Measure and Integration

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Lecture No. # 25

Construction of Product Measures

Welcome to lecture twenty-five on measure and integration. In the previous lectures we had started looking at measure and integration on product spaces. In the previous, we defined the notion of product sigma algebra and today, we will define the notion of product measure.

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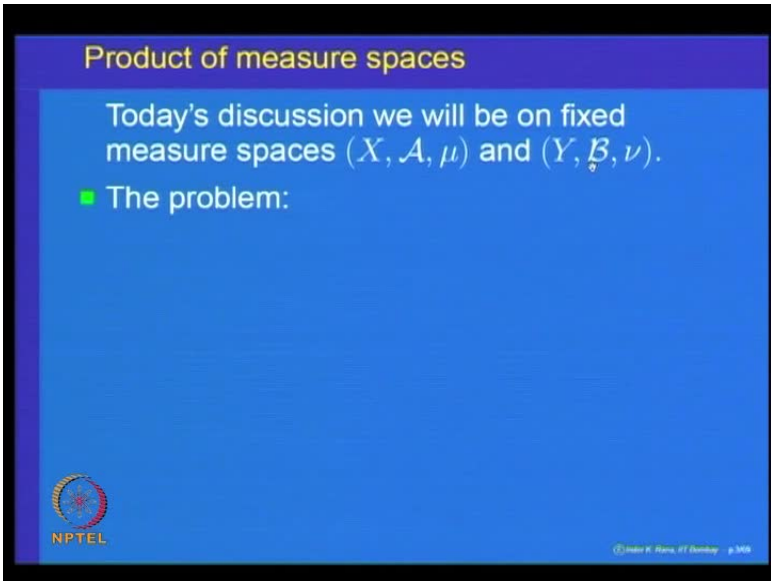

So, let us recall. We will fix for today's discussion 2 measure spaces, X A mu and Y B nu. So, X is a set, A is a sigma algebra of subsets of X and mu is a measure defined on the sigma algebra A. And similarly, for the measure space Y B nu, B is sigma algebra of subsets of Y and nu is a measure on the sigma algebra B.

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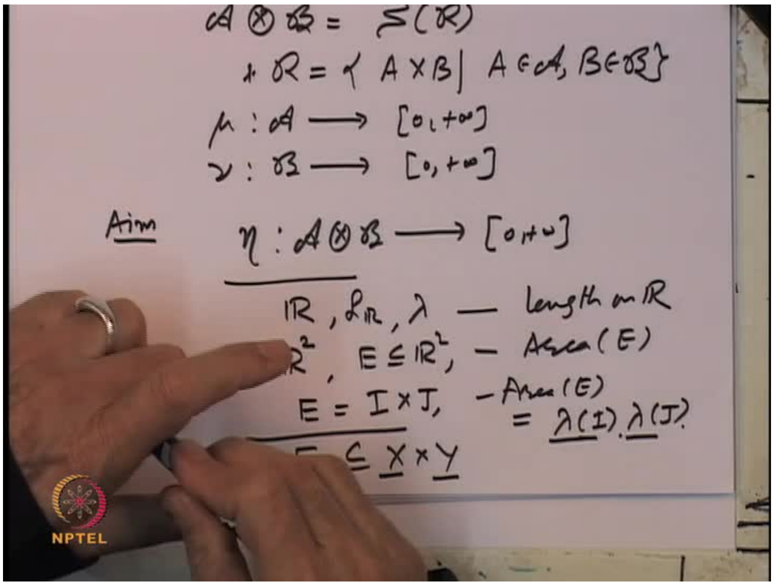

So, we have already defined the notion of the product measure, namely A cross B. So, if you recall, so we defined the notion of A times B, so this is the sigma algebra generated by all rectangles and rectangles were defined as the sets A times B, where A belongs to the sigma algebra A and B belongs to the sigma algebra B.

So, now we have given a measure mu on the sigma algebra A and given a measure nu on the sigma algebra, on the sigma algebra B, so that is a measure on B. So, our aim or the problem is to define a measure eta on the product sigma algebra A times B using the measure on A and using the measure nu on B.

Why such things are important? So, let us just recall, that on real line and the Lebesgue measurable sets, we had defined the notion of the Lebesgue measure. So, that extended the notion of length; so, that extended the notion of length on R for subsets of R, which are not necessarily intervals. So, we want to do the corresponding thing on R 2; so, on R 2, given a set E, we would like to define the notion of area of E and if E is a nice set, for example, E looks like a rectangle I cross J, then we know, that its area, so let us call it as area of E is defined as length of I times length of J.

So, this, this motivates the notion of the product. So, if we have sets in X cross Y, so in general, if E is a set in X cross Y and we have a notion of size here and notion of size here, then we will like to know, define the notion of size for subsets E in X cross Y and for sets, which are nice, which as very simple to describe, we would like to put it as the product of the length into breadth.

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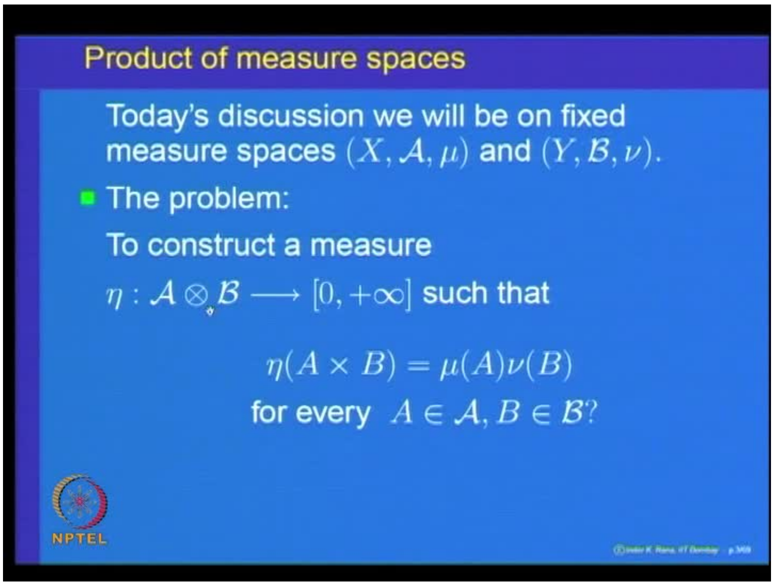

So, the abstract question, the problem, abstract problem is the following, that given 2 measure spaces X A mu and Y B nu, we want to construct a measure, let us call it as eta, on the product sigma algebra A times B, such that for sets, which are rectangles. So, what are the rectangles on abstract measure spaces? They are the sets of the type A times B, where A belongs to the sigma algebra A and B belongs to the sigma algebra B.

So, for such sets we want, that the notion of the size for subsets namely, so our notion of size is the measure. So, measure of a set A times B should look like mu of A, some, something like length of A into nu of B length of the set B.

So, this is the abstract problem, given 2 measure spaces X A mu and Y B nu, how to define a measure in a nice way on the product sigma algebra, such that on rectangles it looks like the product of the corresponding measures.

So, eta of A times B should look like mu of A times nu of B. In fact, this requirement, that eta of A cross B is mu of A times nu of B, itself says a way of doing this. So, that means, this fixes the notion of the measure for rectangles, which are of the type A cross B.

So, if we can show, that this set function eta, which is defined by this equation for measurable rectangles A times B by this equation and if you can show, that is, a measure, it is countable additive, then we know, that measurable rectangles form a semi algebra and they generate the sigma algebra A times B.

So, we can take advantage of our extension theory and then extend this eta, if it is a measure on the semi algebra of all rectangles to the sigma algebra A times B.

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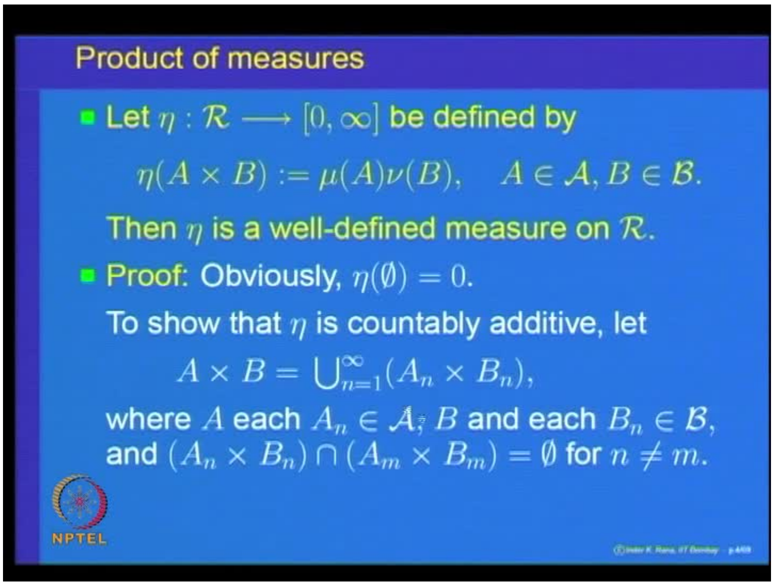

So, that is roughly the route we want to follow. So, to do that, to implement this possibility, so let us write, that eta defined on rectangles, so we are defining 1st eta on rectangles A times B, so eta of A cross B is defined as mu of A times nu of B. Obviously, it is a well-defined set function and we want to claim, that this is actually a measure. So, eta or empty set is 0, that is o.k. because if A or B are empty set, then this is 0. So, we want to show, that it is a measure on R; that means we have to show, that eta is a countably additive set function.

