Measure and Integration

Prof. Inder K. Rana

Department of Mathematics

Indian Institute of Technology, Bombay

Module No. # 07

Lecture No. # 24

Product Measure, an Introduction

Welcome to lecture 24 on measure and integration. In the previous lectures, we had done the basic theory of measures and integration. Today, we will start with measure - the notion of product measures and integration on product measure spaces.

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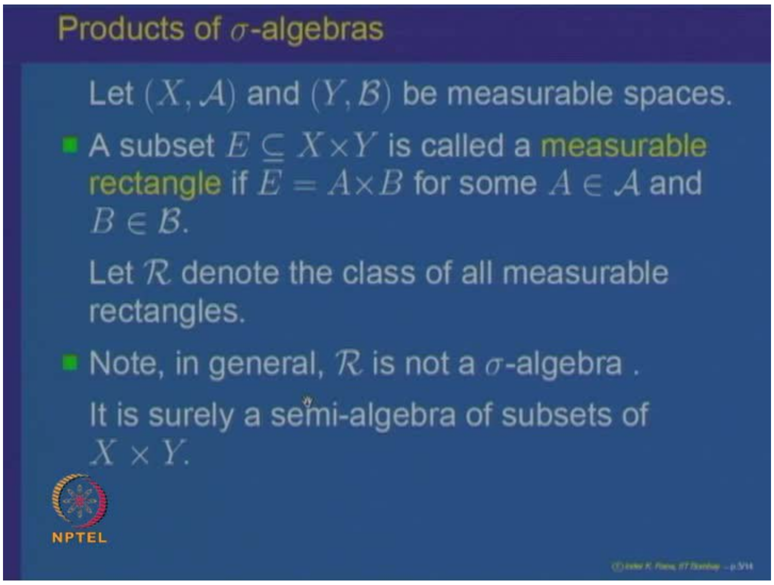

To start with, we will have the notion of product sigma algebra. Let us start with the two measurable spaces; X, A and Y, B are measurable spaces. Then, a subset of the product space X cross Y, a subset E is called a measurable rectangle if it looks like A cross B, where A belongs to the sigma algebra A and B belongs to the sigma algebra B.

The collections of all measurable rectangles or just called as rectangles, will be denoted by the set R. So, the set R denotes the class of all measurable rectangles which are subsets of the set X cross Y and each subset is of the type A cross B, where A is in the sigma algebra A and B is in the sigma algebra B.

In general, we had already observed while discussing the notion of semi algebras and sigma algebras that sets of the type A cross B, where A comes from a sigma algebra and B comes from other sigma algebra. This collection of rectangles, in general, need not form sigma algebra; in fact, it need not be even algebra, but surely A and B are being sigma algebras, they are also semi algebras. Then we had shown that the rectangle sets of the type A cross B surely form a semi algebra. So, the set of all measurable rectangles they surely form a semi algebra of subsets of X cross Y.

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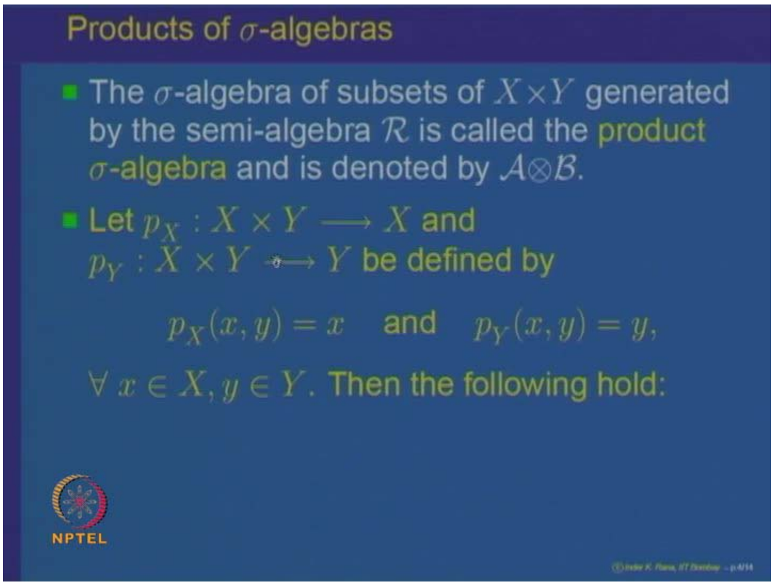

This being semi algebra of subsets of set X cross Y and in general may not be sigma algebra, we can generate sigma algebra by these rectangles. The sigma algebra generated by this rectangles is denoted by A times B. A times B here is the special symbol cross with a circle, so A times B will denote the sigma algebra generated by the rectangles R.

Let us give another characterization of these sigma algebras, the product sigma algebra in terms of what are called the projection maps. Let us look at the map p x p lower x which is defined from X cross Y to x as p x of x comma y, is the first coordinate x. Similarly, p y is a map from X cross Y to Y and is denoted by p y of x y is y, the second coordinate.

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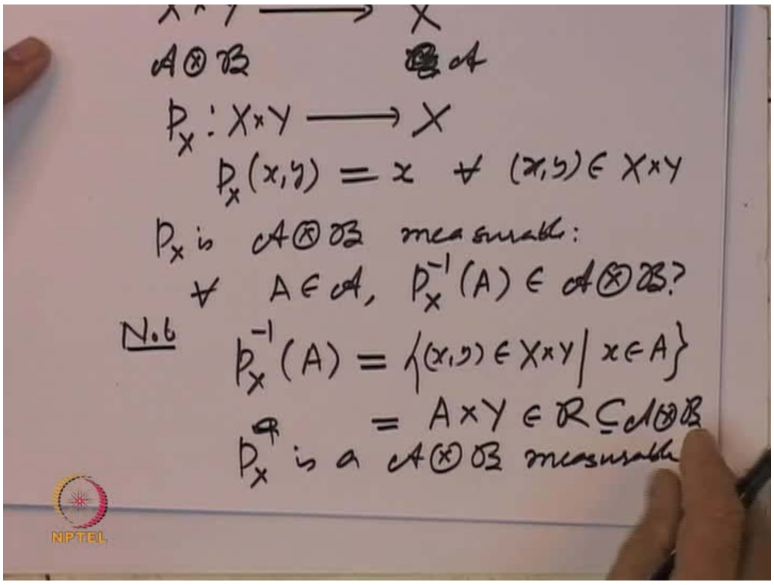

These two maps they are called the projection maps, the projection of X cross Y on to X and on to Y. So, the claim is that in case we give X cross Y the product sigma algebra, then these are measurable maps. Let us prove this, so X cross Y to Y; on X we have got sigma algebra A, on Y we have got the sigma algebra B, on X we have got the sigma algebra A; so let us look at A, so X cross Y to X. So here, we have got the product sigma algebra A times B on X cross Y. On X we have got the sigma algebra A and p x is the map defined on X cross Y to X. It is defined as P x(x comma y) is equal to the first coordinate x for every (x comma y) belonging to X cross Y.

So, our claim is that this p x is measurable map, when we give the product sigma algebra on X cross Y. So, p x is A times B measurable. To show that - what we have to show is the following that means, for every set A belonging to the algebra A, if we look at p x inverse of A then that belongs to the product sigma algebra A cross B, so that is what we have to show.

Let us calculate, we note that p x inverse of A, it is all x comma y belonging to X cross Y such that x belongs to A. So that is a meaning of the set p x inverse of A, but that is same as x belongs to A and y is independent, so this is A cross Y just the set A cross Y. A belongs to the sigma algebra A and the set y belongs to the sigma algebra B, so this is actually a rectangle. So this belongs to p x inverse of A, actually belongs to a rectangle, which generates the sigma algebra A times B.

So that shows the inverse image of every set in the sigma algebra A under p x is in that sigma algebra product sigma algebra A cross B. So that shows that p x inverse that means, p x is A times B a measurable map.

