Measure and Integration

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Lecture No. # 23

Denseness of Continuous Functions

Welcome to lecture 23 on measure and integration. In the previous lecture, we had started looking at the space of Lebesgue integrable functions on the interval a b. We had defined the notion of L 1 metric on it and we had proved that under the L 1 metric, L 1 a b is a complete metric space.

We will continue the study of this space L 1 a b, a bit more and today, we will show that the space of continuous functions on the interval a b is dense inside the space of integrable functions under the L 1 metric.

So, let us just recall the proof of the fact that the space L 1 a b under the L 1 metric is complete.

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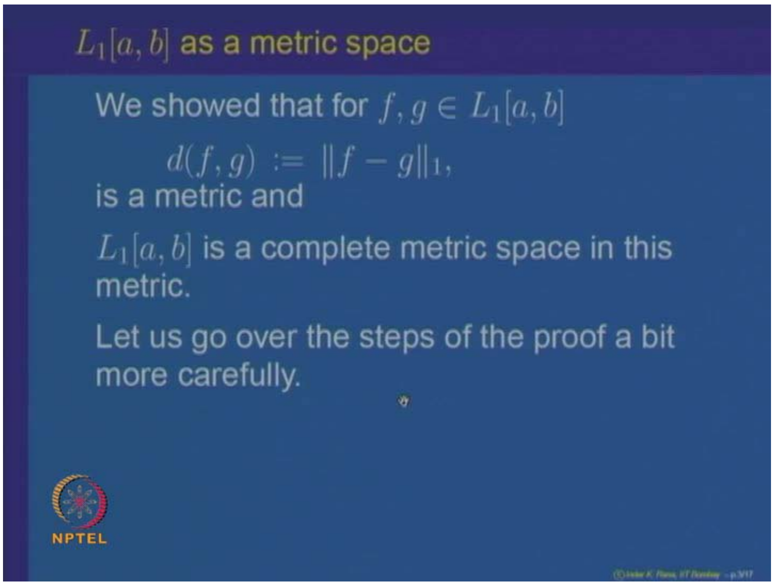

So, we defined the notion of L 1 metric as follows - for the functions f and g in L 1 of a b, we defined the distance between f and g to be the L 1 norm of f minus g; so this 1 indicates what is called the L 1 norm which we had defined last time. So, this is precisely equal to the L 1 norm or the distance between f and g is equal to integral of mod of f minus g d lambda over the interval a b. We showed that, if you identify the functions almost everywhere; that means, if you do not distinguish between functions f and g which are equal almost everywhere; L 1 of a b then this becomes a metric and the space L 1 a b is a complete metric space under this metric.

I just want to go through the over the proof main steps of the proof once again to emphasis something important as follows:

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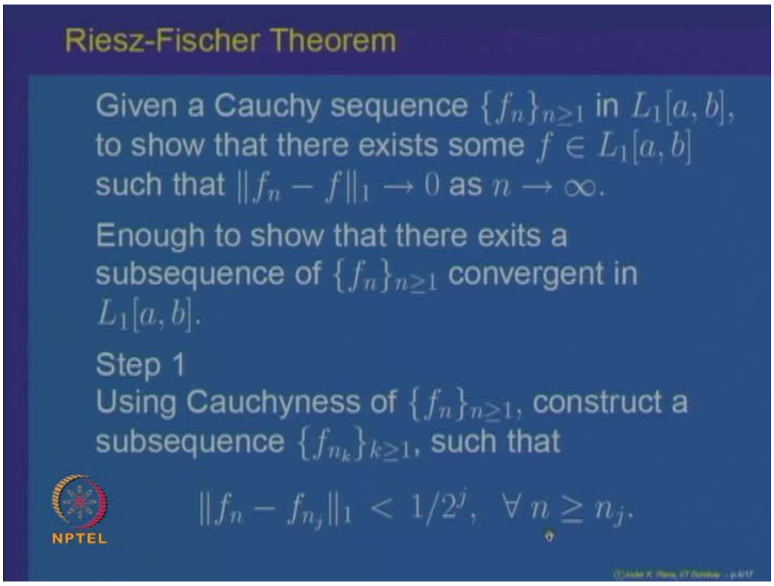

So, let us go through the steps again to show that this L 1 of a b is complete, what we have to show is given a Cauchy sequence f n in L 1 of a b; we have to show that there exist a function f in L 1 of a b, such that f n converges to f in the L 1 norm.

So, to do that, we said it is enough to show that the Cauchy sequence f n converges in L 1 metric; it is enough to show that there is a subsequence of f n which is convergent in L 1.

So, this is a general fact about metric spaces namely - in any metric space given a Cauchy sequence, a Cauchy sequence converges if and only if there is a subsequence of it which is convergent; so this is the fact about metric space - is we are going to use here to prove that L 1 of a b is complete, given the Cauchy sequence f n, we will try to construct a subsequence of f n which is convergent in L 1 norm.

So, as a first step using the Cauchyness property of f n, we construct a subsequence f n k of f n, such that the L 1 norm of f n minus f n j is less than 1 over 2 to the power j for n bigger than or equal to n j. This was done basically the Cauchyness says that the distance between f n and f m goes to 0 as n and m go to infinity so after some stage the difference between f n and f m can be made as small as you want so by using induction we construct this subsequence such that f n minus f n j L 1 norm is less than 1 over 2 to the power j.

So, what we wanted to note down that in this step 1 we have not use anywhere the fact that we the functions are defined over the interval a b or real line we are just use a general fact about Cauchy sequences.

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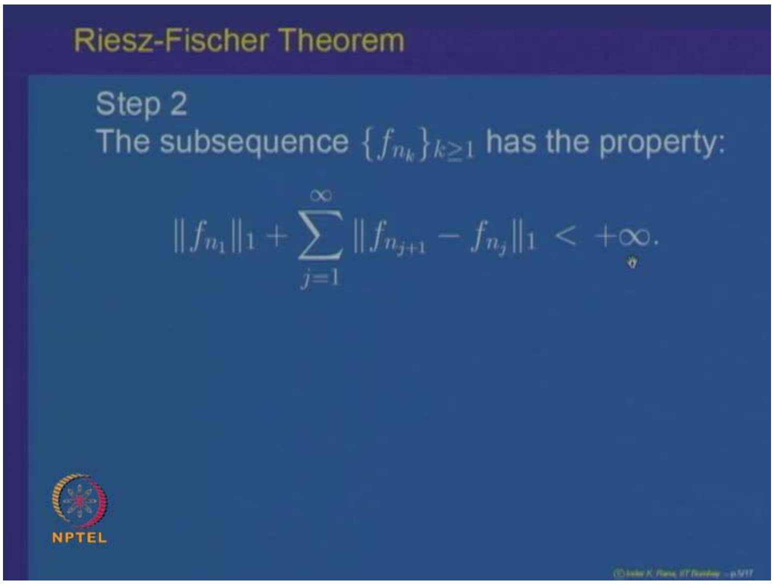

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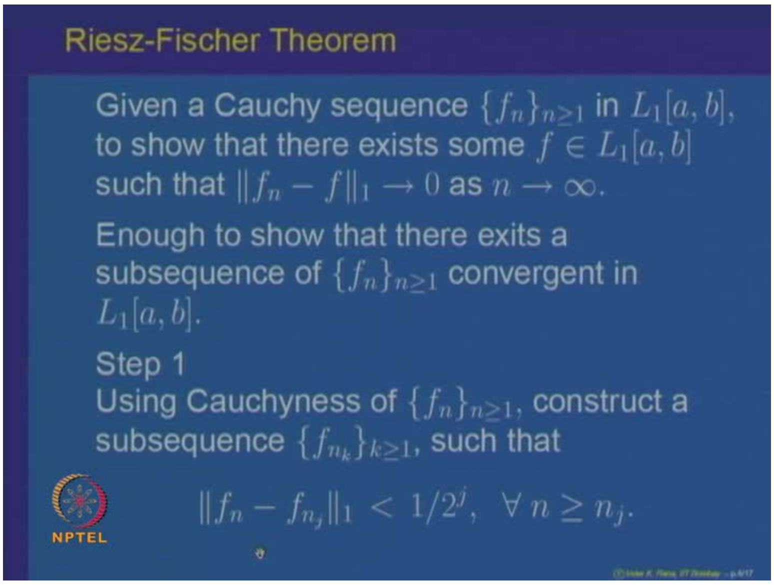

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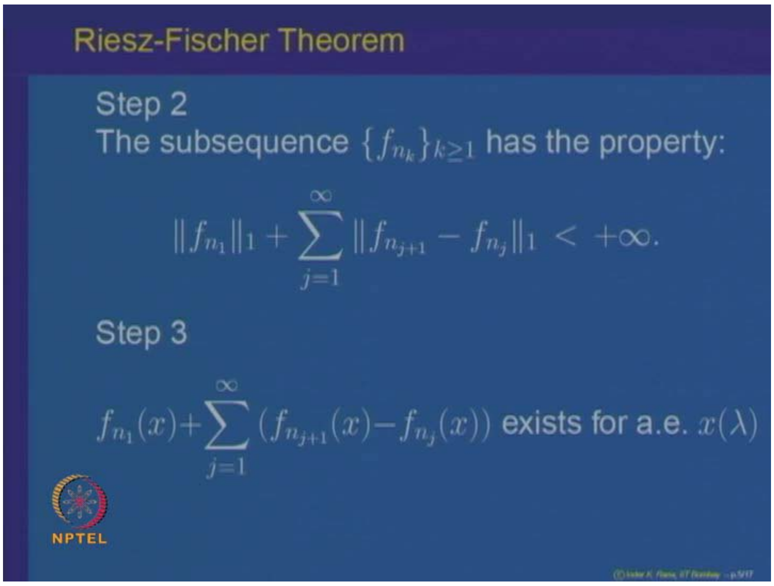

In step 2, we said that the look at the Cauchy sequence f n k that we have just constructed this has the property that the L 1 norm summation of the L 1 norms of f n 1, that is the first term plus the consecutive differences the norm of f n j plus 1 minus f n j is a convergent series. This follows from step 1 because in the step 1 the difference between f n and f n j so f n n plus 1 j minus f n j less than 1 over 2 to the power j so that clearly says that this sum of the norms will be less than summation 1 over 2 to the power j which is finite so again this follows from step 1 and we are not using anywhere the fact that our underlying space is the real line or the interval.

And now in step 3, we want you to conclude that the function f n 1 x plus summation of f n j plus 1 x minus f n j x exists almost everywhere. If you recall the proof of this was from the fact using the series form of the Lebesgue dominated convergence theorem namely, whenever you have given a series of L 1 functions and if the L 1 norms are finite then the functions series itself is convergent almost everywhere so again here we do not use the fact that we are over the real line.