So, to show that let us take a rectangle, A cross B, A times B and suppose, it can be represented as a union of rectangles A n cross B n, which are pair wise disjoint. So, A cross B is written as union n equal to 1 to infinity of rectangles A n cross B n, where all the sets A, A n's are all in the sigma algebra A; the sets B and B n's are all in the sigma algebra B, and this rectangles are pair-wise disjoint, that means, A n cross B n intersection with some A m cross B m is empty whenever n is not equal to m.

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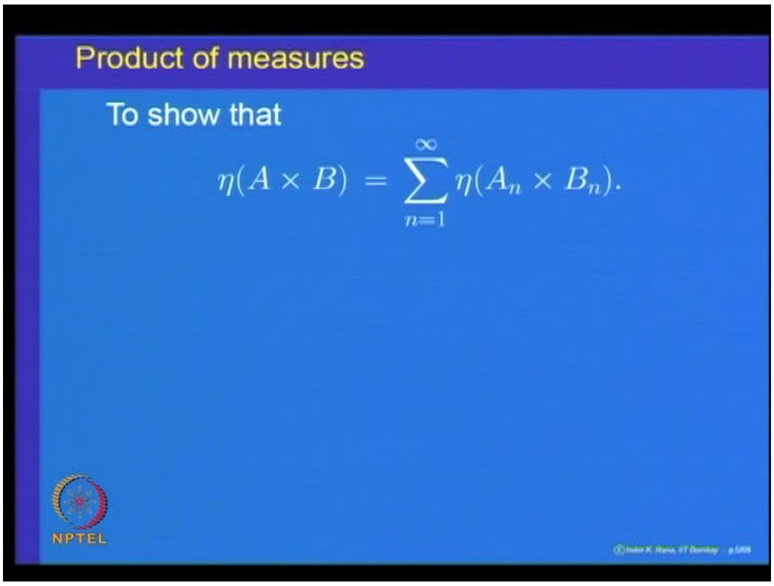

So, if a rectangle is written as a countable disjoint union of rectangles, then we want to show, that eta of A cross B is equal to summation n equal to 1 to infinity of eta A n cross B n. So, this is what we have to show. So, to show that let us proceed as follows, so let us write. So, here is a rectangle A cross B, which is written as a union of rectangles A n cross B n, n equal to 1 to infinity and this rectangles are pair-wise disjoint.

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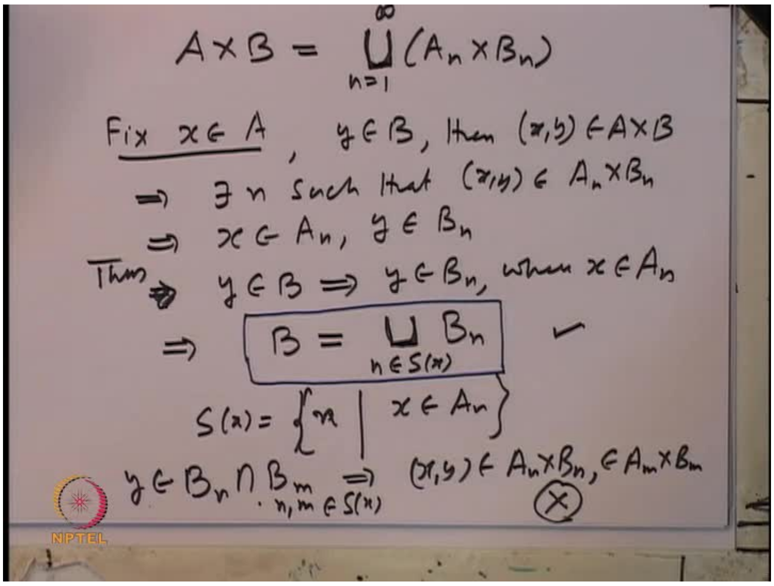

So, that disjointness, we will represent it by putting a square cup, so the notion of union, instead of putting this U we will put it above the square just to indicate that, so that we do not have to write every time, that they are pair-wise disjoint; this symbol itself indicates, that they are pair-wise disjoint.

So, we want to compute eta of A cross B and show it is equal to summation of eta A n cross B n. So, to do, to do that let us proceed as follows, let us fix any point x belonging to A. So, for any point x belonging to A, if you look at y belonging to B, then (x, y) belongs to A cross B. We have fixed A, and take any point y in B, then the ordered pair (x, y) belongs to A cross B; so, that is A cross B.

So, it will belong to one of the sets here, so which set it will belong? It will belong to some A n and B n, so then x cross y belongs, so this implies there exists n, such that (x, y) will belong to A n cross B n, so that is a possibility.

But now, if this happens, so that implies, that x must belong to A n and y must belong to B n. If the ordered pair (x, y) belongs to A n cross B n, then x belongs to A n and y belongs to B n, but then what does that imply? That implies, that if y belongs to B, so that means, implies, so thus let us write what we have. y belonging to B implies y belongs to B n, where, what is this n, where x belongs to A n. So, whenever x belongs to A n, y will belong to some B n. So, that implies, that I can write the set B as union of over sets B n, where n belongs to, let me write S of x. So, what is S of x? S of x is the set of all those indices n, such that x, which is fixed, belongs to A n. So, out of the indices 1, 2, 3, so on, look at those n for which x belongs to A n.

So, if x belongs to A n and then, y will belong to some B n, that means, y belongs to those B n's, such that x belongs to A n. So, this is what we want to claim and not only that, we want to claim, that these B n's, which are involved here, they are pair-wise disjoint. That means, this union is a pair-wise disjoint union, why is that? Because if this is not disjoint, that means, if B n, if point y belongs to B n intersection B m, where both m and n are in S x, x x, that usually, that will imply, that x y belongs to A n cross B n and also, it belongs to A m cross B m, where n and m are in the set; so, n and m are in the set of S of x, so n and m belong to S of x.

Suppose, so I want to show these 2 union is disjoint. So, take 2 elements here, B n and B m, that means, for n and m belonging to S of x, look at the intersection. Suppose, there is a y in the intersection, that will mean what? That x belongs to A n and y belongs to B n and similarly x belongs to A m and y belongs to B m. That means, x y belongs to both of these, which is a contradiction because A n and B m are disjoint.

So, what we are saying is the following, namely, we are saying the following, that for any x fix, I can write my set B as a disjoint union of sets. So, this is what the conclusion is, I can write the set B as a disjoint union of sets B n’s, which are coming in this union, but what, which n’s, those n’s, such that x belongs to A n.

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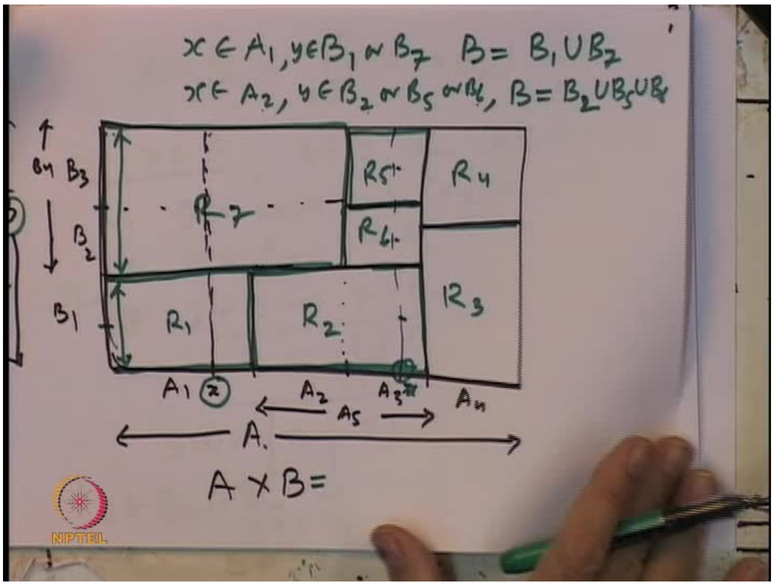

So, to look at a pictorial view of this, let me just take a very simple example, illustration of this. So, this is the set A. This is the set A and this is the set B, so we have got a rectangle A cross B; we have got a rectangle A cross B and it is written as a disjoint union of rectangles. So, here is the disjoint union of rectangles, what are those rectangles? One is this rectangle, the 2nd rectangle is this, the 3rd rectangle is this, the 4th rectangle is this, the 5th is this and the 6th rectangle is this.