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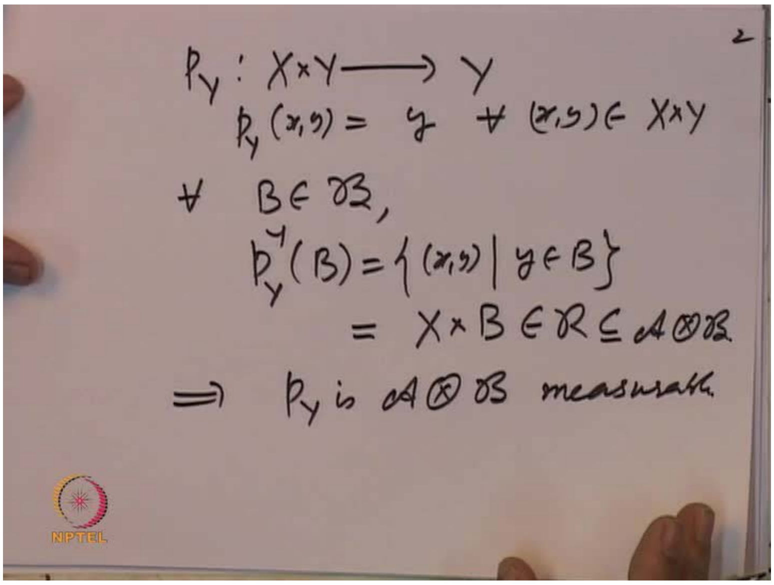

So, p x is A times B measureable. Similarly, we can show that the p y; so p y which is a map from X cross Y to Y, where p y of (x comma y) is y for every x comma y belonging to X cross Y. Then for every set B in the sigma algebra on y that is B, if we calculate p y inverse of B that is all (x comma y) such that p y, so y belongs to B and that is same as X cross B, which belongs - which is a rectangle and hence is in the sigma algebra A times B. For every set in the sigma algebra B p y inverse of B is in the product sigma algebra A times B. So that implies that p y is A times B measurable, so this is a measurable map.

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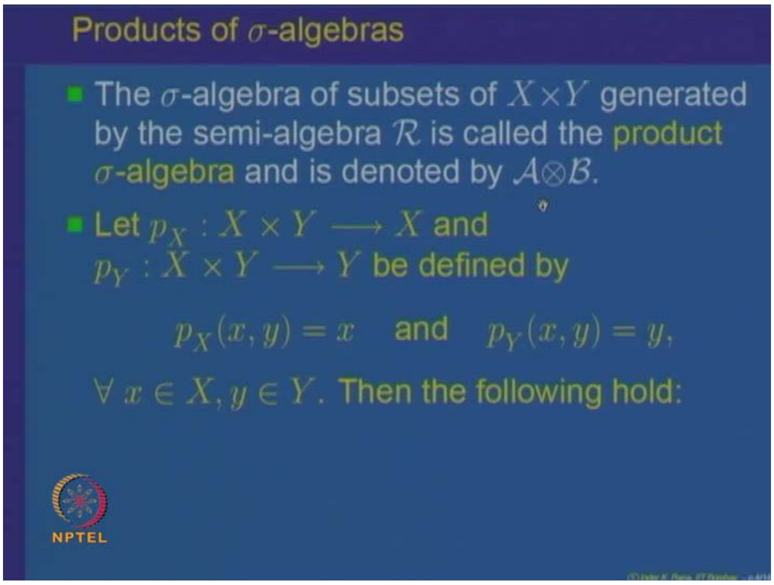

So, what we are saying is that the product sigma algebra A cross B is a sigma algebra on the product space X cross Y, which makes both the projection maps p x and p y measurable, so that is the property we have just now proved.

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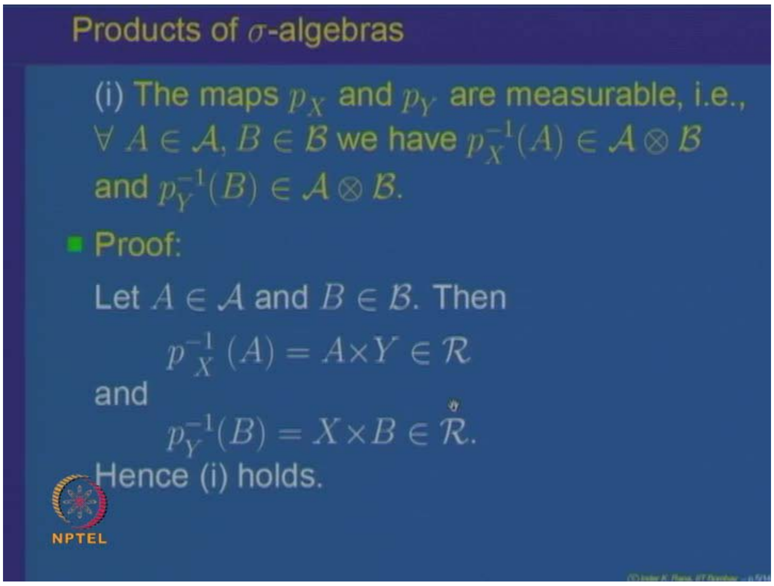

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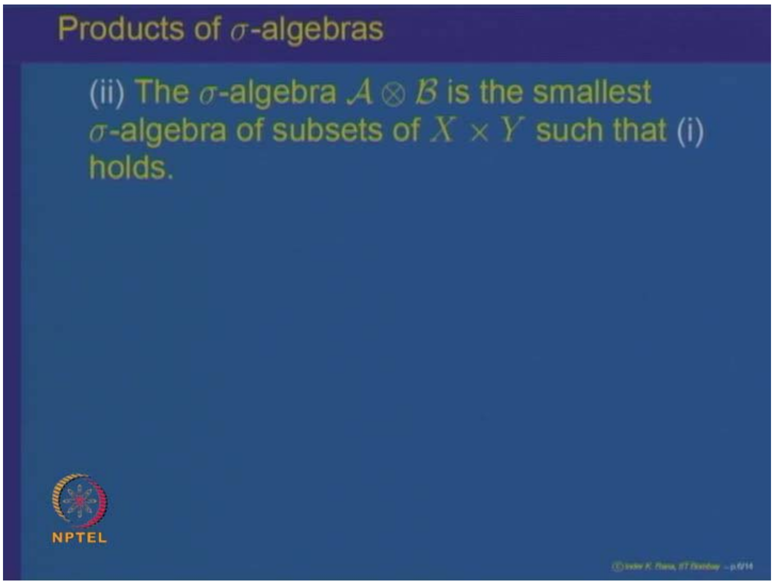

In fact, something more can be said, one can even show. So, this is a proof, let us go through the proof again, if A belongs to A and B belongs to B, then p x inverse of A is just A cross Y, which is a rectangle and p y inverse of B is again a rectangle. Hence, they both belong to the product sigma algebra and hence, p x and p y are measurable. So, what we want to show actually is that A cross B on X cross Y is the smallest sigma algebra of subsets of X cross Y, such that the earlier property holds namely, this is the smallest sigma algebra of subsets of X cross Y such that p x and p y both are measurable.

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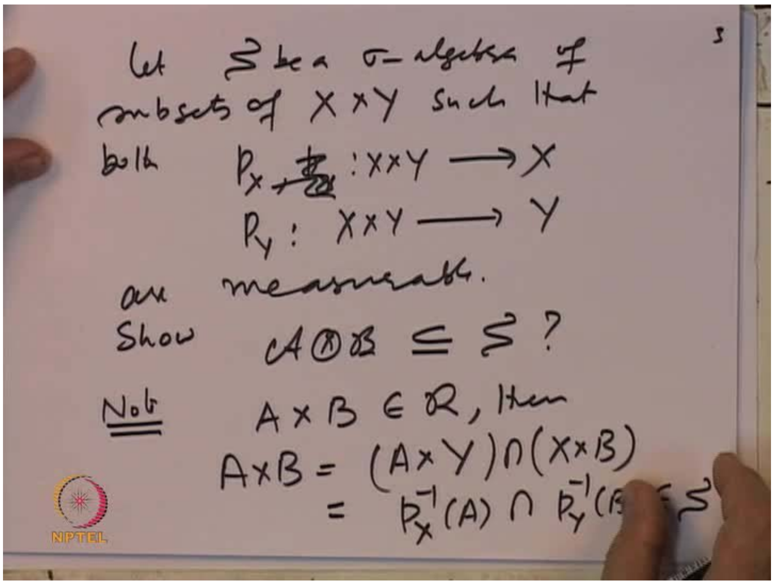

Let us look at a proof of that; so let us assume, so let S be a sigma algebra of subsets of X cross Y such that both p x and p y from X cross Y. Let us write p x will be in X and p y which will be X cross Y to Y, are such that both these maps are measurable. We want to show that A times B; A times B is also a sigma algebra of X cross Y with respect to which both p x and p y are measurable. We want to show this is a small s; that means, if S is any other sigma algebra so that p x and p y are measurable, we want to show that it must be including A times B, so A times B is inside S.

