**Measure and Integration**
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**Module No. # 06**
**Lecture No. # 22**
**Lebesgue Integral and Its Properties**

Welcome to lecture number 22 on measure and integration. If you recall, in the previous lecture we had started looking at the properties of - Lebesgue measure - of Lebesgue integrable functions and we started looking at analyzing when does a function which is Lebesgue integrable on the interval a b and its relation with the Riemann integral of the functions on the interval a b.

We had started looking at the proof of the theorem; namely, that if f is a function defined on an interval a b to R - which is Riemann integrable, then we wanted to show that it is also Lebesgue integrable and the Riemann integral is the same as the Lebesgue integral.

We will continue with the proof of that theorem and then go on to analyze some more properties of the space of Lebesgue integrable functions on the interval a b.

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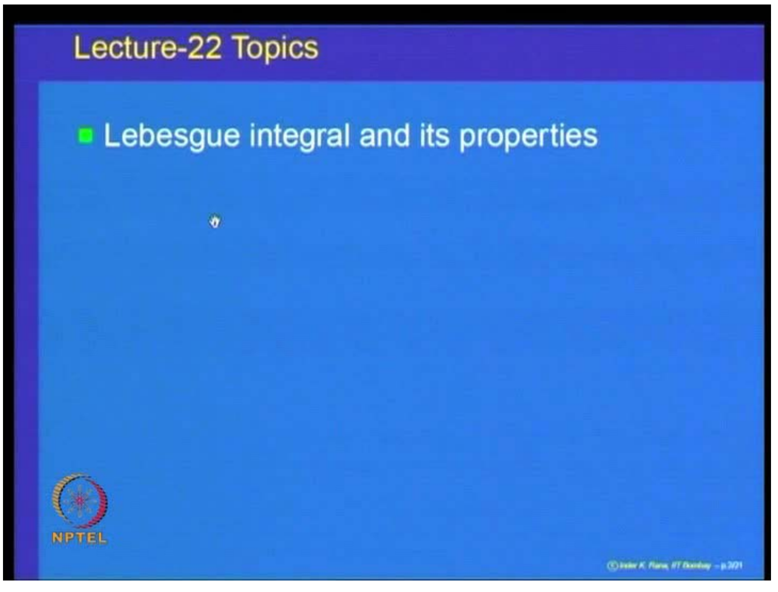

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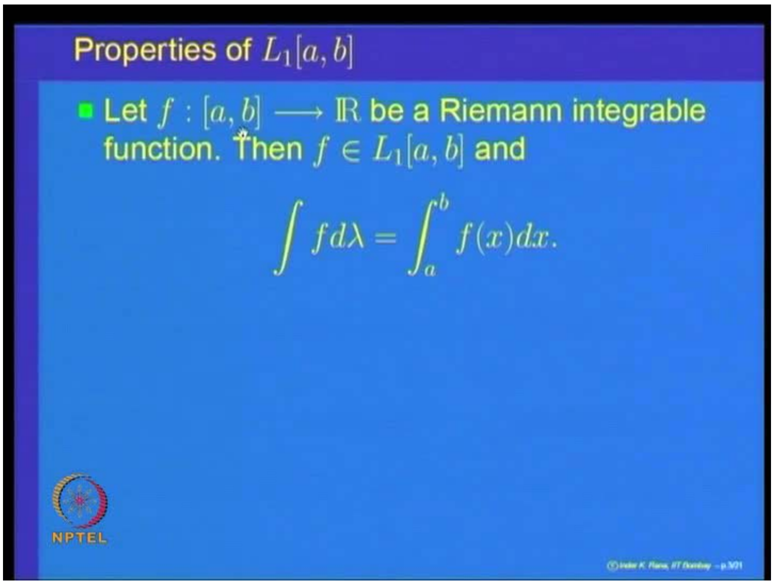


Today's lecture is going to be mainly concerned with Lebesque integral and its properties. The theorem we wanted to prove was that if f is a function defined on an interval a b to R and it is Riemann integrable, then it is also Lebesgue integrable and the Riemann integral of the function is same as its Lebesgue integral.

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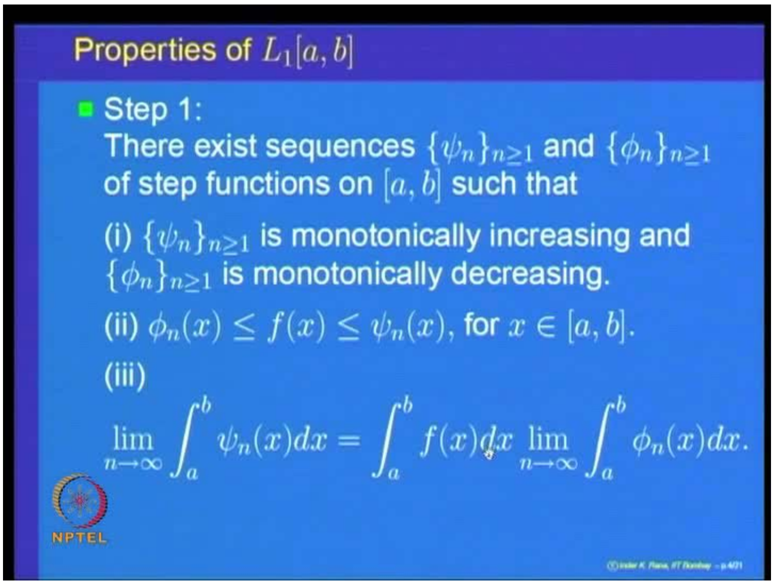


To prove this theorem, we started with this idea that since f is Riemann integrable, there exist sequences psi n and phi n of step functions on the interval a b, such that the sequence psi n is monotonically increasing and the sequence phi n is monotonically decreasing. And the function f is between these two sequences - phi n and psi n, for all points x belonging to a b. The Riemann integral of psi n(s) converges to the same value as the Riemann integral of f and that is the same as a limit of the Riemann integrals of the step functions phi n.

Let us recall these steps which we had proved last time. What we are given is - f is a function defined on an interval a b to R and f is Riemann integrable.

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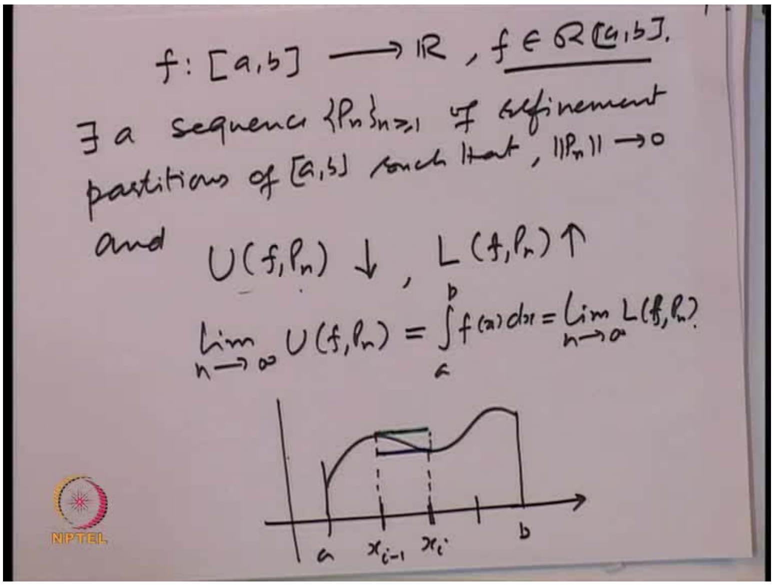

So what does Riemann integrability imply? The Riemann integrability of the function implies the following: namely, there exists a sequence P n of refinement partitions of the interval a b, such that, the norm of these partitions goes to 0 and the upper sums of f with respect to these partitions - that decreases, and the lower sums of f with respect to P ns increases and the common value is the integral.

Limit n going to infinity of upper sums is the same as integral - Riemann integral - of f and that is the same as the limit of the lower sums. This is because f is Riemann integrable.

Now, let us see what the upper sums are and what the lower sums are. We need to analyze them slightly more carefully. Let us draw a picture of the function, say the function looks like this; so this is 'a' and this is 'b' and we get the partition.

With respect to a partition, let us say this is the general interval - x i minus 1 and x i;  in this interval, look at what the smallest value of the function is - that is this, look at this height. Look at the largest value of the function in that interval; largest value of the function is somewhere here, so look at that height.

Lower sums consist of the areas of these rectangles with height as the blue line and the upper sums consist of the sums of all the areas, which are the green lines. Mathematically, what this means is the following - let us write mathematically what it means.

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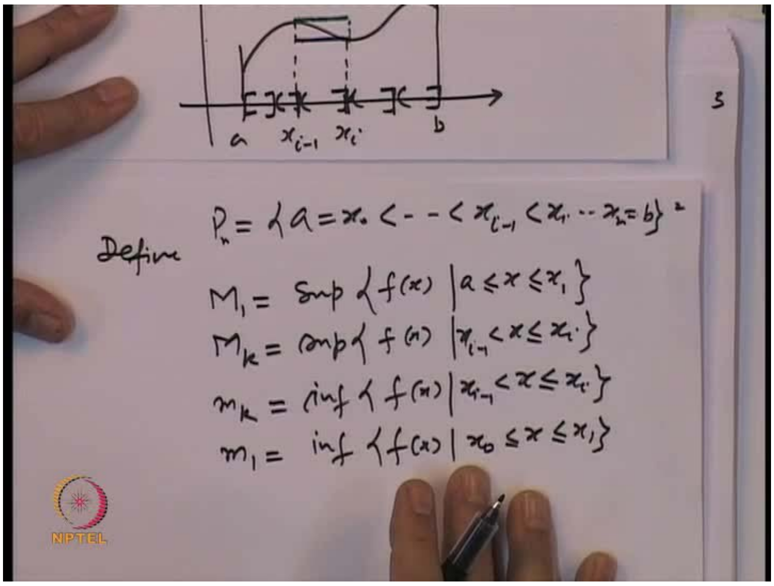


Mathematically, these things mean the following: namely, look at, consider the function, let us write, so consider, let us define, define, let us say, P is the partition, say P n is the partition which looks like ‘a’ equal to x 0 less than x i minus 1 less than x i x n equal to b. Let us say that is the partition. Let us say M 1 to be the supremum of the function f x, x belonging to ‘a’ that is less than or equal to x x 1. Let us write M k to be the supremum of the function in the general interval  f x for x between x i minus 1 and x i.

