Measure and Integration
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Module No. # 06
Lecture No. # 21
Dominated Convergence Theorem and Applications

Welcome to lecture 21 on measure and integration. In the previous lecture, we had started looking at the properties of sequences of integrable functions and we started proving an important theorem called Lebesgue's dominated convergence theorem.

Let us continue looking at that; after that we will start looking at the special case of integration on the real line and that will give us a notion of Lebesgue integral.

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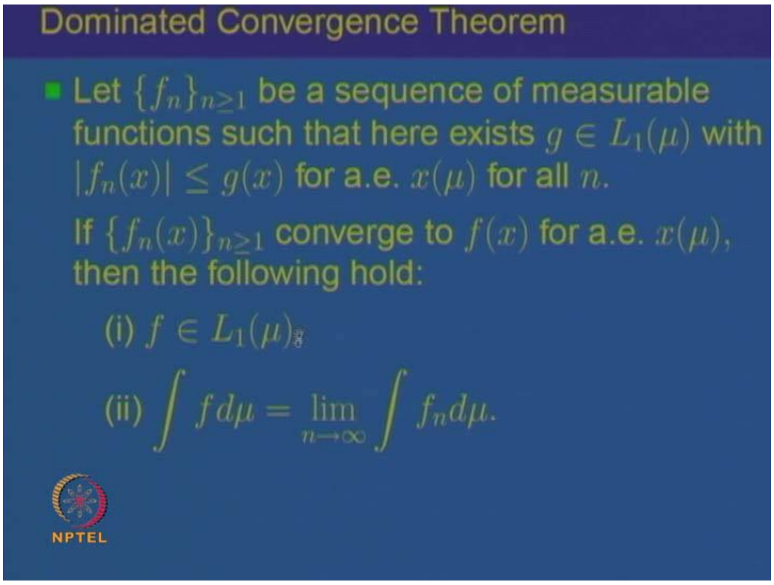

Let us recall what we had started proving, namely, dominated convergence theorem, which says, that if f n is a sequence of measurable functions such that there exists a function g belonging to L 1, say, that all the f ns are dominated by this integrable function g, 'almost everywhere' x for all n and if f n converges to f, then the limit function is integrable and integral of f is nothing but the limit of integrals of f ns.

So, the theorem basically says, that if f n is a sequence of measurable functions, all of them dominated by a single integrable function g, then all the f ns become integrable, of course. And if f ns converge to f then f is integrable and integral of f is nothing but the limit of integrals of f ns.

We had proved this theorem in this particular case when instead of this ‘almost everywhere’, that mod f ns are dominated by g everywhere, and f ns converge to f everywhere. So, to extend this case to ‘almost everywhere’, we have to do only a minor modification.

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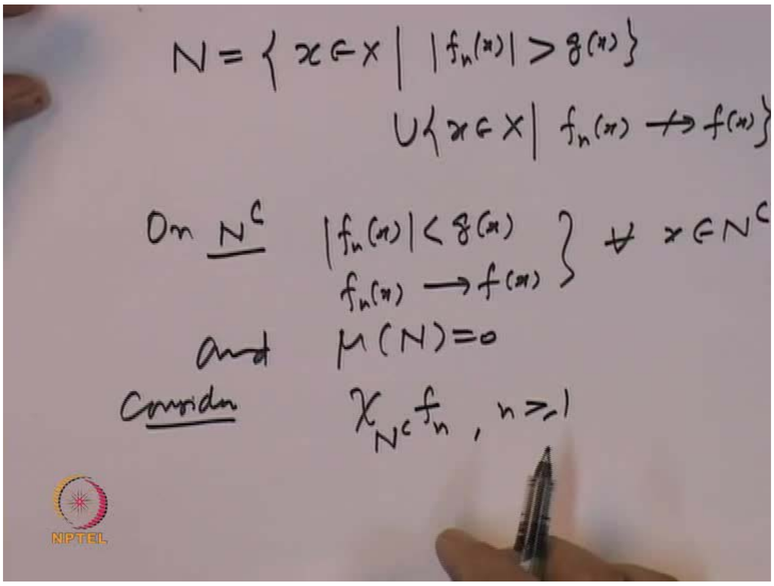

Let us define the set N to be the set of all x belonging to X, where mod f n x is not dominated by g. So, g of x or union the set of all those points x belonging to X, say that, f n x does not converge to the function f of x.

So, on N compliment we have f n x is less than g x and f n x converges to f of x. For every x belonging to N compliment and mu of the set N is equal to 0, because we are saying that this mod f n x is less than g x ‘almost everywhere’ and f n x converges to f of x ‘almost everywhere’. So the set where it does not hold, that is, the set N and that set has - N - has got measure 0. Now, let us consider the sequence indicator function of N compliment times f n.

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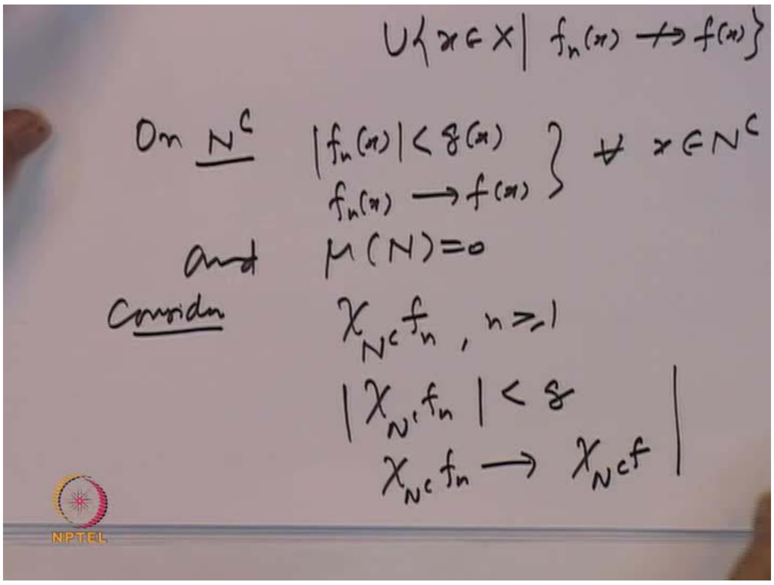

This is a sequence of functions, which are dominated by for n bigger than or equal to 1; they satisfy the property, namely, this, the indicator function of N compliment f n mod of that is less than g for all x everywhere and the functions converge to the indicator function of N compliment f.

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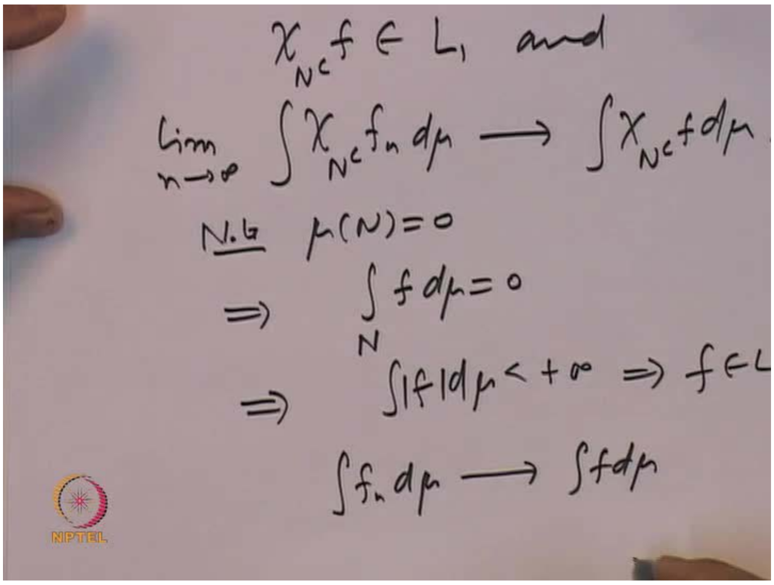

By our earlier case, what we get is the following: namely, that indicator function of N compliment times f is L 1 is integrable and integral of f n limit n going to infinity, indicator function of N compliment times f n, the integral of that converges to the integral of indicator function of N compliment times f. So that is by the earlier case when everything is true for all points.

That means this is the same. But note that mu of N is equal to 0, that implies that the integral of f over N d mu is equal to 0. We already know that on N compliment f is integrable. So, that together with this fact implies integral of mod f d mu is finite, implying that f belongs to L 1. And this equation, which said that integral of f n over N compliment converges to integral of f over N compliment and mu of N being 0 together, gives us the condition that integral of f n d mu converges to integral f d mu, because integral f n d mu is the same as integral of f n over n plus integral over N compliment and integral over N compliment converges to integral over N compliment of f and on n both are 0. This gives us the required result.

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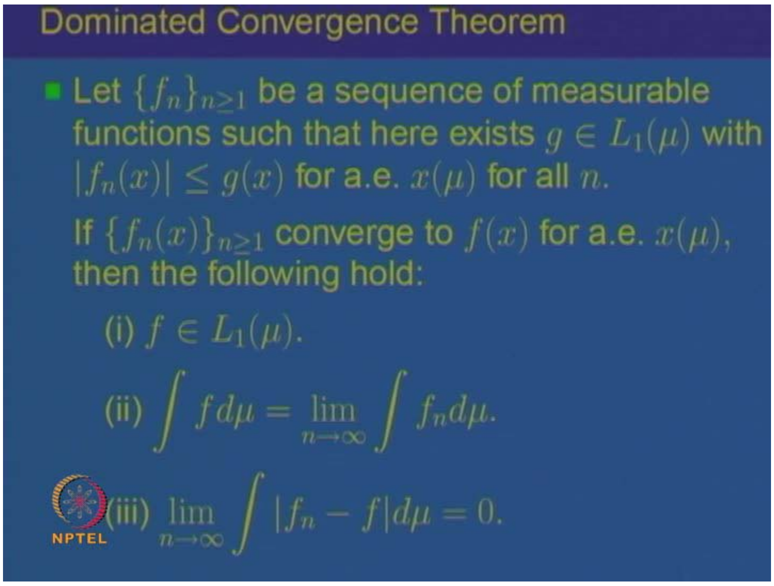

That is how from ‘almost everywhere’, conditions are deduced from the fact that something holds everywhere. This dominated convergence theorem holds for whenever the sequence f n is dominated by g and f n converges to f ‘almost everywhere’, then f limit function is integrable and integral f converges to integral of f n.

As I said, this is one of the important theorems, which helps us to interchange the limit and the integral side. Let us look at some minor modifications of this theorem. One more thing - we can even deduce that integral of mod f n minus f d mu also converges to 0.