So, this A cross B, so the set A cross B is written as a disjoint union of 6 rectangles. So, let me call this 1st rectangle R 1, this is R 2, this is rectangle R 3, this is rectangle R 4, R 5, R 6 and R 7. So, these 7 rectangles and their sides of each one of them, we can write down A 1 cross B 1.

So, this rectangle, this side is A 1 and this side is B 1; for this rectangle, this side is this portion and this is the width, and so on. And now, I wanted to illustrate that point, so let us take a point x belonging to A fixed. So, when x is fixed, what are the points y in B n. So, to find those, let us go vertically. So, if you go vertically, so for any y belonging to this set B, y belonging to B, either y will belong here or y will belong here, so that means, this set B can be written as a disjoint union of this portion and this portion of B. So, that will be B 1 and this will be B 7. So, if x, so if x belongs to A 1, then v, then y can belong to B 1 or B 7 or it can belong to B 7.

So, that means, B will be equal to B 1 union B 7. So, B 1 and union B 7, B 1 is the, width, height of R 1 and B 7 is the height. For example, let us take a point x here, this is a point x, let us take a point x here, which belongs to A 2. So, when I go above, if I fix this then, how does the, how does the y split?

So, if x belongs to A 2, if that is fixed, then to be inside the rectangle, I can be here, I can be here, I can be here. So, it will belong to B 2, B 6 and B 5; so, y can belong to B 2 or B 5 or B 6. So, that means, in that case, B will be equal to B 1 B 2 union B 5 union B 6.

So, what we are saying? So, depending on where the point x is, the set n x, it will be 1 and 7. If x is in A 2, then it will be n, x will be 2, 5 and 6. So, B in either case, B is a disjoint union of rectangles, some of the B is, so this is the important thing, which I wanted to convey.

So, this is what is the conclusion of this argument, that if A cross B is a disjoint union of rectangles A n cross B n, and I fix any point x belonging to A and analyze the points y in B, then (x, y) belongs to A cross B. So, it will belong to some A n cross B n, so y will belong to some B n's.

So, which B n's it will belong? It will belong to only those B n's for which x belongs to A n. So, B can be written as a disjoint union of B n's, where n belongs to S of x. So, this is a disjoint union, so that was the 1st important observation once we have.

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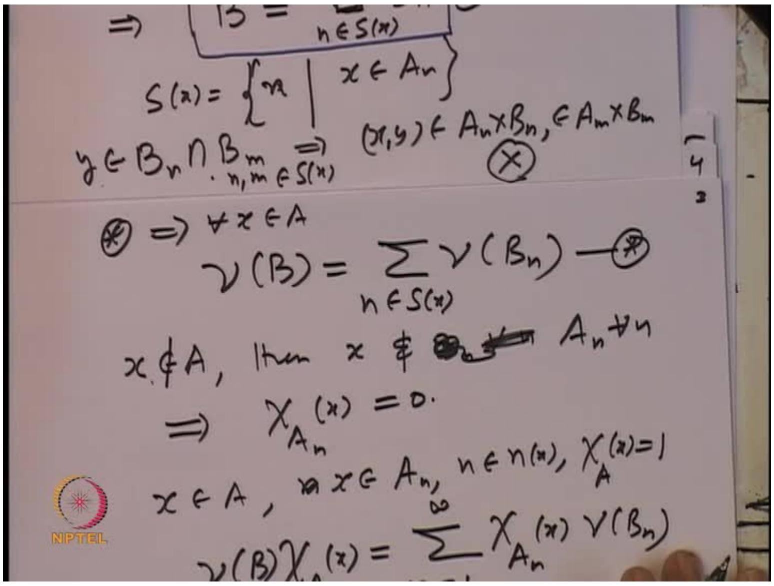

So, once this is a disjoint union and all the B is, the set, the set B n’s are all in the sigma algebra, where nu is defined. So, this being disjoint, that implies, so what we get is that nu of, so this star, so star implies, namely, that nu of B is equal to summation nu of B n’s, where n belong to S of x.

So, now, I would like to transform this equation slightly. So, what was x? x was a point in A, it was, when x is in A, so for every x fix, that means, for every x fix in A, we had this. Now, suppose, now suppose, x does not belong to A. If x does not belong to A, then obviously, x does not belong to B n’s, x does not belong B n for every n, sorry, then, if x does not belong to A, then x does not belong to, sorry, not B n’s, x does not belong to A n for every n.

So, that implies, that chi A n of x will be equal to 0. So, if x belongs to A, then x will belong, then it will belong to some of the A n’s and in that case, for those n, it will be equal to 1. So, if x does not belong to A and if x belongs to A, and x, that means, x will belong to, x will belong to some A n and that means, this n will belong to n x and that means, will have chi A of x will be equal to 1.

So, what I am saying is, this equation nu of B equal to this, I can write it as nu of B times the indicator function of A x is equal to summation over all n equal to 1 to infinity chi of A n x times nu of B n.

So, let us understand this once again, that why is this so? So, if x belongs to A, then the left hand side is, this value of the indicator function is 1, so the left hand side is nu of B, so that is here. And if x belongs to A, then it will belong to some A n. So, if x belongs to A n, this value is 1, so the right hand term is nu of B n. If x does not belong to A n, then this value is going to be 0; if x does not belong to A n, then this value, so if x belongs to A and x belongs to A n, this value is 1, otherwise this value is 0.

So, on the right hand side in this summation, only those terms will be non-zero for which x belongs to A n and that means, the value of the right hand side terms will be indicator function, will be 1 and nu of B n. So, this will be this equation, otherwise both sides are equal to 0, so that holds.

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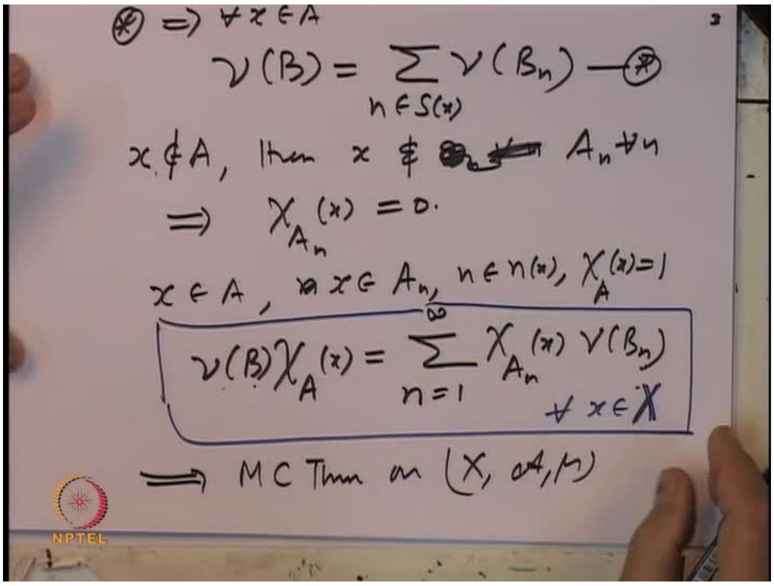

So, what we are saying is that, so from the earlier equation we have come down to the 2nd conclusion, namely, this holds for every x belonging to A. And so, for every x belonging to X, so for every x this equation holds, so that is what we have proved because when x belongs to A. It is the earlier equation star, when x does not belong to A both sides are equal to 0, so this equation holds.

So, this is the 2nd crucial step in the arguments, namely if A times B is a rectangle, which is written as a countable disjoint union of rectangles A n and B n, then indicator function of A times x into nu of B is summation over n indicator function of A n times nu of B n. And now, this is an equation involving nonnegative, this is an equation involving nonnegative, simple nonnegative measurable functions, so own the measure space x A mu, so this is a sequence.

So, now, I can apply monotone convergence theorem. So, this, apply monotone convergence theorem on X A mu, so this is a nonnegative function, which is a limit of, so the sum means, it is a limit of the partial sums. So, it is a limit of nonnegative measurable functions. So, monotone convergence theorem will give me that the integral of this is equal to limit of integrals of this.