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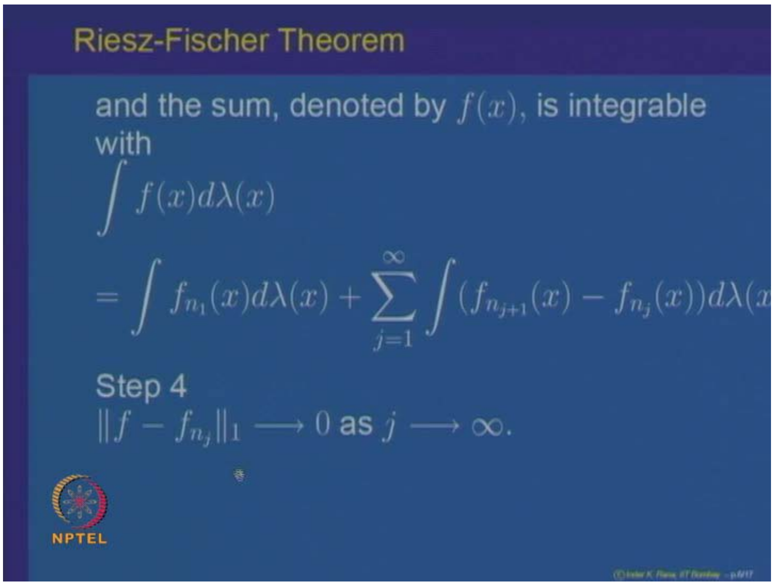

So, this step 3 also is valid and hence as a consequence of that theorem of Lebesgue dominated convergence theorem in the series form we get that f is L 1 and the L 1 norm or L 1 and the integral of f is equal to integral of sum of the corresponding integrals and as a consequence of this it follows that f n j converges to f in L 1.

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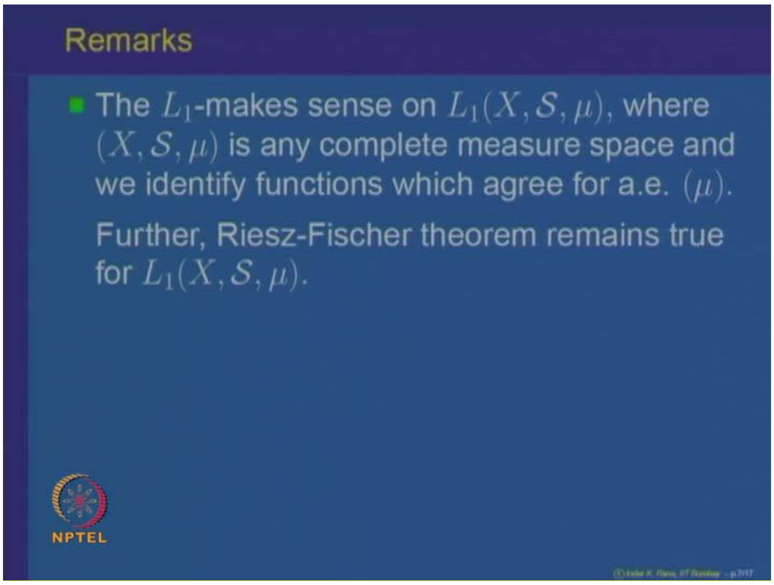

So, what we are saying is in all this steps we have not used anywhere the fact that we are working over the real number system so this proof carries over to any measure space, complete measure space X S mu and that means we can replace the real line by any set X and the sigma algebra Lebesgue measurable sets by a sigma algebra of subsets of X and a measure mu such that X S mu is a complete measured space. We can define the space of mu integrable functions, we can define L 1 of X the space of integral functions and the notion of the L 1 norm make sense for any function f on the measures on the space X if it is mu integrable, we can define the L 1 norm of this

So, what we are saying is that the L 1 norm make sense for any L 1 metric make sense on any in the space of Lebesgue on the space of integrable functions on any measured space X S mu which is complete and as we have seen just now in the proof of the theorem, we do not use anywhere the fact that we are over the real line we use general statements about metric spaces or we use the series form of the Lebesgue dominated convergence theorem.

So, as the result I am saying that the same proof which we have worked out that saying L 1 a b is complete works very well for the space L 1 of X S mu where, X S mu is any measure space so that gives us the riesz-fischer theorem. For a complete measure space X S mu saying that the space of integrable functions on a complete measure space under the L 1 metric is always complete

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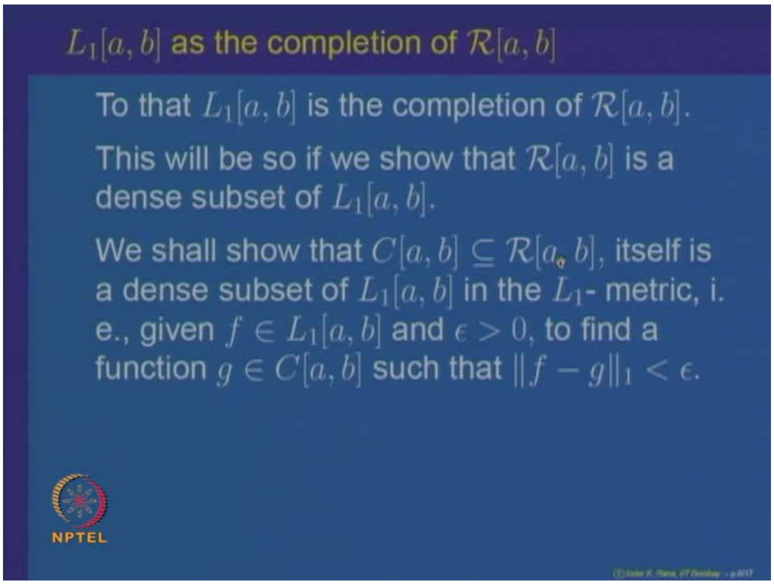

and so that is one observation and now, let us go over to the fact we wanted to prove that L 1 a b which is complete is in fact the completion of the space R a b of Riemann integrable functions on a,b.

So, for to do that what we have already observed that R a, b is a subset of L 1 of a, b. We had proved the theorem that any function which is Riemann integrable is also Lebesgue integrable and the Riemann integral is same as the Lebesgue integral.

So, R a, b is a subset of L 1 of a, b L 1 of a, b is complete to show that this is the completion of R a, b; we want to show that R a, b is a dense subset of L 1 of a, b in the L 1 metric so the denseness of R a, b is to be proved in L 1 of a, b.

In fact, we will prove something much stronger remember that every continuous function on the interval a b is also Riemann integrable so the space C a, b of continuous functions on the interval a ,b is a subset of the space of Riemann integrable functions and we will show that C a, b itself is dense in L 1 of a ,b.

That means for any function f in L 1 of a, b and any number epsilon bigger than 0 we want to show that there exist a function g belonging to C a, b a continuous function such that the norm of f minus g is less than epsilon so that will prove that c a, b is complete in L 1 of a, b.

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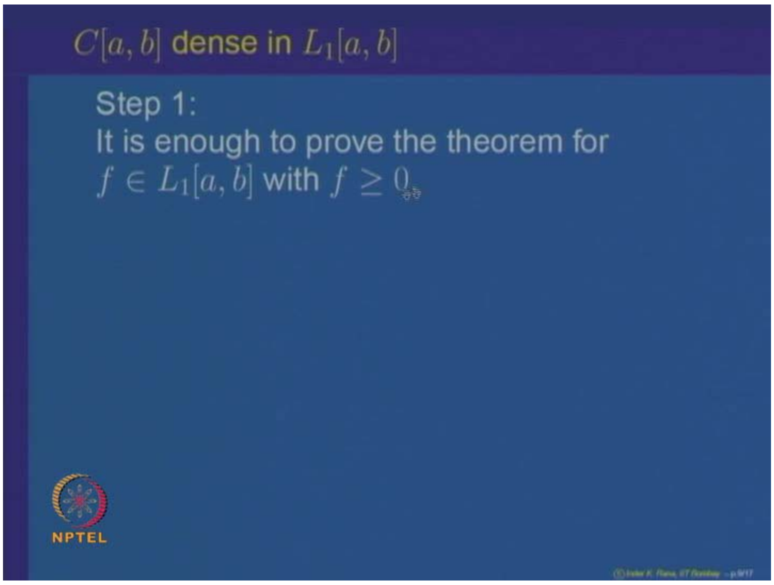

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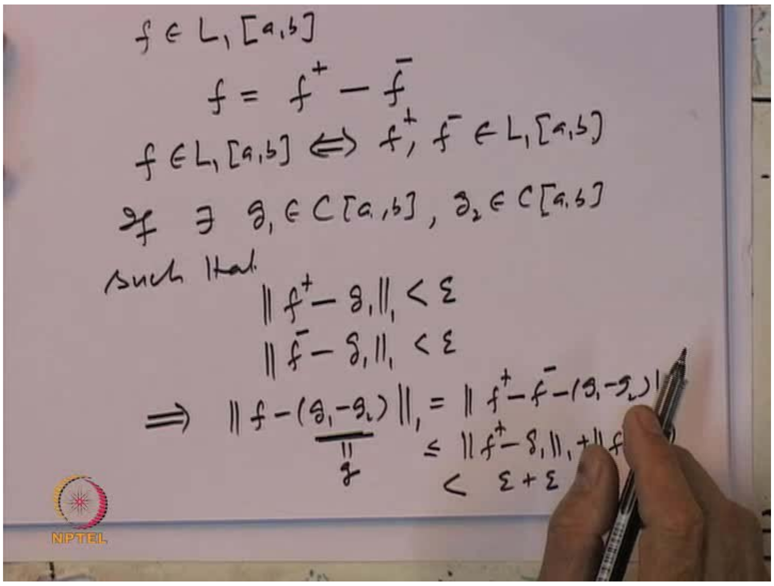

So, we will do it in steps, step 1 is that given that function f in L 1 of a, b which we want to approximate by a continuous function, it is enough to prove the theorem for functions in L 1 a, b such that f is bigger than or f is a nonnegative and that is because if f belongs to L 1 of a, b then, we know that f can be written as f plus the positive part minus the negative part of the function and f belongs to L 1 of a, b if and only if both f plus and f minus belong to L 1 of a, b.

So, in case if nonnegative functions in L 1 of a, b can be approximated, so if there is a function g 1 belonging to C a, b and a function g 2 belonging to C a, b continuous functions such that the norm of f plus minus the continuous function g 1 L 1 norm is less than epsilon and norm of f minus g 1 is also less than epsilon, then this will imply that the norm of f minus g 1 minus g 2 L 1, which will be equal to norm of f plus minus f minus minus g 1 g 2 and that will be less than or equal to norm of f plus minus g 1 plus norm of using the triangle inequality property of the norm so f minus minus g 2 and that will be less than epsilon plus epsilon.