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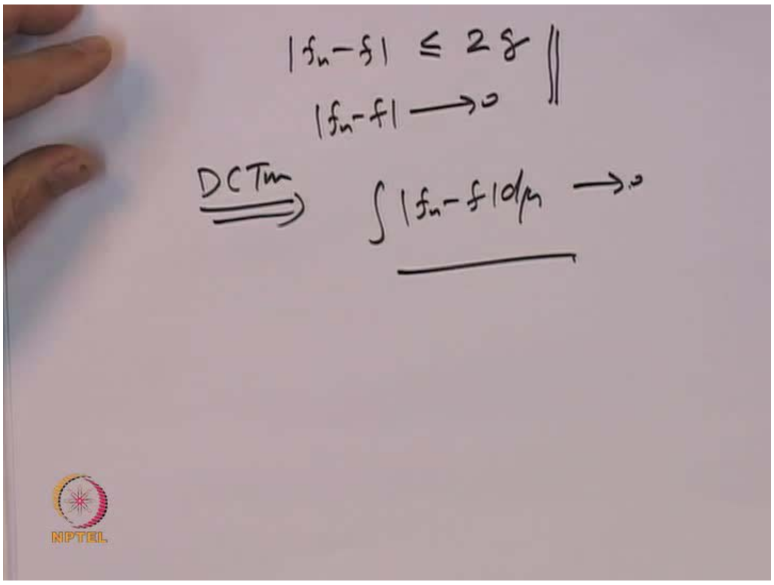

To deduce that part, we just have to observe that mod f n minus f is less than or equal to twice of g and mod f n minus f goes to 0. So, again, an application of dominated convergence theorem - which we proved just now - implies that integral of mod f n minus f d mu goes to 0. That is another modification, another consequence of the dominated convergence theorem.

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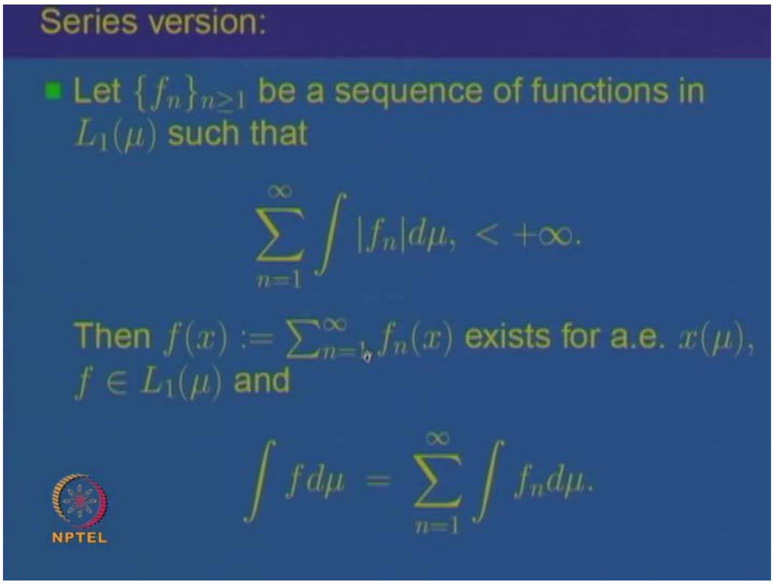

Let us prove what I call as the series version of this theorem; namely, that if f n is a sequence of functions which are integrable and integrals of f n summation 1 to infinity, so sum of all the integrals of mod f ns are finite. Then, the conclusion is that the series f n x converges 'almost everywhere' and if you denote the limit, the sum giving f of x, then the function is integrable and integral f is equal to summation of integral f ns.

So essentially, this theorem says that if the summations of mod f ns are finite then this, the series f n x is convergent 'almost everywhere' and integral of f is equal to integral of summation of integral f ns, that is again interchange of limit essentially.

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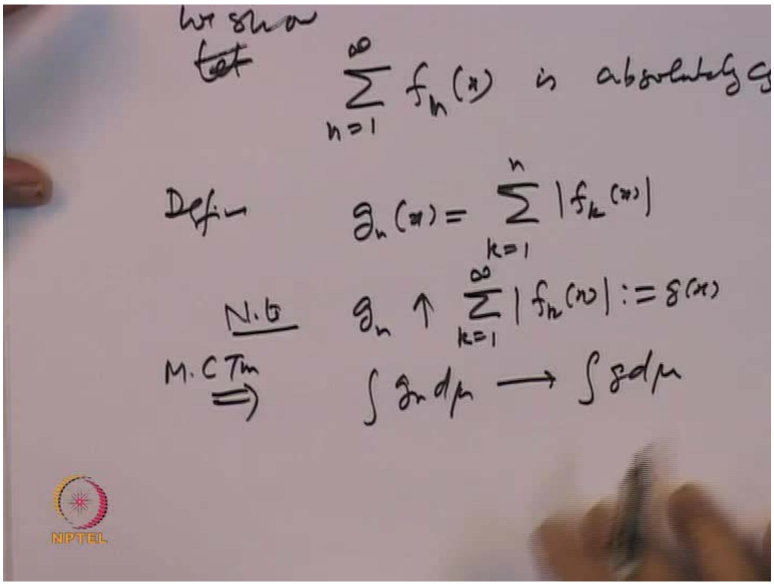

Let us see how from dominated convergence theorem we can get this. To show that this series is convergent 'almost everywhere' we will actually show that it is absolutely convergent. For that, let us define g n of x to be equal to summation mod f k of x k going from 1 to n, the partial sums of the absolute values 1 to n.

Let us observe this sequence g n. Note g n is a sequence of nonnegative measurable functions and g ns are increasing to some function, that is, they are going to increase to k equal to 1 to infinity mod of f k of x, and let us call that as g x, and they are increasing to the function g of x. That implies, by monotone convergence theorem, we have integral of g n d mu must converge to integral of g d mu.

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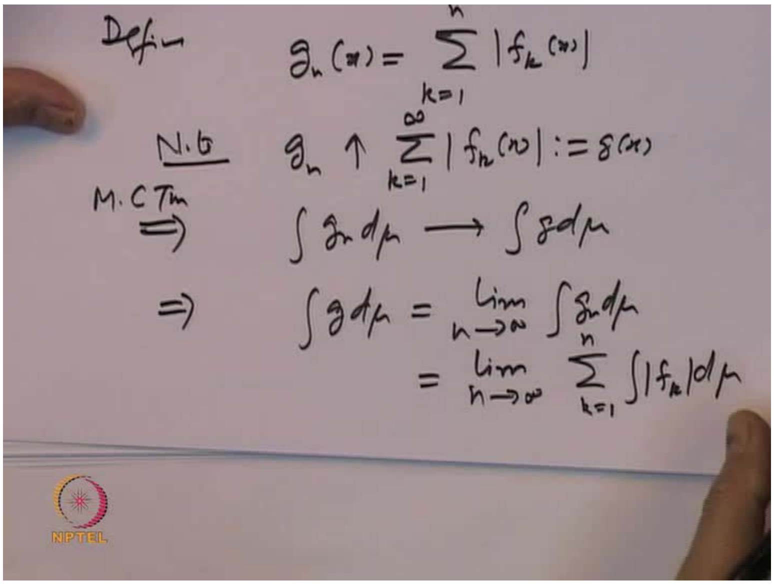

But integral of g n d - that is the same as saying integral g d mu is equal to limit n going to infinity of integral g n d mu. But what is integral of g n? It is the sum of absolute values of f k 1 to n. So, by linearity property, this is nothing but limit of n going to infinity of summation 1 to n k equal to 1 to n of integral mod f k d mu. And this limit is nothing but 1 to infinity and that is given to be finite. Let us write that.

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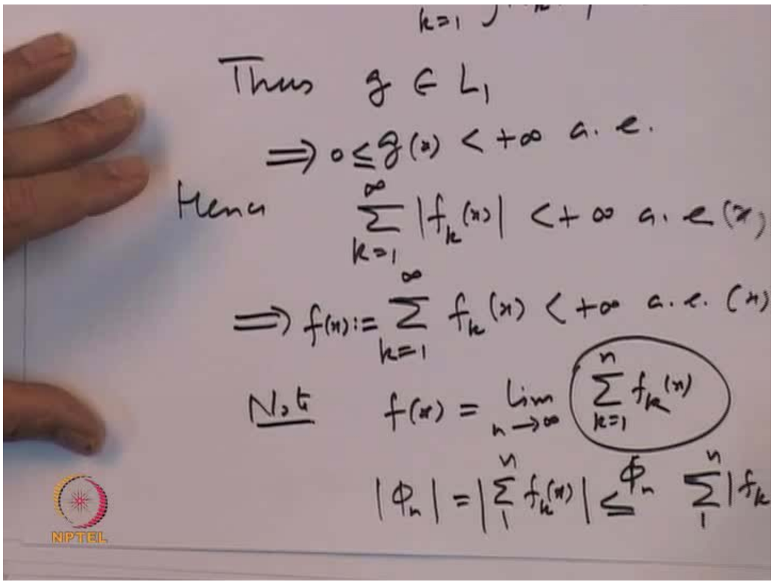

This is equal to summation k equal to 1 to infinity of integral mod f k d mu, which is given to be finite. Hence, what we get is - thus g is integrable - g is an integrable function. Saying that g is integrable implies - recall that if a function is integrable and g is a nonnegative function - g of x is finite ‘almost everywhere’, that is, nonnegative function which is an integrable function must be finite ‘almost everywhere’.

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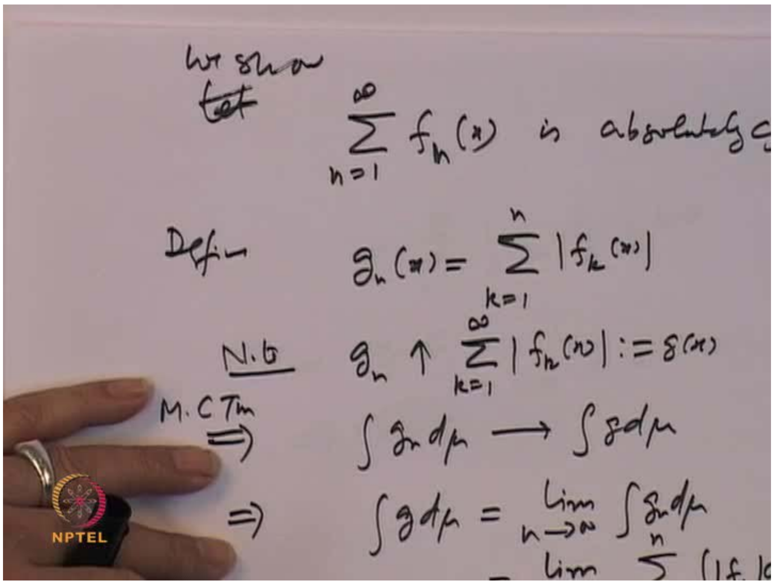

We get that g is finite ‘almost everywhere’. And what is the function g? So, g is nothing but the limit of the absolute values of f k x. That means, that proves the series; hence, sigma k equal to 1 to infinity mod f k of x is finite ‘almost everywhere’ x. Once the series is absolutely convergent, it is also convergent, that implies that sigma k equal to 1 to infinity, f k x is finite for ‘almost everywhere’ x.

Let us denote this limit by f of x; this is f of x. As observed earlier, note f of x, we can also write f of x as the limit n going to infinity of summation k equal to 1 to n f k x. And if these functions are called something, say, phi n, then note that mod phi n - what is mod phi n? It is the absolute value of 1 to n f k x. Absolute value of that and that is less than or equal to summation 1 to n mod of f k 1 to n and that is nothing but our g n which is less than or equal to g.