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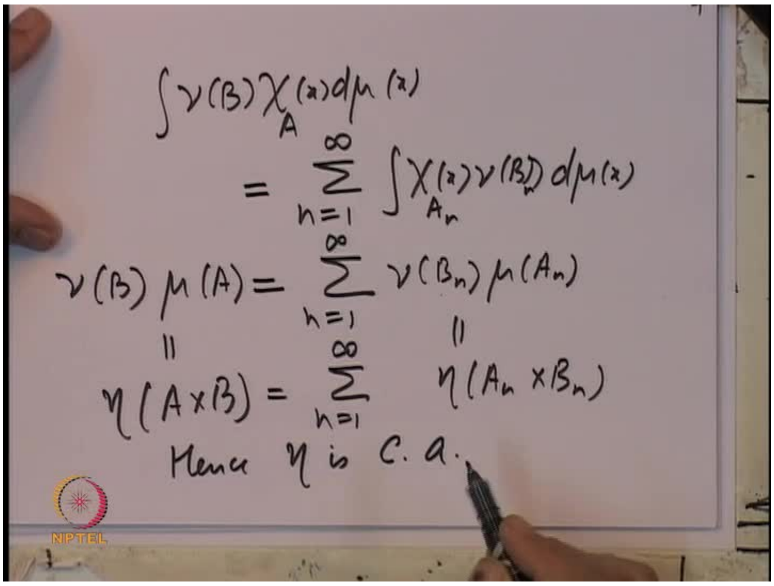

So, an application of monotone convergence theorem to this equation star gives me, that integral chi of B indicator function of A x d mu x can be written as summation n equal to 1 to infinity integral of indicator function of chi A n nu of B n d mu x.

So, this is a straightforward application of the monotone convergence theorem. Left-hand side is a nonnegative measurable function, which is a limit of nonnegative measurable functions on this measure space. So, integral of the left-hand side with respect to mu must converge to the integral is equal to the limit of the integrals on the right-hand side; so, this is an application of monotone convergence theorem.

Now, let us compute the right-hand side, the left-hand side, the integral of nu of B, nu of B is a constant, so that goes out; integral of the indicator function of A with respect to mu, so that is mu of A is equal to summation n equal to 1 to infinity and this integral again, nu of B n is a constant, so nu of B n goes out of the integral sign, and integral of A n with respect to mu, so that is mu of A n.

So, what we have gotten is nu of B into mu of A is summation nu of B n into mu of A n. But this thing is nothing but eta of A cross B and this is nothing but, each term is nothing but eta of A n cross B n. So, what we get is eta of A cross B is equal to summation n equal to 1 to infinity eta of A n cross B n and that proves, that, that, so hence eta is countably additive. So, that proves that the eta is a measure.

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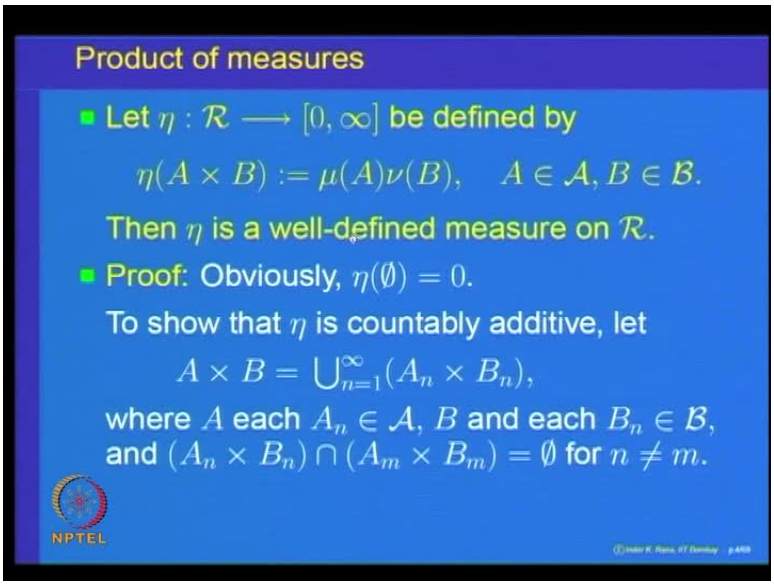

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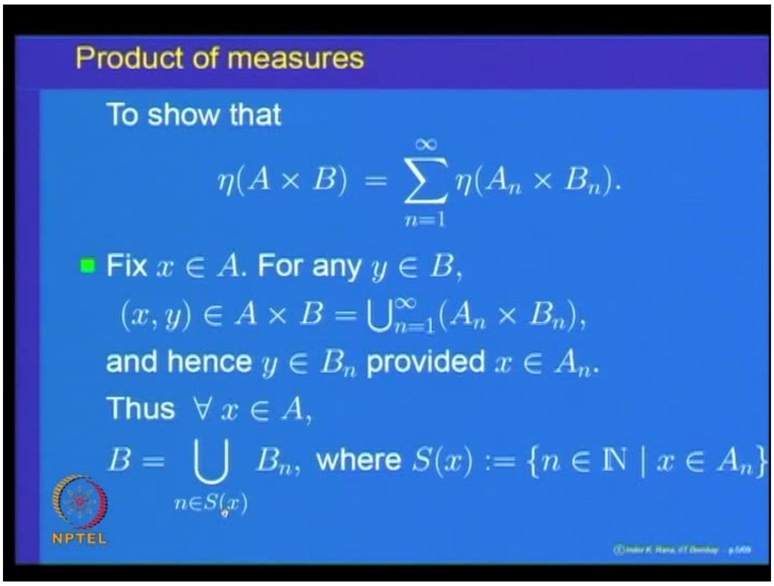

So, this is how one proves, that eta which is defined, so eta which is defined as eta of a rectangle equal to rectangle A cross B to b mu of A into nu of B is countably additive.

So, let me slowly go through the proof once again. There is only one small idea involved in it and other is, rest is straight forward applications of the earlier results. So, let us write A cross B, so I am going through the proof once again, that A cross B is written as a countable disjoint union of rectangles A n cross B n. And what we want to show, that eta of the rectangle A cross B is equal to summation of the measures of the each rectangles, so summation over n of eta A n cross B n.

So, to prove this, what we do is as follows. Look at the set A cross B, so fix any element x belonging to A, then for any y belonging to B, we know, that (x, y) belongs to A cross B, which is nothing but union of A n's.

So, that means, (x, y) will belong to exactly one of them, but which one of them, so (x, y) will belong to that (A n, B n). Whenever this x belongs to that A n, because (x, y) belonging to A n cross B n implies, x must belong to A n and y must belong to B n. That means, what we are saying is (x, y) belongs to A cross B if and only if x belongs to A n and for that, x, the y should belong to B n because x is fixed, so that n is fixed. So, what are those n's, which are fixed? So, y belongs to B n provided x belongs to A n. So, for a fixed x collect together those n's, so find the set S of x, all those indices n, such that x belongs to A n. See, A n's are not disjoint, so x can belong to more than one of the A n's.

So, look at those. If x belongs to A n, but for a fix x it will belong to only one of them, so if x belongs to A n, then y will belong to B n. So, as x varies, so as x varies over A for every fix x, you will get a collection of B n's. So, what are those B n's? Those B n's are index by n belonging to S of x, such that x belongs to A n and this union is a disjoint union.

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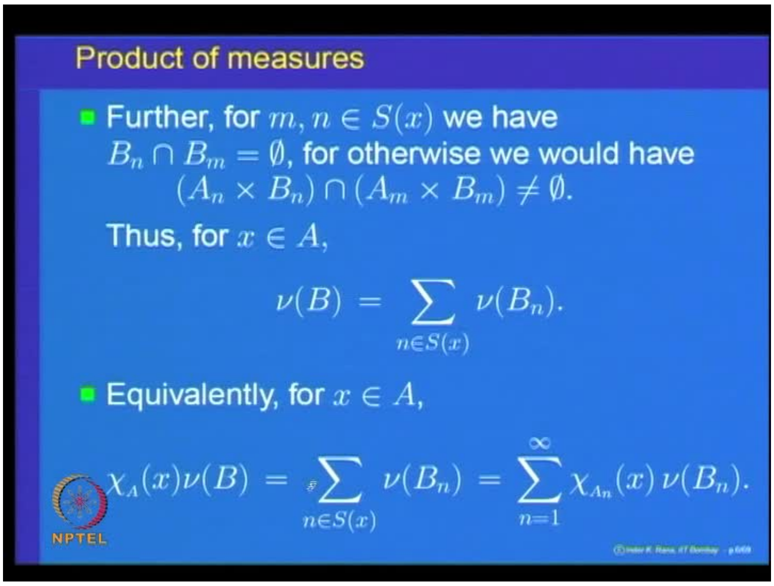

So, for every x fix in A we can decompose B into a disjoint union of B n's over those n's, such that n belongs to S of x, this being a disjoint union because A n's, B n's are disjoint. We get our 1st equality, 1st equality, that for any fix x in A nu of B is summation nu of B n's over those n's, which belong to S of x and now, we observed, that equivalently this thing we can write it as nu of B. I can multiply it by the indicator function of A because x belongs to A, so this will be equal to 1 and this nu of B n, I can multiply if x belongs to A n, that means, n will belong to S of x, so I can multiply here by the indicator function of A n, if n belongs to S of x. And if x does not belong to A n, that means, it cannot belong to any one of the A, that this all the remaining terms here will be 0 and all the remaining this side is also equal to 0.