Let us prove this, let us take a set. So note, if you take a rectangle, so if you take a set A cross B which is a rectangle, then this rectangle A cross B - I can write it as A cross B can be written as A cross Y intersection with X cross B. So, this is simple set theoretic fact that A cross B - I can write it as A cross Y intersection with X cross B, because the first component A cross X will give me A and the second component Y intersection B will give me B. This set A cross Y, just now we saw is nothing but p x inverse of A and the second set X cross B is p y inverse of the set B.

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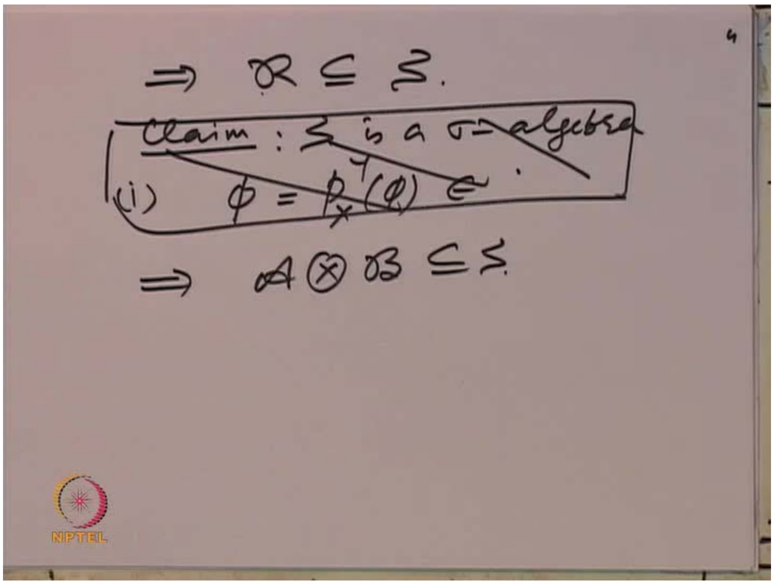

The set A times B can be written as p x inverse of A intersection with p y inverse of B. We are given that the sigma algebra S has the property that both p x and p y are measurable, so as a consequence of this for every set A in the sigma algebra A; p x inverse A belongs to the sigma algebra S and p y inverse also belongs to p y inverse of B also belongs to S, because p x and p y are both measurable, so this belongs to S.

So, what we have shown is that if S is a sigma algebra with respect to which both p x a p y are measurable then S must include all rectangles. Our analysis shows, so implies that all rectangles are inside the sigma algebra S and we wanted to show that the product sigma algebra is inside S, so it is enough to show that S is sigma algebra. Let us claim and try to prove that S is sigma algebra.

What we have to show that first, if you look at empty set then I can write empty set as equal to either p x inverse of empty set or p y inverse and hence this belongs to - we want to show that S is a sigma algebra, so that means, sorry S, is already given to be a sigma algebra, sorry, we do not have to prove this, so what we have shown all the rectangles are inside S and S is a sigma algebra, so that implies that A times B is also inside S.

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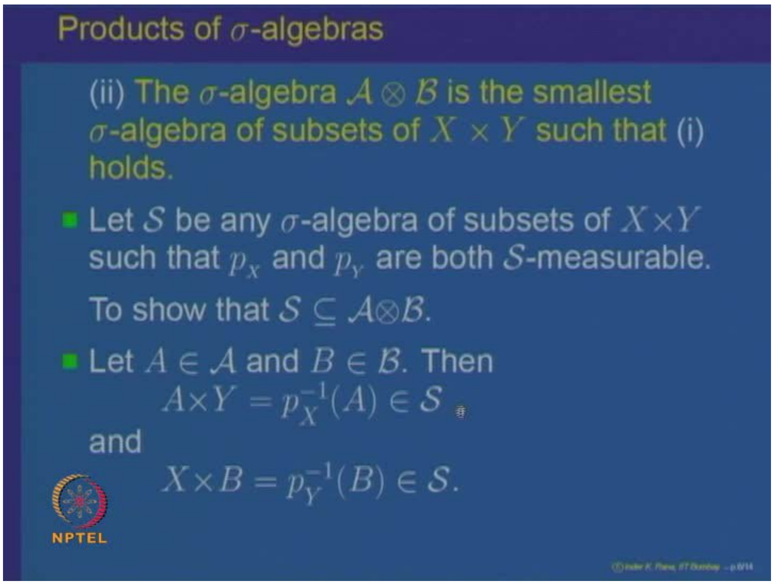

So, A times B is the smallest sigma algebra with respect to which both the projection maps are measurable. Let us go through the proof again, so if S is any other sigma algebra of subsets of X cross Y such that both p x and p y are measurable in that case, we want to show that S, this is a type we should show that S includes A times B.

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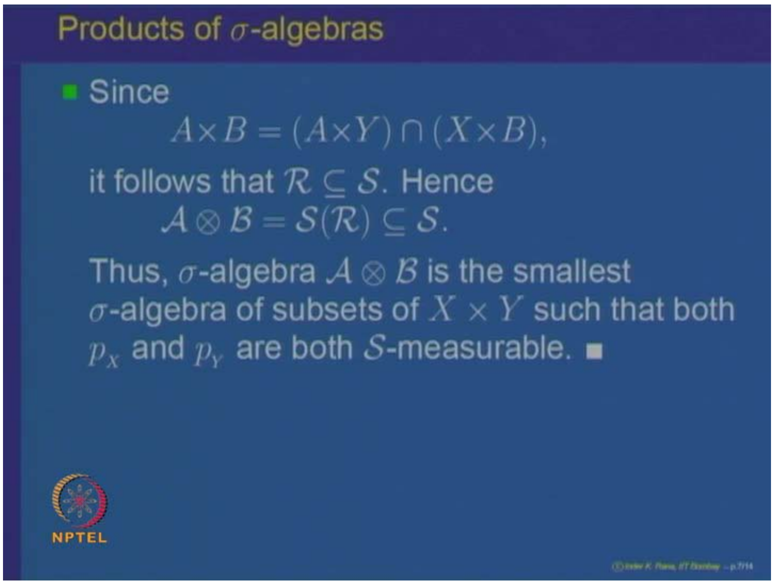

Let us take a set A in A and B in B then, A cross Y is p x inverse of A as just now observed and X cross B is p y inverse of B. Both belong to S because p x and p y are measurable, so A cross Y and X cross B are both sets in the sigma algebra S. So, their intersection must belong to the sigma algebra that means A times B, which is A cross Y intersection with X cross B belong.

All the rectangles are inside S and S is a sigma algebra, so all the sets in the product sigma algebra that is the smallest sigma algebra including rectangles must all come inside, so that shows that the product sigma algebra is inside S. Product sigma algebra is a smallest sigma algebra of subsets of X cross Y with respect to which both the maps p x and p y are measurable.

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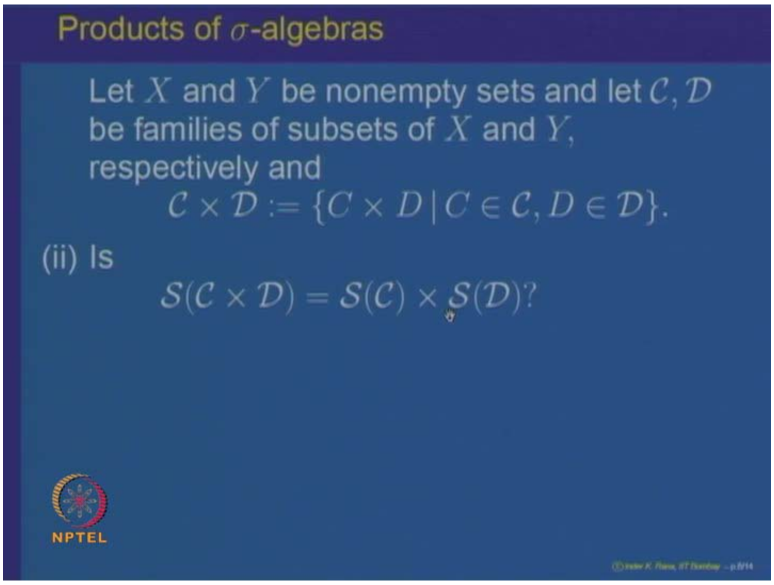

Let us look at some more properties of generating sigma algebras on product spaces. Let us look at this problem, let us look at two sets: X and Y of course, nonempty sets. Let us look at two families of subsets of X and Y, so C is a family of subsets of X and D is a family of subsets of Y, then we can form rectangles by elements of these families. Let us denote C times D to be the collection of all sets of the type C cross D, where C is in the collection C and D is in the collection D. Now, this is a collection of subsets of X cross Y and we can generate a sigma algebra out of it.