So, if you call this function as g so what we are saying is that if nonnegative functions in L 1 can be approximated by continuous functions, then any function f in L 1 can be approximated because f can be split as a difference of two nonnegative functions in L 1.

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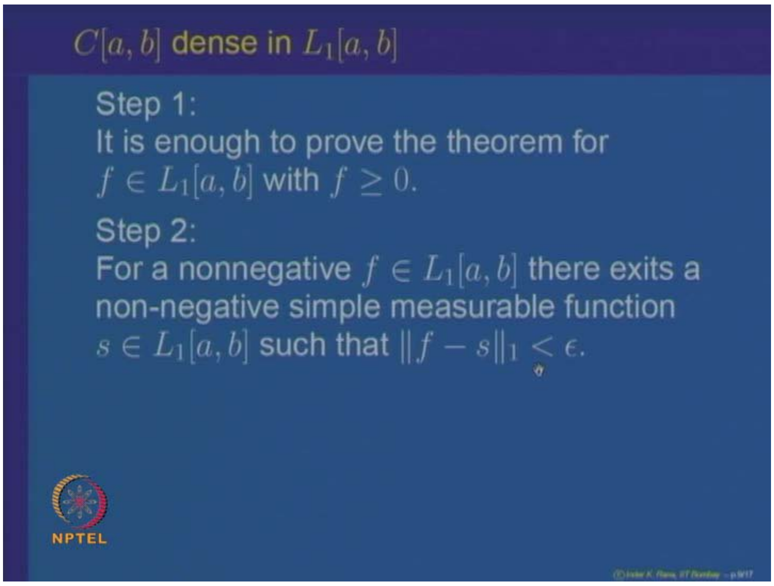

So, this is step 1 namely that it is enough to prove the theorem for functions which are integrable and which are nonnegative so this is the first observation that showing that C a, b is dense in L 1 of a, b; we can assume that the function f in L 1 a b is a nonnegative function so this is a first simplification or first step.

The second step says that for a nonnegative function f in L 1 a, b. So, observation is that for a nonnegative integrable function there exists a non-negative simple measurable functions in L 1 a, b such that the norm of f minus s is less than epsilon.

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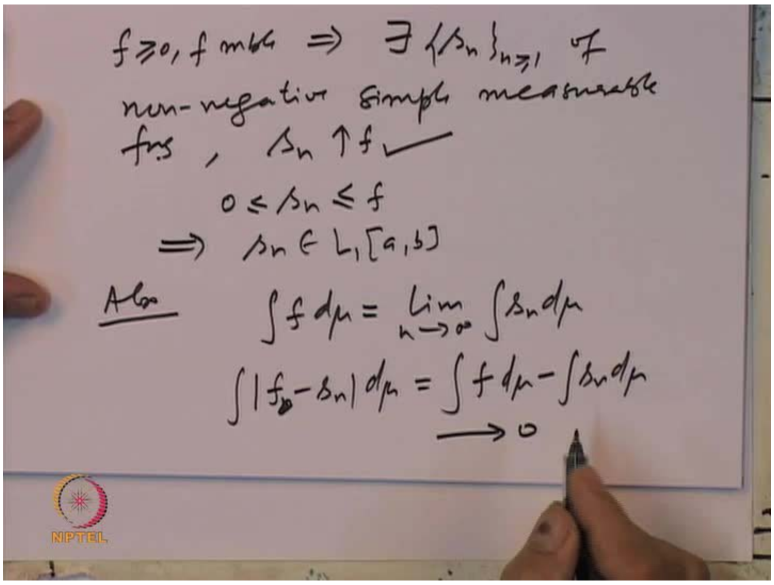

So, what we are saying is that if f is a nonnegative integrable function, then it can be approximated by a nonnegative simple measurable function, which is integrable so let us prove this step. How does we do that so? We are given that f is nonnegative and f belongs to L 1 of a b

now, because f is nonnegative and it is integrable. So, f is nonnegative measurable so f bigger than or equal to 0, f measurable implies there exist a sequence s n of nonnegative simple measurable functions, simple measurable functions such that s n increases to f but then s n is less than or equal to f and all are nonnegative so that implies that s n also belongs to L 1 of a, b.

So, because s n is dominated by f they are nonnegative functions so that implies as, we have seen earlier that s n also will belong to L 1 of a, b and also because s n is increasing to f so integral of f d mu can be written as limit n going to infinity integral of s n d mu. right That is by the definition of the integral, for a nonnegative measurable function, the integral is the limit of the approximating sequence of nonnegative simple measurable functions

but note that each s n integral of s n is less than or equal to integral of f. So, we can write that actually as absolute value of f minus s n d mu that will be equal to integral of f d mu minus integral of s n d mu because f minus s n is nonnegative, so its absolute is same as f minus s n so and that integral is equal to this and that goes to 0.

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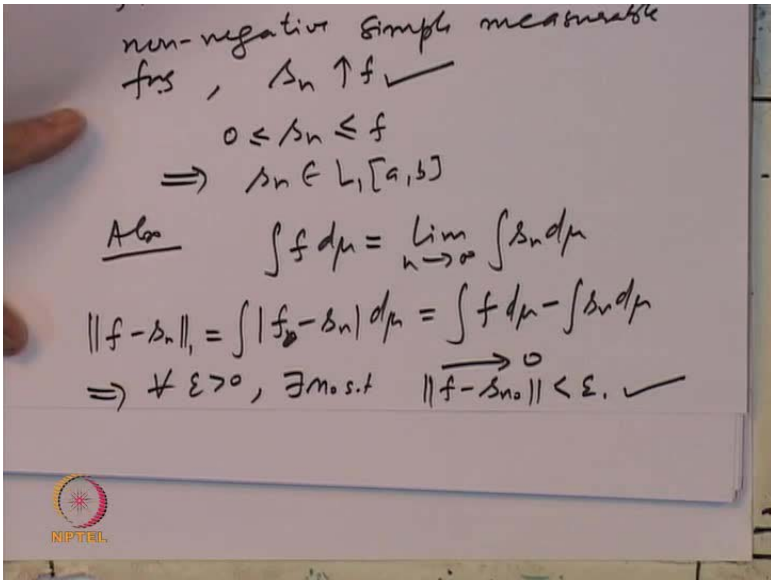

So, that means we have got a sequence of simple measurable functions nonnegative simple measurable functions, which are in L 1 and so this is L 1 norm; so integral of mod f minus s n goes to zero that means the norm of f minus s n L 1 norm goes to 0.

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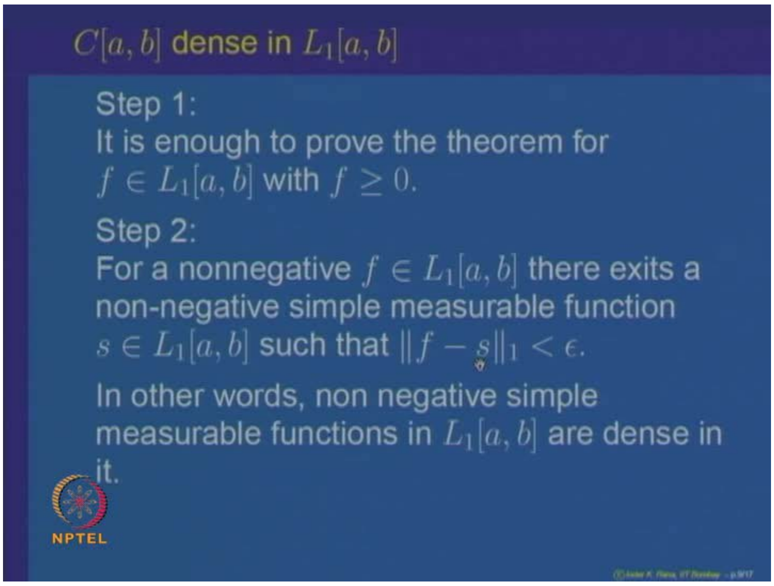

So, once it happens so that means for any epsilon we can choose a n naught such that f minus s n naught will be less than epsilon so implies for every epsilon bigger than 0 there is a n naught such that norm of f minus s n naught will be less than epsilon so that proves the second step that close to a integrable function f which is nonnegative there is a nonnegative simple measurable function close to it close in the sense of L 1 norm.

So, as step 3 so that means what that means in order to approximate f by a continuous function we can approximate it is enough to approximate nonnegative simple measurable functions in L 1 by a continuous function because f can be approximated by a simple nonnegative simple measurable function in L 1 and if nonnegative simple measurable function can be approximated then will be through(())

So, what we have shown till now is that the nonnegative simple measurable functions in L 1 are dense in L 1 that itself is of interest a result or of interest is independent result of interest that means for a integrable functions the nonnegative simple functions are dense close to it and using positive negative part this will give you that in the space of L 1 of a b if you look at the simple integrable functions they are dense in L 1 norm

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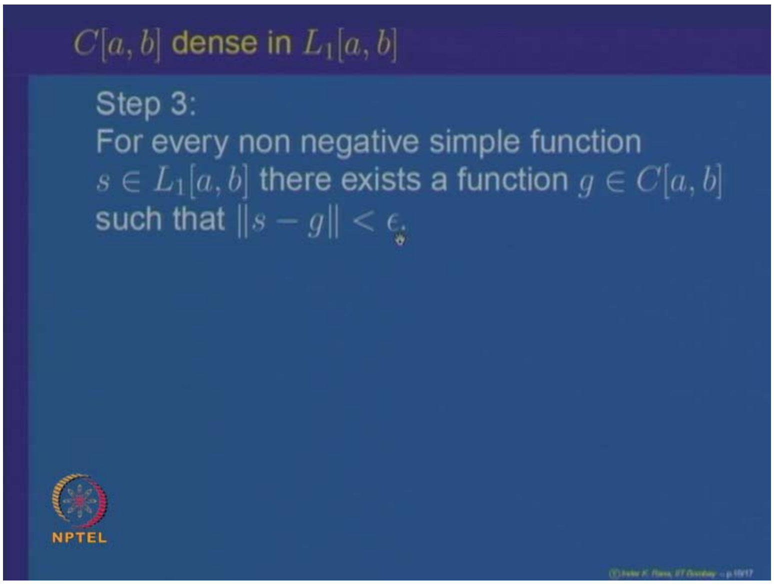

So, for us our theorem, so it is enough to prove that for a simple non negative simple integrable function L 1, of a b there exists a function g continuous function close to it so that is what we have to prove our theorem.