So, what we are saying is, for any x in A, I can write this is equal to this and this equation make sense whenever x does belong to A. Also, because v of x does not belong to A, this side is equal to 0 and that side x does not belong to A, so it does not belong to any one of the A n's, so all the terms are 0.

So, this equation, first we can write it as indicator function of A times x nu of B is equal to summation of chi A n's. And now, we realize, that not only this equation is valid for x belonging to A, this is, equation is valid for all x in x.

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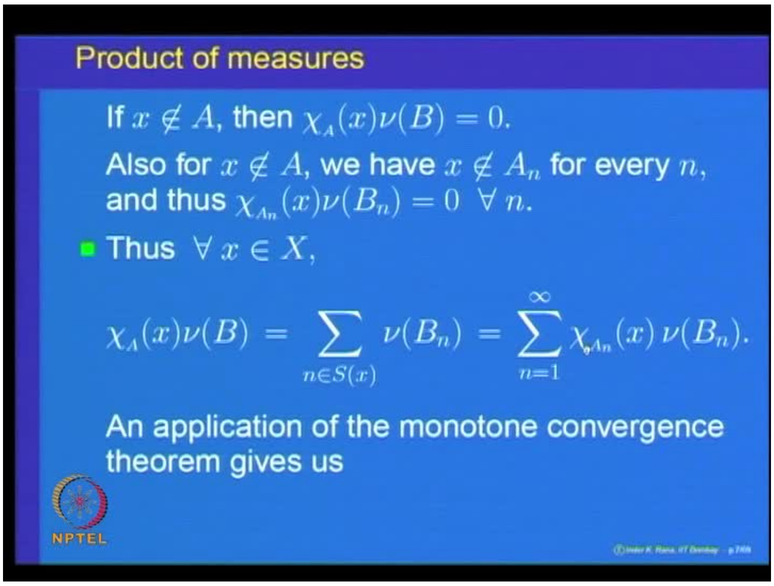

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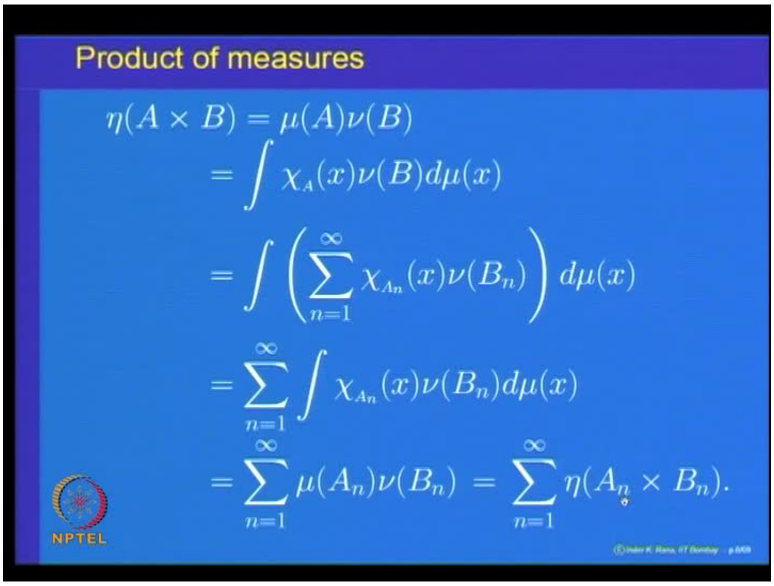

So, once that is observed, so that is what is observed here. So, what we get is that the equation chi of A x nu of B is equal to the summation chi of A n nu of B n for all x. And now, this is an equation about nonnegative measurable functions, so left-hand side is a nonnegative measurable function, which we can realize as a limit of nonnegative measurable functions, namely the partial sums of this series and apply monotone convergence theorem. So, that will give us that the integral of the left-hand side is equal to summation of the, so I can take the integral sign inside by monotone convergence theorem and say, that integral of chi A x nu B d mu y is nothing but integral of the summation. And so, here is the application of the monotone convergence theorem, I can take this integral inside, so that is equal to summation of integral of indicator functions. And now, it is just a matter of writing down the values of this nu of B n is a constant, so goes out, so this integral is nothing but mu of A n so and that nu of B n and the left hand side, this was integral of chi of A nu of B nu of B is a constant. So, that is integral of chi of A with respect to mu, so that is mu of A.

So, that gives us, that eta of A cross B is equal to summation eta of A n cross B n whenever A cross B is a disjoint union of rectangles. So, that proves, that this is eta is a, eta is a countably additive function.

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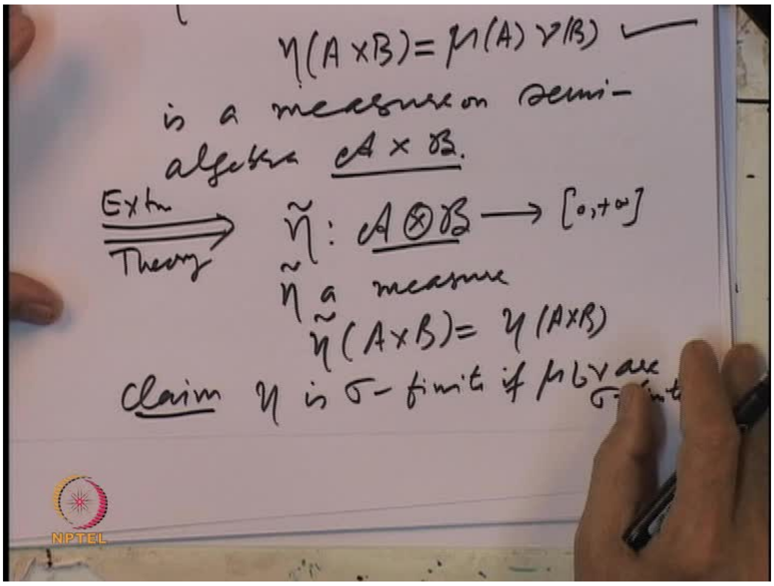

So, what we have gotten is eta is a countably additive function, so let us just observe. So, what we got is, we got eta defined on A cross B 0 to infinity by eta of A cross B equal to mu of A nu of B is a measure. We got, that this is a measure on the semi algebra, so this is important on the semi algebra A times B. So, implies by our general extension theory via outer measures and so on.

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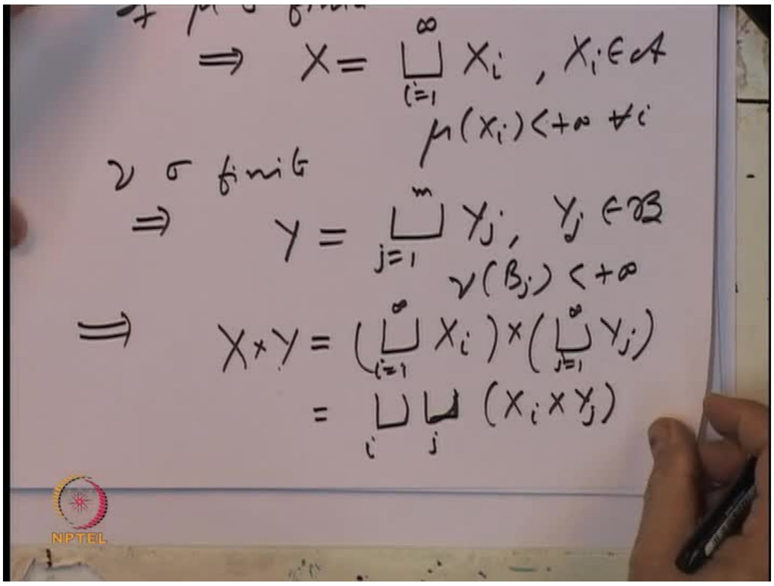

We can extend, we can define eta tilde on A times B, define eta tilde a measure and eta tilde of A cross B to be equal to eta of A cross B. That means, this eta can be extended via outer measures to the sigma algebra generated by A cross B, the semi algebra A cross B and if you recall, we had said, that this extension will be unique provided this eta is a sigma finite measure.

So, we claim eta is sigma-finite if mu and nu are sigma-finite, so we want to show next, that if A and B, if mu and nu are sigma-finite, so let us assume, so if mu sigma-finite, so that implies, I can write X as a disjoint union of sets X i 1 to infinity, each X i in the sigma algebra A and mu of X i finite for every i. And similarly, nu sigma- finite implies, I can write Y as disjoint union of sets Y j, where each Y j is an element in the sigma algebra B and nu of B j is finite.