Keep in mind, here I am taking left open and right close; here, at the end point, both sides are closed. These intervals are disjoint intervals. What I am doing is in the first part I am looking at this, then I am looking at left-open right-closed, left-open right-closed, left-open right-closed.

I am partitioning the interval a b according to the partition point's P n and then looking at the supremums in the respective intervals.

Similarly, let us write m k to be equal to infimum of f x in x x i minus 1 less than or equal to x i. And m 1 to be the infimum in the first interval, so that is, f x x in x 0 x 1. So what is this value? This M 1 and M k are corresponding to the height, which is the green line; that is the maximum value of the function in the interval x i minus 1 to x i and w 1s correspond to small m ks.

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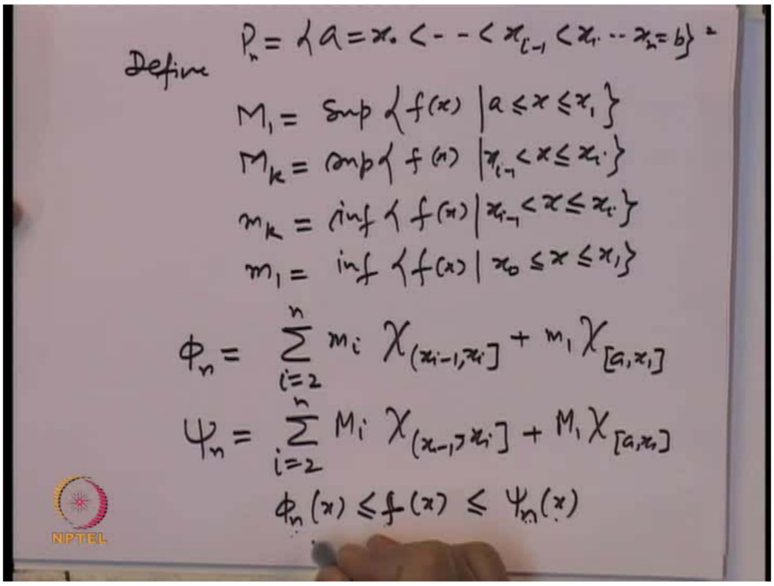


Once we have done this mathematically, let us define the required functions. Let us define phi n is the function, which is summation m i indicator function of x i minus 1 to x i, i equal to 2 to n. In the first one - let us put that value as m 1 - the indicator function of a x 1.

Similarly, let us write psi n to be sigma i equal to 2 to n, capital M i - the maximum value in the interval, x i minus 1 to x i and capital M 1 in the first interval; that is indicator function of a to x 1. These are the functions we defined. They correspond to the function, which is phi n. Small phi n will look like the minimum values, like this, and it will look like this. The capital and psi n(s), they will look like maximum value; so they look like this and look like this and look like this.

So, quite clearly, these functions phi n and psi n are step functions and phi n of x is less than or equal to f of x and less than or equal to psi n of x.

So, as a first step, the Riemann integrability of the function f over the interval a b gives us a sequence of functions phi n and psi n where, each phi n is a step function, each psi n is a step function, and phi n is less than or equal to f of x is less than or equal to psi n of x. Since our requirement was that Riemann integrability implies that the sequence P n of partitions is a sequence of refinement partitions, it implies that the sequence psi n will be a decreasing sequence and phi n will be an increasing sequence.

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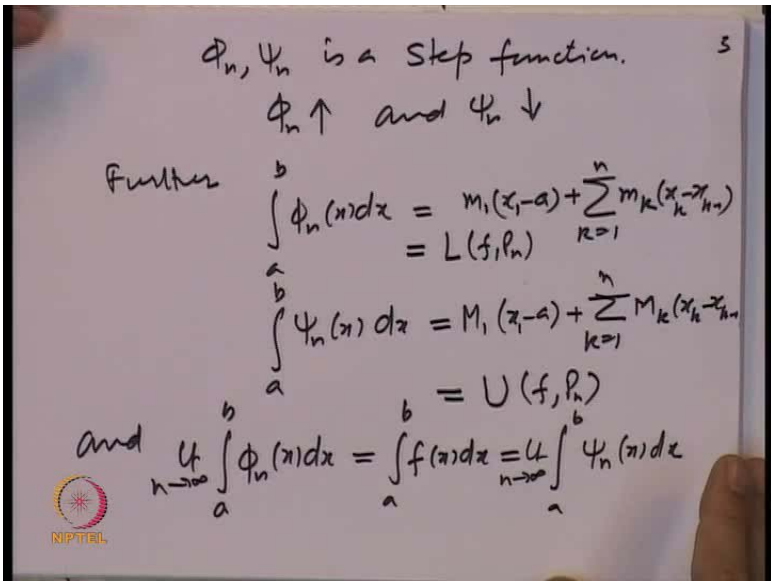


Let us write that observation; namely, - that we have phi n(s) - each phi n psi n is a step function. Phi ns are increasing and psi n(s) are decreasing. Moreover, further, let us look at the Riemann integral of the function phi n x d x, a to b. Because phi n is constant on each subinterval of the partition, this is nothing but equal to m 1 times the length of the first interval, that is - x 1 minus a plus the sums of those rectangles. That is small m k times x k minus x k minus 1 k equal to 1 to n. Similarly, the Riemann integral of the functions psi n x d x is equal to capital M 1, in the first interval times width of the interval, that is, x 1 minus a plus k equal to 1 to n - the areas of the other rectangles - so that is capital M k times x k minus x k minus 1.

Those are the Riemann integrals. By the definition of the upper and the lower sums - this is precisely the lower sum of f with respect to P n and this is precisely the upper sum of the function f with respect to the partition P n. Saying that the function is Riemann integrable implies that the upper sums and the lower sums converge to the same value. The Riemann integrability implies that the Riemann integral a to b of phi n x d x limit n going to infinity is the same as the Riemann integral a to b f x d x, and that is the same as the Lebesgue same as the Riemann limit of the upper sums, that is, limit n going to infinity the Riemann integral a to b of psi n x b x.

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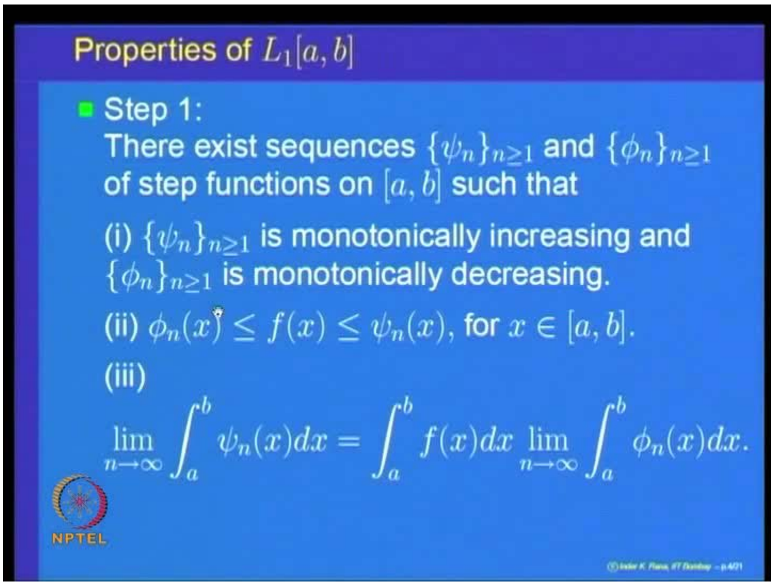


That proves our first step. As a consequence of Riemann integrability of the function f, we have constructed two sequences of step functions psi n and phi n; where, psi n is monotonically increasing and phi n is monotonically decreasing, and - limits of both - the integrals of both of them converge to the Riemann integral of f.

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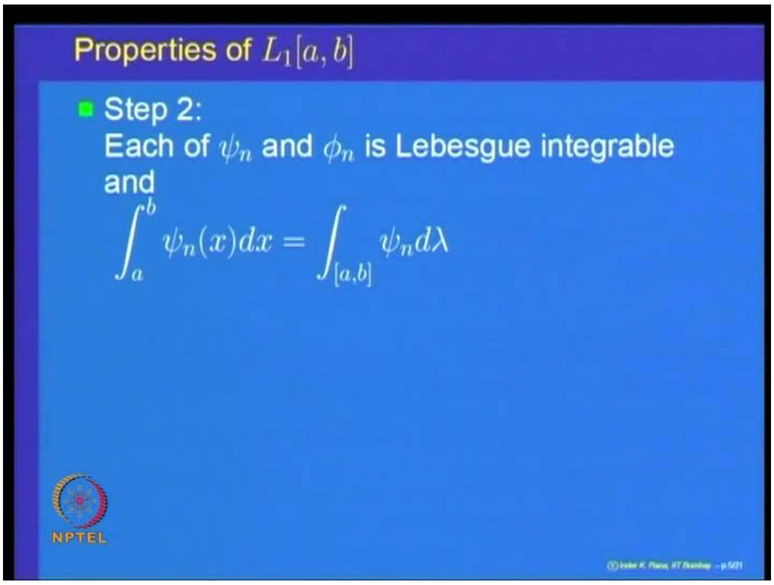


Now, let us observe at this stage - in some sense, the step functions and the Riemann integrals are the building blocks for the Riemann integral.

