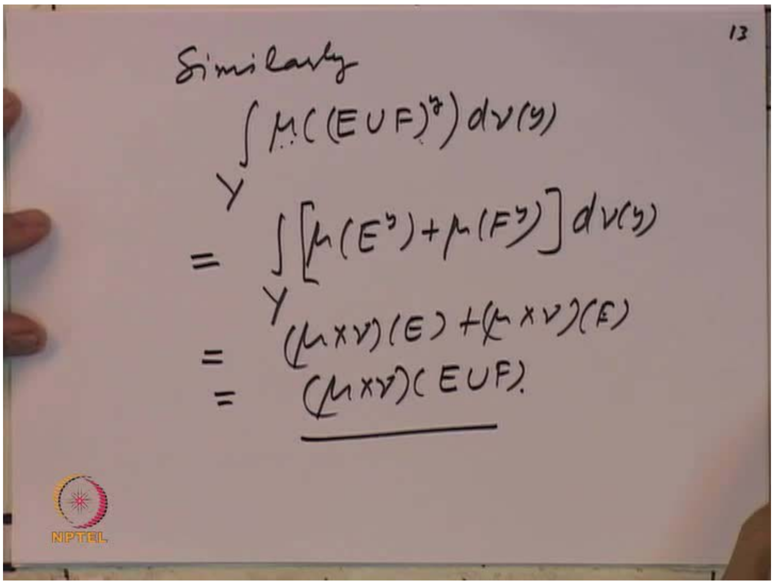

Similarly, if I integrate over y and mu of E union F section at y d nu of y, if I take the section of E union F with respect to y take its mu measure that is a measurable function and its integral with respect to nu that just now we observed that this section is nothing but mu of E y plus mu of F y. That was because this section E union F section is same as E section union F section and they are disjoints. So, measures add up and that is equal to d of nu y. Now, once again as before we can write this as mu cross nu of E plus mu cross nu of F and by again using the property of that mu cross nu is a measure E and F are disjoint this is mu cross nu of E union F.

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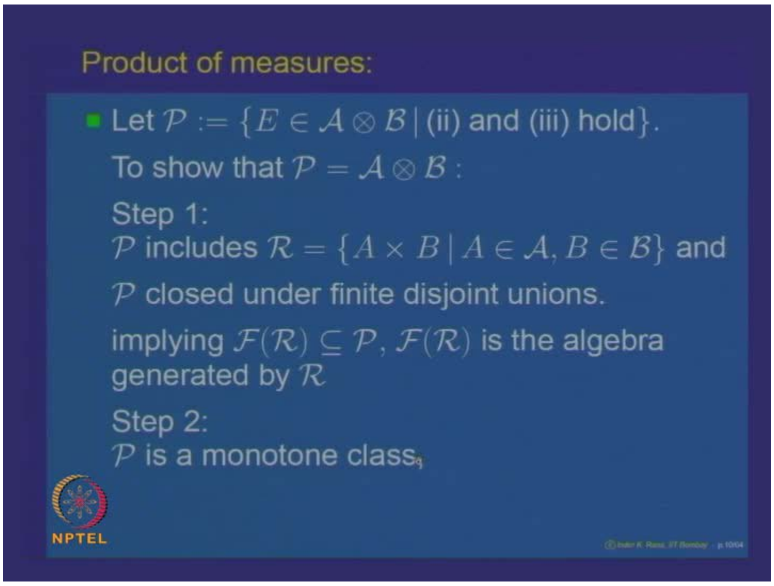

That proves the second part of the property namely, not only P includes rectangles in fact P is closed under finite disjoint unions. As a consequence of this because finite disjoint union of elements of a semi algebra give us the algebra generated by that semi algebra. As a consequence of step 1, we have got that the algebra generated by rectangles is inside the class P, where F R is the algebra generated by rectangles.

Our next step should be trying to show that this P is actually sigma algebra, but one tries to show that P is sigma algebra, one land into problem and is not able to show that it is closed under arbitrary unions. So that will be a problem, one modifies the arguments. Instead of showing that P is a sigma algebra one tries to show that P actually is at least a monotone class.

Once one tries to show that P is a monotone class it includes algebra - it will includes the monotone class generated by the algebra - which is the sigma algebra that is the route we will follow. From here onwards our technique will be the monotone class technique. So, we will try to show that P is a monotone class; it will include the monotone class generated by the algebra F of R, which is same as the sigma algebra generated by R and that will complete the proof.

The second step, we will do it in the next lecture. Today's lecture, we have just concluded that the class P for which we want to prove the required claim hold includes the algebra generated by rectangles. So, we will continue the proof in the next lecture. Thank you.