But then, that, this implies we can write X cross Y as disjoint union of X i's cross disjoint union of Y j's. And now, it is a just a simple matter to set, that equality, namely, this is same as the unions over i unions over j of rectangles X i cross Y j because if (X, Y) belongs here, that means, X belongs to the union X i's and Y belongs union Y j's. So, that means, X will belong to only 1 of X i and only to 1 of Y j, so it will belong here. And conversely, so this is a disjoint union and now we only have to observe the fact, that…

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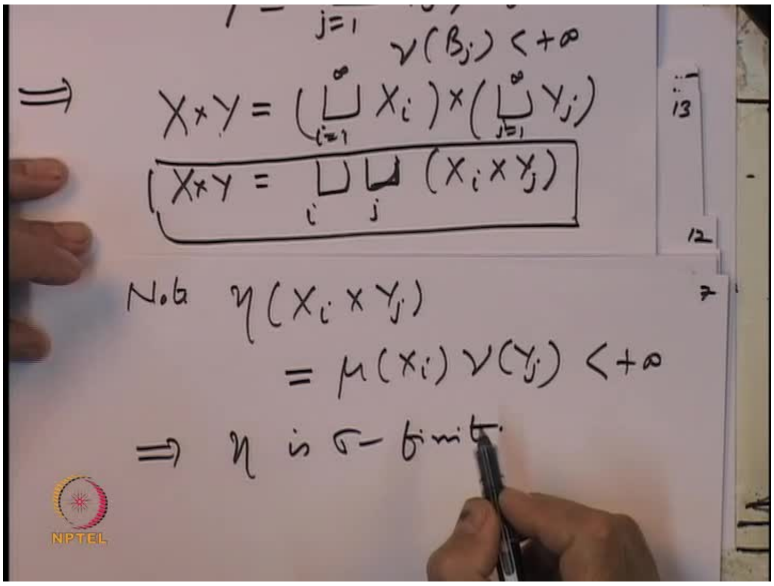

So, X cross Y has been decomposed, has been decomposed into a disjoint union of sets X i cross Y j and we only note, that eta of X i cross Y j, it is a rectangle, so its measure is mu of X i times nu of Y j and both of them being finite, so this is a infinite quantity.

So, X cross Y is written as a disjoint union of sets X i cross Y j and each piece has got finite measure, so that implies eta is sigma-finite. So, the measure eta is sigma-finite on the rectangles and hence as a unique extension to the sigma algebra.

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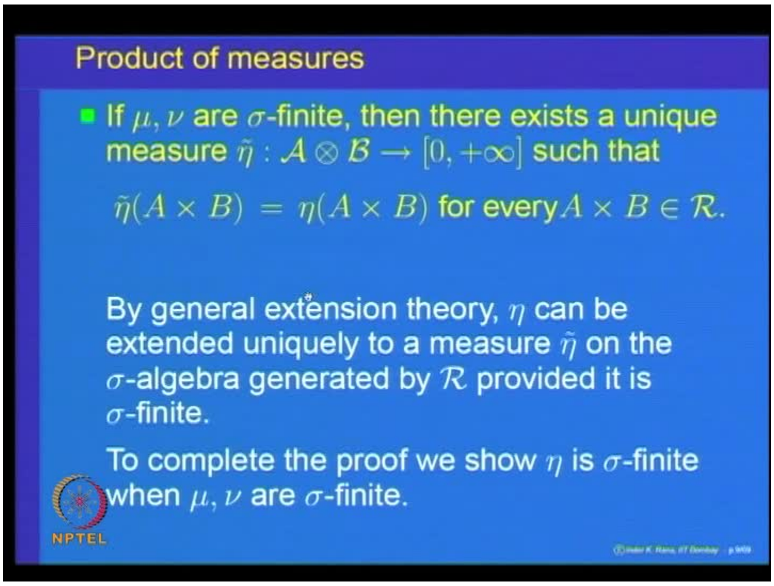

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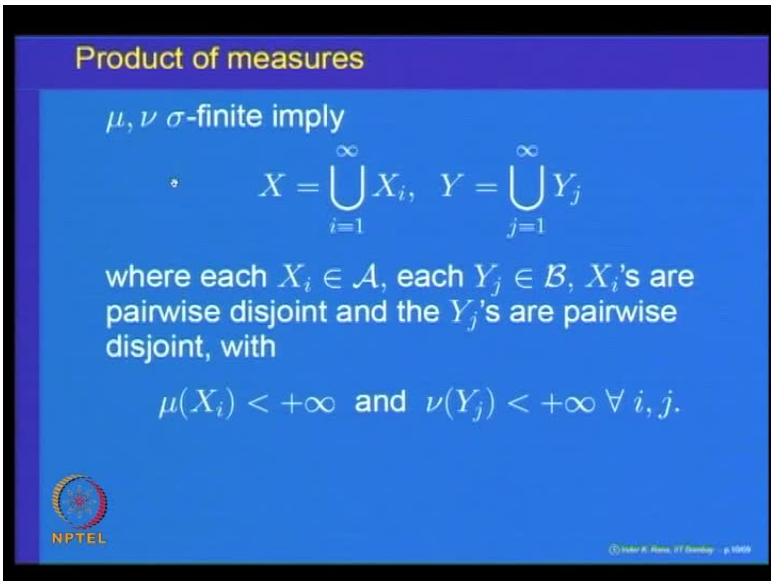

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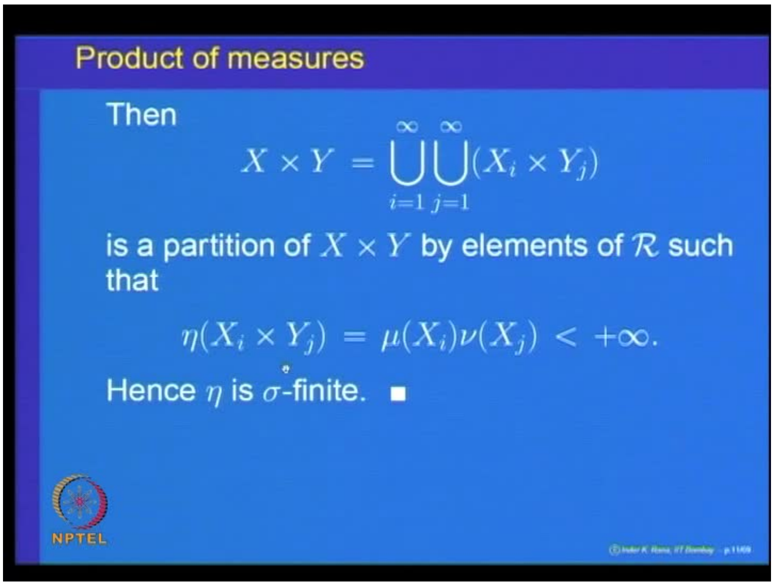

So, this is what we wanted to prove, that eta, that extension is also sigma-finite. So, general extension theory gives me a unique, so that mu and nu are sigma-finite, so that implies X is a disjoint union, Y is a disjoint union, so we can write X cross Y as a disjoint union of the rectangles X i cross Y j, that is what I just now illustrated and each piece has got a finite measure.

So, by that process, we get, eta is sigma-finite on rectangles. So, by extension theory eta can be extended uniquely. So, that is the important thing, eta can be extended uniquely to a measure on the product sigma algebra, so that for rectangles it is the product.

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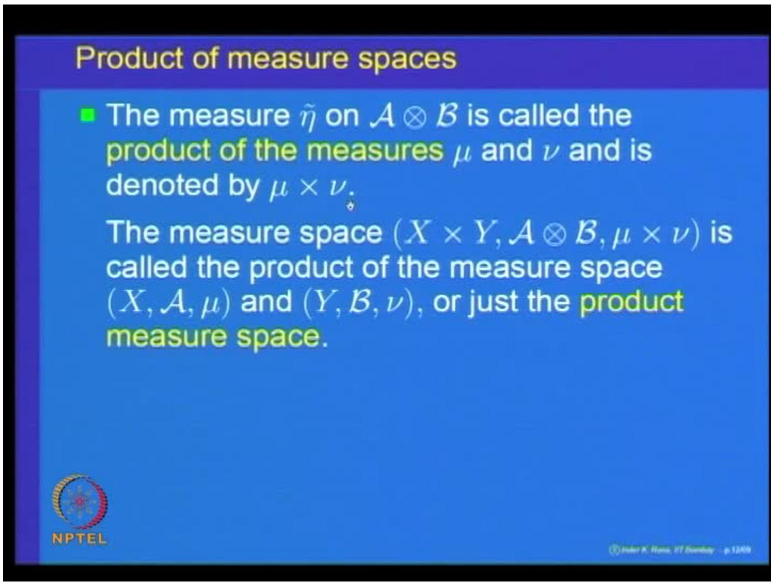

So, this is the measure eta, which is defined on A times B on the product sigma algebra is called the product measure and is normally denoted by mu cross nu.