So, on the other hand, we can first generate sigma algebra out of the collection C and then generate the sigma algebra out of the collection D. There are two ways of generating sigma algebras of subsets of X cross Y; the first is look at C cross D that is a collection of subsets of X cross Y and look at the sigma algebra of subsets of X cross Y generated by them.

On the other hand, look at the sigma algebra generated by C, so call it as S of C that is the sigma algebra of subsets of X and generate the sigma algebra by the collection D. So, call that as S of D and then take the rectangles generated by these two sigma algebras, so that is S C cross S of D.

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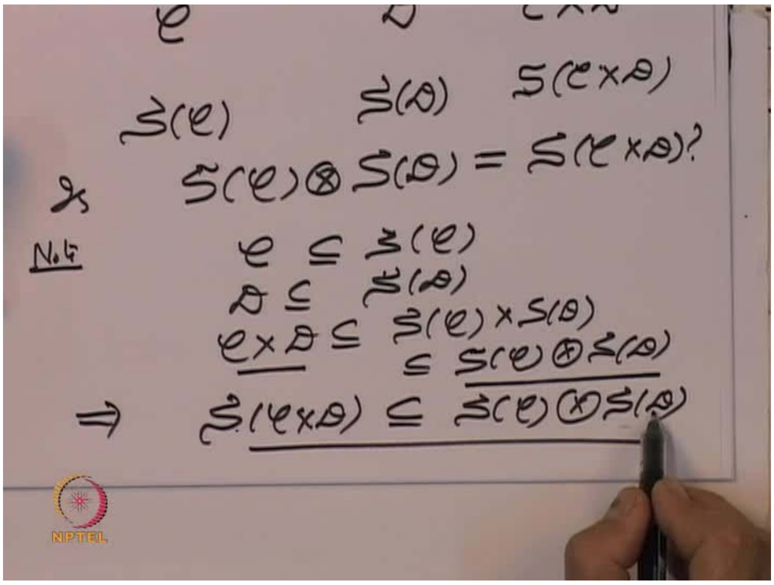

So question is, are these two things equal? Let us observe that since C cross D is already contained in S of C cross D, so the sigma algebra generated by them will be inside it. The first observation is that in general S of C cross D will be inside S of C cross S of D. So, this follows from the fact, so we have got a set X, we have got a set Y, we have a collection of subsets of X, we have a collection of subsets of Y that is D and so we form a collection of subsets of X cross Y that is C cross D.

We can generate the sigma algebra by C, we can generate the sigma algebra by D and also we can generate the sigma algebra by C cross D. The question we are analyzing is; is S of C cross S of D equal to S of C cross D.

The first observation is we are going to note is the following, so C is contained in S of - so the question is, whether this product is equal to this. The question first observe that C is subset of S of C and D is contained in S of D, so C cross D is going to be a subset of S of C cross S of D. We can generate the sigma algebra by these rectangles, so that will be a subset of S of C times S of D.

So, C cross D is always in the product sigma algebra; product of the sigma algebras S of C times S of D, so that implies the smallest one that is the sigma algebra generated by C times D must be inside S of C times S of D.

So, this is always true that the sigma algebra, so first take the product of the family C and D and generate the sigma algebra out of it and that is always a subset of first generate the sigma algebra S of C and S of D and then that take their product S of C cross S of D.

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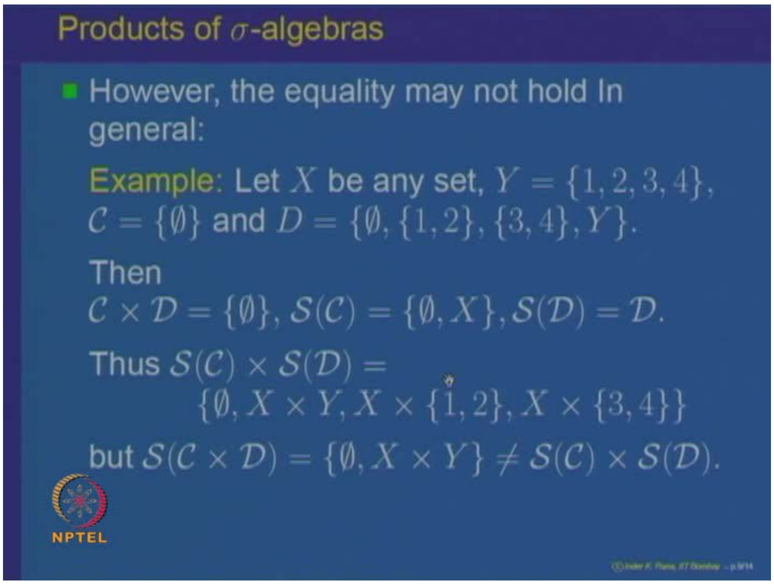

The question is the other way round equality true and we will show by an example that in general, the other way around equality may not hold namely that S of C times S of D may not be a subset of S of C cross D. For that a simple example works, so let us take x to be any set and Y to be the set consisting of four elements 1, 2, 3 and 4.

Let us look at C; the collection which consists of just the empty set and the collection D, which consists of the empty set; the set 1 and 2 and the set 3, 4 and the whole set Y. The collection D consists of four sets: the empty set, the whole space, the subset with two elements 1 and 2, the subsets with two elements 3 and 4.

Note; if you generate the sigma algebra by C that is same as the algebra generated by C, so that will be just empty set and the whole space, so S of C is empty set and the whole space X. The sigma algebra generated by D is equal to D itself, because D itself is a algebra; the complement of 1 and 2 that is 3 and 4 that is here, complement of 3 and 4 in the set Y that is here, 5 and empty set and the whole space are there, so D actually in itself is a algebra. So, the sigma algebra generated by D is equal to because it is a finite collection, so the sigma algebra is the algebra itself that is equal to D and if we look at the product the rectangles generated by C and D, so the first component is always going to be empty set; that means C cross D is just the empty set. So, the sigma algebra generated by the collection C cross D will be the empty set and the complement is that the whole space that is X cross Y.

So, S of C cross D consists of just two elements: namely empty set and the whole space X cross Y. On the other hand, if you look at S of C cross S of D then it consists of the empty set - the whole space of course, and then it will consist of sets of the type X cross the set in Y that is 1 cross 2. Of course, the set X cross 3 comma 4, so there are at least the empty set, the whole space and the sets of the type X cross the two elements set 1, 2 and X cross the two element set 3 and 4 and of course, this is not going to be equal to S of C cross D. Even S of C cross D is not equal to even the rectangles, so it cannot be actually equal to the whole of S of C times S of D also.

In general, we cannot expect that if you take two classes of subsets: one of X and one of Y, so C is a collection of subsets of X and D is a collection of subsets of Y. We can take the rectangles formed by taking elements from C and D, so that is the sets denoted by C cross D. Then, we generate the sigma algebra out of this collection so that the sigma algebra S of C cross D that may not be always equal to the sigma algebra generated by C times, the sigma algebra generated by D. However, would like to find some conditions under which we can ensure these two are equal and that is our going to be our next theorem.

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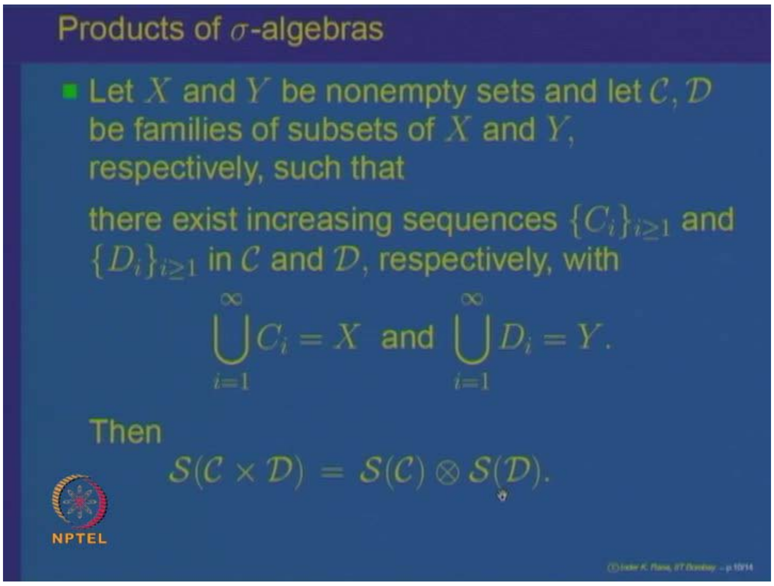

Theorem says, let X and Y be non-empty sets and C and D be families of subsets of X and Y, such that the whole space X can be represented as a union of elements from that collections C and the space Y also can be represented as union of elements from that collection D.