All these partial sums, which we have called phi n(s) are all dominated by g and phi n(s) converge to f by dominated convergence theorem.

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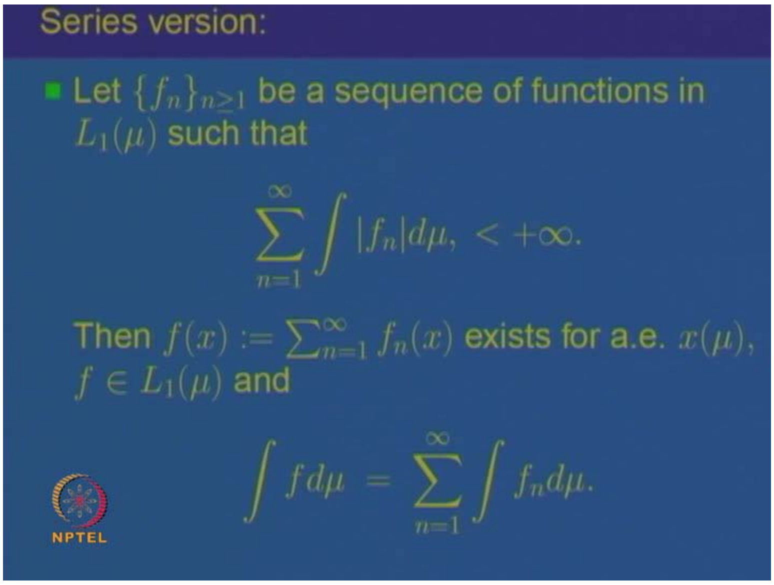

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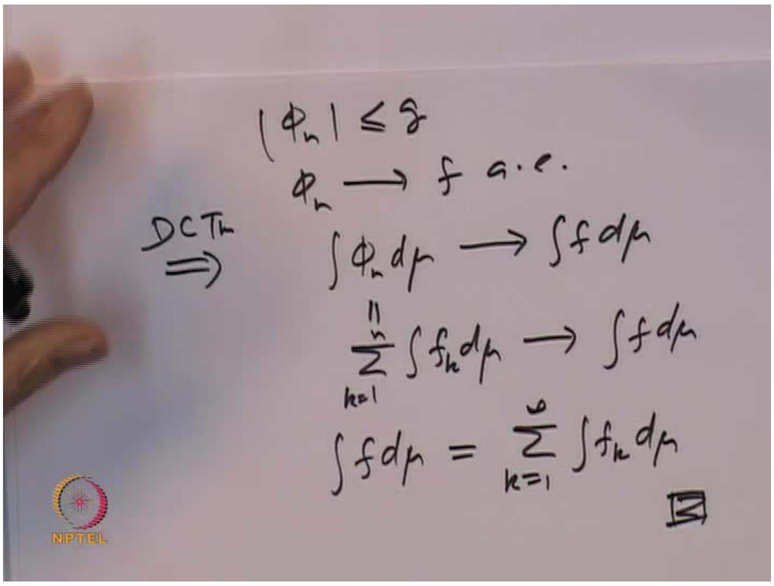

What we have got is: all the phi n(s) are less than or equal to g and phi n(s) converge to f ‘almost everywhere’. That implies by dominated convergence theorem, that integral of phi n(s) d mu must converge to integral of f d mu.

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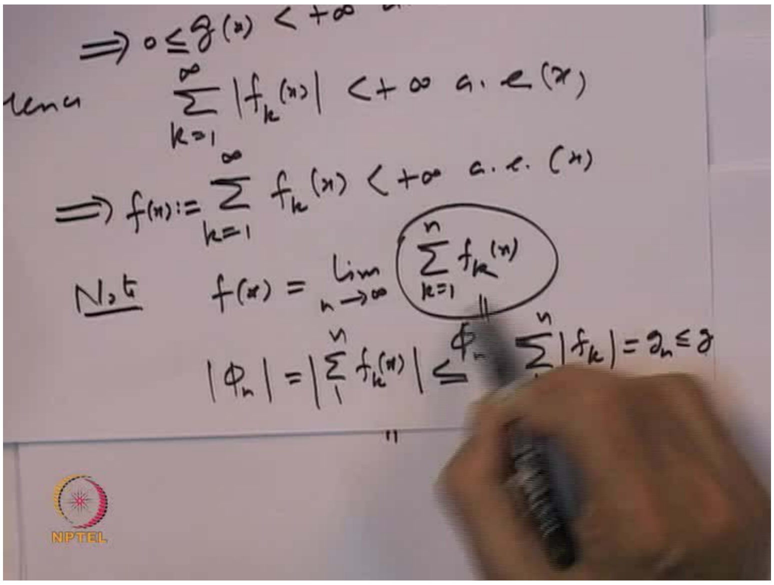

This is nothing but - this phi n - this is what we called phi n, that is, summation 1 to n. So this is nothing but summation of 1 to n of integral k equal to 1 to n of integral f k d mu must converge to integral f d mu and that is same as saying that integral f d mu is equal to summation k, equal to 1 to infinity integral f k d mu.

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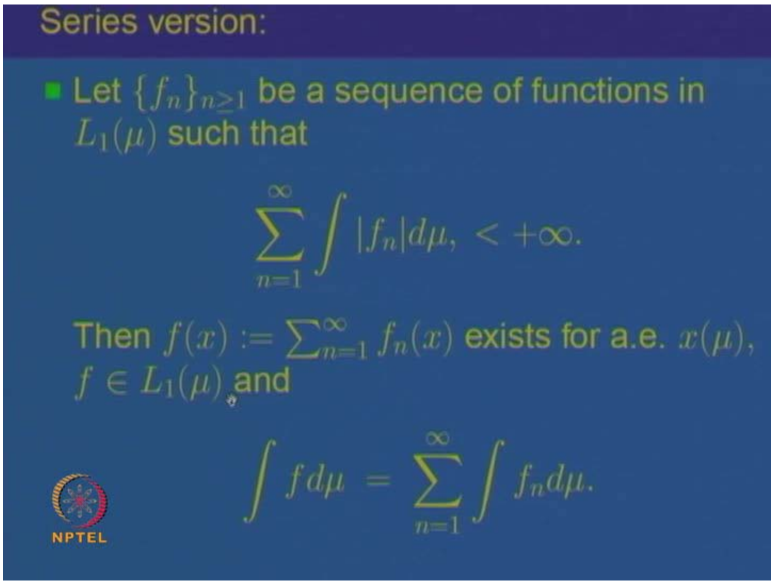

That proves the theorem, namely, if f k is a sequence of functions which are integrable and the sum of the integrals is finite, then the series f n x n 1 to infinity itself is convergent 'almost everywhere' and the limit function is integrable and integral of the limit function is equal to summation of integrals of f ns.

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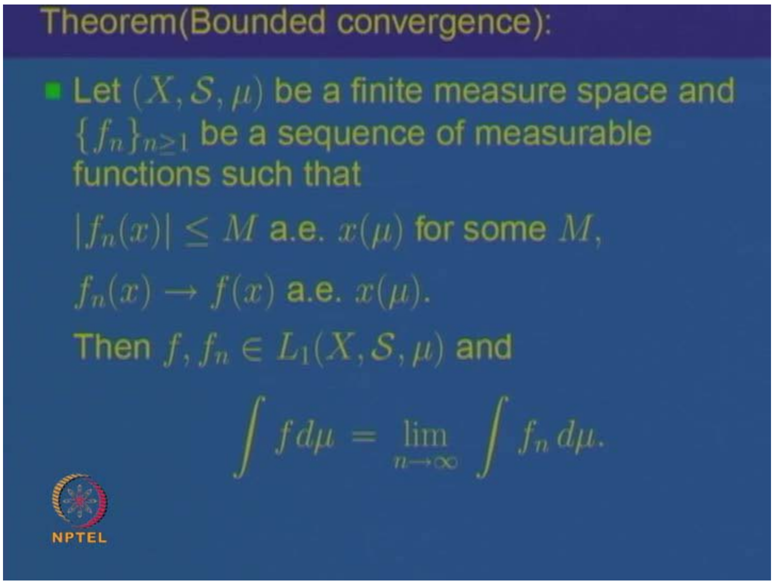

This we will refer to as the series version of dominated convergence theorem. There is another interpretation of the dominated convergence theorem, when the underlying measure space is a finite measure space, then one has that if X S mu is a finite measure space and f n is a sequence of measurable functions, such that all of them are dominated by a single constant M 'almost everywhere' and f n x converges to f of x then integral f ns converge to integral f.

This is a particular case of dominated convergence theorem when the underlying measure space is a finite measure space and the only thing to observe here is that because let us see how does this follow from our is a dominated convergence theorem.

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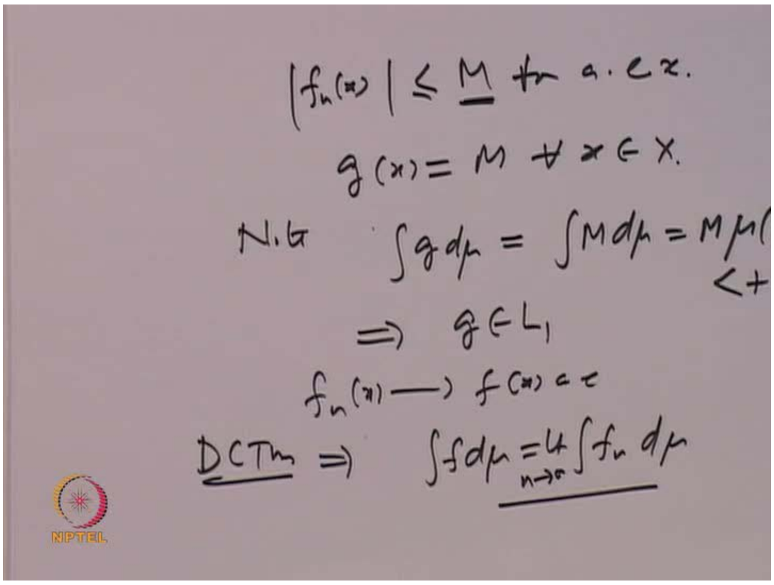

We are given that mod f n x is less than or equal to M for ‘almost everywhere’ x.

Now look at this constant function M. Look at the function g of x which is equal to M, for every x belonging to X. The constant function is measurable, note that g is a nonnegative measurable function because it is a constant function. Note, that its integral g d mu is equal to integral the constant function M d mu and that is equal to M times the measure of the whole space x which is finite. What we are saying is that on finite measure spaces a constant function is always integrable. This implies g is L 1.

So, f n x bounded by M- that is a constant function, that is an integrable function. Once we have that and f n x converges to f of x ‘almost everywhere’. So, now, dominated convergence theorem is applicable and that implies integral f d mu is equal to integral f n d mu limit n going to infinity.