So, let us summarize what we have done. We started with the 2 measure spaces, X A mu and Y B nu and for the product set X cross Y, we first define the rectangles, namely sets of the type A cross B, where A belongs to the sigma algebra A and Y belongs to the sigma algebra B, so that gives us, sets of, subsets of X cross Y called measurable rectangles, they only form a semi algebra. So, we extend, we generate the sigma algebra by this semi algebra of rectangles and call that as the product sigma algebra, denoted by A with circle cross A times B.

And now, given measures mu on the sigma algebra A and a measure nu on the sigma algebra B, we want to define a measure on the product sigma algebra. So, that is done by defining the product for a, defining the nu measure first on rectangles. So, eta of the rectangle A cross B is defined as the product of mu of A and nu of B and we show that this is a measure.

So, this becomes a measure on the semi algebra of rectangles and if it is sigma-finite, that means, if we assume, that the given measures mu and nu are sigma-finite, then this extends uniquely to a measure on the product sigma algebra A cross B and that measure is called the product of the measures mu and nu.

So, given 2 measure spaces X A mu and Y B nu, which are sigma-finite, we get the product measure space X cross Y. The sigma algebra A cross B generated by the rectangles and the product measure mu cross nu go obtained via the extension theory, so this is the product measure space constructed, as just now said.

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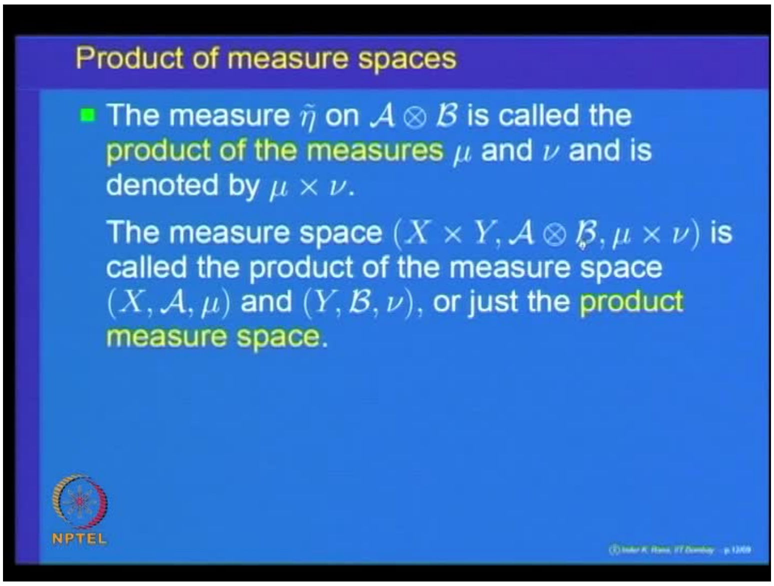

So, now, the next problem we want to analyze is the following, namely, this product measure mu cross nu, that we have gotten, is obtained via extension theory, but it does not tell us how does one compute the product measure mu cross nu of a set in A cross B. So, that is not indicated because we are making use of the extension theory.

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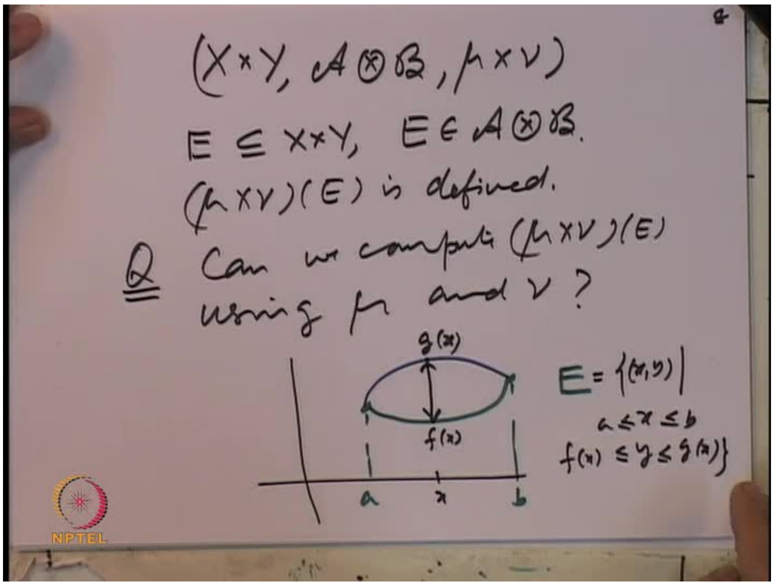

So, next problem that we want to analyze is the following. So, namely, so we have got the product measure space X cross Y, the product sigma algebra A times B and the product measure mu cross nu.

So, let us take a set E contained in X cross Y, which is of course E is an element in A times B. So, mu cross nu of this set E is defined. So, the question is can we compute, can we compute this quantity mu cross nu of E using mu and nu? So, that is the question?

And there, let us just recall something from our elementary calculus. Supposing in the plane we have got a set, which looks like the following, it looks like this, is a set. So, this is a set E, which looks like the following, namely, here is a point A and here is a point B. So, the set E looks like, so let us just write what does E look like? E is equal to all, x cross, (x, y), such that x belongs, x belongs between a and b and y. So, at any point x, if I look at y, this is the portion of y, so it starts with a green boundary, so y is bigger than or equal to some function f of x, that is a green curve, and less than or equal to, here is g of x. So, this is what we call in calculus or elementary analysis sets of type 1 and for such sets, for such sets we can find out what is the area.

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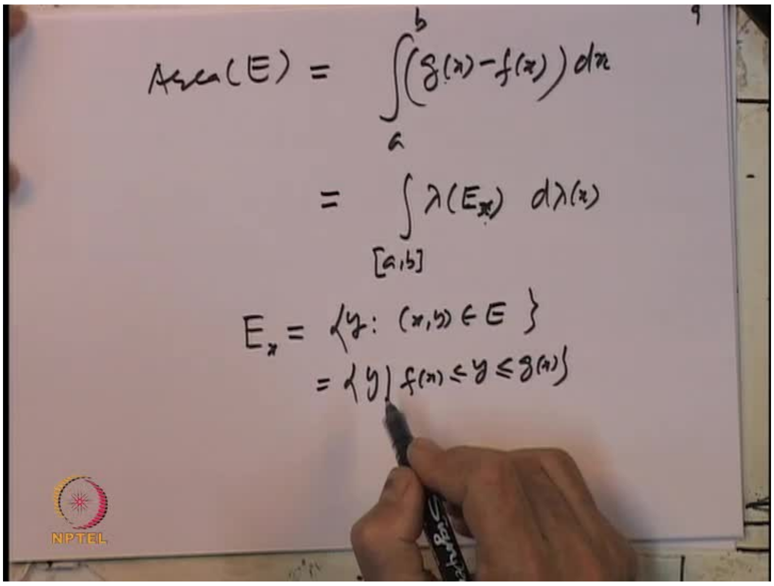

So, area of the set E, if you recall from calculus, it can be obtained. As you look at this difference height, what is this, what is this height? So, that is nothing but g of x minus f of x and integrate that from a to b dx. So, Riemann integral as an application of Riemann integration, we do that, we define it equal to this.

But now, let us rewrite this, this I can write it as this Riemann integral. So, Riemann integral, I can write integral over a b of d lambda with respect to the Lebesgue measure and what is g x minus f x, that is precisely the Lebesgue measure, the Lebesgue measure of this height.

So, Lebesgue measure of…; let me write as E x, what is E x? E x is equal to all y, such that (x, y) belongs to E, which is same as all y, such that y is between f x and g x. So, that set I am writing it as follows, so I am writing as Lebesgue measure of a notation called E x.

So, you can think of that, look at this set x, let us look at that, look at that set E to find its area. We are just adding up the areas of these small strips, so I can think it as that way, that is what, this integral seems to indicate.