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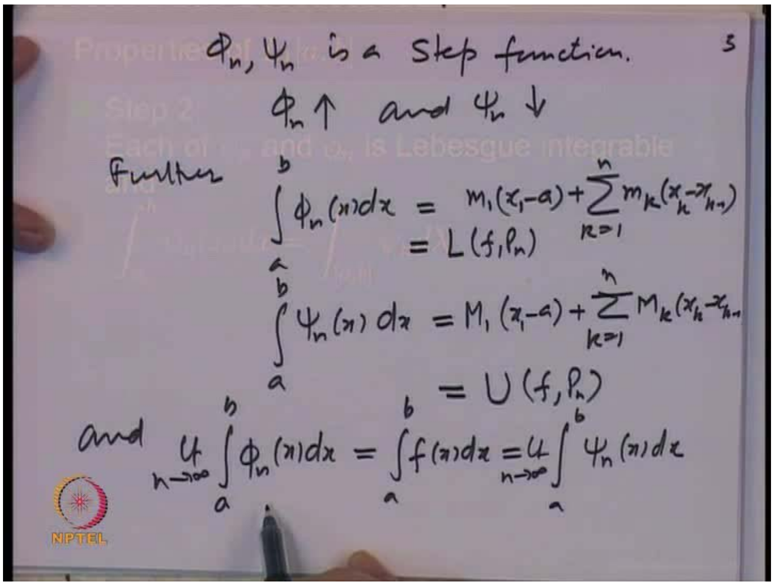


Let us observe, that the lower sum of the function f with respect to the partition which was the Riemann sum - which was the Riemann integral of phi n - is also nothing but the Lebesgue integral of the function phi n with respect to the Lebesgue integral.

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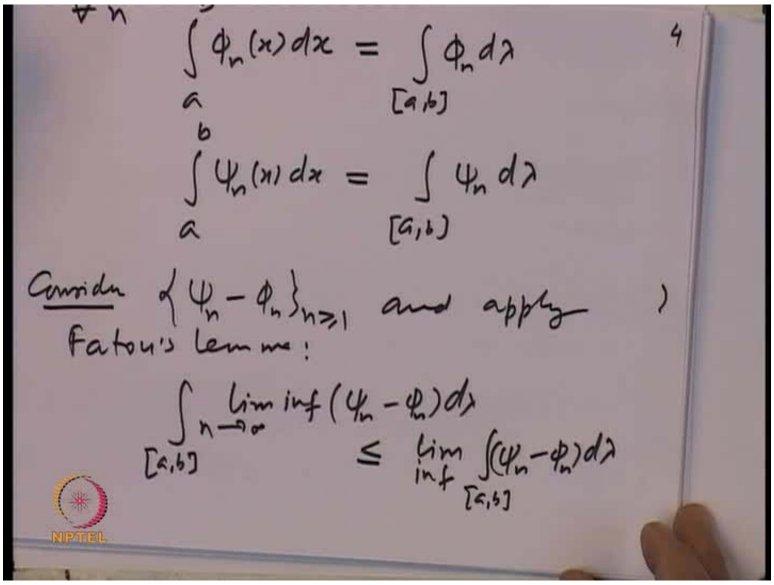

Because, this function phi n is a - simple function - simple measurable function and its integral is nothing but this integral. So, the observation is that, for every n, integral a to b of phi n x d x - the Riemann integral, is the same as the Lebesgue integral of the function phi n with respect to the Lebesgue integral over the interval a b. This observation is simply by the fact that phi n is a step function, hence it is a simple measurable function, that is, integral is nothing but the value times the Lebesgue measure of the portion on which that value is taken and that wing hair subintervals - it is the same as the Riemann integral. Similarly, the Riemann integral a to b of psi n x d x is equal to integral over a b of psi n d lambda.

This is the observation, which is going to play an important role for us. Now, let us define - let us consider - because we have got the observation that the function f n the function phi n is always dominated by f x is less than or equal to psi n, let us look at the function psi n minus phi n.

Look at the function, consider the sequence psi n minus phi n. Look at the sequence of the step functions. These are step functions as well as simple measurable functions and they are nonnegative. They are nonnegative simple measurable functions and by Fatou's lemma - so we are going to apply Fatou's lemma to this - consider this sequence and apply Fatou's lemma.

What will that give me? That will give me that - limit inferior of psi n minus phi n d lambda limit n going to infinity integral over a b, so, the Lebesgue integral of the limit inferior is less than or equal to limit inferior of the integral psi n minus phi n d lambda.

That is by Fatou's lemma. Let us observe here that integral of psi n over a b, minus integral of phi n over a b, the limit inferior of that is equal to 0. Because integrals - the Lebesgue integrals - of psi n are same as the Riemann integrals of psi n and Lebesgue integrals of phi n are same as the Riemann integrals of phi n, and even those integrals converge to the Riemann integral of f.

So, this right hand side is equal to 0. That means that the limit inferior of psi n minus phi n - that function is a nonnegative function and its integral - Lebesgue integral - is equal to 0. That we know implies that the function must be 0 'almost everywhere'. This implies that limit inferior of the sequence, which is psi n minus phi n x, must be equal to 0 'almost everywhere' x.

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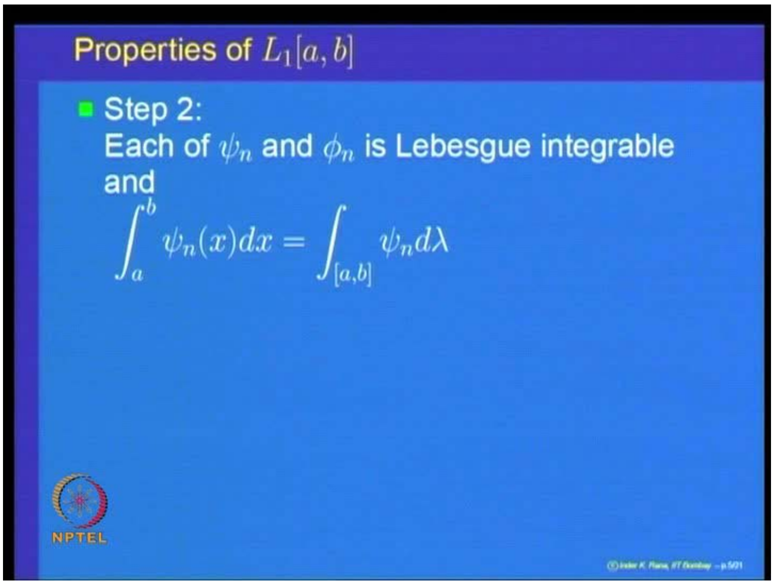

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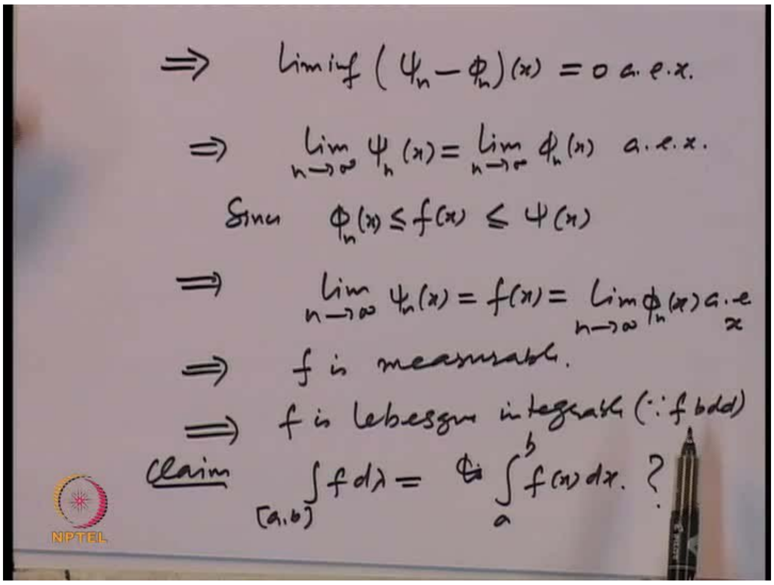


But, we know that psi n(s) are decreasing and phi ns are increasing. So, this limit inferior exists. That implies, that limit n going to infinity of psi n(s) is equal to limit n going to infinity of phi n x, 'almost everywhere' x.

But, we know, since f of x is between phi n and psi n of x, this implies - along with the earlier effect - this implies, that limit n going to infinity of psi n x must be actually equally to f of x, actually equal to limit n going to infinity of psi n of phi n of x, for almost all x. So this must happen.

But, that implies - because each psi n is a measurable function, each phi n is a measurable function - this implies, f is measurable.

But recall, f was Riemann integrable function so obviously it is a bounded function, this implies, that f is Lebesgue integrable, because, f bounded. So by boundedness of f, every bounded measurable function and - all this is defined on a b which is a finite measure space, that must be integrable. That we had observed earlier.

So f is Riemann integrable. We only have to prove now. Claim that integral f d lambda over a b is equal to integral a to b of f x d x. This is the only thing left to be shown. What we have shown till now is that f is Lebesque integrable, so this integral exists. Now, let us observe one thing, not only is f Lebesque integrable, it is limit of a sequence of functions phi n(s) - phi n(s) converge to f of x - the sequence phi n is converging to f of x.

Psi ns are decreasing to f of x. Whichever fact we require we can use. Let us use the fact that psi n(s) are - limit of psi n(s) converge to f of x. They are decreasing and each psi n is an integrable function, so we can apply Lebesgue's dominated convergence theorem to conclude.