We are putting this condition, this collection C and D are such that there is a sequence of elements of C which gives you the whole space X. There is a collection of elements from collection of sequence D i in the collection D such that its union is again equal to Y. Under this condition we are going to show that the sigma algebra generated by C cross D is same as the sigma algebra generated by C times the sigma algebra generated by D, so this equality holds.

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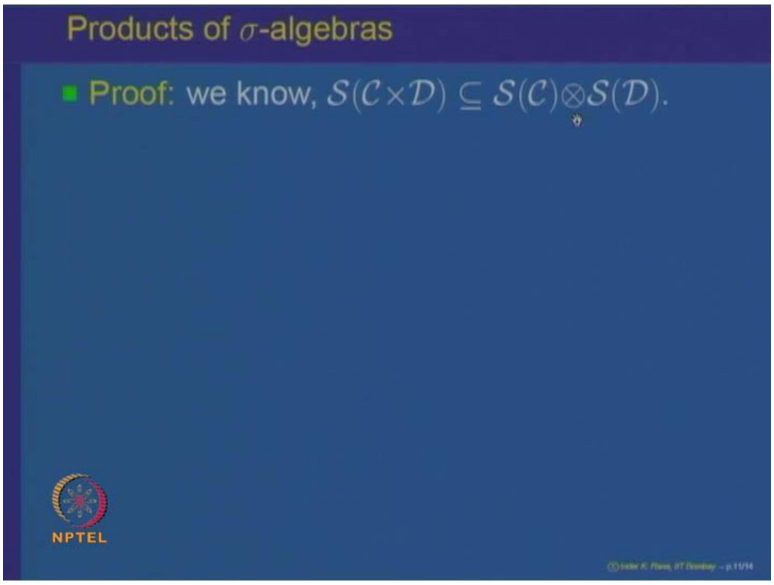

Of course, we have already proved that S of C cross D is the subset of the product sigma algebra S of C times S of D. We only have to prove the other way around inequality. What we have to show is the following, namely; we are given, so this is the fact which is given that X can be written as union of C i; i equal to 1 to n, where the C is belong to C and Y also can be written as a union of elements D j; j equal to 1 to infinity, where D j is belong to D.

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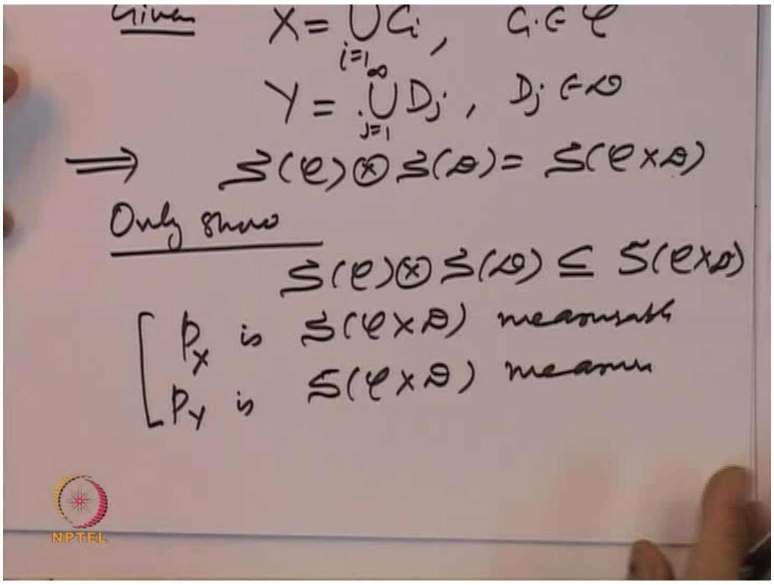

These two conditions we want to show, imply that S of C times S of D is equal to S of the sigma algebra generated by C cross D. We have already shown that the sigma algebra generated by C cross D is a subset of this, so we have to only show that left hand side that S of C times S of D is a subset of S of C cross D. So, this is what we have to show.

To show this, we will follow the previous proposition which said that the product sigma algebra is the smallest sigma algebra with respect to which the projection maps are measurable. Supposing, we are able to show that the projection map p is S of C cross D measurable and p of y is also S of C cross D measurable. So, if we show this, then what do we mean? So, p x and p y are both measurable with respect to this sigma algebra S of C cross D, so it must include the product sigma algebra namely, S of C cross S of D.

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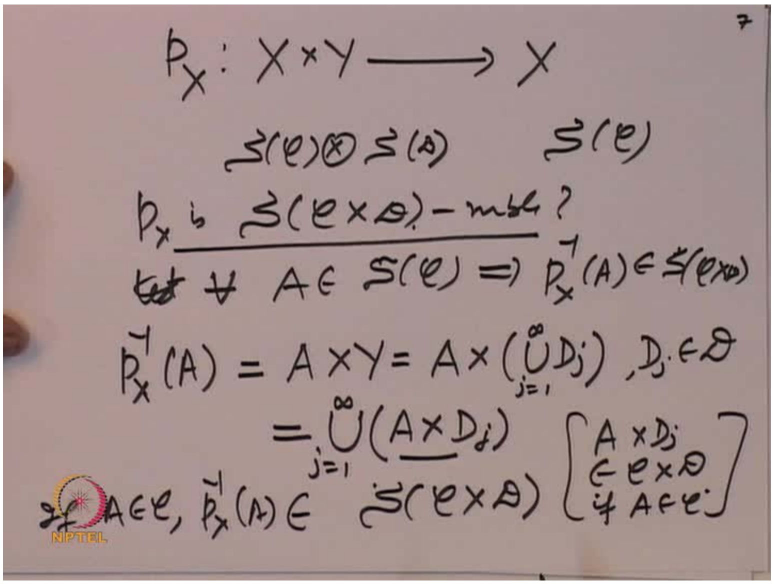

Let us show that these two maps are measurable with respect to the sigma algebra S of C cross D. For that what we have to show is the following, so let us look at the case of the projection map p x; so p x is a map from X cross Y to X. Here, we have the product sigma algebra S of C times S of D that is a product sigma algebra on X, we have the sigma algebra S of C. So, what we want to show is that the p x is in fact measurable with respect to the sigma algebra S of C times S of D. So, p x is measurable this is what we want to show.

To show that let us take a set. What we have to show is, we have to show that for every set say A belonging to S of C should imply that p x inverse of A belongs to S of C cross D, so that is what we have to show.

Let us first observe, what is p x inverse of A? If A is a subset of X and belonging to - so let us not do bother at present, where it belongs. Let us look at p x inverse of A that is going to be equal to A cross Y by definition, because A is a subset of X so the projection lies in A that means the inverse image is A cross Y. Now, by the given condition the set Y is representable as a union, so this is A cross union of D js; j belonging to 1 to infinity where each D j belongs to the collection D.

So that means, I can write this as union j equal to 1 to infinity of A cross D j. This implies that p x inverse of A is written as union of rectangles, which look like A cross D j. Now, D js are inside the collection D and if A belongs to C then this will belong to C cross D. So, the union will belong to the sigma algebra generated by C cross D, why? Because A cross D j will belong to C cross D if A belongs to C.

So, if A belongs to C then p x inverse of a just now represented as S belongs to C cross D, but what we want to show is not only for C, for S of C also this property is true. We apply the usual sigma algebra technique to prove this. Let us write, u to be the collection of all the subsets A belonging to S of C, such that p x inverse of A belongs to S of C times D.

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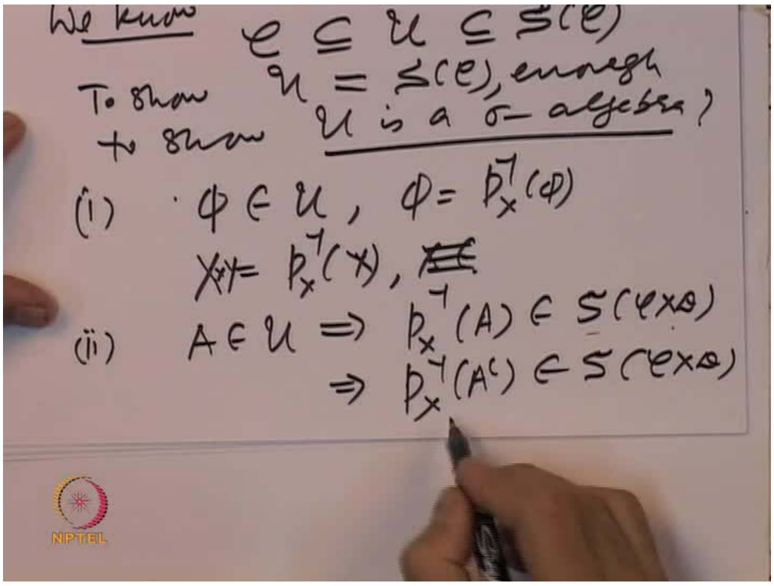

Let us look at this collection, what we have proved just now, so we know that C is contained in u and u is a collection of subsets of S of C, so that is contained in S of C, but what we want; we want for every set A in S of C p x inverse of A belongs to S of C that means we want to show that this collection u is actually equal to S of C and u is a subset of S of C and C is inside u to show that u is equal to S of C. It is enough to show that the collection u is to show that u is equal to S of C. It is enough to show, so enough that u is sigma algebra; because once u is sigma algebra it include C, so the smallest one that is S of C will come inside so everything will become equal. We have to only show that this is sigma algebra.