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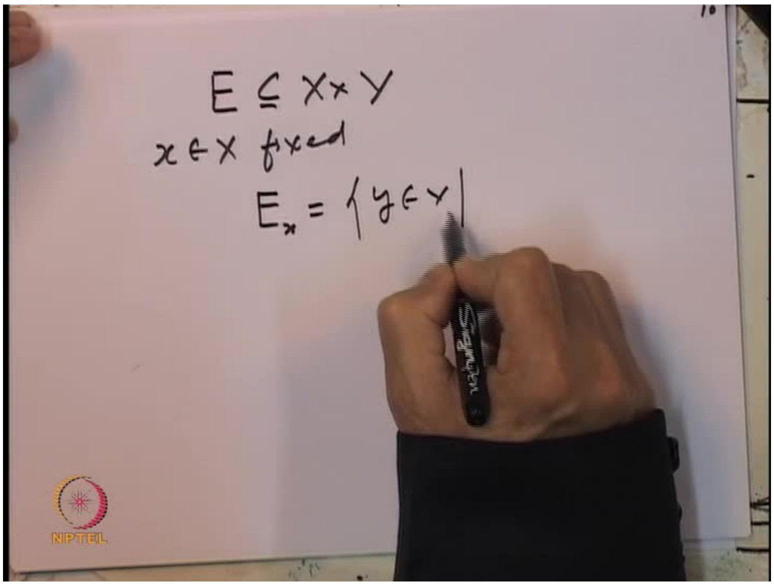

So, we would like to generalize. This is in the case of our construction, the same idea we want to generalize it, so here is what we want to do. So, given a set E in X cross Y for x belonging to X fix, let us look at E x, that is, so here is abstract, now X is an abstract set, Y is any some abstract set.

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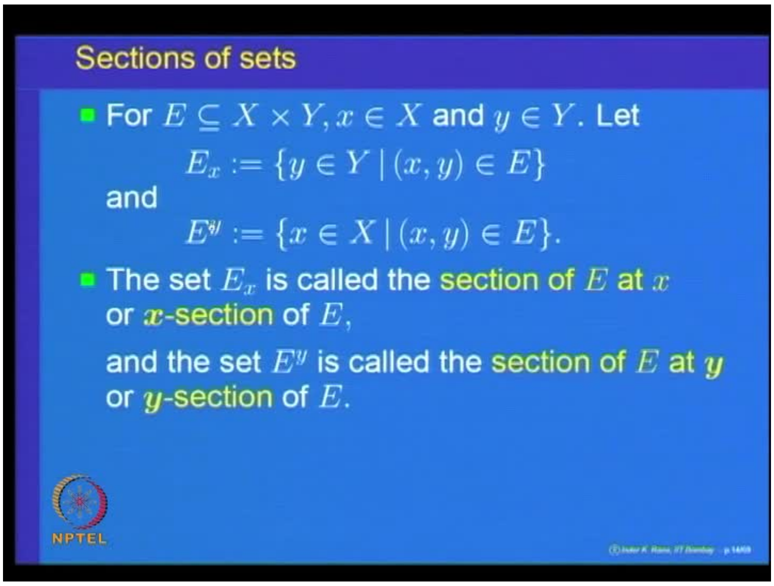

So, look at all those points y, belonging to Y section is the part of the horizontal line and the x section is part of the vertical line. So, as I said, E x is called the section of E at x or just the x section of E and similarly, E y, this set E y is called the section of E at y or just the y section of E.

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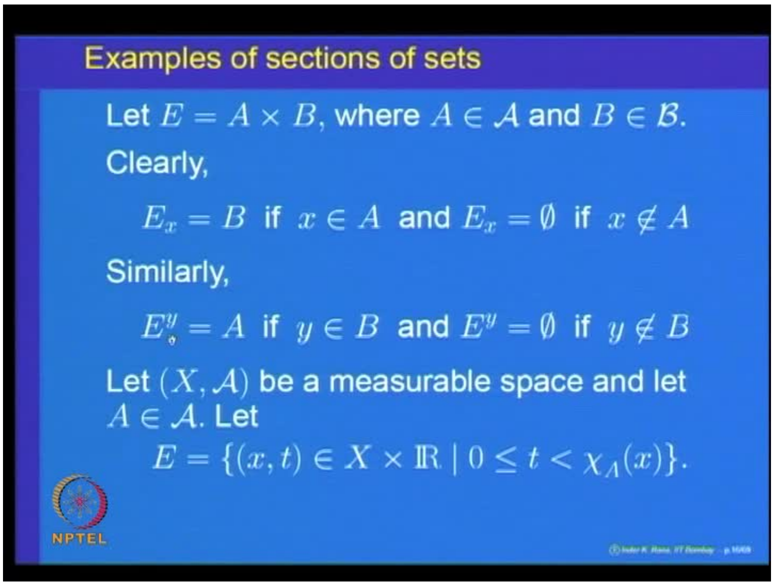

So, here are some simple properties we want to verify for these sections. So, first of all we want to verify and let us look at some examples, 1st let us take a set E, which actually looks like a rectangle, so in the x cross y let us take actually a rectangle A cross B, where A belongs to A and B belongs to B.

So, then for any x in A if, for, what are the points, why such that (x, y) will belong to E, that means, y must belong to B. So, E x, the x section of E for a rectangle A cross B is nothing but B, the set B itself. If x belongs to A and if x does not belong to A, then the point (x, y) is never going to belong to E, so the x section is empty set. So, here is the simple observation, that for a rectangle A cross B, the x section is equal to the set B if x belongs to A, and it is an empty set if x does not belong to A.

Similarly, the y section of E or the section of y at a point y in y, so all x, such that (x, y) belongs to, so if y belongs to B, then for all x in A, (x, y) is going to belong to E, so that means, the y section of E is equal to A if y belongs to B, and it is empty set if y does not belong to E. So, for rectangles, these are very easy to compute. What are the sections for a rectangle A cross B? The x section for x belonging to A is B, otherwise empty. Similarly, the y section is equal to A if y belongs to B, otherwise it is empty.

Now, let us look at another example, so let us take a measurable space X A and look at the ordered pairs (x, t). So, t belongs to R, such that this t lies between the evaluated indicator function of A at x. So, we are looking at the ordered pairs (x, t), such that for every x, t lies between 0 and A, and X belongs to X.

So, what are the, what are the sections of this set E? This is a subset of A cross B and where B is, y is the real line, so it is a subset of X cross R, we want to find its sections. So, let us observe, that for a point x in A, if x belongs to A, then this indicator function of A, the value will be equal to 1, so t will be between 0 and 1. So, if x belongs to A, then t will be between 0 and 1, so the section is going to be the interval 0 1, 1 naught included. And if x does not belong to A, then this is going to be 0. So, t is going to be the singleton 0.

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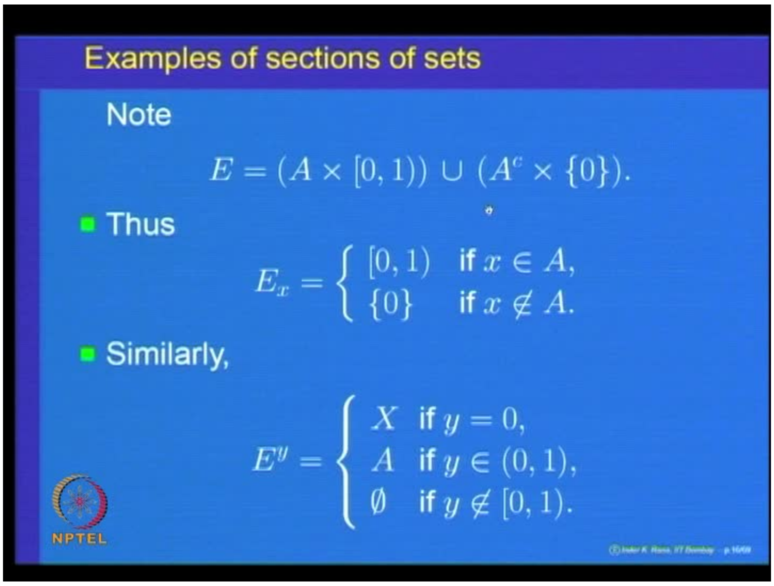

So, the section, if x belongs to A, so the section depends on whether x belongs to A or not. So, the section of E at a point x is equal to the interval, closed interval 0 open at 1 in R if x belongs to A, otherwise it is the 0 set, or another way of looking at this is the following, that the set E, I can write it as A cross, the interval closed at 0 and open at 1 union A, complement of this A complement cross the singleton 0, this is another way of writing the same set E, as I explained just now.

So, the section, now is union of 2 disjoint rectangles. So, section in the 1st case, when x belongs to A, section is going to be 0 1 and in the 2nd case, the section is going to be the singleton 0, if x does not belong to A; so, these are the x sections.

We can similarly find the y sections. So, for y belonging to 0 to 1, that means, y is the real line. So, for a real number between the closed at 0 and open at 1 interval, it is going to be A at 0, so and if y is equal to 0, then this is going to be the whole space X and empty set. So, this is easy computation, from this it follows, so this is how one computes the sections of these sets. So, these sections are going to play important role in computing the measure of a set E in the product space.

So, in the next lecture, we will analyze the x sections, the y sections, various properties of these sections under complements, intersections and unions and then show, that each section for a set E in the product sigma algebra, each section is again an appropriately measurable set whose measure can be defined and then, you can take the measure of that and define the functions and compute the integral of the product, set product, compute the product measure of the set E.

So, we will continue this study of sections and their implications for product measures in the next lecture. Thank you.