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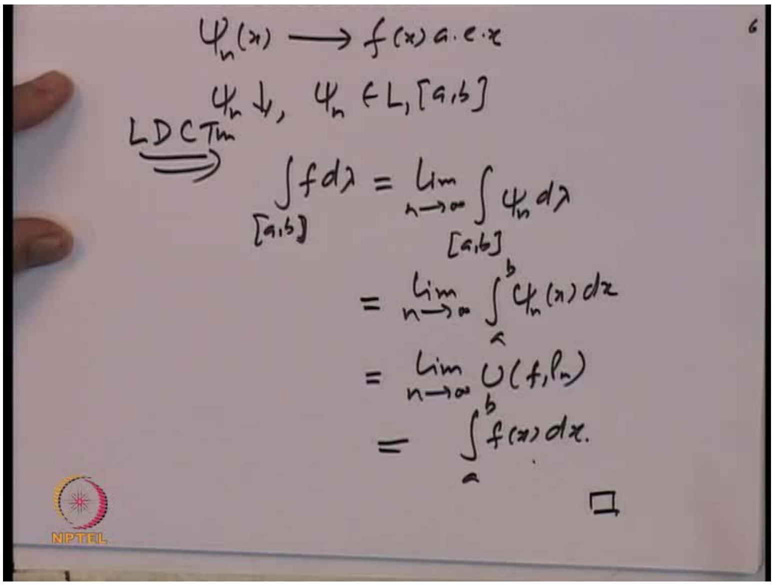


By dominated convergence theorem, since psi n x converges to f of x 'almost everywhere' x, and psi n(s) are decreasing and integrable psi n(s) decreasing, psi n(s) integrable, implies, by Lebesgue's dominated convergence theorem, that integral of f d lambda over a b must be equal to limit n going to infinity of the Lebesgue integral of psi n d lambda.

That is the observation we have already made; that the Lebesgue integral of the step function psi n is same as the Riemann integral. So, n going to infinity, that is, a to b of psi n x d x. This Riemann integral, we have observed, is the upper sum - and which limit is equal to - this is the limit of the upper sums of f with respect to P n and that converge to integral d to b f x d x.

So, that will prove that Lebesgue integral of f is same as the Riemann integral of f over d lambda.

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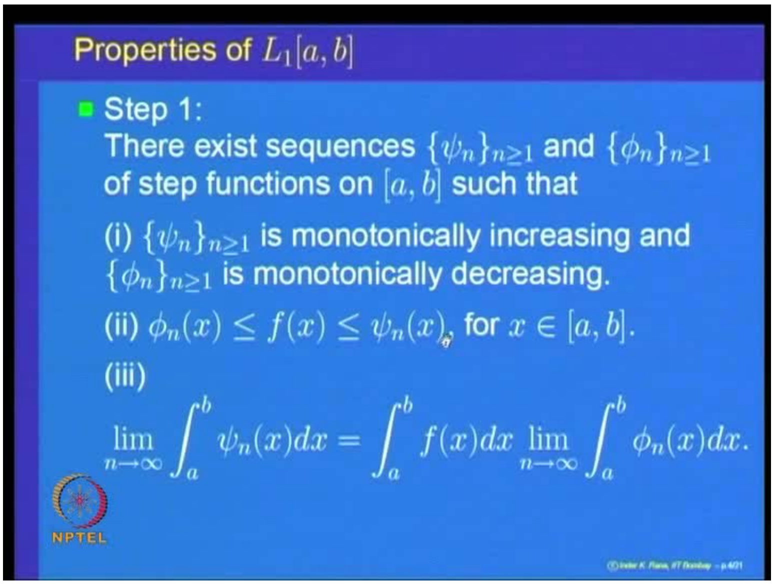


That proves the theorem. Let us go back and recall the proof. What is the first step? As a first step in the proof - using the Riemann integrability of the function we construct two sequences of said functions - phi n and psi n. Say that, f is trapped in between them. The upper sums are nothing but the Riemann integrals of psi n(s) and the lower sums are nothing but phi n(s). That means, the Riemann integral of psi n converges to the Riemann integral of f, which is also equal to - so here is a equality sign - equal to, the Riemann integral of phi n.

This construction is purely from the fact that f ns - that the function f - is Riemann integrable. The second observation is that each phi n and psi n being a step function is also measurable and Lebesgue integrable. The Lebesgue integral of phi n is the same as the Riemann integral of phi n, and the Lebesgue integral of psi n is the same as the Riemann integral of psi n.

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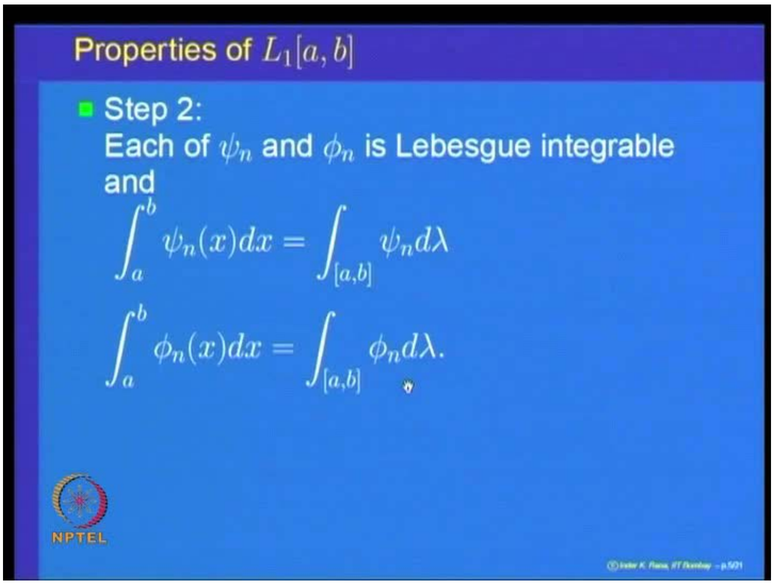


That is the second observation wake. These two integrals - Riemann integrals - of the step functions are same as the Lebesgue integrals. Now, because of the earlier consequence that the Riemann integrals of phi n(s) and psi n(s) converge to the Riemann integral of f, look at the difference phi n minus psi n. Look at that sequence of measurable functions - phi n minus psi n; that is a measurable nonnegative measurable function.

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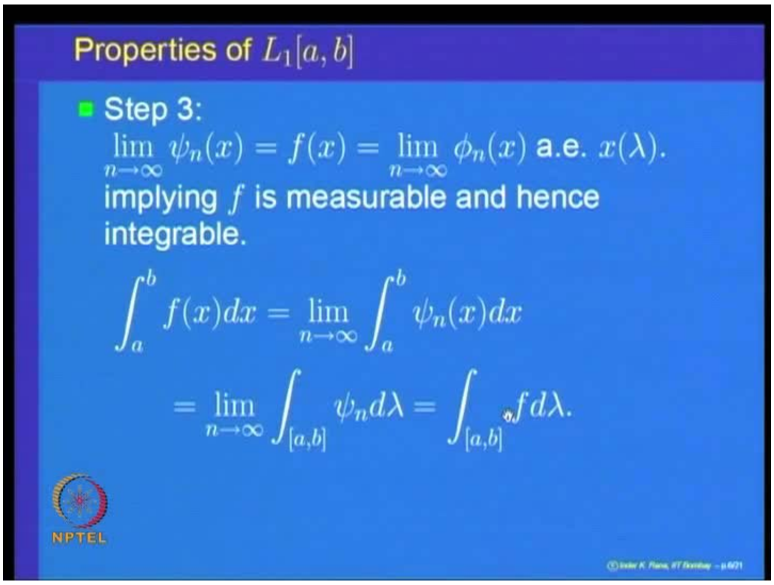

An application of Fatou's lemma will give us that these psi n(s) and phi n(s) both must converge to the same value and that is f of x ‘almost everywhere’.

That will imply that f is measurable. Being a bounded measurable function on a finite measure space, it becomes integrable. Now, the psi n(s) are decreasing to f of x, so an application on Lebesgue’s dominated convergence theorem gives that the Riemann integral of f is same as the limits of the Riemann integrals of psi n(s), which are equal to the Lebesgue integrals of psi n. That dominated convergence theorem gives you, that is, the Lebesgue integral of f over a b.

That proves the theorem completely. - This is the step we wanted to - This was the beginning of our lectures. We want you to say that to remove the defects of Riemann integral - namely, that the fundamental theorem of calculus may not hold and that the space of Riemann integrable functions may not be complete under what is called the L1 metric. We want you to extend the notion of integral from Riemann integrable functions to a bigger class.

So, here we have constructed a class of functions on a b, which are called the Lebesgue integrable functions on a b. We have shown that, if the class of Riemann integrable functions is a subset of the class of Lebesgue integrable functions, and the notion of Lebesgue integral extends the notion of Riemann integral beyond the class of Riemann integrable functions.

That was the first step in our extension theory. Lebesgue integral is an extension of the Riemann integral.

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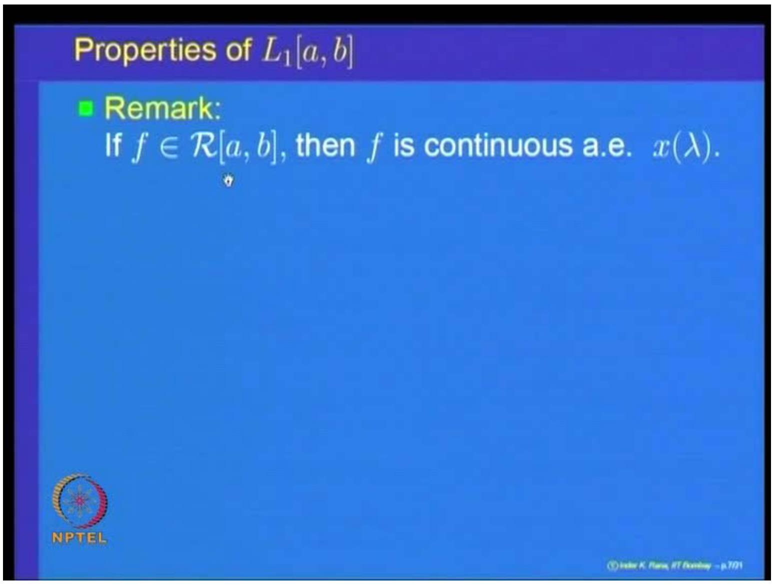


As a next step we want to look at - the space of Riemann integrable functions are inside the class of Lebesgue integrable functions. Here is an observation which one can observe from the proof of this theorem, that, if a function is Riemann integrable then it must be continuous ‘almost everywhere’.