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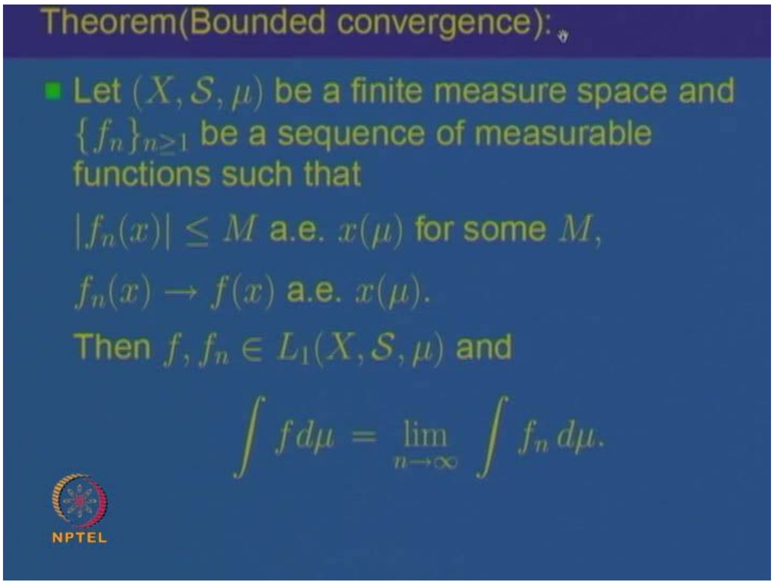

The main thing is on finite measure spaces, a constant function becomes integrable because of this reason. This is what is called bounded convergence theorem and it is quite useful when underlying measure space is a finite measure space.

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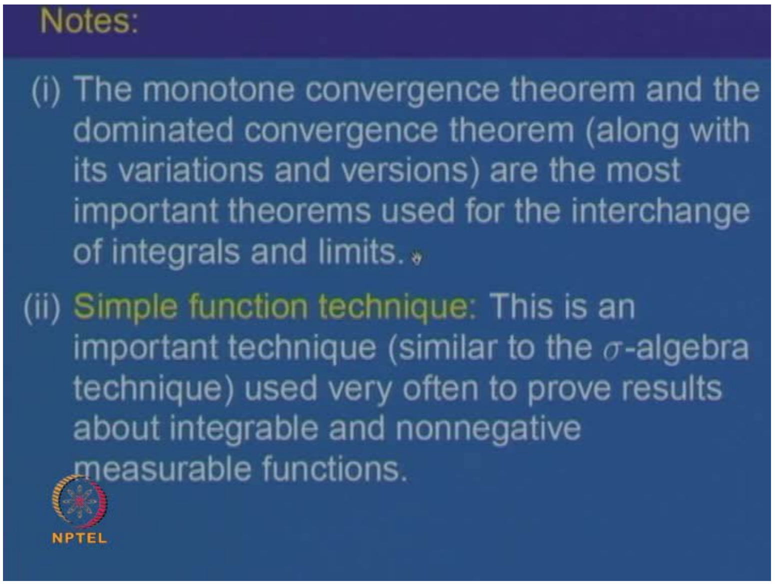

Let us look at what we have proved till now. We have looked at the space of integrable functions, proved linearity property and an important theorem called dominated convergence theorem.

If you recall, for nonnegative measurable functions we had two theorems. One was monotone convergence theorem; namely, that was a theorem when f n is a sequence of nonnegative measurable functions increasing to a function f. Then, integral of f is equal to limit of integral. That means interchange of limit and integration is possible by monotone convergence theorem whenever the sequence f n is monotonically increasing and a sequence of nonnegative measurable functions.

The second theorem, which involved sequences of measurable functions was again for nonnegative measurable functions and that was called Fatou's lemma.

There we do not emphasize, we do not require that the sequence f n be nonnegative and measurable. We only want the sequence f n to be a sequence of nonnegative measurable functions- they need not be increasing. For such a sequence we had that integral of the limit inferior of the sequence f n is less than or equal to limit inferior of the integrals of f n.

That was Fatou's lemma. Now we have the third theorem- dominated convergence theorem, which again helps you to interchange the notion of integral and the limiting operation under the condition that all the f ns are dominated by a single integrable function.

So, these are the three important theorems, which help us to interchange limit and the integral signs.

Let us at this stage emphasize one more point about this technique of integration. So basically, for integral, we started with simple functions and then we go about nonnegative functions and then we defined it for integrable functions.

This process of step-by-step defining the integral can be useful in proving many results and I call it - the simple function technique. This is a technique, which is used very often to prove some results about integrable functions and nonnegative measurable functions. What is the technique? Let me outline that and then I will give an illustration of this.

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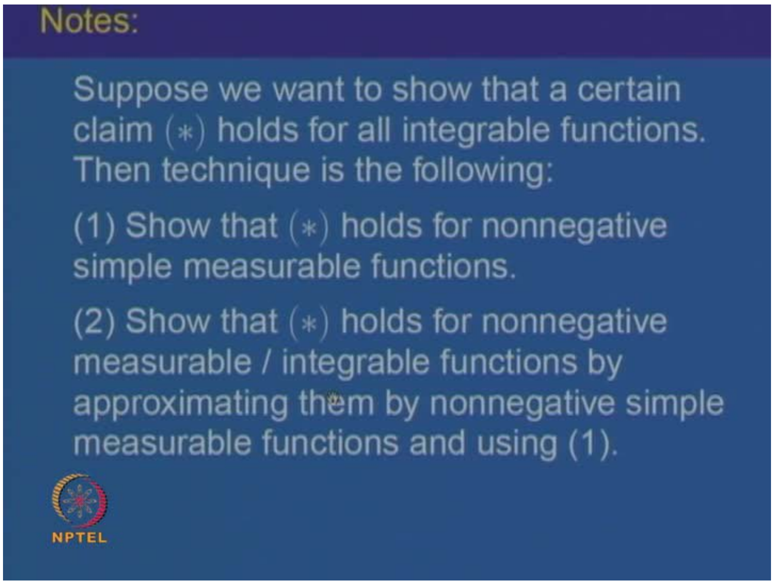

Suppose you want to show that a certain property - let us call that property, star - holds for all integrable functions.

To prove that the property holds for all integrable functions, the technique is as follows; basically, show that this property, star, holds for all nonnegative simple measurable functions.

If you want to show that a property holds for all integrable functions, first show that it holds for the class of nonnegative simple measurable functions. Next, show that star holds for nonnegative measurable integrable functions by using the fact that nonnegative measurable functions are limits of increasing limit of simple measurable functions.

There one normally uses the monotone convergence theorem. Using monotone convergence theorem, one extends the property star from simple measurable functions to nonnegative measurable functions or nonnegative integrable functions.

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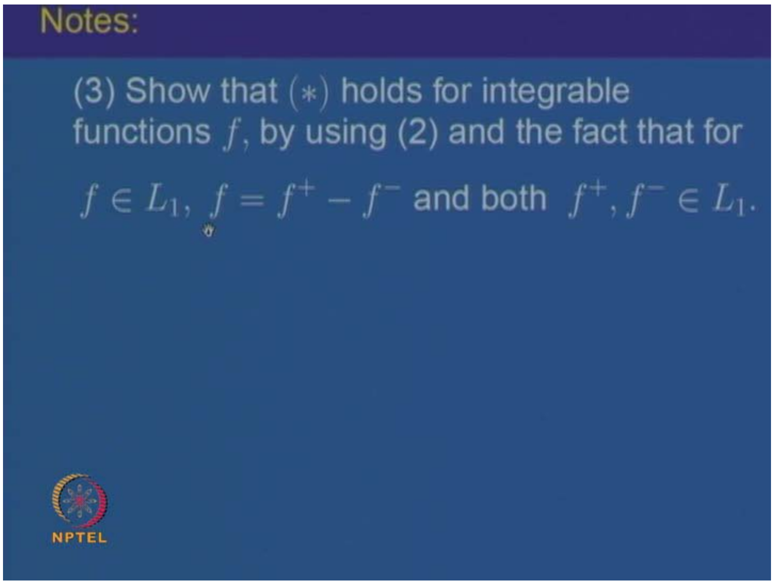

Then, keeping in mind that for a function f it can be split into positive part and negative part: f can be written as f plus minus f minus and if a property holds for nonnegative functions - about integrals - f for f plus that will hold for f minus hold, then conclude from there that it will hold for f also.

So, this is what I call as the simple function technique to prove results about integrable functions.

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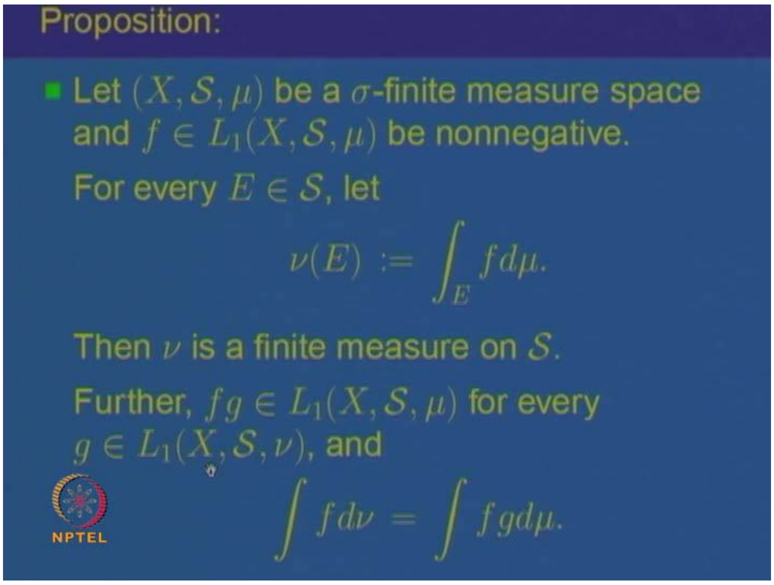

To give an illustration of this, let us look at the following result. Let us take a measurable space X S mu, which is sigma finite measure space and let us look at a function f which is integrable on this measure space and is nonnegative. So we have got a sigma finite measure space and f is a nonnegative integrable function on this measure space.

Let us define nu of E for every set in the sigma algebra. Let us define nu of E to be integral of f d mu over the set E - integral of f over the set E - is denoted by nu of E for every set E in the sigma algebra S.

We had already shown that this nu, the set function nu, is in fact a finite measure on S. We have already proved this. But what we want to prove now is that, if g is any integrable function on the measure space X S nu - this nu is the new measure. If g is integrable on X with respect to nu, then the product function f into g is integrable with respect to mu and this relation holds integral f d mu - so integral of f with respect to nu is equal to integral of f into g with respect to mu.

What we have done is, by fixing a function f, which is nonnegative, we had defined a new measure on the measurable space by nu of E to be equal to integral of E f d mu. And we are saying that if we want to integrate a function with respect to a function g, with respect to this new measure, then it is the same as integrating the function - the product function f g - with respect to the old measure mu.

So let us see how the simple function technique is used to prove this result.