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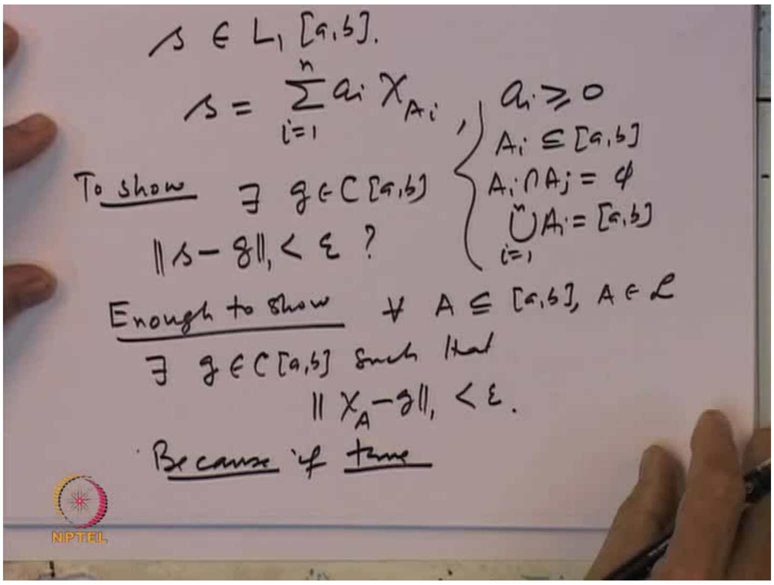

So, let us we have got a simple s, which is simple integrable non negative function so that means what that means this s will look like sigma a i indicator function of sets a i i equal to sum 1 to n because this is simple nonnegative. So, a is bigger than or equal to 0 and this sets A i's are subsets of a b of course they are Lebesgue measurable; they are disjoint so A i intersection A j is empty and the union of A i's is equal to a b i equal to 1 to n.

So, saying that s is a nonnegative function which is a nonnegative simple measurable function, which is in L 1, so nonnegative simple means it is of this type and obviously this becomes integrable so this must be of this form right.

So, what we want to show is that to show there exist a continuous function g on a b such that norm of s minus g is less than epsilon so this is what we have to show.

Now, s is a linear combination of indicator functions of A I, so our claim is that enough to show. So, to prove that this claim it is enough to show that for every A inside a b of course A Lebesgue measurable there exists a function g which is this C a b such that the L 1 norm of the indicator function of a minus g is less than epsilon, saying this is enough because if this is true for every i so the reason is because if true then what we will do for every i we will approximate the indicator function of A .i

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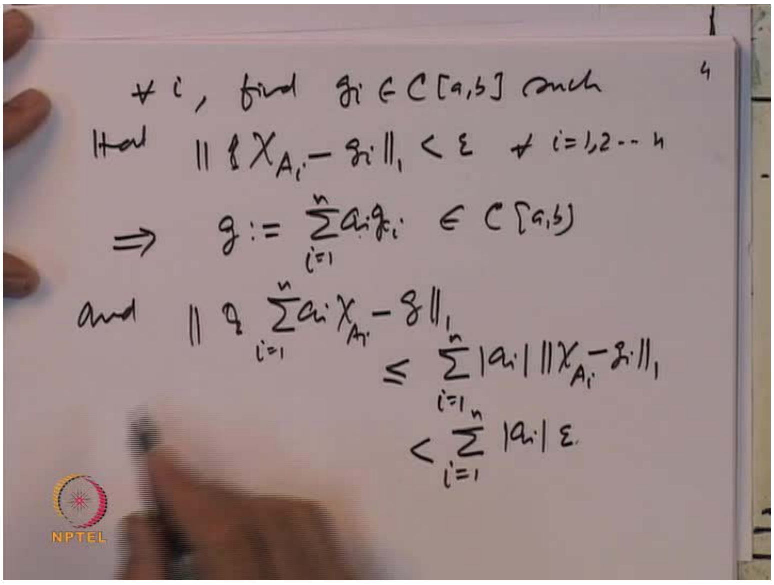

So, for every i find g i belonging to C a b such that the L 1 norm of the indicator function of A i minus g i is less than epsilon. So, we will do that for every i equal to 1 to up to n, then will imply that if i define g to be equal to sigma a i of g i a i times the function g i;then this function bill is a continuous function because it is a finite linear combination of continuous function. So, this is a continuous function and the L 1 norm of sigma a i 1 to n of indicator function of a i minus this g and this is s so the norm of s minus g L 1 norm will be less than or equal to sigma mod a i norm of L 1 norm of indicator function of A i minus g i.

So, that is by triangle inequality, g is a sum of six sigma a i g i. So, i can say it is less than or equal to absolute value 1 to n of mode a i times this and each one of them. So, this less than sigma i equal to 1 to n mod a i and this each one of them is less than epsilon; so it is less than this which is a small quantity, so you can modify this epsilon suitably.

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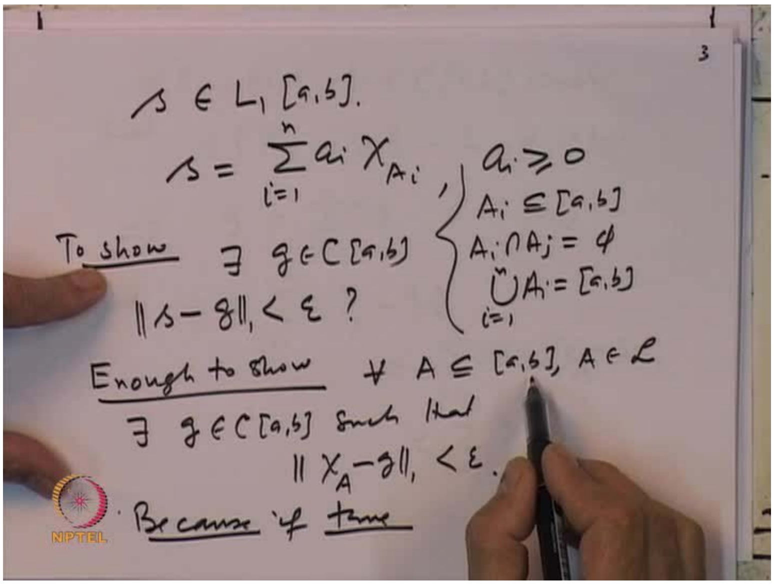

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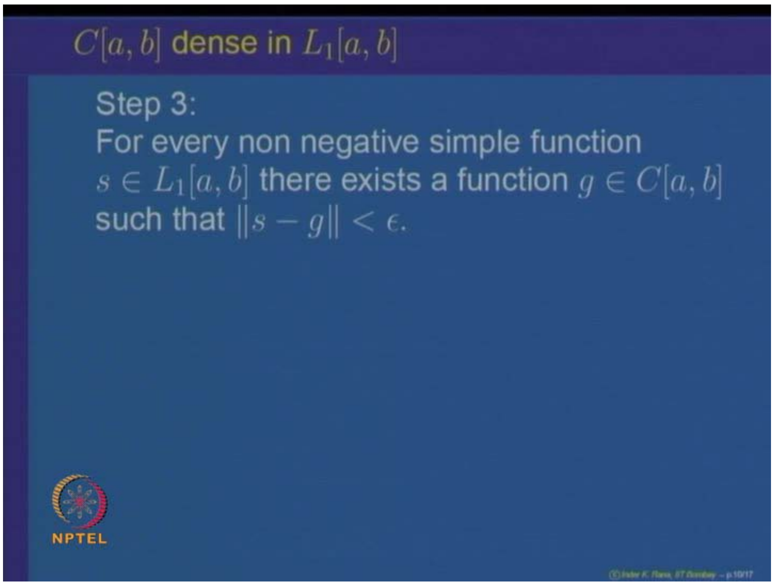

So, what we are saying is to prove that for every nonnegative simple measurable function s, there is a continuous function g close it in the L 1 metric. It is enough to show that for every indicator function of a set A in a, b, there is a function, continuous function close to it so that is what we have to prove so let us do that.

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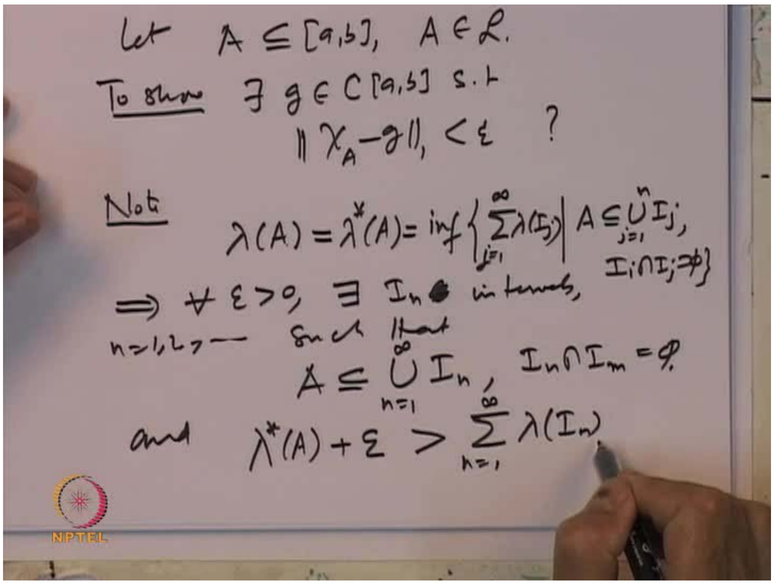

So, let A be a subset of a, b and A Lebesgue measurable to show there is a continuous function close to it. So, to prove that let us make a observation here to show that there exists a continuous function g close to that so that means, C of a b such that norm of indicator function of a minus g is less than epsilon so this is what we have to show.

So for that let us observe one thing, so note which you can call it as a lemma, which we already proved while dealing with the Lebesgue measure but let us recall the proof of this once again that look at the Lebesgue measure of the set A. A is a subset of a, b so it is a finite quantity and the Lebesgue measure what is it equal to Lebesgue measure of the set A because A is Lebesgue measurable, it is same as the Lebesgue outer measure of A by definition and that is equal to the infimum of given the set a cover it by disjoint intervals I j some finite number of them I j is our pairwise disjoint

and look at the sums of these intervals, lengths of these intervals I j 1 to infinity and take the infimum of all such things so lambda star of A, which is same as the Lebesgue measure of A is nothing but the infimum of the sums of the lengths of those intervals I j's which cover A and we can assume that they are pairwise disjoint.

So, that is what is being that is so the definition and now this being finite because it is inside a, b so implies for every epsilon bigger than 0 I can find intervals I ns there exist intervals I n's intervals n equal to 1 to and so on such that the set A is contained in union of I n's n equal to 1 to infinity and I n's are pairwise disjoint so I n intersection I m is equal to empty and it is the infimum so I can make lambda of lambda star of A which is same as lambda of A plus epsilon should be bigger than summation of sorry lambda of A plus epsilon cannot be the infimum so that must be bigger than summation of lambda of I n's right

So, that is by the definition that lambda star of A or lambda of A is the infimum of certain things, so lambda of A plus epsilon cannot be the infimum, so it must be bigger than some terms over which you are taking the infimum so this is true.