Let us look at empty set belongs to u because it is just empty set is equal to p x inverse of empty set. So, empty set belongs to u. What about X; X is equal to p x inverse of Y and Y belongs to S of C, so this also belongs to - so because y belongs to sorry p x we want to show that the whole space A belongs to S of C, so what we want to show is that x belongs to - so look at p x inverse of X that is X cross Y and that belongs to S of C cross D.

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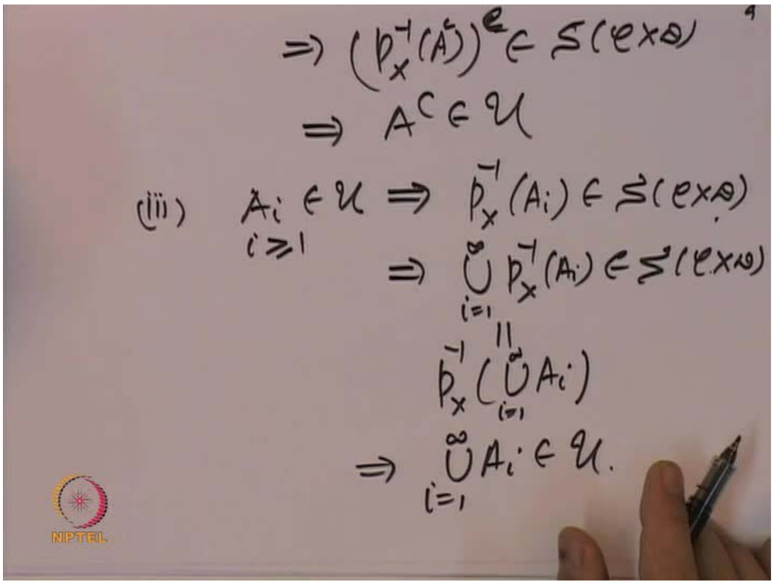

So that implies X belongs to this collection, so X is also inside it. So empty set and the whole space belong to it; obviously, let us look at the second condition that if A belongs to u then that implies p x inverse of A belongs to S of C cross D. So that is given to us, but S of C cross D is a sigma algebra, so it is closed under complements, so that implies that p x inverse of A complement also belongs to S of C cross D, but p x inverse of A complement is nothing but, so that implies this set is nothing but p x inverse of A and complement of that.

So that belongs to S of C cross D and that implies that A complement, so that means that A complement belongs to u because sorry it should be other way around p x inverse of A belongs - that means this complement belongs to S C cross D, so if a set belongs its complement belongs and this is nothing but the complement of p x inverse of A complement.

So, whenever a set has the A has the property that p x inverse of A belongs to the sigma algebra p x inverse of A complement also belongs to the sigma algebra S cross C cross D that means A complement belong. So, u is closed under complements and finally, let us look at suppose A i is belong to u for a sequence i bigger than or equal to 1 belongs to u that means p x inverse of A i is belong to the sigma algebra S of C cross D.

These are usual techniques for proving the sigma algebra, so that implies and this is sigma algebra S of C cross D and p x inverse of A i is belong to it, so union of them p x inverse of A i is the union 1 to infinity also belongs to S of C cross D. This union of the inverse image is the inverse image of the union, so that is p x inverse of the union i 1 to infinity. So that belongs to S of C cross D that means, we have shown whenever p x inverse of A i is belong to the sigma algebra the p x inverse of the union also belong, so that implies that the union A i is 1 to infinity also belong to u.

So that proves that u is a sigma algebra of subsets of X, so u is a sigma algebra include C, so u is equal to S of C that means p x inverse of A is measurable. What we have shown is that if we look at the map p x; X cross Y to X then it is S of C cross D measurable and the product sigma algebra S of C cross S of D is the smallest one with respect to this is measurable. So that will prove that S of C cross D includes the sigma algebra S of C times S of D.

So that is how we prove that whenever, the family C and D are the property that you can write, so whenever X is a union of C i s and Y is a union of D j s, where C i s you know in C and D j s in C, whenever this property is true then whether you take the classes C and D and take the rectangles and generate the sigma algebra that is going to be same as generating the sigma algebras first and then taking the product sigma algebra. So, this is a useful theorem, which gives us a dividend namely the following.

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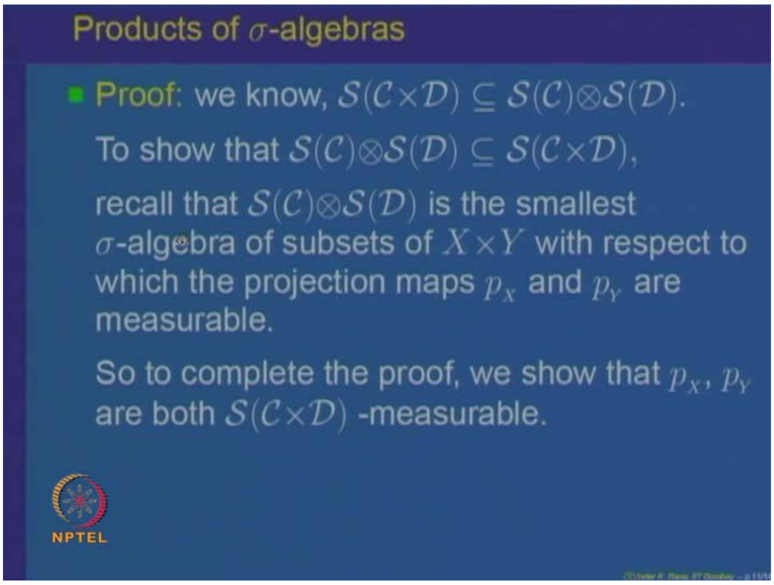

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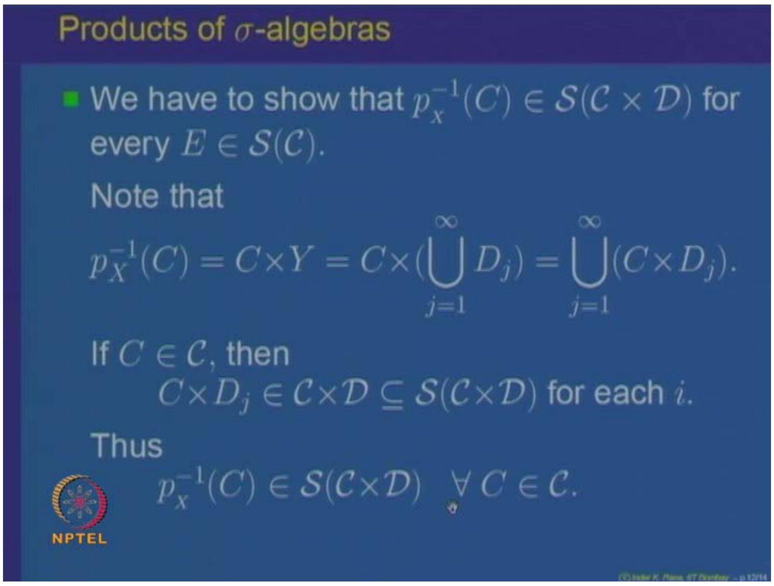

We look at a consequence of this, which says that we have this is a repetition of what the ideas that I have said to show that p x inverse is measurable, we have to show this and that can be written as p x inverse of C because Y is a countable union, so you write this as a union. This union splits into this union of C cross D j s and if C belongs to C because this belongs to C cross D, which is instead has of C cross D. So, p x inverse of C is an element in S of C cross D, whenever C belongs C, so p x inverse of C is in S of C cross D for every C.