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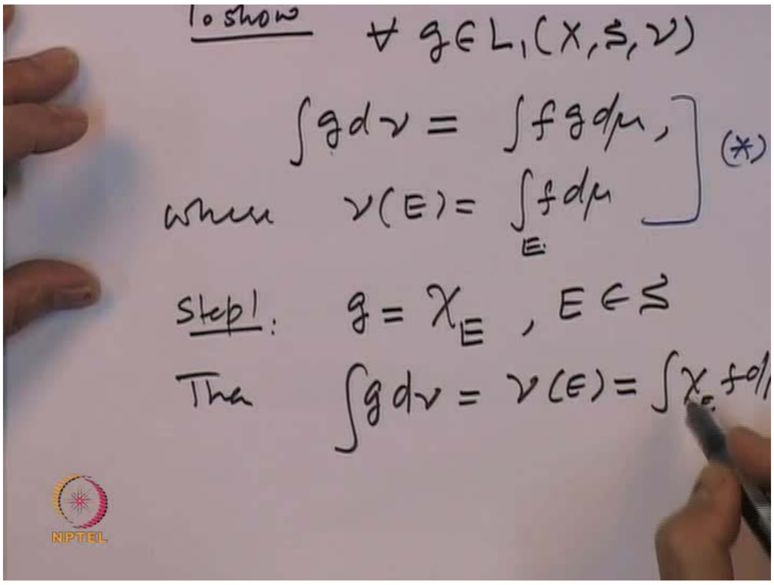

Let us start. We want to show that for every g belonging to L 1 of X S nu integral of g d nu can be represented as integral f g d mu. Recall, we defined nu of E to be equal to integral of f d mu over E. This is the property, star, we want to prove for every function g.

As we said, let us first check this property. Step 1 - let us take g is a L 1 function. Let us say g is a function which is an indicator function of E, let us take g as the indicator function of E - E belonging to S. In that case, the integral g d mu, the left hand side is nothing but nu of E because g - this is the indicator function - so this is integral of nu of E which by definition is equal to integral chi E of f d mu.

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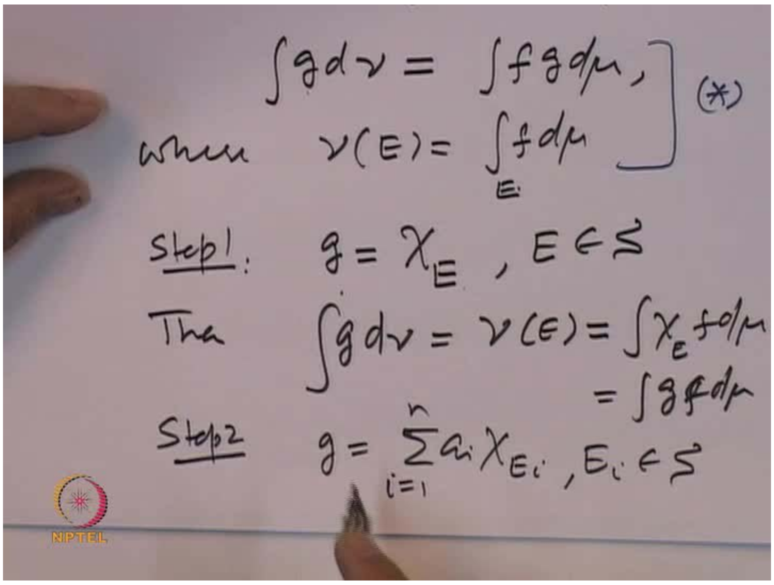

So chi E is g, this is equal to integral g f d mu. What does it say? It says that the required property, star, holds, when g is the indicator function. Now let us take a nonnegative simple function - that is step 2, let us take g sigma a i chi of E i, i equal to sum 1 to n, where ‘E i’s belong to S.

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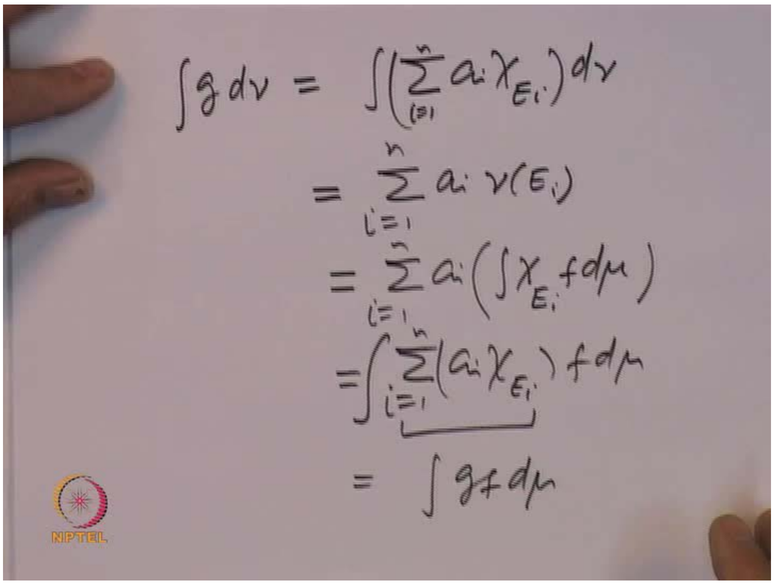

Our claim is that this property holds for this g also. We are saying that the next step is to verify the required property. Integral of g d nu, by definition, is equal to integral of sigma a i indicator function of E i d nu. What is that? By inheritive property of the integral, it is sigma i equal to 1 to n of a i - that is scalar times nu of E i. Because integral of the indicator function is the measure. That is equal to sigma i equal to 1 to n a i. And nu of a i by definition is integral chi E i of f d mu - that is the definition of mu of E i. Which I can again write as sigma i equal to 1 to n, you can take this a i out and again by the linearity property that is integral a i chi E i times f d mu. But, this is nothing but my function g, this is integral of g f d mu. What we are saying is that if g is summation a i chi E i, then using linearity property this is the same as integral - goes in - so that is a i integral of the indicator function of E i, that is nu of E i - nu of E i by definition is integral over E i of f d mu and again using linearity property of the integral I can shift it outside.

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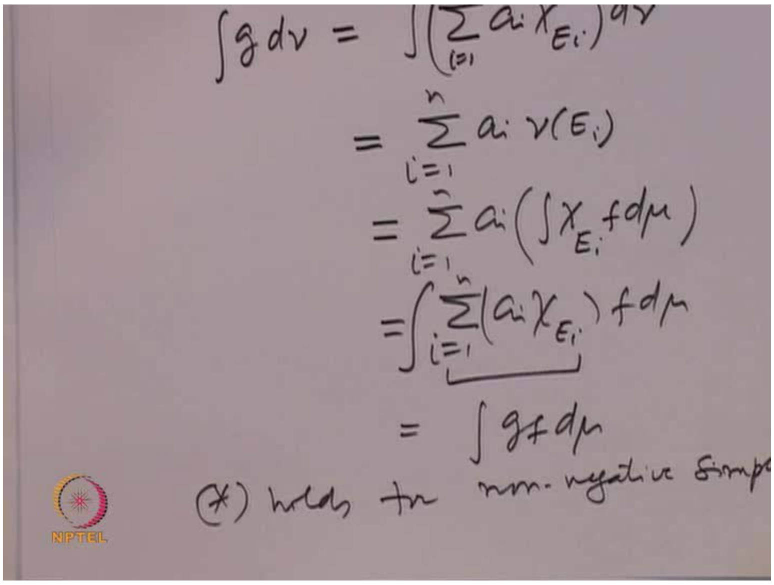

So, it is integral of summation a i chi E i times f d mu, which is g. The required property holds, so, star, holds for nonnegative simple functions g. That is what I said - a simple function technique. Now, let us try to prove that this property also holds when g is a nonnegative measurable function.

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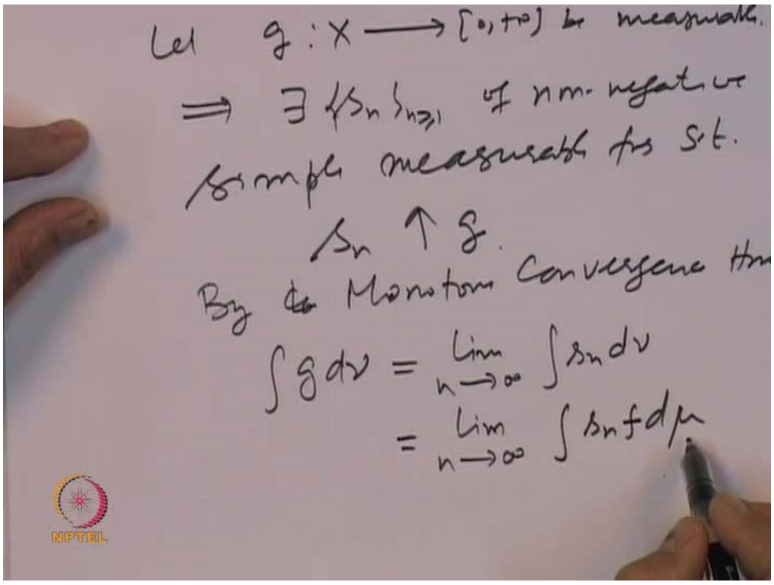

Now, let us look at g. Let g on x be measurable. Then we know by the property of measurable functions it implies that there exists a sequence s n of nonnegative simple measurable functions such that s n increases to the function g.

Then by Lebesgue’s, by monotone convergence theorem, integral of g d nu with respect to nu must be equal to limit n going to infinity integral s n d nu. But, for nonnegative simple functions - we just proved this - the star holds. That means this can be written as limit, so by step 2, I can write this as integral of s n times f d mu. With the integration of a nonnegative, simple measurable function with respect to nu can be converted into the nonnegative simple measurable function multiplied by f d mu.

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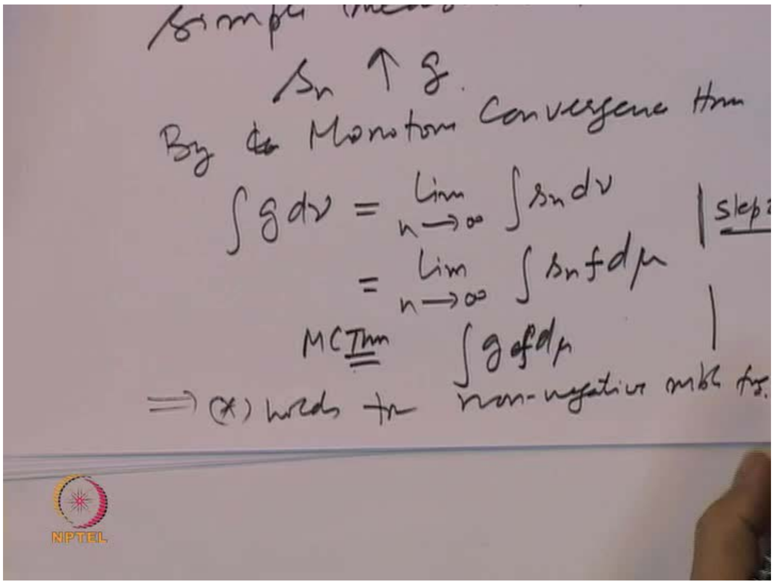

At this stage we observe that if s n is increasing to g then s n times f will be increasing to g times f. All are nonnegative simple measurable - all are nonnegative measurable functions. Once again, monotone convergence theorem is applicable and this limit is nothing but integral of g f d mu. Once again we have used this step was by our step 2 that the property holds for nonnegative simple functions integral with respect to nu is integral with respect to mu of product function. Now, once again we are applying monotone convergence theorem.