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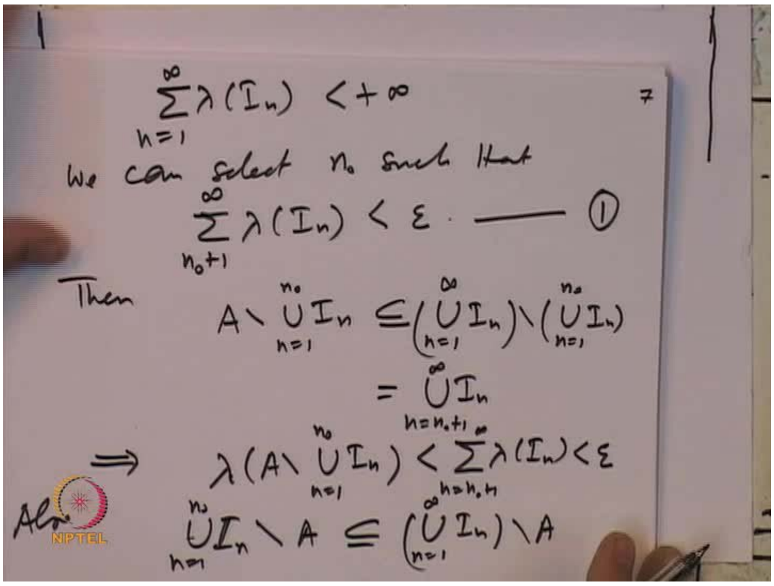

So that means and this is finite so that implies- so here is the observation that this sigma lambda of I n's n equal to 1 to infinity is finite right because lambda star of A which is lambda of A, which is finite is inside a b so this is finite.

So, we can choose so this is a series, which is convergent. We can choose a stage say n naught such that the sum from n naught plus 1 to infinity of lambda I n is small, so let us say it is less than say again epsilon.

So, now define so then let us look at the set A minus union I n n equal to 1 to n naught. Look at this so this is a subset of A, is contained in; so recall A is contained in the union of I n's. So, I can say this is subset of union of n equal to 1 to infinity of I n's because A is contained in this minus union n equal to 1 to n naught of I n's about that means, this is equal to union of I n's n equal to n naught plus 1 to infinity, so this is that is same as this.

So, that implies by the sub additive property that lambda of A minus, the union of the intervals I n from 1 to n naught will be less than summation of lambda I n n equal to n naught plus 1 to infinity and that we know is less than, so here is the property 1 so by 1 this is less than epsilon.

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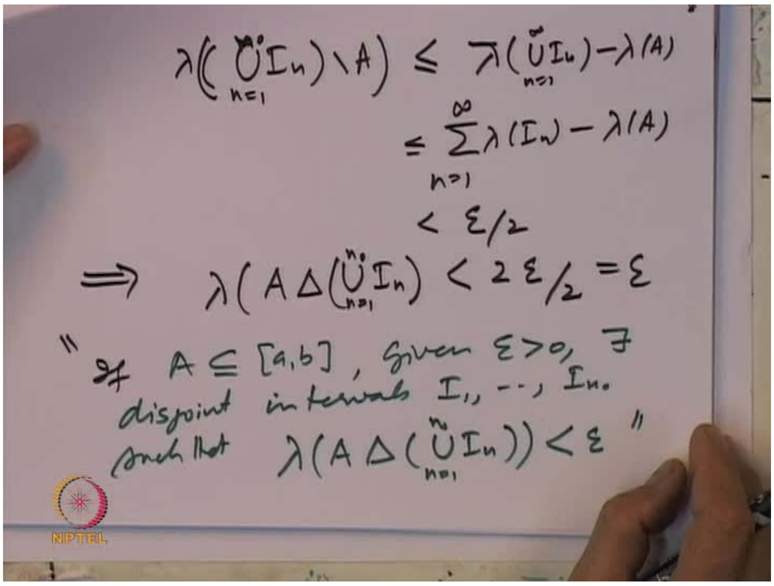

So, the difference of the measure, Lebesgue measure of the difference of A minus this finite union of disjoint intervals is less than epsilon also. say look at the sets union of 1 to n naught the intervals I n minus A that is contained in union of n equal to 1 to infinity I n because instead of n naught, we will take it to infinity minus A and everything is finite. So, this implies all I got finite Lebesgue measure. So, this implies that the Lebesgue measure of union of I n's n equal to 1 to n naught minus A. So, Lebesgue measure of this set will be less than or equal to the Lebesgue measure of union of I n's n equal to 1 to infinity minus Lebesgue measure of A, which is equal to summation less than or equal to summation of lambda of I n n equal to 1 to infinity minus lambda of A. and that we know that we know is less than epsilon because that is how we constructed the sequence I n I n's cover A and the difference between the summation of lambda I n's minus lambda of A is less than epsilon.

So, we have got that the Lebesgue measure of Lebesgue measure of A minus, the finite union of intervals 1 to n naught is less than epsilon. Lebesgue measure of the finite union minus A is less than epsilon, so that together they imply that the Lebesgue measure of A symmetric difference between this set which is a finite union of intervals disjoint intervals 1 to n naught is less than 2 epsilon.

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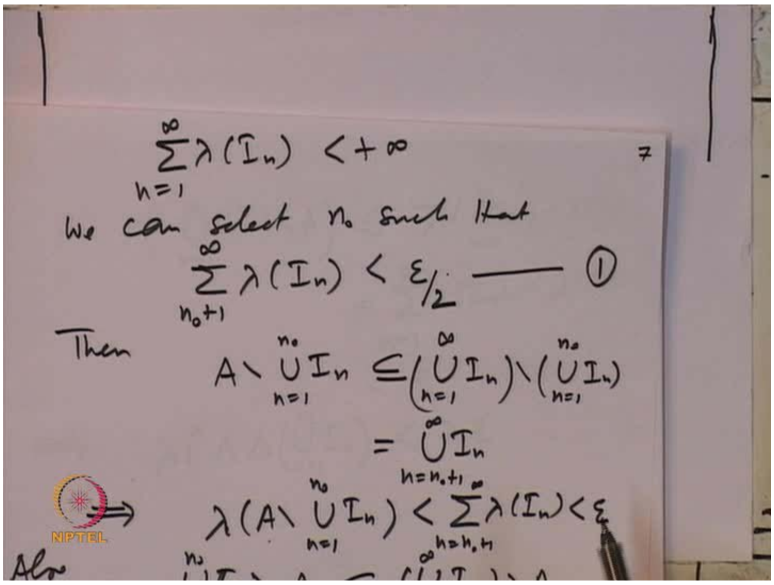

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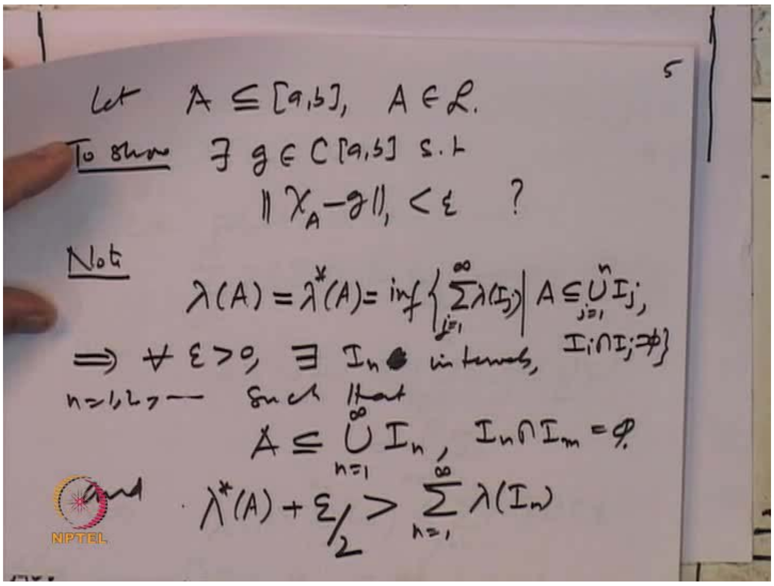

So, to make things look very nice what we could have done is given an epsilon. we kind of selected here when we got this tail of the series, we could have made it less than epsilon by 2 and in the beginning also when we got when we took outer measure being finite and covered by so we could have met that inequality less than epsilon by 2 in the starting itself so saying lambda of A is finite so there is a sequence of intervals covering such that lambda star of A plus epsilon by 2 instead of epsilon

Similarly in the second one also you could have met epsilon by 2, then we would have gotten this as epsilon by 2 and we would have gotten this by epsilon by 2, So, we have gotten it epsilon that is only a cosmetic change in our proof but the basic fact is what we are saying is so what we have shown is the following that if so. This is the important thing that we have shown, so we have shown that if A is contained in a to b given, epsilon bigger than 0 there exist disjoint interval I 1 some I n naught such that the Lebesgue measure of the set A symmetric difference between this union of the intervals. I n's 1 to n naught is less than epsilon, so this we had proved earlier also for general measure, so I have repeated the proof for the Lebesgue measure because it is good to revise things anyway.

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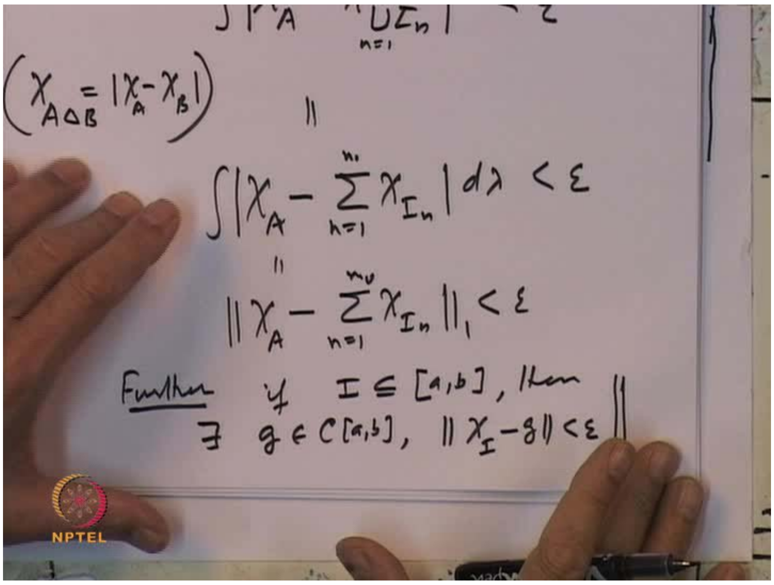

So, using this but now, let us look at what does this statement last statement mean; that means, this thing is nothing but integral of the indicator function of A minus the indicator function of sets a I n's which are pairwise disjoint 1 to n naught absolute value of this. So, this quantity is precisely equal to this so what we are saying is the indicator function of A delta B is absolute value of indicator function of A minus indicator function of B. So, this is general fact, so I am using that here so Lebesgue measure of a set is the integral of the indicator function and the indicator function of the symmetric difference is the indicator function of the difference of absolute value of difference so that is less than epsilon.