To conclude this observation from the proof itself, basically, what we have to look at is that, the function f is the limit of those step functions. If we leave aside those partition points - remember we concluded that f is limit of those step functions phi n(s) and psi n(s) - so with that we concluded.

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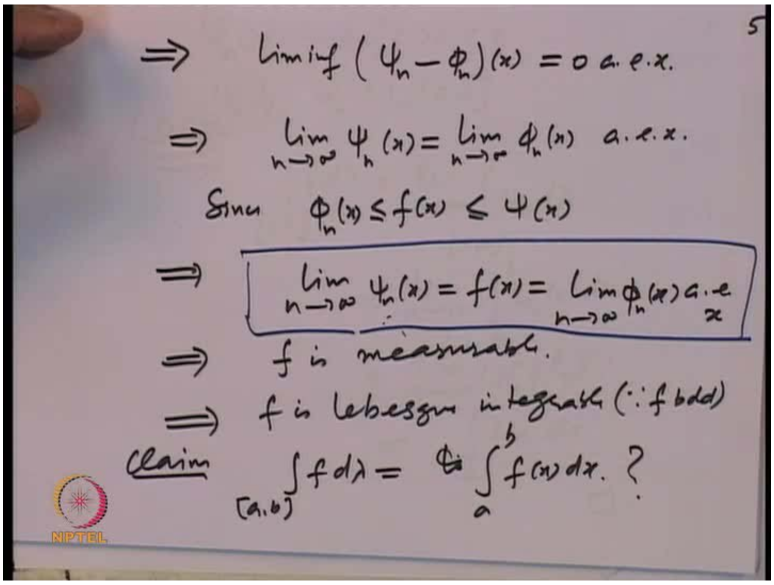


We concluded that - this is the fact that we proved in our theorem - that the limit of the step functions phi n and psi n is f of x ‘almost everywhere’. So, ‘almost everywhere’, f is limit of psi n(s) and psi n(s) are peacewise continuous functions - they are step functions. So, the points where psi n(s) may not be continuous are possibly the points of the partition points of p ns.

If we pool together all the partition points - there will be at the most countably many. If you remove them along with this ‘almost everywhere’ set - so outside a set of measures 0 - this function f will become continuous.

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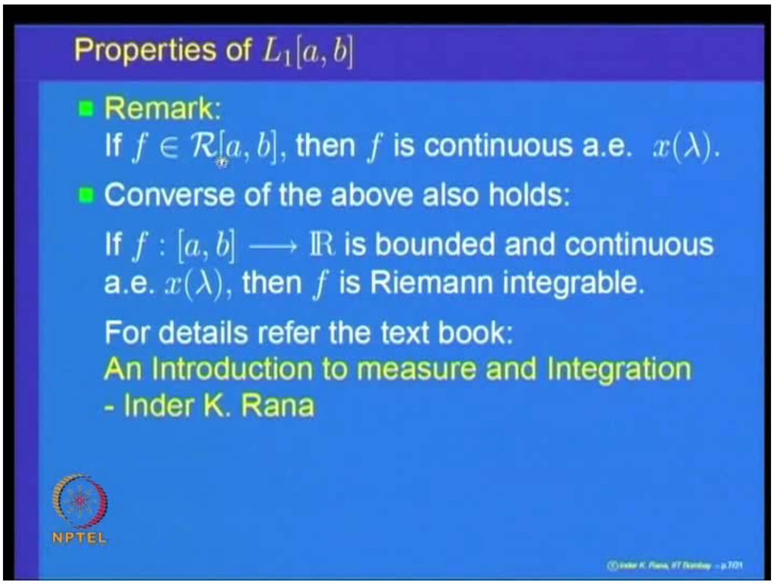


I am just indicating the possibility of a proof of that. Those interested should probably look into the textbook. In fact, the converse of this theorem is also true; namely, that if f is a bounded continuous function, which is - bounded function which is continuous almost everywhere, then f is Riemann integrable.

That means, there is a characterization of Riemann integrable functions in terms of continuity. Namely, a function f is Riemann integrable if and only if it is continuous ‘almost everywhere’.

We are not given the proof of this, those interested should probably look into the textbook as mentioned above - An Introduction to Measure and Integration, in which the complete proof is given. But, what I wanted to indicate is that the proof of one part of the theorem, namely, for a Riemann integrable function it should be continuous ‘almost everywhere’ is already included in the proof of the fact that we proved just now, that, Riemann integrable implies it is Lebesgue integrable.

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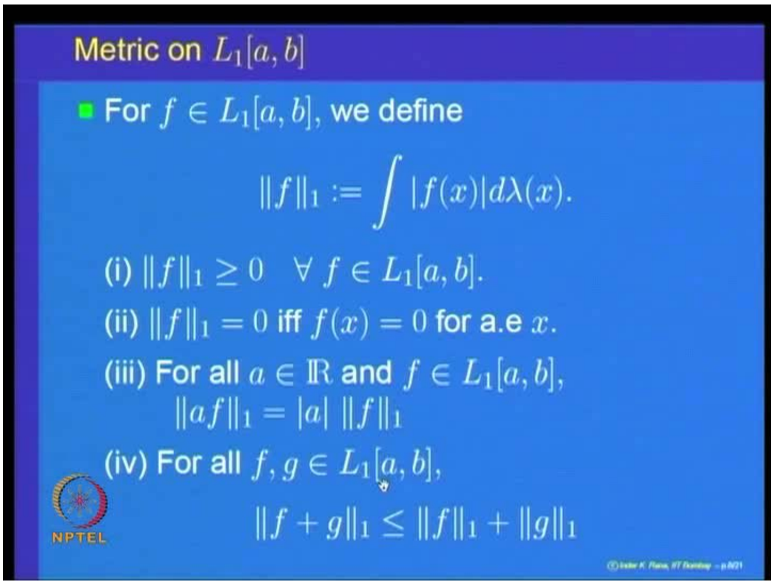


That is one part of the properties of the space L 1 a b, that, R a b - the space of Riemann integrable functions is in L a b. Now, let us look at a metric on L 1 of a b. Recall, we have already shown that L 1 of a b is a vector space. We showed that if f and g are integrable functions, then f plus g is integrable, f into g is integrable and alpha times f is integrable. It is a linear space, so it is a vector space over the field of real numbers. On this, we are going to define a notion of a magnitude.

For a function f in L 1, we define, what is called the L 1 norm of f to be the integral of the absolute value of f of x d lambda x. This is called the L 1 norm of the function f which is in L 1. Clearly, it is a finite number because f is integrable, so this right hand side exists and is finite. You are integrating a nonnegative function, so the first property, namely, L 1 norm of a function is bigger than or equal to 0 for all functions f in L 1 of a b, that is obvious.

The second property, namely, supposing this function f is 0 'almost everywhere', then clearly integral of the function will be equal to 0 - we have already observed that - so norm will be equal to 0. Conversely, if the L 1 norm is equal to 0, then the function, the absolute value of f of x being a nonnegative function, its Lebesgue integral 0, that implies that the function must be 0 'almost everywhere'.

So, the second property, namely, the norm is equal to 0 if and only if the function is 0 'almost everywhere'. The third property, which is that if you multiply the function f by alpha, the norm of alpha f is equal to the absolute value of alpha times norm of f. That is obvious, because, in the definition, if we replace f by alpha f then this being a constant - the property of the integral, so the integral of alpha f is equal to mod alpha times integral mod f. That gives you the property that the L 1 norm of alpha f is equal to the absolute value of the constant with which you are multiplying - mod alpha times norm of f 1.

Finally, the triangle inequality property; namely, if f and g are integrable functions then we have already shown that f into g is also an integrable function. Integral of the absolute value of f plus g will be less than or equal to integral of absolute value of f, plus integral of absolute value of g, by the triangle inequality of the numbers. That will give us: L 1 norm of f plus g is less than or equal to L 1 norm of f plus L 1 norm of g.

These are the properties of this magnitude or this norm of f 1. At this stage, I just want to point out to you that this is very much similar to the Euclidean metric or Euclidean norm on R n.

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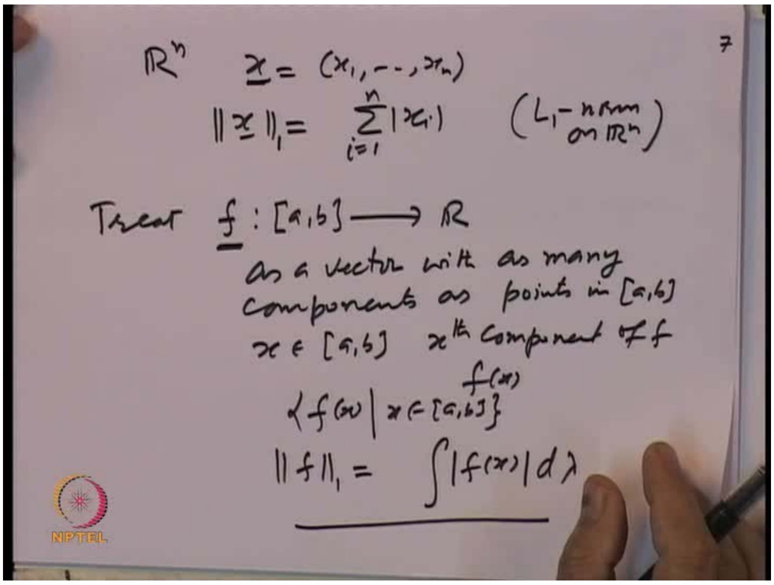


See, on R n, for a vector x with components x 1 x n, we normally define the norm to be equal to - you can take the norm to be equal to - sigma mod of x i, i equal to 1 to n.

This is what is called the L 1 norm on R n. What we are saying is that if x is a vector with n components then this must be the L 1 norm.

Now, treat a function f defined on an interval a b to R. Treat this as a vector with as many components as points in a b. I want to treat this f as a vector with as many components as points in a b. So, for a point x in a b, what is the xth component of f? It is nothing but f of x.