First, integral of g is equal to limit of integral s ns d nu by monotone convergence theorem. Now, by our earlier step, this is equal to integral of s n f d mu. Again, by monotone convergence theorem it goes back. That means - this implies - that star holds for nonnegative measurable functions.

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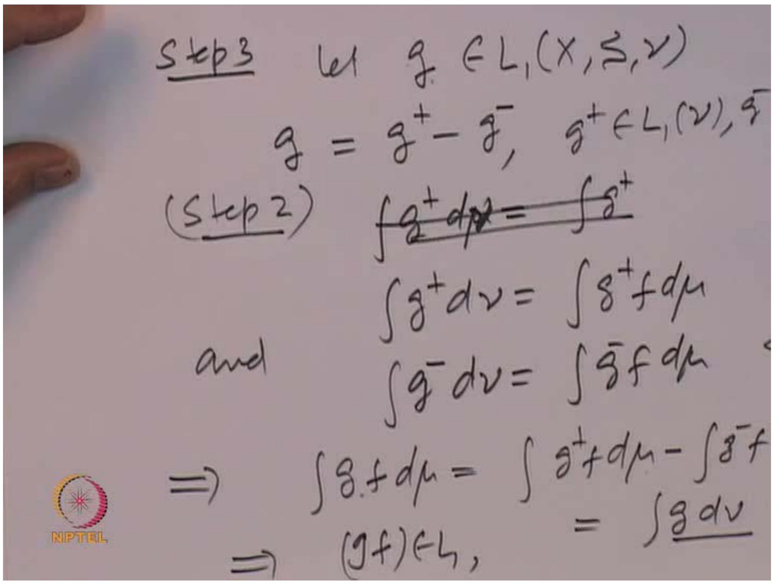

Now, let us come to the last part, namely, final step 3. Let g belong to L 1 X S nu - g be a integrable function.

Then what is g equal to? It is, g plus minus g minus - where g plus is a nonnegative measurable function, g minus is a nonnegative measurable function. By step 2 we know that integral g plus d mu is equal to integral of g plus - sorry - d nu - let me write it again - integral g plus d nu is equal to integral g plus of f d mu and integral of g minus d nu is equal to integral g minus f d mu, that is by step 2. Now, because g is L 1, that implies g is equal to g plus g minus. So g plus is in L 1 of nu and g minus also belongs to L 1 of nu.

A function g is integrable if, and only if, its positive part and negative parts are integrable; that means these quantities - they are all finite.

These are all finite quantities. That means what? And f is nonnegative - that implies integral of g f d mu is equal to integral of g plus f d mu minus integral g minus f d mu. By definition of the positive part and the negative part of the function g f - f is nonnegative - the positive part of the function g f is same as g plus times f and the negative part is nothing but g minus f - both of these are finite quantities. That implies that g f is L 1 and by these two, this is the same as integral of g d nu.

For step 3 - for a g which is integrable - we have deduced that this property is true. This is step 3. This is what I call the simple function technique. Let me go back and show you once again what we have done.

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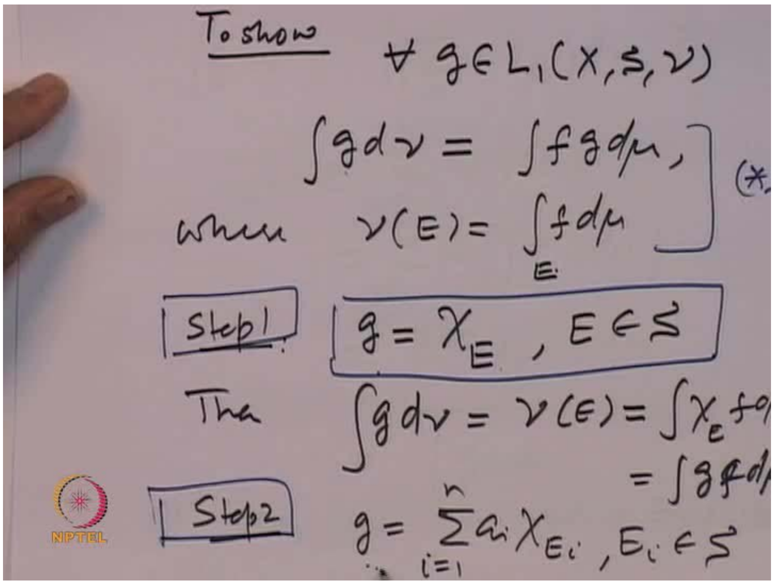

We wanted to show that – this is property star - we wanted to show for every function g, which is L 1. This is my step 1, that - look at the functions g which are indicator functions - I want to verify this for the indicator function g to be the indicator function. When g is the indicator function, this left hand side is integral of over E of the constant function 1. This is equal to integral d nu is [int/integral] nu of E, which by definition is integral f over E, which I can write as integral f E. So, that is true.

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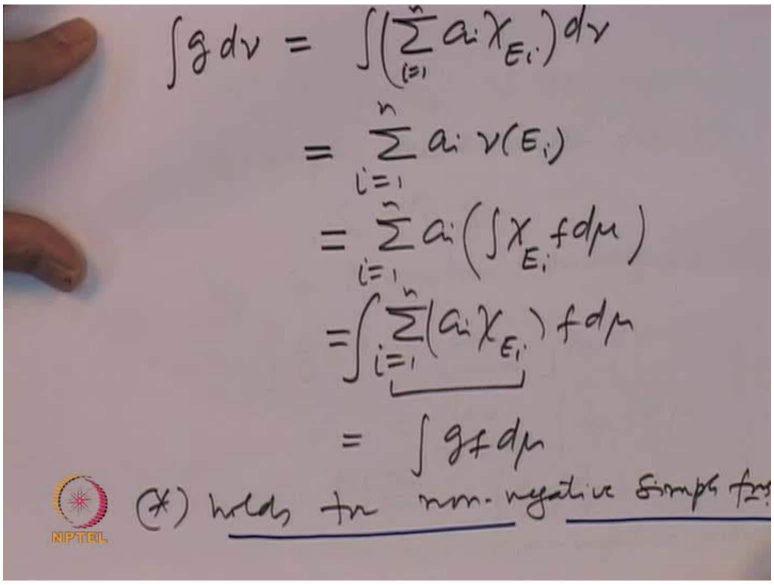

Step 1 is to verify the required thing holds for characteristic function. And Step 2 - by using the property that the integral is linear, we show that it is true for every nonnegative simple functions. Take g - a nonnegative simple measurable function and apply. So, g is equal to integral of nonnegative a i indicator function E i and interchange and show that required property holds. Step 2 was that the required property holds for nonnegative simple measurable functions.

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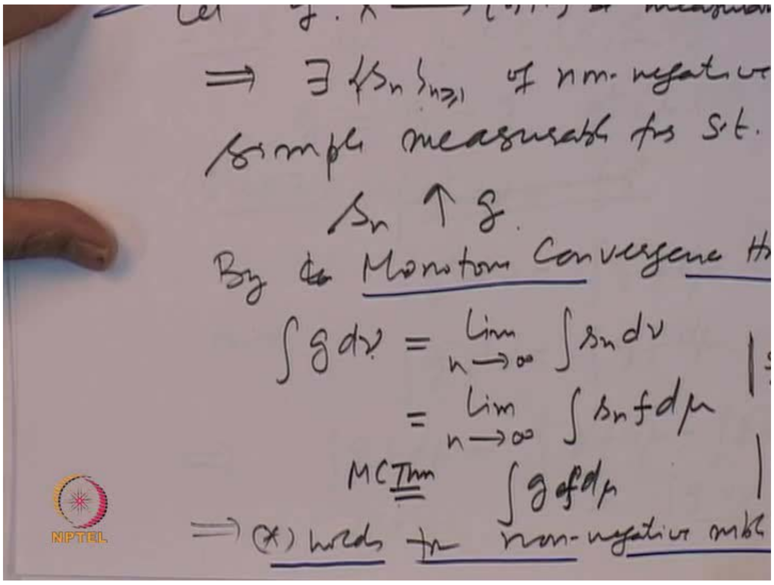

Then, using an application of monotone convergence theorem - that is step 3, that, if g is a nonnegative measurable function, then we know that it is a limit of nonnegative simple measurable functions increasing limit. So, an application of monotone convergence theorem together with the earlier step gives us that integral of g d nu is equal to integral of g f d mu.

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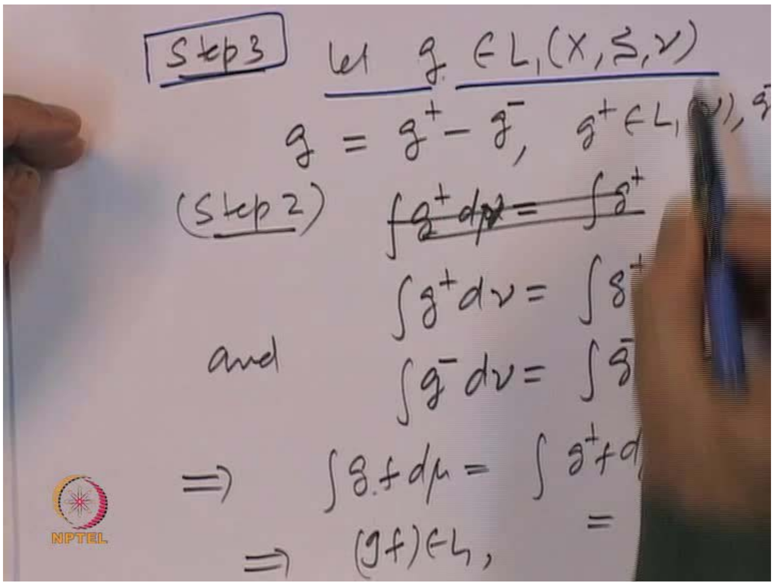

That is the next step - to show that it holds for nonnegative measurable functions. Once that is done, the final step - that it holds for all integrable functions, is via splitting the function g into the positive part minus the negative part. And g integrable means both are integrable and for each one of them the required claim star, holds. So by putting them together we get that the required claim holds - property star, holds - for all functions g, which are L 1. This is what I normally call as the simple function technique.

While proving results about integrable functions, one quite often uses the simple function technique and while proving some properties about subsets of sets, recall, we had the sigma algebra monotone class theorem technique.

For proving properties about sets, one uses monotone class - sigma algebra monotone class - technique and for proving results about integrals one normally uses what is called the simple function technique.

With this, we have defined and proved general properties about integral of functions on sigma finite measure spaces.

Now - we will try - we will specialize this property, this construction, when x is real line.