So, as a consequence, what we are saying is that given a set A, inside the interval a, b, we can find finite number of disjoint intervals I n such that this property is true but now, note that these are disjoint intervals. So, this is equal to the indicator function of A minus the summation indicator functions of I n's n equal to 1 naught n naught d lambda so that is less than epsilon.

So, what we have proved that if you look at the indicator function of a set A, inside the interval a, b then there are disjoint intervals I n such that the indicator function of A minus the sums of the indicator function of these intervals I n's will be less than epsilon.

Now what we want you to approximate so that so this is same as the L 1 norm, so we have got L 1 norm of the indicator function of A minus the function, which is a sum of the indicator functions of I n's 1 to n naught that is less than epsilon L 1 norm;

but our aim was to approximate the indicator function of A by a continuous function. We are saying that the indicator function of A is close to sum of indicator functions of intervals, so what does that mean that means, if I can approximate each one of these functions, which are indicator functions of intervals inside the interval a, b and a finite number of them by a continuous function., then I am through So, the next step is to show that so let us write it as further, we claim that if I is a interval; if say I is a interval inside the interval a, b then there exist a continuous function g C a b such that with the property that norm of the indicator function of the interval I minus the continuous function is less than epsilon.

So, once we are able to prove this fact for each interval I n will have a continuous function take the sums of those so they will approximate the sum of the indicator functions of intervals and that in turn approximates the indicator function of A and our proof will be complete.

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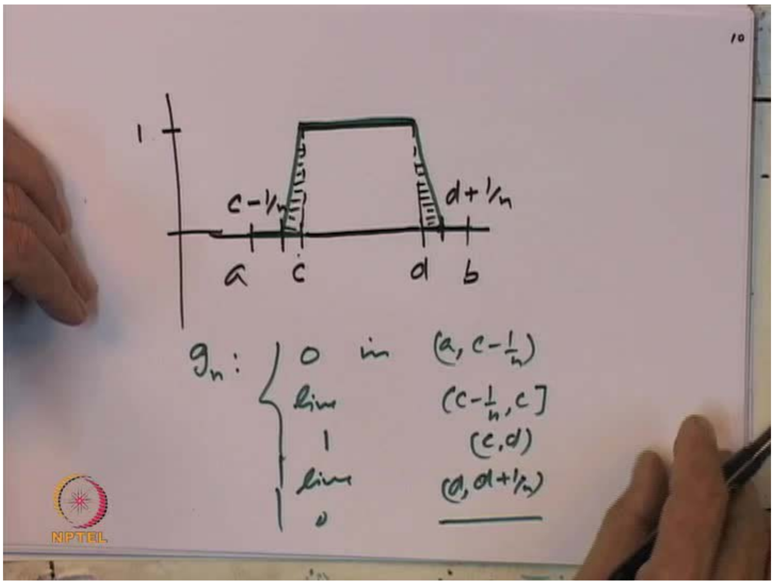

So, what we want to show is that given a interval inside, so given the interval I so here is a and here is b; we are given a interval inside it so I just want to draw a picture and show, what is the proof going to be so here is the interval I inside it let us say it is c to d.

So, the indicator function of A indicator function, so we have got the indicator function of c to d. So, let us look at so this is going to be the indicator function of so this height is 1 so let us just draw this is 1.,

So, the indicator function of the interval c to d looks like it is 0 here in a to c and c to d it is going to be 1 and d to b it is going to be 0 again here it does not matter what are the values at the point c and d.

Now, we want to approximate this by a continuous function, so it is obvious what we should do to make this function continuous and such that the area below the graph of this function does not exceed too much.

So, let us take a point here, which is c minus 1 by n for any n and let us take a point here, which is d plus 1 by n so take a this point and now what we do is we take the function. So, I am going to define a function g n. What is the function g n? It is 0 in a to c minus 1 over n in this portion, it is 0 and in the portion between c minus 1 over n to c it is going to be the line joining, so that is the line segment so I am describing the graph of this so the line segment and then it is 1 in the interval c to d and then again from d where the discontinued is coming I join it by the point d plus 1 over n.

So, it is the again the line joining that between d to d plus 1 over n and 0 remaining so what I am saying is if you are given the indicator function of a sub interval of a b then I can always make it continuous. I can approximate by a continuous function so what will be the extra thing, we will be adding we will be adding the areas of these two rectangles.

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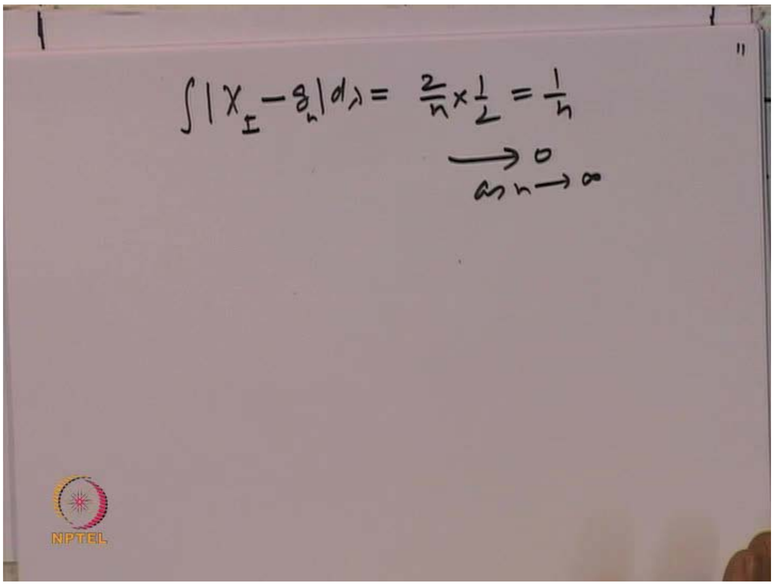

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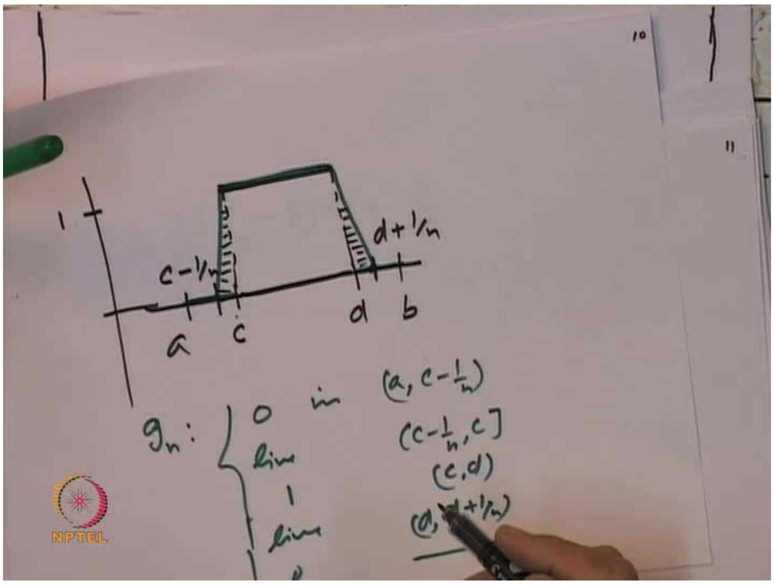

So, if I define this as a continuous function g n then, what is that L 1 difference so the integral of mod of the indicator function of the interval I which is c to d minus this continuous function g n d lambda will be equal to the areas of these two rectangles. Height is 1, so it is 2 by n because each triangle because this length is 1 by n, this height is 1 so half base into height; so it is n by 2 actually so multiply by 1 by 2 so that is 1 by n and that goes to 0 as n goes to infinity.

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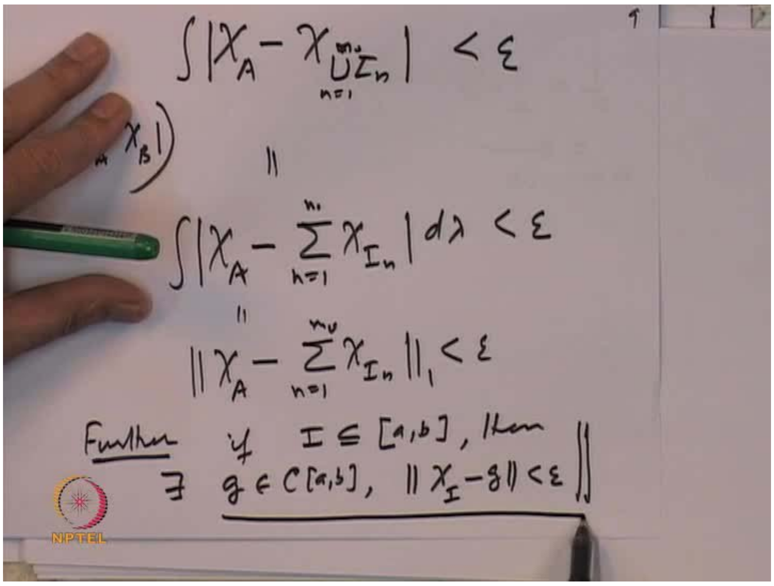

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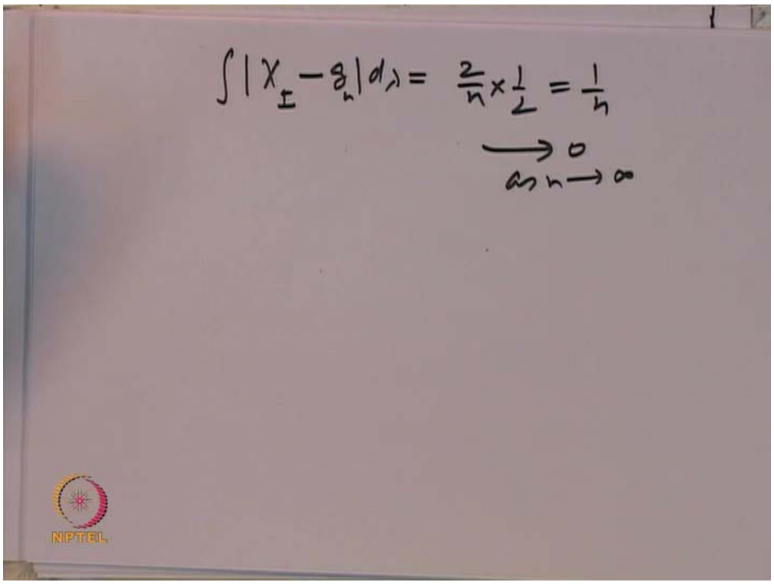