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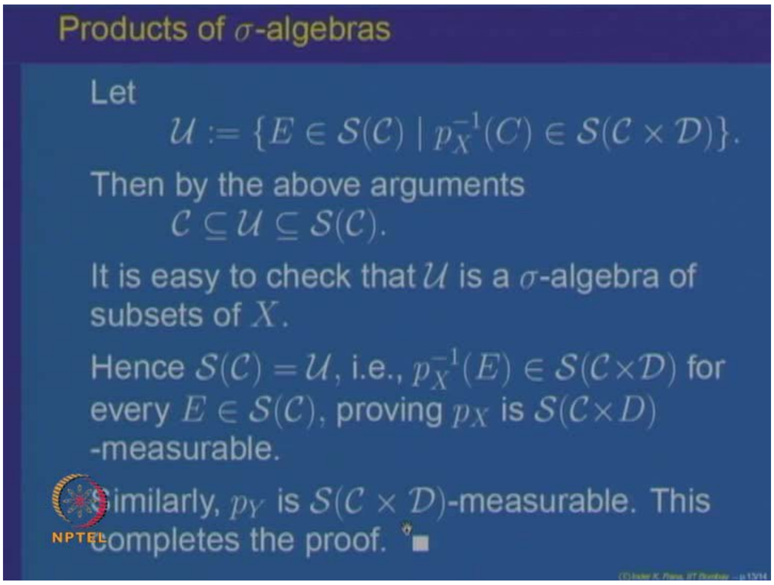

To prove it is equal for all S of elements of S of C this property is true, we use the sigma algebra technique namely, look at the set u, which is the collection of all the sets in S of C which have this property. Then show that we already know that C is inside u and u is inside S of C, so to prove the equality one just has to show that u is sigma algebra.

So that is easy, we have just now shown it is sigma algebra. So that proves the fact that whenever C and D are there, two classes of subsets of X cross Y then S of C cross D is same as S of C times S of D.

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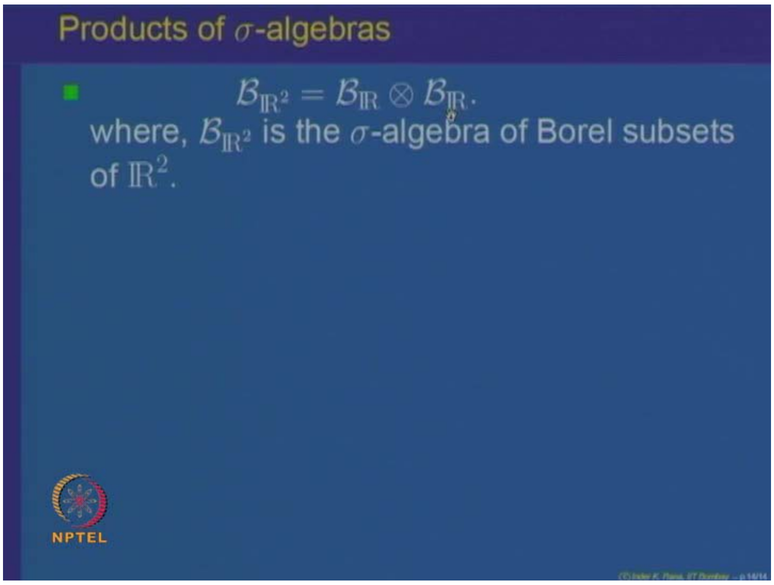

As a consequence of this, let us prove the fact that on the plane B R 2 that is a sigma algebra generated by Borel subsets of R 2 is equal to the Borel sigma algebra of R times the Borel sigma algebra of R.

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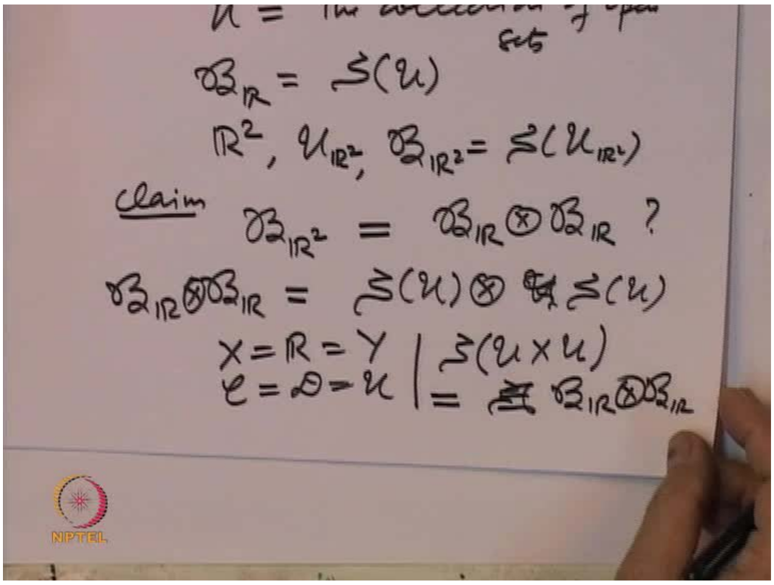

To prove this fact, let us just observe the following. So, on the real line we have got u the collection of open sets and B R 2 B of R the Borel sigma algebra of R is nothing but the sigma algebra generated by open subsets of real line. Let us look at the set R 2; on R 2 we have got the collection of open sets and the sigma algebra generated by them. So, let us write u of R 2 the collection of open subsets of R 2 and generate the sigma algebra; so B of R 2 is the sigma algebra generated by open subsets of R 2.

We are claiming the following, look at the Borel sigma algebra of R and look at the product of this with the Borel sigma algebra of R. So that gives you a sigma algebra of subsets of R 2 and on the other hand, you have got the sigma algebra of subsets of R 2 called the Borel sigma algebra of subsets of R 2 and what we want to show is that these two are equal.

So, note on the left hand side, so B of R cross B of R is nothing but the sigma algebra generated by open sets times the sigma algebra i in the same sigma algebra S of u. It is the product of the same sigma algebra with itself, the Borel sigma algebra with itself. The Borel sigma algebra is generated by open subsets of the real line, so this is a perfect setting for applying our previous theorem, so we have got X equal to R equal to Y and C equal to D equal to open sets.

So, our previous theorem will imply that if you look at C cross D so if you look at u cross u and then look at the sigma algebra generated by it, that must be equal to the sigma algebra B R cross B R.

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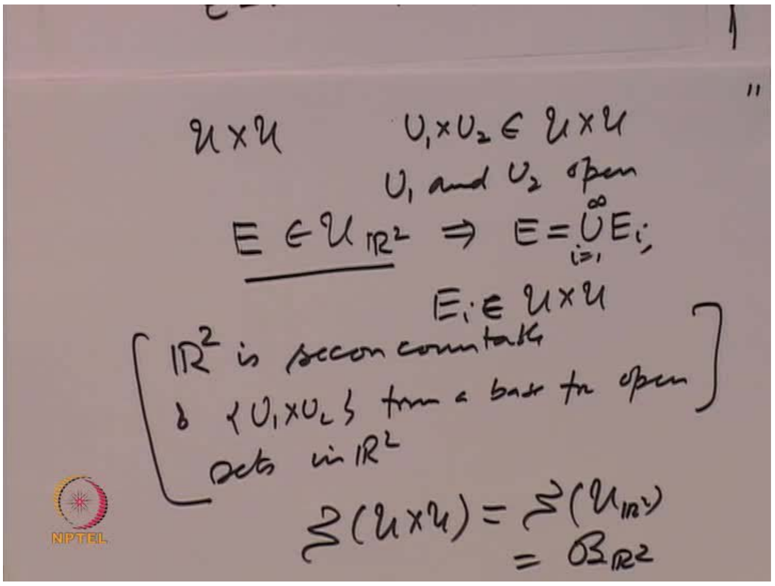

So that is from our previous theorem, but what we want to observe here is that if we look at u cross u, so if you look at the sets of the type u cross u then these are sets of the type, so what is the set in the type u cross u? So that is a set of a type that open set u 1 cross an open set u 2. These are the type of sets, which belong to u cross u, so u 1 and u 2 both open. Now, if you take any open set say, E and any open set in R 2 then this is effect from the basic matrix basis that the open sets in R 2. The sets of the type u 1 cross u 2 forms a base for the topology of open sets for the topology of R 2.

What we are saying is, if this implies that E can be written as a countable union of sets E i s 1 to infinity, where each E i is a set belongs to u cross u. So, this set is from basic topology namely, R 2 is second countable and the sets u 1 crosses u 2 forms a base for open sets in R 2.