You can treat f of x, x belonging to a b, as a vector. If you treat it that way, then what is the L 1 norm of f you would like to define? Keeping in mind, take the absolute values of the components - this is the component, take its absolute value - and you want to sum it. The summation is nothing but integral d lambda. In that sense, this is perfect generalization of the ordinary L 1 norm on R n.

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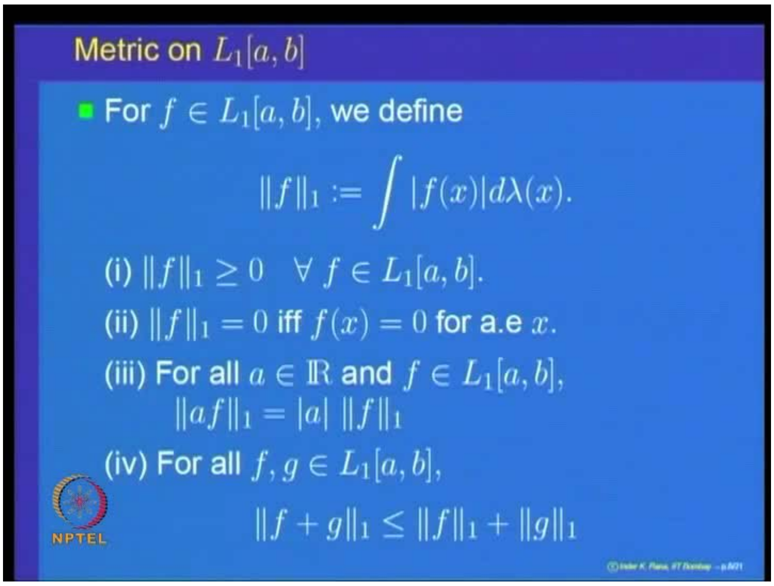


The only problem in this is that we do not have the property that L 1 norm is equal to 0 if, and only if, f is equal to 0. We have only got that the L 1 norm is equal to 0 if, and only if, f of x is equal to 0 ‘almost everywhere’. But let us keep in mind that the L 1 norm does not change if we change the function on a null set - on a set of measure Lebesgue measure 0.

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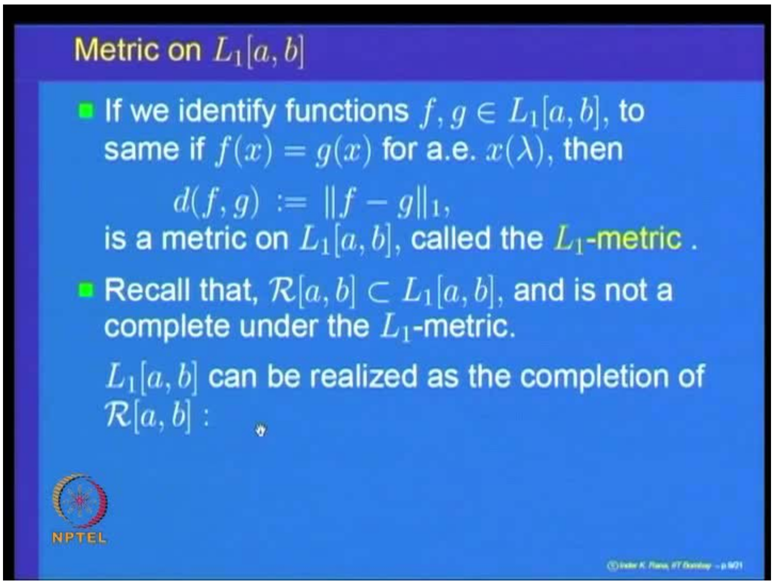


With that observation in mind, from now onwards what we do is: in L 1 of a b we identify functions which are equal 'almost everywhere'. So, if two functions are equal 'almost everywhere' - if f and g in L 1 are same, are equal 'almost everywhere' lambda - then we treat these functions to be same.

With that understanding, this becomes a metric d f g, which is equal to norm of f minus g L 1 norm of f minus g, becomes a metric on L 1 of a b and is called the L 1 metric. The observation I want to point out is that R a b is a subset of L 1 of a b, so R a b as a subspace with L 1 metric is not complete in the L 1 metric.

That is an observation, which you should read from that textbook already mentioned. This was one of the defects of Riemann integral that motivated the development of Lebesgue integral; namely, the space R a b under the L 1 metric is not a complete metric. This is not a complete metric under the L 1 metric, but there is a theorem; namely, every metric space can be completed.

There is an abstract theorem in metric spaces that every metric space can be completed. For example, look at the set of rational numbers that is not complete under the usual distance of absolute value as a distance, and its completion is the real numbers. That is the important property of real numbers, that it is complete under that metric.

Similarly, R a b under the L 1 metric is not complete and there is an abstract theorem that R a b can be completed, it can be put inside a complete metric space.

What we are going to show is that completion is nothing but L 1 of a b. We are going to prove that L 1 of a b - the space of all Lebesgue integrable functions on a b - can be realized as the completion of the space R a b under the L 1 metric.

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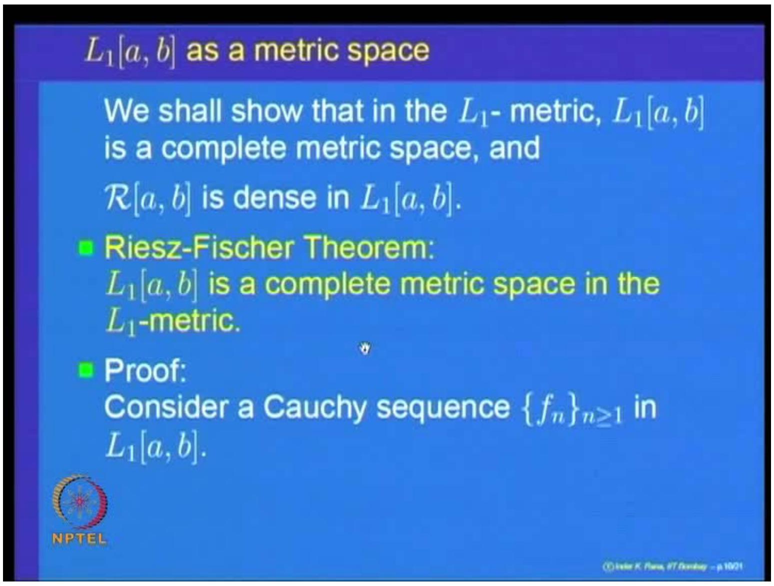


We will be doing this in two steps. One, we will show that in the L 1 metric the space L 1 a b is complete - that is one. And, for it to be a completion of R a b, we will show that R a b sits inside L 1 of a b as a dense subset. So, R a b sits inside L 1 of a b as a dense subset in the L 1 metric and L 1 is complete. That will show that L 1 of a b is a realization of the completion of the space R a b. Like rationals are dense in the real line, the field of real numbers is complete. That is why we say that real numbers is the completion of the field of rational numbers.

In a similar way, we want to prove that L 1 of a b is the completion of this space of R a b. That means we want to prove that L 1 a b in the L 1 metric is complete and R a b is dense in it.

The completion part is called Riesz Fischer theorem. Riesz Fischer theorem states that L 1 of a b is a complete metric space in the L 1 metric.

Let us look at a proof of this. To prove that it is a complete metric space, what do we have to show? We have to show that every Cauchy sequence f n in L 1 of a b converges to a value in L 1 of a b.

Every Cauchy sequence in L 1 of a b is convergent, and convergent to a point in L 1 of a b; that is what we have to show.

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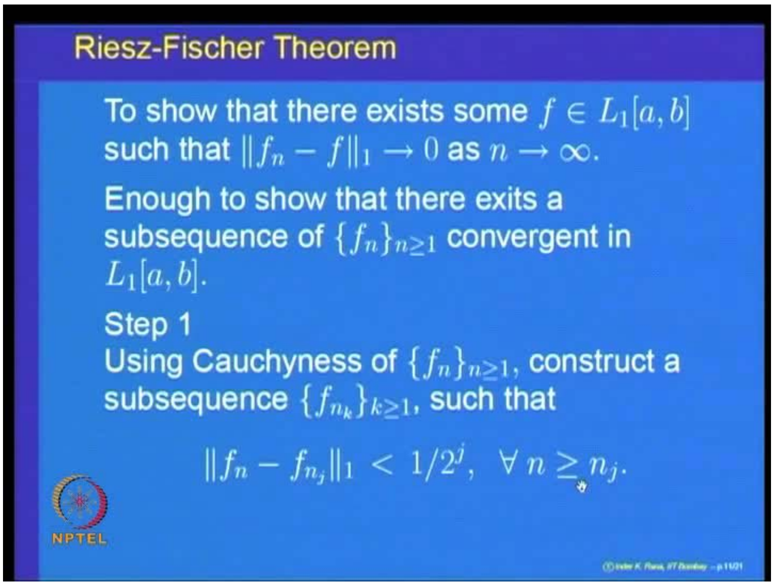


This is what we have to show; that there exists some function f in L 1 of a b such that, f n minus f L 1 norm converges to 0 as n goes to infinity. To prove this fact, here is an observation; what we will show is - it is enough to show - that there exists a subsequence of f n, which is convergent in L 1 norm.

To show that f n is convergent in L 1 metric, it is enough that f n is a Cauchy sequence. To show that a Cauchy sequence is convergent, it is enough to show that there exists a subsequence of the Cauchy sequence, which is convergent. That will prove that the sequence is convergent. To do that we had to produce a subsequence of f n, which is convergent to a function in L 1.

Let us look at the first step of our construction. To construct that subsequence, we are going to use the Cauchyness of the sequence f n. So, f n is Cauchy; the first step is, saying that the sequence f n is Cauchy implies that I can pick up a subsequence f n k of f n such that the norm of f n minus f n k is less than 1 over 2 to the power k for all n bigger than or equal to n k.