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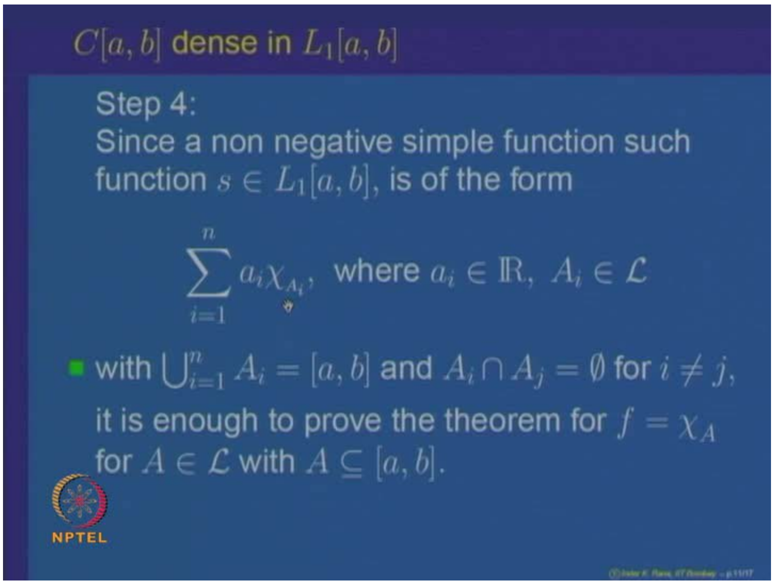

So, that will prove the fact that the close to indicator function of a interval there is a continuous function and hence close to the indicator function of any set inside the interval a, b there is a continuous function. and that will and Finite linear combinations of the indicator functions are the simple function, so that will prove the fact that for a nonnegative simple function there is a continuous function close to it.

So, step 4, which was that if is a nonnegative simple function of this form, then close to it right so is a linear combination. So, indicator function a i chi A I, I am just repeating the last step again so because each one of them can be approximated if each one of them can be indicator function of a intervals can be approximated then the simple function can be approximated.

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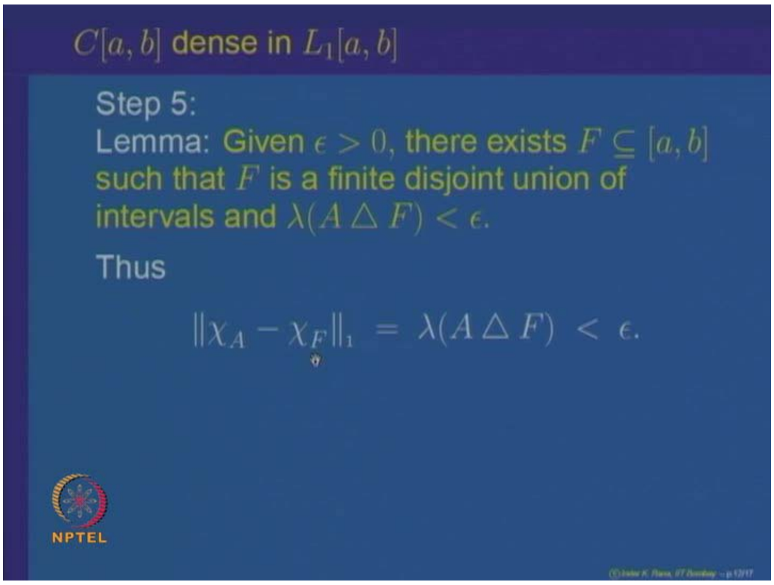

So, the problem came to approximating indicator functions of sets inside the interval a, b and for that we said we will use a lemma which says that given epsilon bigger than 0 there exist a set F, inside a, b, which is a finite disjoint union of intervals such that the Lebesgue measure of A symmetric difference f is less than epsilon.

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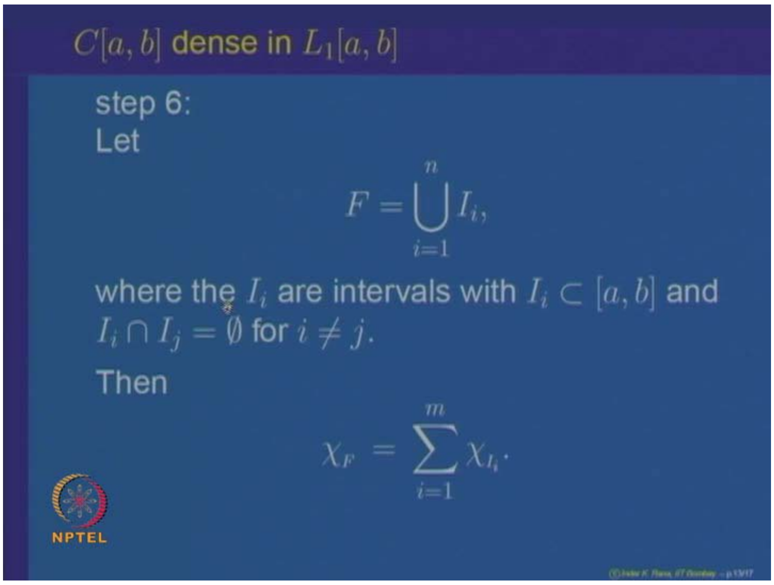

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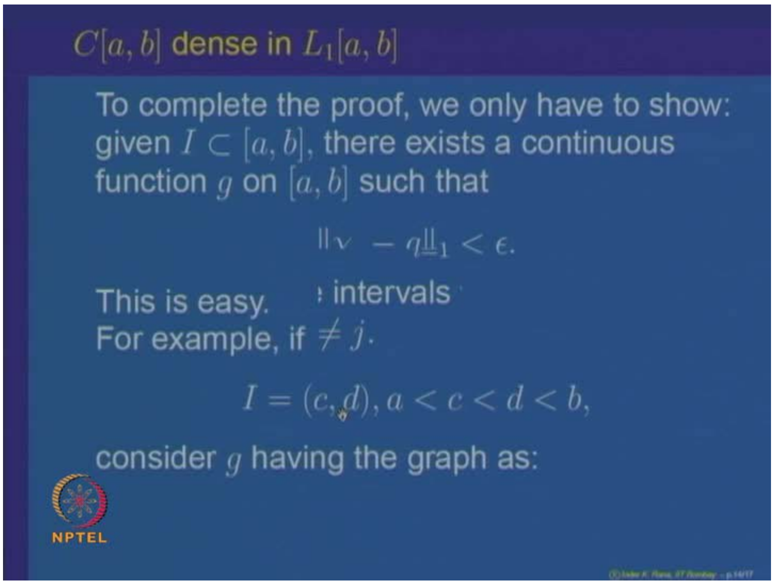

So, using that we will get that indicator function of A minus the indicator function of this set F, which is a finite disjoint union of intervals and so problem reduces to approximating indicator functions of intervals inside the given interval a, b and for that we just extrapolate by piecewise linear functions and get the required thing.

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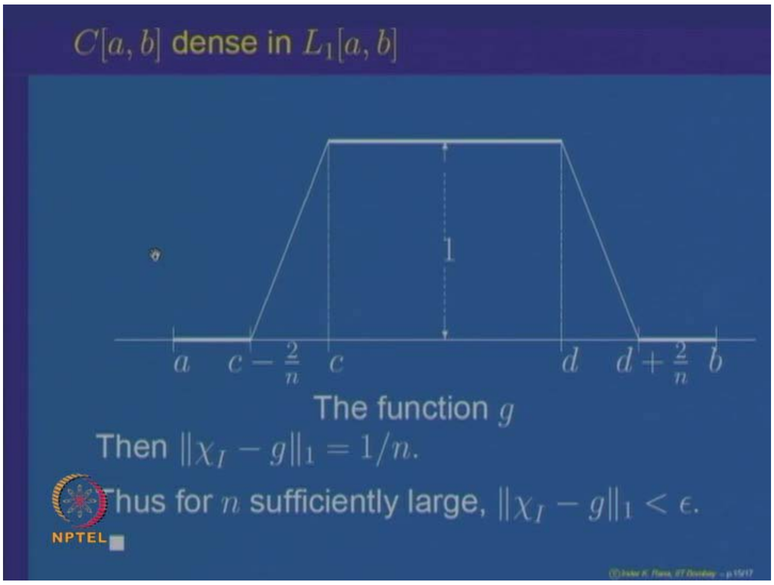

So, indicator function of a interval there is a continuous function close to it so for that we just now said look at the graph of so here is a to b. So, look at some point close it c minus 1 over n or c minus 2 by n and take this linear piecewise linear function, which is a continuous function and that will approximate in L 1 norm the given function the indicator function of a interval.

So, that will prove the fact namely that close to L 1 function there is a continuous function. I just want to discuss the proof of this theorem, in fact, we have almost use all the results on measures and on integration that we have proved till now so and this is the technique which we use to prove things about integrable functions.

So, what we wanted to show was that given a function f, which is integrable on the interval a, b there is a continuous function on the interval a, b such that the L 1 norm of the difference is small that means every integrable function on the interval a, b can be approximated by a continuous function.

So, what are the first step of our first step in our proof was that since every function f can be written as the positive part minus the negative part and f integrable implies if and only if the positive part and the negative part are integrable ;we want to approximate f by a continuous function, which is the difference of two nonnegative integrable functions f plus and f minus and if you can approximate each one of them separately then we can combine them to approximate the function f.

So, the first step was namely, we can assume without any loss of generality that our integrable function is nonnegative that is 1 so the next step is so if we take the function f to be nonnegative and integrable how is the integral of nonnegative functions defined. They are defined by looking at limits of increasing sequences of simple measurable functions and taking their integrals so we go back to the definition of integral of a nonnegative function.

So, f nonnegative integrable implies f is nonnegative measurable, so as a consequence or the definition, there exist a sequence of nonnegative simple measurable functions. Such that call that s n such that the sequence s n of simple nonnegative simple measurable functions increases to the function f, but if s n is increasing to f that means s n is less than or equal to f and f is nonnegative so that implies each s n is nonnegative and integrable.

So, s n is a sequence of simple nonnegative integrable functions on the interval a b and the integral of s n increases to integral of f that means that is equivalent to saying that s n converges to f in L 1 norm.