So, this together imply that every open set in R 2 can be written as a countable union of sets E i and these E i s are open rectangles, you can call them each E i is a open set cross another open set, so by that fact that will follow that the sigma algebra S of u cross u is same as the sigma algebra on R 2 generated by open sets and that is B of R 2.

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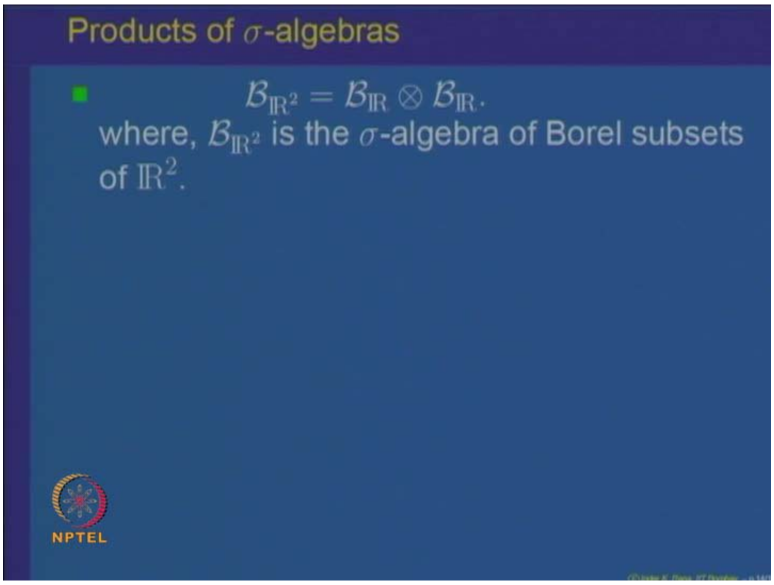

So that will prove that the Borel sigma algebra in R 2 is same as - so if you want to generate Borel subsets in R 2, what you can do is you can generate Borel subsets in real line and then take the product sigma algebra Borel subsets cross Borel subsets and that will give you the sigma algebra of Borel subsets in R 2. With that we come to the conclusion of the description of sigma algebras or the product of sigma algebras on R 2.

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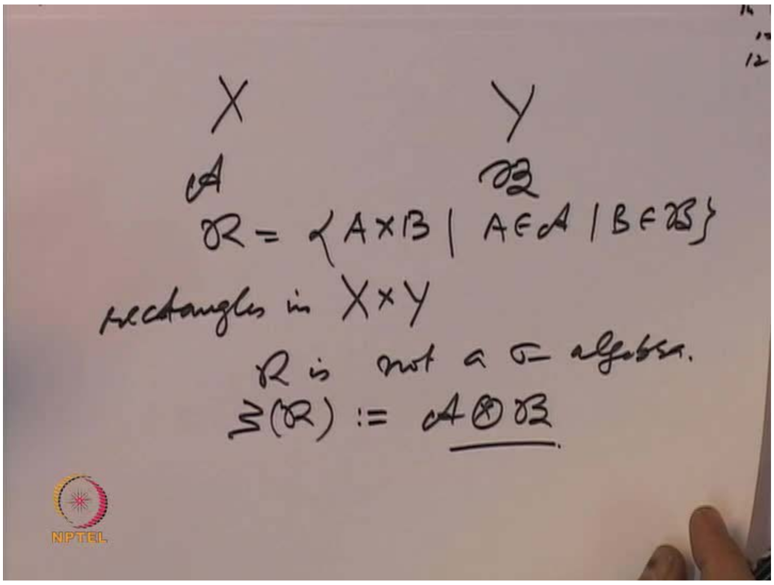

The main thing is that to remember the following namely, given X and given Y, we can have the product set X cross Y. Here, we have got sigma algebra A; here, we have got a sigma algebra B, so we take sets A in A and B in B, so that gives the sets of the type A cross B. So, these kinds of sets are called measurable rectangles.

So, sets R equal to A cross B, where A belongs to A and B belongs to B give you subsets of X cross Y, so they are called rectangles in X cross Y. In general, these rectangles do not form, so R is not sigma algebra; in general it is not sigma algebra.

You can generate; so generate the sigma algebra out of this rectangles and that is denoted by A times B. So, this is called the product sigma algebra, so the product sigma algebra is the sigma algebra generated by all rectangles. Another way of generating product sigma algebras is by generating the sigma algebras.

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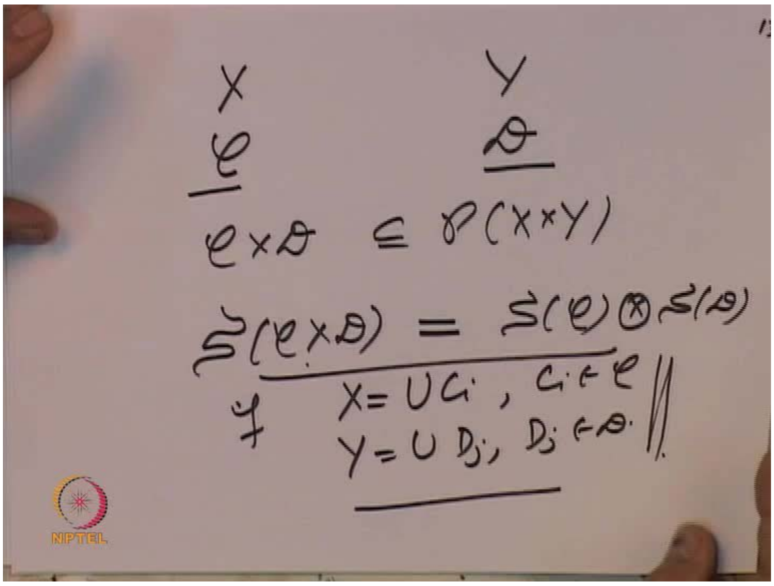

So here is X, here is Y; there is a collection C here, there is a collection D here, so we get C cross D is a collection of subsets of X cross Y, so one can generate the sigma algebra by this collection. On the other hand, one can generate first the sigma algebra S of C here and generate the sigma algebra by this S of D here. Then look at the product of them, so these two are equal if we can write X as union of C i s; C i s belonging to C and Y can be written as union of D j s; D j s belonging to D.

Under these conditions, these two are equal and if this condition is not true, then only you can say it is the left hand side is a subset of the right hand side, so these are the product sigma algebras.

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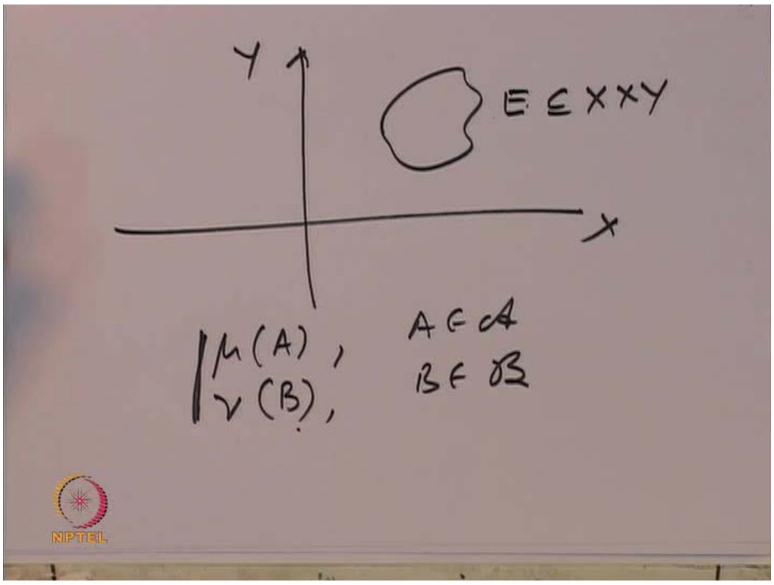

What we want to do in the next lecture is the following; so here is the set X, here is the set Y, so this is a subset E in the product sigma algebra, what we want to do is that if we have got a notion of size of sets A in the sigma algebra A belonging to A and a notion of size for sets B; B belonging to B, we want to know, what are the sets for which E contained in a X cross Y for which we can define the notion of size.

In a sense, what we are trying to do is that, we will try to construct a measure on the product of the sigma algebras A cross B, by looking at the measures on X and on Y. We will do that in the next lecture.

In the next lecture, we will look at measures on the products spaces. How to construct given a measure on the space X on sigma algebra A and a measure nu on the sigma algebra B of subsets of Y? How to construct a measure in a natural way on the product sigma algebra A times B? That will also generalize the notion of areas in R 2 and volume in R 3 and so on. So, we will continue the study of construction of measures on product spaces in our next lecture, thank you.