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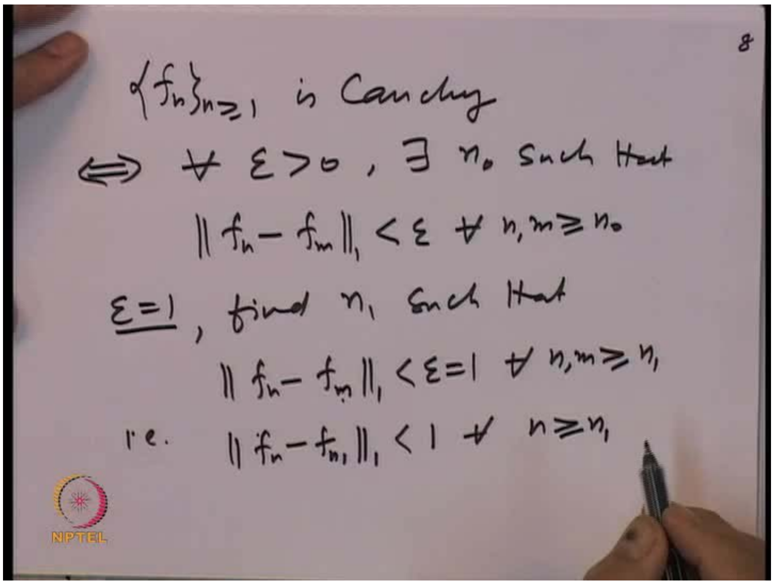


To do that, we start with - what does Cauchyness mean? Saying that the sequence f n is Cauchy means - it is same as - for every epsilon bigger than 0, there exists some stage, n naught, such that norm of f n minus f m is less than epsilon for every n and m greater than or equal to n naught; that is Cauchyness.

To start our construction - take epsilon equal to 1 and find n 1 such that norm of f n minus f m is less than epsilon equal to 1 for every n and m bigger than n 1. In particular, when m is equal to n 1 that will give me, that is, f n minus f n 1 will be less than 1 for every n bigger than or equal to n 1.

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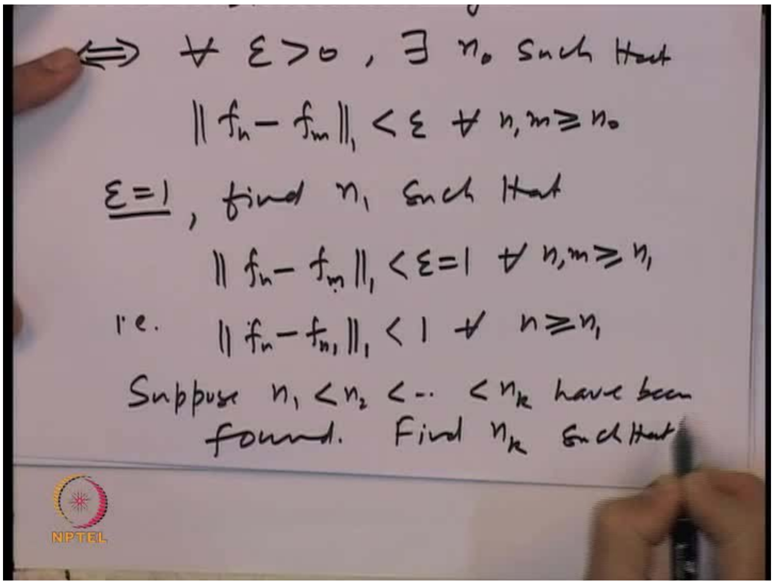


That is the first stage. Suppose, n 1 less than n 2 less than n k have been constructed, have been found.

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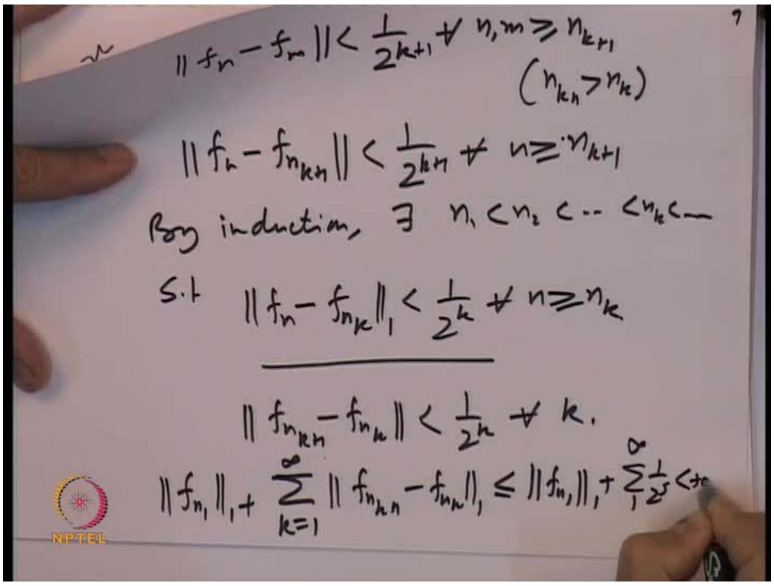


Now, use Cauchyness, so, find nk such that - how do I find n k? I will find n k such that norm of f n minus f m will be less than 1 over 2 to the power k for every n and m bigger than or equal to - find n k plus 1, we have to construct this next one, n k plus 1. I will find n k plus 1 such that n k plus 1 is greater than n k and this property holds.

Then, for m equal to n k plus 1 I will have, f n minus f m k plus 1 will be less than 1 over 2 to the power - I can put it as k plus 1 also that does not matter, k plus 1 - for every n bigger than or equal to n k plus 1.

By induction, there exists n 1 less than n 2 less than n k less than so on, such that norm of f n minus f n k is less than 1 over 2 to the power k, for every n bigger than or equal to n k.

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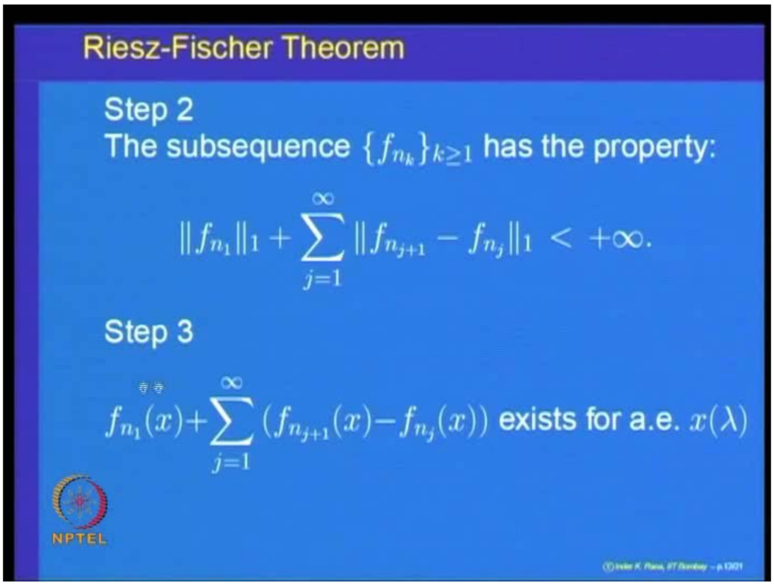


That is step 1 - we do that. Once that is done, we go to step 2 and step 2 is showing that this subsequence that we have constructed has the property that the sum of the L 1 norms of f n 1 plus the L 1 norm of 1 to infinity of this, is finite.

The sums of the L 1 norms of f n 1 plus the L 1 norms of f n j plus 1 minus f n j j 1 to infinity, that is all finite. That follows from the fact that - we looked at this just now - that means norm of f n k plus 1 minus f n k is less than 1 over 2 to the power k. If I specialize n to be equal to n k plus 1, then I have this property for every k. This implies that if I look at the summation: norm of f n 1 plus sigma norm of f n k plus 1 minus f n k, k equal to 1 to infinity will be less than or equal to norm of f n 1 plus sigma 1 to infinity 1 over 2 to the power j, and that is finite.

So, that will prove that the required property, step 2, namely, sums of norms of these quantities f n 1 and this are finite.

Once we have that, now I can apply my series form of the Lebesgue's dominated convergence theorem. As step 3 - as the next step - I want to conclude that if I look at the corresponding functions f n 1 x plus summation j 1 to infinity f n j plus 1 x minus f n j x, then, this series is convergent almost everywhere. That precisely follows from the series form of the Lebesgue's dominated convergence theorem. If I look at this series then I know that L 1 norms of this series is finite.

That is precisely the perfect situation where the series form of the Lebesgue's dominated convergence theorem applies and says that these functions - they must - this series must converge almost everywhere; so, this sum must exist almost everywhere.

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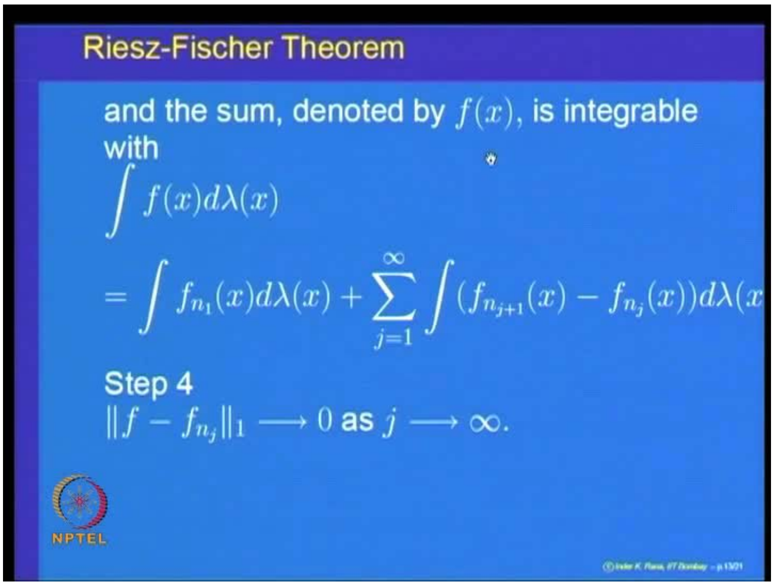


If we denote the sum by f of x then, f of x equal to this exists 'almost everywhere' and it also says that, that function is actually an integrable function and integral of f is equal to integral of f n 1 plus integral of the sums of the corresponding series. So, these integrals of f are equal to integral of f n 1 plus this series.