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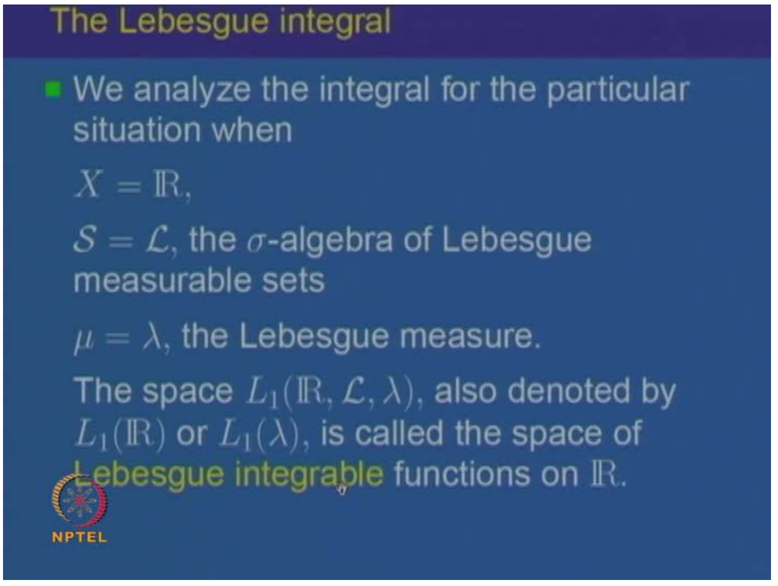

We want to specialize this thing for the real line - let us see what we get. You will be looking at the special case when X is real line; the sigma algebra is L - that of Lebesgue measurable sets and the measure mu will be the lambda - the Lebesgue measure.

So, we will be working with the measure space X S mu, which is the same as real line Lebesgue measurable sets and Lebesgue measure.

The space of all integrable functions on this measure space - R L and lambda, is called the space of all Lebesgue integrable functions and is also denoted by L 1 of R or L 1 of lambda.

This is the space of all Lebesgue integrable functions. We want to study this space of Lebesgue integrable functions in some more detail.

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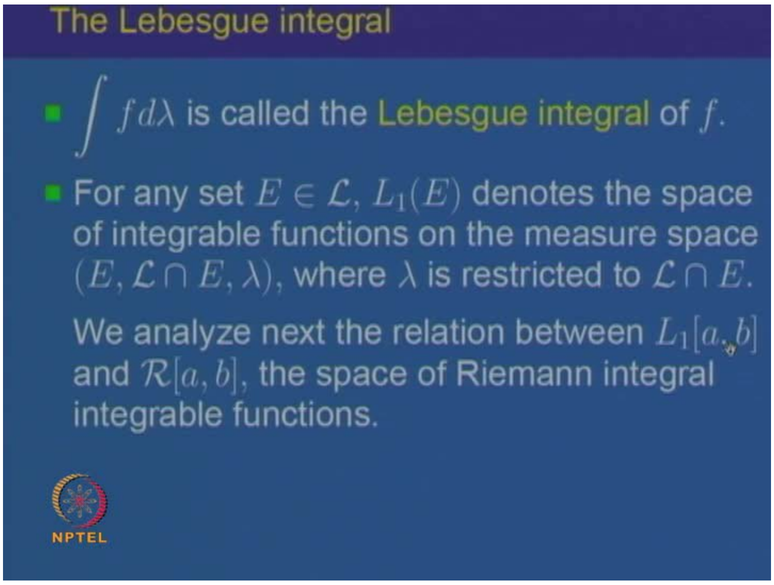

Let us first agree to call integral f d lambda to be the Lebesgue integral of the function f. So, whenever f is integrable or nonnegative integral, f d lambda will be called the Lebesgue integral of f.

Sometimes, we have to look at functions which are defined on subsets of E. So, for any subset E which is Lebesgue measurable, L 1 of E will denote the space of all integrable functions on the measure space E - so the underlying set is E.

L intersection E is the collection of all Lebesgue measurable sets inside E, and lambda is the Lebesgue measure restricted to subsets of L intersection, the sigma algebra L intersection, E. Of particular interest for the time being, is going to be the set: when E is a close bounded interval a b.

We will start looking at the space L 1 of a b. That is the space of all Lebesgue integrable functions defined on the interval, close bounded interval, a b and we also have the space R a b, namely, the space of all Riemann integrable functions on a b.

So, we want to compare these two spaces. On one end we have got the space of Lebesgue integrable functions on a b, on the other hand we have got the space of Riemann integrable functions on a b; we want to see the relation or establish a relationship between the two. That was one of the starting points for our discussion of the subject, namely, the space of Riemann integrable functions had some difficulties, some problems, some drawbacks, for which we want you to extend the notion to a larger class - this is the larger class L 1 of a b.

What we are going to show is: R a b, the space of all Riemann integrable functions is a subset of L 1 of a b, and the notion of Riemann integral is the same as the notion of Lebesgue integral for Riemann integrable functions.

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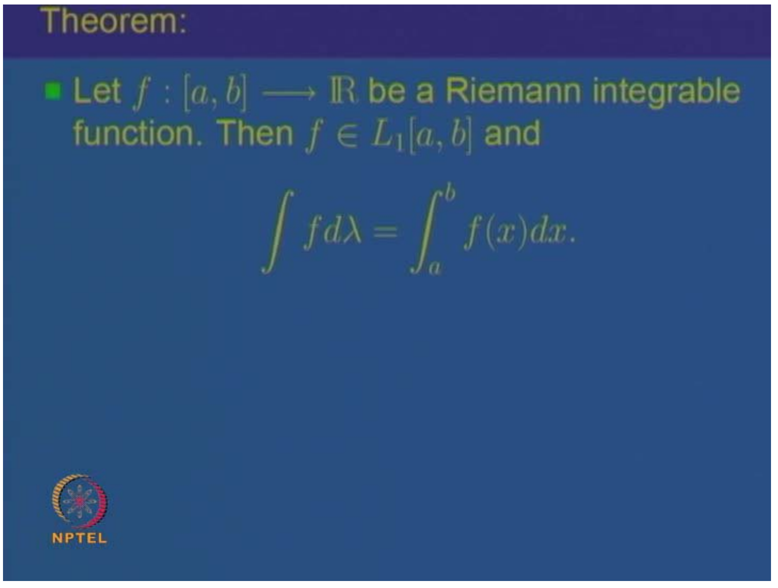

That is called the relation between the Riemann integral and the Lebesgue integral. To be more specific, we want to prove the following theorem: namely, if f is defined on a close bounded interval a b is Riemann integrable function then f is also Lebesgue integrable. And the Lebesgue integral is the same as the Riemann integrable of the function f; this is what we wanted to prove.

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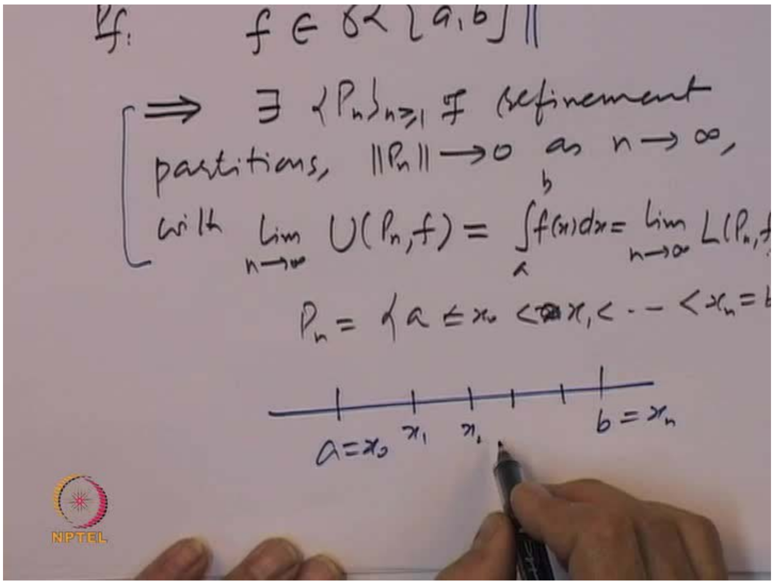

So, let us start looking at how we prove this. The proof of the theorem - we are given that the function f belongs to R a b. It is a Riemann integral function. Let us recall how the Riemann integral of a function is defined - it is defined via limits of upper sums and lower sums of partitions.

It implies that, there exists a sequence P n of refinement partitions with norm of P n going to 0 as n goes to infinity - partitions n going to infinity. With the upper sums of P ns with respect to f, limit of that is same as the Riemann integral of f, is the same as the limit of the lower sums L P n of f.

That is the meaning of saying that a function f is Riemann integrable. We can find that Riemann integrable implies, that there exists a sequence of partitions P n - which are refinement partitions. Refinement means P n plus 1 is obtained from P n by adding one more point. And norm of these partitions - the maximum length of the subintervals - goes to 0. And, integrablity means that the upper sums and the lower sums both converge to the same value and that is the Riemann integral of the function f.

This is the property of saying that f is Riemann integrable. Now, from here, let us look at what is U P n f upper sum. Let us write down the partition P n as something. Let us say, P n looks like 'a' so interval is a to b so, 'a' the point x naught less than x 1 less than x n which is equal to b.

Let us say that is the partition P n. In the picture it will look like - here is 'a' here is 'b' this is x 0, this is x n, and here is x 1, x 2 and so on.

To construct the upper sums - what one does to construct the upper sums? One looks at the maximum value of the function in this interval, and the minimum values in this interval.

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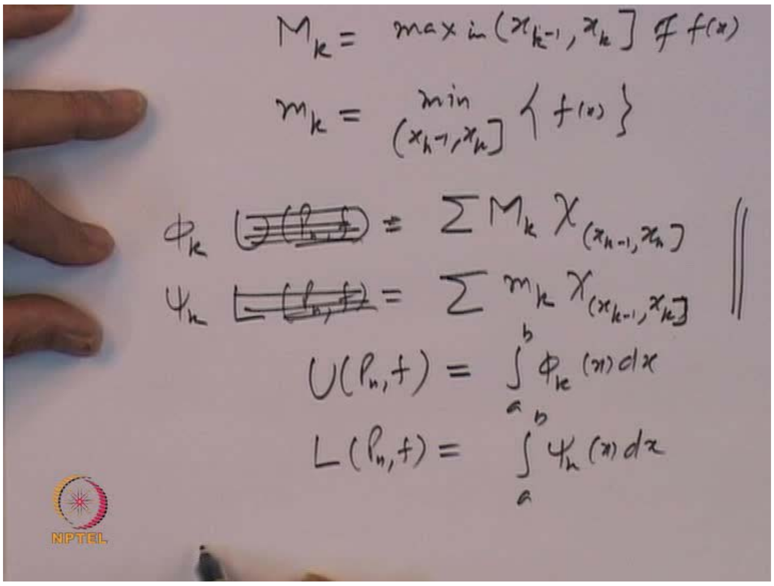

Let us write M k to be the maximum value of the function in the interval x, say x i minus. Let us write x k minus 1 to x k.