So, given f a nonnegative integrable function, we have a sequence of nonnegative simple integrable functions in converging to it in the L 1 norm. So, this is a fact that simple functions in L 1 of a b are dense but we want to go a step further so look at a simple function. Now, so to approximate f, which is nonnegative, we have got a sequence of simple functions converging to it in L 1 that means close to f, which is nonnegative integrable. There is a simple integrable function, so if you can integrate if you can approximate simple nonnegative simple integrable functions by a continuous function then we are through

But a nonnegative simple function is a finite linear combination nonnegative linear combination of indicator functions of sets so and if you can approximate each indicator function by a continuous function, then the corresponding linear combination of those continuous functions will approximate the simple function.

So, the next step is that to show that close a simple integrable functions can be approximated by a continuous function ; it is enough to show that the indicator function of a set A inside a b is can be approximated by a continuous function.

So, it comes down to saying that look at a set A, which is Lebesgue measurable inside the interval a b and we want to approximate the indicator function of this set by a continuous function and here comes the property of the Lebesgue measure; that the Lebesgue measure of a set A inside a b that means it is a finite set of finite Lebesgue measure.

We can approximate this set by finite disjoint union of intervals and what does that approximation mean, it means that given a set A, inside the interval a b, which is Lebesgue measurable there exist a set, which is a finite disjoint union of intervals such that the Lebesgue measure of the set A and the symmetric difference of this finite disjoint union is small is less than say epsilon.

And but saying that The Lebesgue measure of the symmetric deference A with a finite disjoint union of is small is same as saying that the L 1 norm of the indicator function minus the difference between the L 1 norm of the indicator function and the linear combination of the indicator functions of those disjoint intervals is small.

We want to approximate the indicator function by a continuous function so and close to it is a finite linear combination of indicator functions of intervals so the problem reduces to approximating the indicator function of an interval, inside the given interval a b by a continuous function and that is achieved by making the indicator function piecewise linear.

So, this is the step in effect we have used all most all the theory in proving this theorem, so thus theorem prove that C a b is dense. So, that proves the fact that C a b is dense in L 1 of a b and C a b is the subset of R a b. So, that will prove that R a b is dense in L 1 of a b and hence as the result we get that L 1 of a b is the completion of the space of R a b of of course of C a b so that proves the theorem.

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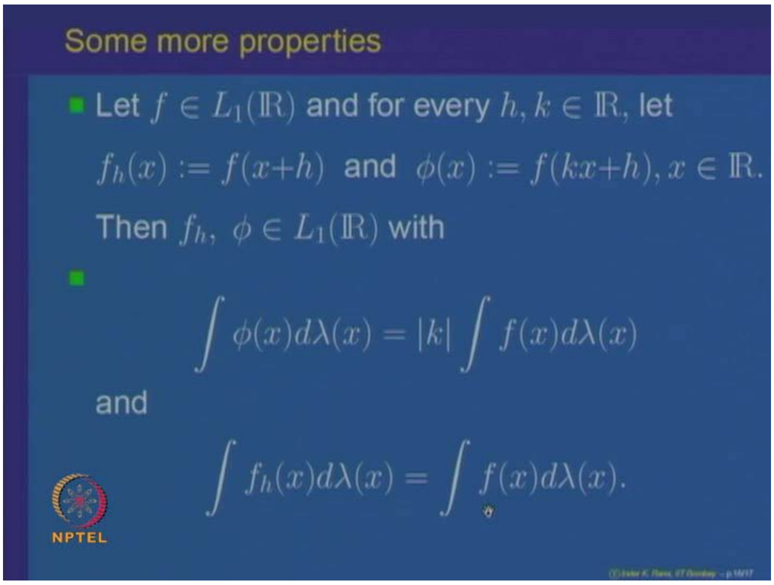

So, very important result that L 1 of any interval a b is complete and as we observed L 1 of any subset also will be complete if you look at the proof and it is a completion of R a b.

Let us look at some more properties of these functions, so let us look at a function f in L 1 of R so f is a integrable function.

For real number h and k, let us define f lower h of x namely equal to f of x plus h ,so the value of this new function f lower h at x is the value of at the translated point x plus h. So, this will call as a translation of the function f and similarly let us define the function phi, which is defined as phi of x to be f of k times x plus h for any x that means you multiply the number x by k and add h to it translate and then take the value of this so claim is both these functions f h and phi are again integrable. The integral of this function phi is absolute value of k times the integral of f and the integral of the translated function f h is same as the integral of the original function f.

So that means the space L 1, if you make a translation or a magnification then these are again leave the functions in the space L 1 and with these properties so these can be easily proved on the lines that we have proved just now.

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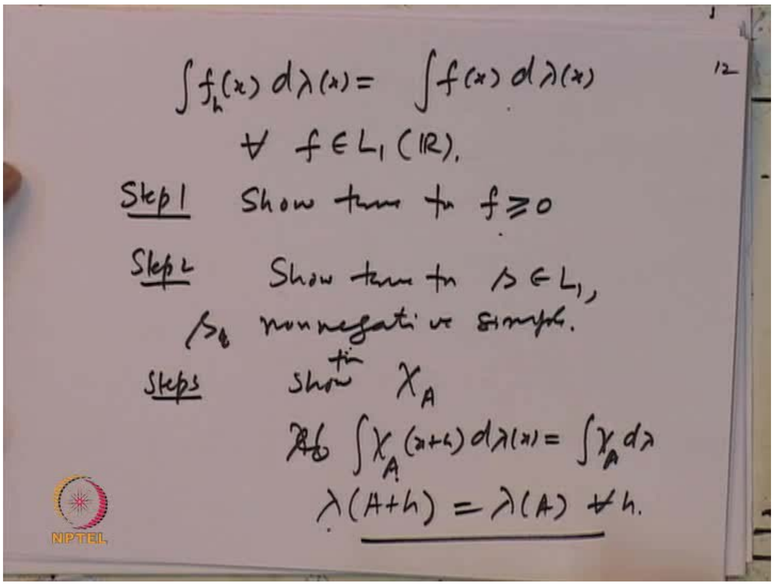

So, let us try to prove that integral of f h x d lambda x is equal to integral of f x d lambda x so we want to prove for every f in L 1.

So, again we will that simple function technique, so we want to prove for every f in L 1. So, note, we can assume so step 1 show true for f nonnegative show it is true for f nonnegative because once it is true for f nonnegative I can look at the positive part and the negative part.

So, show it is for and do how do you show it is true for nonnegative. So, step 2 show true for a simple function s belonging to L 1 s nonnegative simple because every function can be approximated by nonnegative simple functions. Nonnegative simple functions are indicator functions of sets, so step 3 show for f equal to the indicator function show for indicator function and what is that that means we want to show that lambda so indicator function of A x plus h d x d lambda x is equal to integral of indicator function of A d lambda but that is same as saying showing that lambda of A plus h or A minus h does not matter is equal to lambda of a for every h and that is the property of the Lebesgue measure that it is translation invariant.

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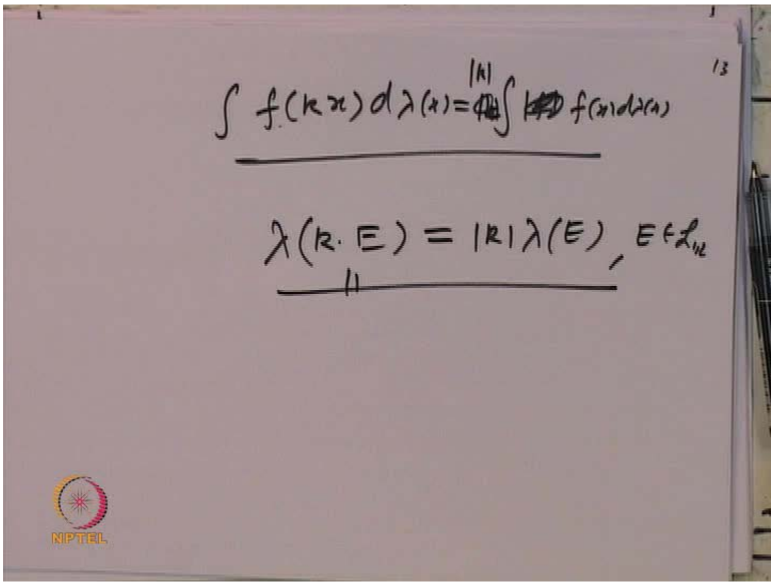

So, what we are saying is this property is true for indicator functions, so it will be true for nonnegative simple functions. So, by taking limits it will be true for nonnegative integrable functions and then by positive and negative part it will be true for all functions in L 1 and a similar result will work for second identity namely if I multiply so f of k x let us just look at d lambda x is equal to mod k times mod f of mod k times f of x d lambda x so that again by the same technique let us look at what happens when it is a indicator function so lambda of multiplication k times a set E what is it equal to and we look at this via outer measures one can show that this is same as mod k times lambda of E.

So, you show it for all sets E, which are Lebesgue measureable, then this is equal to the indicator function and then finite linear combination of indicator functions and so on.

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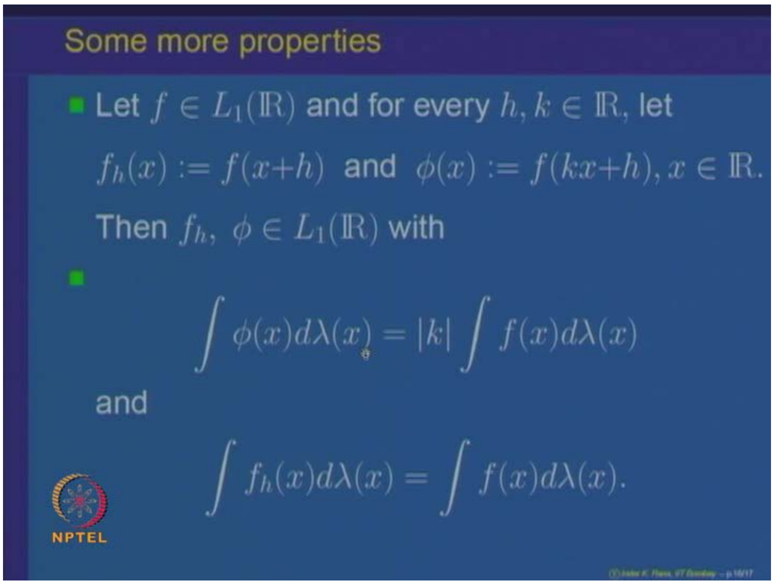

So, ask the reader to verify this by the simple function technique, so these are properties of Lebesgue integrable function. So, what we have done is that we specialized the space of integrable functions on the real line and deduced some nice properties ;one of the properties was that the space of Riemann integrable functions is dense in the space of Lebesgue integrable functions and so in one sense this is very nice and so.

So, this completes the process of extension of measures and defining integrals with respect to measures and their properties in the next few lectures. We will start looking at how does one construct, what are called product measure spaces and how does one integrate on product measure spaces.

So, this is an important part of measure theory that means measure and integrations on product spaces, we will start looking at in the next lecture.

Thank you.