So, we have located a function f and now it is only to claim that the difference f minus f n j goes to 0; that, f is the limit of that subsequence.

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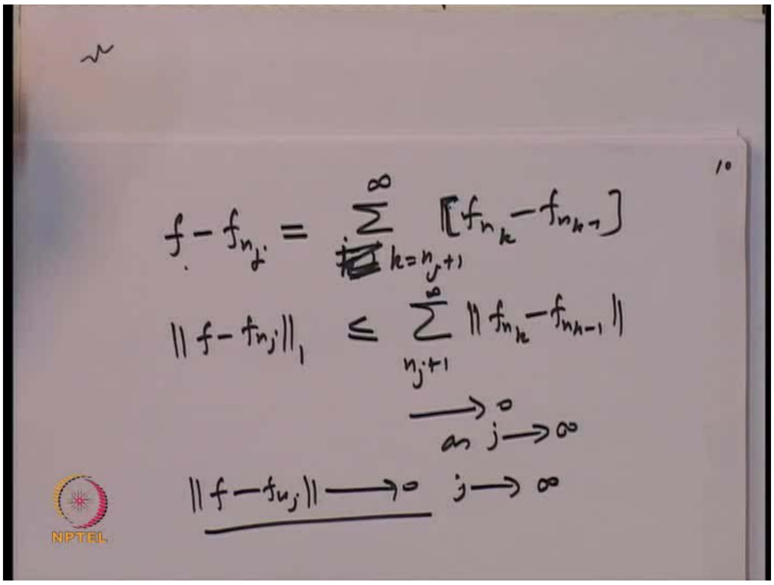


Note that, if we look at f minus f n j - that is precisely equal to summation j equal to, k equal to j, sorry let me, nj - that means, k equals to n j plus 1 of f n k minus f n k minus 1. So, the difference between f and f n j is nothing but the tail of the series with the terms f n k minus f n k minus 1.

Norm of f minus f n j L 1 norm is going to be less than or equal to summation norm of f n k minus f n k minus 1 from k from the stage n j plus 1 onwards, and that is a tail of the convergent series, geometric series 1 over 2 to the power j, so, that goes to 0 as j goes to infinity.

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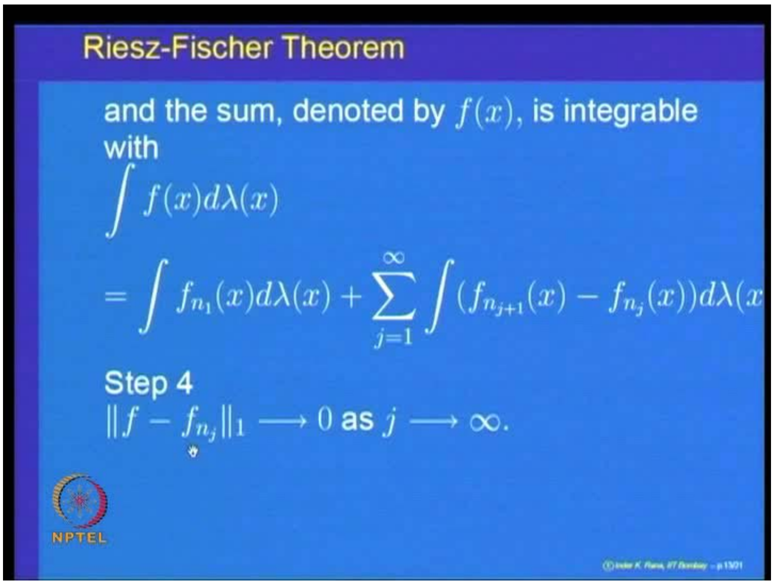


That will complete the proof that, hence, f minus f n j goes to 0 as j goes to infinity. That will prove that the subsequence f n j is convergent.

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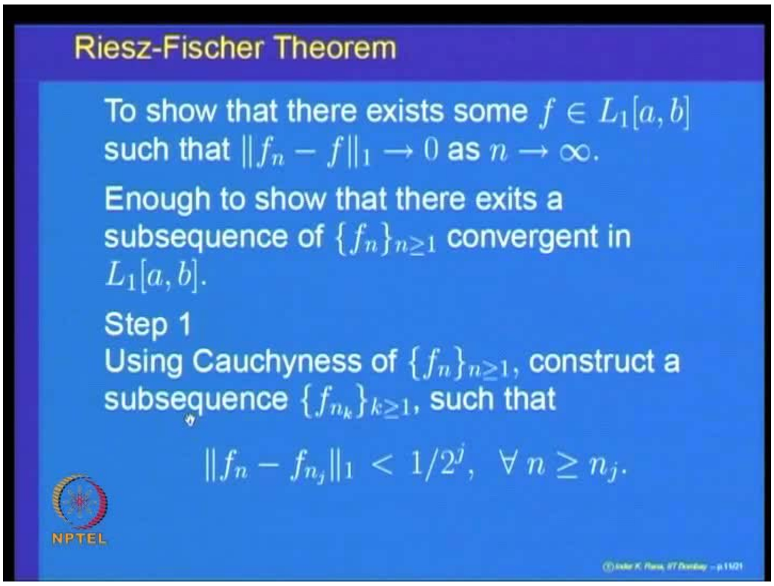


Let us just go back to the proof once again. Basically, the proof requires one observation; namely, to show that a Cauchy sequence is convergent; it is enough to produce a subsequence, which is convergent. And that subsequence is produced in such a way that by using the Cauchyness, we produce our subsequence with a property that the norms of the consecutive terms of that subsequence are less than 1 over 2 to the power j.

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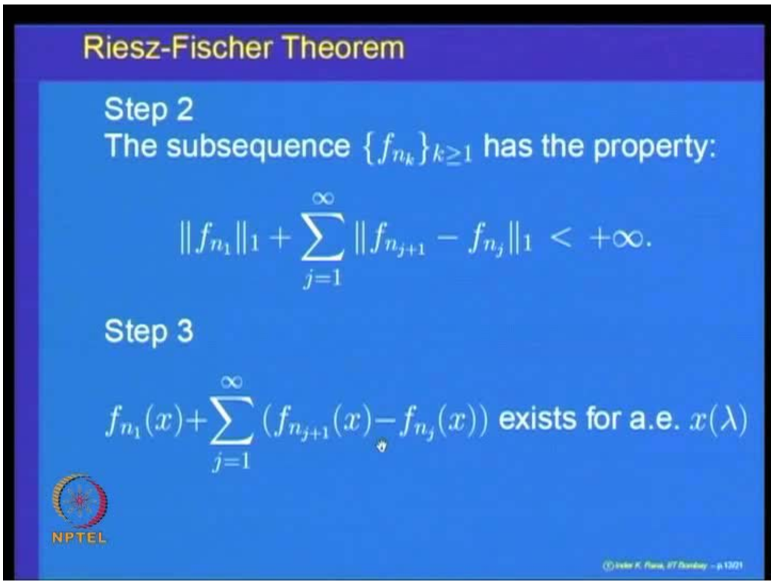


Once that is done, we apply the series form of the Lebesgue's dominated convergence to conclude that the sum of f n 1 x plus summation f n j plus 1 x minus f n j that exists 'almost everywhere', that is an integrable function and the integrals converge.

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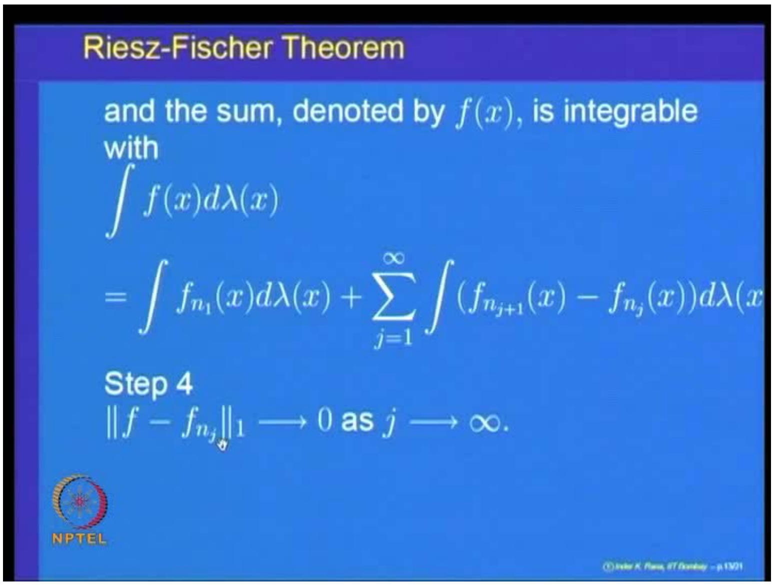


Once we have that, it is obvious that norm of f minus f n j goes to 0; because once we subtract the remaining thing is a tail of the series 1 over 2 to the power j, and that must go to 0; with that, we prove the Riesz Fischer theorem.

Today, we analyzed two important things; namely, one, that the space of Riemann integrable functions is inside the space of Lebesgue integrable functions - Lebesgue integrable - on the interval a b, and that the notion of Lebesgue integral extends the notion of Riemann integral - that was one property. The second property - we observed that the space of Lebesgue integrable functions on the interval a b under the L 1 metric is a complete metric space.

We will continue this analysis tomorrow and we will show that the space of Riemann integrable functions which is inside the space of Lebesgue integrable functions sits inside as a dense subset. Hence, that will prove that L 1 of a b is the completion of the space of Riemann integrable functions.

Thank you.