I am just trying to make the intervals disjoint maximum in this interval of maximum in this interval maximum of maximum of f of x maximum in of f of x. Similarly, M k, let us write - it is the minimum in the interval x k minus 1 to x k of f of x. Only at the end points do you have to make it closed, but that is not going to matter much.

Then, we define what is U P n f. That essentially looks like, summation of the maximum value into the indicator function of that subinterval. The lower sum with respect to P n f looks like summation small m k - the minimum value of the function in that subinterval, x k minus 1 and x k.

Let us do one thing. This is not the upper sum, let us call this - when in the interval x k minus 1 to x k the value is capital M k, let us call that as the function phi k and when you are taking the minimum value in that interval and summing up let us call that as psi k.

These are functions because they are linear combinations of indicator functions and the upper sums and lower sums are nothing but - the upper sum P n f is nothing but - Riemann integral a to b of phi k x d x, and the lower sum P n f is equal to - the integral of - Riemann integral of this function psi k of x d x. These functions phi k and psi k, which are linear combinations of indicator functions are in fact nonnegative measurable functions on the measure space a b - the interval a b. That is the observation that we should note.

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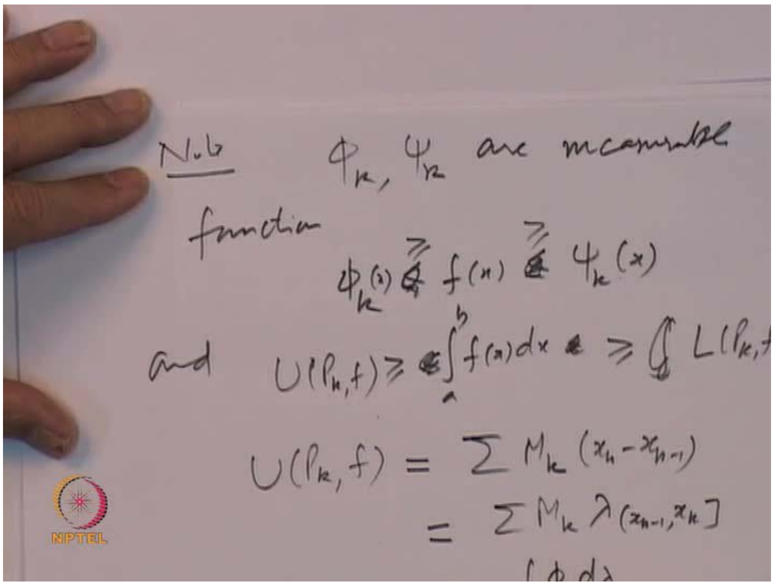

Let us note down, phi k and psi k are - measurable functions, non negative, sorry, - simple measurable functions. Say, that phi k is less than or equal to - at every point x is less than or equal to - f of x, is less than or equal to psi k of x. As far as the integral is concerned the integral a to b of f x d x is between the upper sum and the lower sum. That is, the phi k was maximum, so this should be bigger than or equal to like this - because phi k is taken as the supremum. This is the upper sum P k of f and that is bigger than or equal to the - upper sum, sorry - lower sum with respect to the P k of f and in the limit both of them are converging.

Here is the second observation: that the upper sum with respect to the partition of f is the same as - so what was it? - That was equal to sigma M k into the length of the interval x k minus x k minus 1, that is the upper sum - that is also the Riemann integral. In fact, this is also equal to length - so this is the length - so you can write this as a length.

M k times the length of x k minus 1 and x k, which is same as the Lebesgue integral of the function phi k d lambda.

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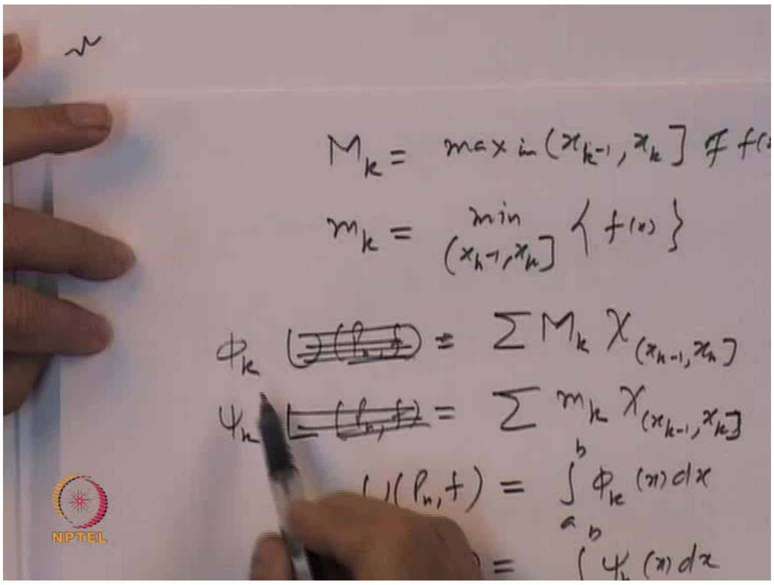

So, this is the important observation that we should keep in mind that the building blocks for Riemann integral, which are these step functions, are also Lebesgue integrable and the Riemann integral of the step functions phi k and psi k are same as the Lebesgue integrals of phi k and psi k.

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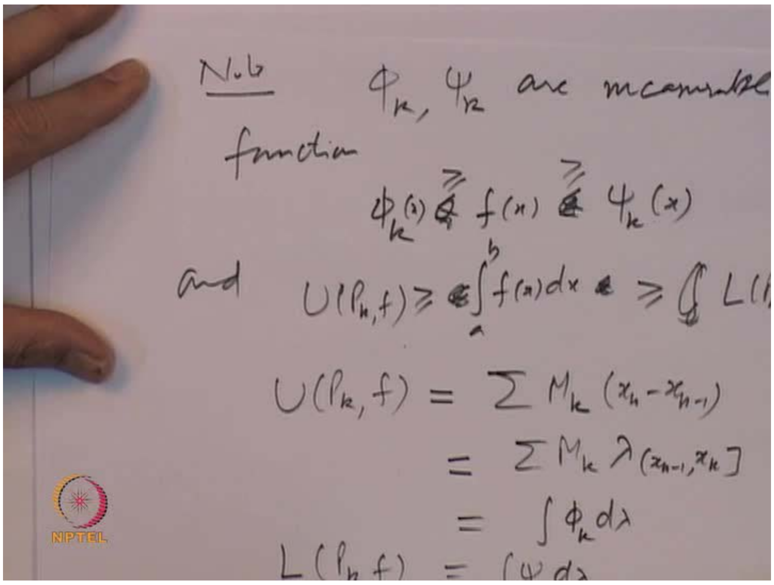

Similarly, the lower sum P k f is equal to integral of psi k d lambda. Now, essentially, the idea is to put them together, because phi k and psi k - they are between these two.

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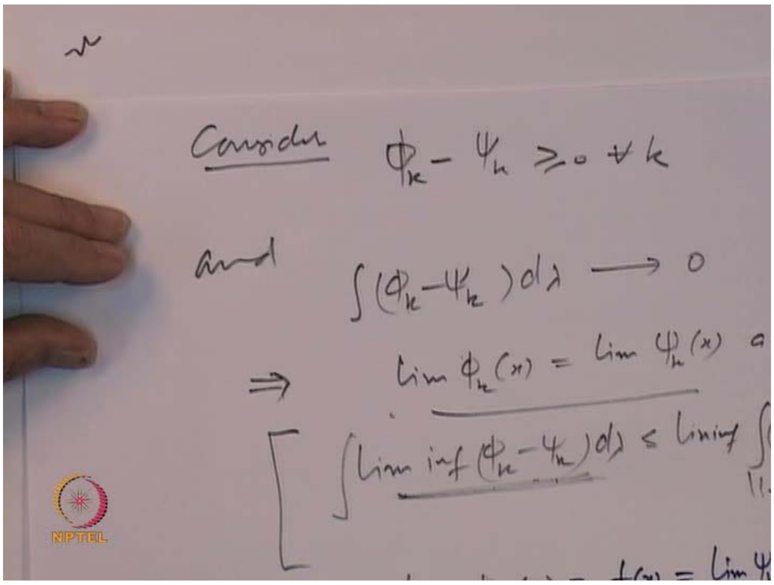

Let us look at integral of - look at the sequence. Consider the sequence - psi k minus - phi k minus psi k. Recall phi k is bigger than f x less than psi k, so phi k minus psi k is nonnegative for every k. Saying that the upper sums and lower sums converge to the same value is saying that the integral of phi k minus psi k d lambda, that goes to 0. So, that goes to 0 because the phi k d lambda is the upper sum, this is the lower sum and that goes to 0.

So that implies that limit so that means this implies that the limiting function f is trapped in between. That means limit phi k x is equal to limit psi k x 'almost everywhere'. Why is that? That we can deduce from the fact that applying Fatou's lemma. To deduce this look at the limit inferior of phi k minus psi k integral d lambda will be less than or equal to limit inferior of integral phi k minus psi k and that is 0 - so this is 0 - so this says that integral of a nonnegative function is 0, so the function must be 0 'almost everywhere' and that is the same as saying this must be 0 'almost everywhere'.

And f is trapped in-between. That implies that limit phi k x - limit of phi k x - is equal to f of x is equal to limit psi k x for 'almost everywhere' x.

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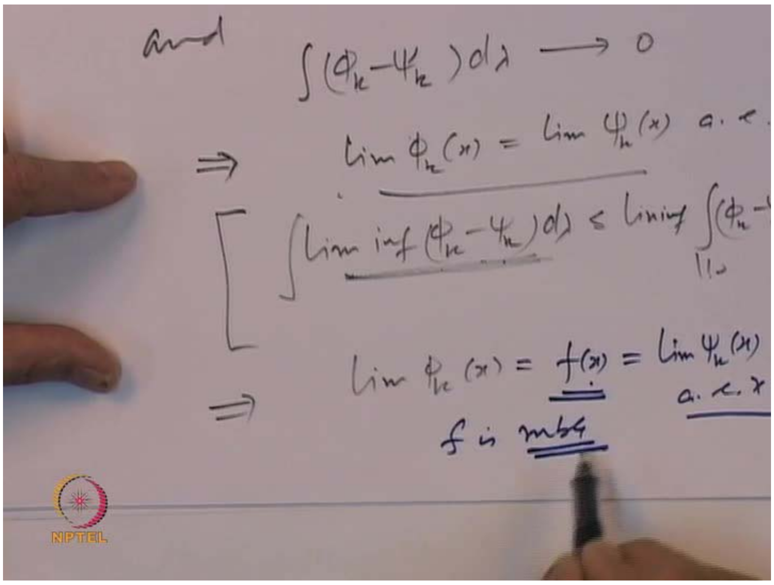

So, that proves - is equal to - so we are falling short of time - that means that the function f is measurable.

So we will continue the proof of this tomorrow, in the next lecture. Our aim is to prove that the space of Riemann integrable functions is inside the space of Lebesgue integrable functions and the Riemann integral is the same as the Lebesgue integral.

We will continue the proof in the next lecture.

Thank you.